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Reflexivity, the dual Radon-Nikodym property, and continuity of adjoint semigroups

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In this note for certain Banach spaces we give characterizations of reflexivity and the dual Radon-Nikodym property in terms of continuity of adjoint semigroups. Some applications outside the realm of semigroup theory are given.
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0. INTRODUCTION

Let $T(t)$ be a C_0-semigroup on a Banach space X. It is well-known that the adjoint semigroup $T^*(t)=(T(t))^*$ need not be strongly continuous on X^*. However, if X is reflexive, it is; this is a theorem of R.S. Phillips [14]. In this note we will prove the following converse.

THEOREM A. *Let X be a Banach space with a Schauder basis. The following statements are equivalent:*

(1) *X is reflexive;*

(2) *For every C_0-semigroup $T(t)$ on X, the adjoint semigroup $T^*(t)$ is strongly continuous;*

(3) *For every C_0-semigroup $T(t)$ of X, the second adjoint semigroup $T^{\odot *}(t)$ is strongly continuous.*

The definition of $T^{\odot *}(t)$ is given below. The idea of this theorem consists in showing that every Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$ (or more generally, with a Schauder decomposition) admits C_0-semigroups $T(t)$ with the property that $T^*(t)$ is strongly continuous if and only if $\{x_n\}_{n=1}^{\infty}$ is shrinking.

It follows from the proof of Theorem A that Grothendieck spaces with the Dunford-Pettis property cannot have a Schauder decomposition. This was first observed by D.W. Dean [3]; see also [12].

The Radon-Nikodym property is in many ways a close analogue of reflexivity. Here we will show that a weak*-continuous semigroup on a dual Banach space with the Radon-Nikodym property is strongly continuous for $t>0$. In this setting it turns out to be useful to consider Banach spaces with an unconditional basis, since on them C_0-semigroups can be constructed in a canonical way such that, when X^* is nonseparable, the adjoint semigroup fails to be strongly continuous even for $t>0$. These observations, together with the fact that separable duals have the Radon-Nikodym property, indicate what ideas lie behind the following theorem.

THEOREM B. *Let X be a Banach space with an unconditional basis $\{x_n\}_{n=1}^{\infty}$. The following statements are equivalent*:

(1) *X^* has the Radon-Nikodym property*;

(2) *Every adjoint semigroup on X^* is strongly continuous for $t>0$.*

In fact, if $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis for X, we will show that (1)-(2) hold if and only if X^* is separable if and only if $\{x_n\}_{n=1}^{\infty}$ is shrinking, which by a theorem of R.C. James (see [11]) is the case if and only if X does not contain a subspace isomorphic to l^1. More generally, H.P. Lotz proved that for Banach lattices X, X^* has the Radon-Nikodym property if and only if X does not contain a subspace isomorphic to l^1; see [7].

This note is organized as follows. In section 1 we will give some definitions and standard results which will be used afterwards. After that, sections 2 and 3 are concerned with Theorems A and B, respectively. In section 4 our results are applied to bases in c_0.

1. PRELIMINARIES

A one-parameter family $\{T(t)\}_{t\geq 0}$ (briefly, $T(t)$) of bounded linear mappings from a Banach space X into itself is called a *semigroup* if the following two conditions are satisfied:

(1) $T(0)=I$ (I the identity map of X);

(2) $T(t)T(s)=T(t+s)$ for all $t,s\geq 0$.

A *strongly continuous* semigroup (also called a C_0-semigroup) is a semigroup that satisfies

(3) $\lim_{t\downarrow 0}\|T(t)x-x\|=0$ for all $x\in X$.

The *generator* A of a C_0-semigroup $T(t)$ is defined by

$$D(A)=\left\{x\in X:\lim_{t\downarrow 0}\frac{1}{t}(T(t)x-x)\text{ exists}\right\};$$

$$Ax = \lim_{t \downarrow 0} \frac{1}{t}(T(t)x - x) \quad (x \in D(A)).$$

A C_0-semigroup is called *compact* if for every $t>0$ the operator $T(t)$ is compact.

A semigroup $T^*(t)$ on a dual space X^* is called an *adjoint* semigroup if there is a C_0-semigroup $T(t)$ on X such that $(T(t))^* = T^*(t)$ for all $t \ge 0$. An adjoint semigroup need not be strongly continuous. Therefore it makes sense to define

$$X^{\odot} = \{x^* \in X^* : \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0\}.$$

Of course, $X^{\odot}$ depends on the particular semigroup under consideration. It is easy to see that $X^{\odot}$ is invariant under $T^*(t)$; hence the restriction $T^{\odot}(t)$ of $T^*(t)$ to $X^{\odot}$ defines a C_0-semigroup on $X^{\odot}$; its adjoint on $X^{\odot *}$ will be denoted $T^{\odot *}(t)$.

We will need the following properties of C_0-semigroups and their adjoints [2, 9, 17].

PROPOSITION 1.1. *Let $T(t)$ be a C_0-semigroup on a Banach space X.*

(1) *There exist real constants $M \ge 1$ and ω such that $\|T(t)\| \le Me^{\omega t}$.*

(2) *The adjoint semigroup $T^*(t) = (T(t))^*$ is weak*-continuous, that is,*

$$\lim_{t \downarrow 0} \langle T^*(t)x^* - x^*, x\rangle = 0$$

for all $x \in X$.

(3) *$X^{\odot}$ is a norm-closed, weak*-dense subspace of X^*.*

PROPOSITION 1.2. *Let $T(t)$ be a semigroup on a Banach space X.*

(1) *If the map $t \to T(t)x$ is measurable for all $x \in X$ then $T(t)$ is strongly continuous for $t>0$.*

(2) *If $T(t)$ is weakly continuous* (*that is,* $\lim_{t \downarrow 0}\langle x^*, T(t)x - x\rangle = 0$ *for all* $x^* \in X^*$) *then $T(t)$ is strongly continuous.*

A countable collection of closed subspaces $\{X_n\}_{n=1}^{\infty}$ of a Banach space X is called a *Schauder decomposition* of X if for every $x \in X$ there is a unique sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x = \sum_{n=1}^{\infty} x_n$ and for each $n, x_n \in X_n$. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called a *Schauder basis* (briefly, basis) if for every $x \in X$ there exists a unique sequence $\{\alpha_n\}_{n=1}^{\infty}$ of scalars such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$. A basis $\{x_n\}_{n=1}^{\infty}$ is called *normalized* if $\|x_n\| = 1$ for all n. It is well-known that the *coordinate functionals* x_N^* defined by $\langle x_N^*, \sum_{n=1}^{\infty} \alpha_n x_n\rangle = \alpha_N$ are continuous. From this it is easy to see that the maps π_N and P_N defined by

$$\pi_N \sum_{n=1}^{\infty} \alpha_n x_n = \sum_{n=1}^{N} \alpha_n x_n, \qquad P_N \sum_{n=1}^{\infty} \alpha_n x_n = \alpha_n x_n$$

are projections and $C = \sup_N \|\pi_N\| < \infty$. Hence if $\{x_n\}_{n=1}^{\infty}$ is normalized, then $\|x_n^*\| \le 2C$ for all $n = 1, 2, \ldots$. The constant C is called the *basis constant* of

$\{x_n\}_{n=1}^{\infty}$. Analogous definitions exist for Schauder decompositions. For instance, define π_N by $\pi_N \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{N} x_n$. In this case the constant C will be called the *decomposition constant*.

A basis $\{x_n\}_{n=1}^{\infty}$ is called *shrinking* if the coordinate functionals $\{x_n^*\}_{n=1}^{\infty}$ form a basis of X^*. This is the case if and only if $\lim_{N\to\infty} \|x^* |_{[x_N, x_{N+1}, \ldots]}\| = 0$ for every $x^* \in X^*$. Here $x^* |_{[x_N, x_{N+1}, \ldots]}$ denotes the restriction of x^* to the closed linear span $[x_N, x_{N+1}, \ldots]$ of $\{x_n\}_{n=N}^{\infty}$.

$\{x_n\}_{n=1}^{\infty}$ is called *boundedly complete* if the following holds: whenever the sequence $\{\|\sum_{n=1}^{N} \alpha_n x_n\|\}_{N=1}^{\infty}$ is bounded, then $\sum_{n=1}^{N} \alpha_n x_n$ actually converges to some $x \in X$ as $N \to \infty$.

$\{x_n\}_{n=1}^{\infty}$ is called *unconditional* if for every $x \in X$ the expansion $\sum_{n=1}^{\infty} \alpha_n x_n$ of x converges unconditionally, that is, for every permutation σ of the positive integers, $\sum_{n=1}^{\infty} \alpha_{\sigma(n)} x_{\sigma(n)}$ converges.

As an example, note that the standard unit vector basis of c_0 is unconditional and shrinking but not boundedly complete.

PROPOSITION 1.3. *Let $\{x_n\}_{n=1}^{\infty}$ be a basis of a Banach space X.*

(1) *$\{x_n\}_{n=1}^{\infty}$ is boundedly completely if and only if $\{x_n^*\}_{n=1}^{\infty}$ is a shrinking basis for its closed linear span $[x_n^*]$;*

(2) (*M. Zippin* [18]) *A Banach space X with a basis is reflexive if and only if every basis of X is shrinking if and only if every basis of X is boundedly complete;*

(3) *If $\{x_n\}_{n=1}^{\infty}$ is unconditional, then there is a constant $K > 0$ such that for every $t \in l^{\infty}$ and $x = \sum_{n=1}^{\infty} \alpha_n x_n \in X$,*

$$\|\sum_{n=1}^{\infty} t_n \alpha_n x_n\| \leq K(\sup_n |t_n|) \|\sum_{n=1}^{\infty} \alpha_n x_n\|.$$

Proofs may be found in [11] and [16].

A Banach space X is called a *Grothendieck space* if weak*-sequential convergence and weak sequential convergence in X^* coincide. Every reflexive space is trivially Grothendieck. It follows from Theorem A combined with Prop. 1.1 (2) and 1.2 (2) that Grothendieck spaces with a Schauder basis are reflexive. More generally, W.B. Johnson [10] proved that Grothendieck spaces with a Markusevich basis are reflexive, hence in particular separable Grothendieck spaces are reflexive.

A Banach space is said to have the *Dunford-Pettis property* if the following holds: whenever $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ are sequences in X and X^* respectively, such that $x_n \to 0$ weakly and $x_n^* \to 0$ weakly, then $\langle x_n^*, x_n \rangle \to 0$.

PROPOSITION 1.4. (*H.P. Lotz* [12]) *Every C_0-semigroup on a Grothendieck space with the Dunford-Pettis property has a bounded generator.*

Let (Ω, Σ, μ) be a finite measure space. A Banach space X is said to have the *Radon-Nikodym property with respect to* (Ω, Σ, μ) if for every μ-continuous vector-valued measure $G: \Sigma \to X$ of bounded variation there exists $g \in L^1(\mu; X)$ such that

$$G(E) = \int_E g d\mu$$

for all $E \in \Sigma$. X has the *Radon-Nikodym property* if it has the Radon-Nikodym property with respect to every finite measure space.

A bounded linear operator $S: L^1[0,1] \to X$ is called *Riesz-representable* if there exists a $g \in L^\infty([0,1]; X)$ such that

$$Sf = \int_0^1 f(t)g(t)dt \quad \text{for all } f \in L^1[0,1].$$

We will need the following result [4, Thm III.1.5; Cor. V.3.8].

PROPOSITION 1.5. *X has the Radon-Nikodym property if and only if each bounded linear operator $S: L^1[0,1] \to X$ is Riesz-representable.*

2. REFLEXIVITY AND SCHAUDER DECOMPOSITIONS

The main result of this section is Theorem 2.2 below. It asserts that in a Banach space with a Schauder decomposition there exist C_0-semigroups with properties reflecting those of the decomposition in terms of which they are defined. Their construction is based on the following lemma, which is in [16, Thm. II.15.4].

LEMMA 2.1. *Let X be a Banach space with a Schauder decomposition $\{X_n\}_{n=1}^\infty$ with decomposition constant C. Let (γ_n) be a sequence of scalars such that*

$$\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| < \infty.$$

Put $\gamma = \lim_n |\gamma_n|$. Then for all $x = \sum_{n=1}^\infty x_n \in X$ we have

$$\left\| \sum_{n=1}^\infty \gamma_n x_n \right\| \le C \cdot \|x\| \cdot \left(\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| + \gamma \right).$$

Let P_N be the canonical projection defined in section 1 and let $[P_n^* X^*]$ denote the closed linear span of the spaces $P_n^* X^*: n = 1, 2, \dots$.

THEOREM 2.2. *Let X be a Banach space with a Schauder decomposition $\{X_n\}_{n=1}^\infty$ with decomposition constant C. Let $0 \le k_1 < k_2 < \cdots \to \infty$ be any sequence of numbers. Then*

$$T(t)x_n = e^{-k_n t} x_n \quad (x_n \in X_n)$$

defines a compact C_0-semigroup on X wich moreover satisfies:

(a) $\|T(t)\| \le C$ *for all* $t > 0$;

(b) $X^\odot = [P_n^* X^*]$.

PROOF. Fix $x \in X$ of norm $1, x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$. Let $\varepsilon > 0$ be arbitrary and take N such that

$$\| \sum_{n=N+1}^{\infty} x_n \| \leq \varepsilon.$$

Let $t_0 > 0$ be so small that

$$1 - e^{-k_N t_0} \leq \frac{\varepsilon}{N}.$$

Since $0 \leq k_1 < k_2 < \cdots$ also

$$1 - e^{-k_n t} \leq \frac{\varepsilon}{N}, \quad (1 \leq n \leq N;\ 0 \leq t \leq t_0).$$

Then for $0 \leq t \leq t_0$ we have, using Lemma 2.1,

$$\begin{aligned} \|T(t)x - x\| &\leq \| \sum_{n=1}^{N} (1 - e^{-k_n t}) x_n \| + \| \sum_{n=N+1}^{\infty} (1 - e^{-k_n t}) x_n \| \\ &\leq N \cdot \frac{\varepsilon}{N} \cdot \max_{1 \leq n \leq N} \|x_n\| + C\varepsilon (1 - e^{k_{N+1} t} + \sum_{n=N+1}^{\infty} |e^{-k_n t} - e^{-k_{n+1} t}| + 1) \\ &\leq 2\varepsilon C + 2\varepsilon C = 4\varepsilon C. \end{aligned}$$

This shows that $T(t)$ is a C_0-semigroup on X.
Note that by Lemma 2.1 we have

$$\|T(t)\| \leq C \cdot \sum_{n=1}^{\infty} (e^{-k_n t} - e^{-k_{n+1} t}) = C \cdot e^{-k_1 t} \leq C.$$

This is (a).
It is obvious that $[P_n^* X^*] \subset X^{\odot}$ since on $P_n^* X^*$ we have $T^*(t) x_n^* = e^{-k_n t} x_n^*$. To prove the reverse inclusion, let $x^* = \text{weak*} \sum_{n=1}^{\infty} x_n^*$, with $x_n^* \in P_n^* X^*$. We claim that the weak*-sum $T^*(t) x^* = \text{weak*} \sum_{n=1}^{\infty} e^{-k_n t} x_n^*$ is actually strongly convergent for every $t > 0$. Indeed, for every $x = \sum_{n=1}^{\infty} x_n$ we have by Lemma 2.1

$$|\langle \sum_{n=N}^{M} e^{-k_n t} x_n^*, \sum_{n=1}^{\infty} x_n \rangle| = |\langle \sum_{n=N}^{M} x_n^*, \sum_{n=N}^{M} e^{-k_n t} x_n \rangle| \leq 2C \|x^*\| \cdot 2e^{-k_N t} \|x\|.$$

Hence

$$\| \sum_{n=N}^{M} e^{-k_n t} x_n^* \| \leq 4C e^{-k_N t} \|x^*\|.$$

Since $k_N \to \infty$ as $N \to \infty$ we have shown that for $t > 0$ the sequence

$$\{ \sum_{n=1}^{N} e^{-k_n t} x_n^* \}_{N=1}^{\infty}$$

is Cauchy in X^*. From this it follows that $T^*(t) x^* \in [P_n^* X^*]$ for $t > 0$. Now should $x^* \in X^{\odot}$, then $x^* = \lim_{t \downarrow 0} T^*(t) x^*$ and by the closedness of $[P_n^* X^*]$ it follows that we must have $x^* \in [P_n^* X^*]$. This shows $X^\odot \subset [P_n^* X^*]$ and (b) is proved.

Finally note that for fixed $t > 0$,

$$T(t) = \lim_{N \to \infty} \sum_{n=1}^{N} e^{-k_n t} P_n$$

in the uniform operator topology. This is shown in the same way as we did in (b), again using Lemma 2.1. Since each P_n is compact it follows that $T(t)$ is a compact semigroup.

In 2.3, 2.4 and 4.2 we will give examples how information on bases may be derived from the semigroups defined in the above theorem.

COROLLARY 2.3. *(D.W. Dean* [3]) *Grothendieck spaces with the Dunford-Pettis property do not admit a Schauder decomposition.*

PROOF. If X has a Schauder decomposition then the C_0-semigroup $T(t)$ defined in Theorem 2.2 has a generator A given by

$$Ax_n = -k_n x_n$$

which is unbounded, since the sequence (k_n) is unbounded. Now apply Prop. 1.4.

REMARK 2.4. A countable collection of subspaces $\{X_n\}_{n=1}^{\infty}$ of a Banach space X is called a *weak decomposition* of X if for every $x \in X$ there is a unique sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $x_n \in X_n$, such that $x = \text{weak} \sum_{n=1}^{\infty} x_n$. If moreover the canonical projections π_n are weakly continuous, then $\{X_n\}_{n=1}^{\infty}$ is called a *weak Schauder decomposition*. It is well-known (see [16]) that a weak decomposition is a weak Schauder decomposition if and only if each X_n is closed. If X is a Banach space with a weak Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ it is possible to define weakly continuous semigroups on X as we did in Theorem 2.2. One must be somewhat more careful since for weak decompositions one cannot use Lemma 2.1. Define $\varepsilon_m = 1/(m \cdot 2^m)$ $(m = 1, 2, \ldots)$. Put $k_1 = 1$. Let $t_1 > 0$ be defined by

$$e^{-k_1 t_1} = 1 - \varepsilon_1.$$

Suppose $k_1, k_2, \ldots, k_{m-1}$ and $t_1, t_2, \ldots, t_{m-1}$ have been chosen. Choose $k_m \in \mathbb{N}$, $k_m \geq k_{m-1} + 1$ such that

$$\frac{e^{-k_m t_{m-1}}}{1 - e^{-t_{m-1}}} < \frac{1}{2^m}.$$

Let t_m be defined by

$$e^{-k_m t_m} = 1 - \varepsilon_m.$$

Observe that $t_1 > t_2 > \cdots \to 0$ and $1 = k_1 < k_2 < \cdots$. It is not difficult to check that

$$T(t)x_n = e^{-k_n t} x_n$$

defines a weakly continuous semigroup on X. By Proposition 1.2 this semigroup is actually strongly continuous. But then straightforward estimates show that

$$\|x - \sum_{n=1}^{N} x_n\| \leq \|T(t_N)x - x\| + \|T(t_N)x - \sum_{n=1}^{N} x_n\| \to 0 \quad (N \to \infty).$$

In fact we have shown that $\{X_n\}_{n=1}^{\infty}$ is actually a (*strong*) Schauder decomposition. This is a result of W.H. Ruckle. [15]

THEOREM A. *Let X be a Banach space with a Schauder basis. The following statements are equivalent*:

(1) *X is reflexive*;

(2) *For every C_0-semigroup $T(t)$ on X, the adjoint semigroup $T^*(t)$ is strongly continuous*;

(3) *For every C_0-semigroup $T(t)$ on X, the second adjoint semigroup $T^{\odot *}(t)$ is strongly continuous.*

PROOF. (1) $\Rightarrow$ (2) is Phillips's theorem, from which also (1) $\Rightarrow$ (3) follows. We have to prove (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). Suppose X is nonreflexive. Applying Prop. 1.3 (2), let $\{x_n\}_{n=1}^{\infty}$ be a nonshrinking basis of X; let $T(t)$ be the C_0-semigroup on X as in Theorem 2.2. By (b) of Theorem 2.2 and the definition of a shrinking basis we have $X^{\odot} = [x_n^*] \neq X^*$, that is, the adjoint semigroup $T^*(t)$ is not strongly continuous on X^*. This gives (2) $\Rightarrow$ (1). Next, again assume that X is nonreflexive and let $\{x_n\}_{n=1}^{\infty}$ be a nonboundedly complete basis of X; let $T(t)$ be the C_0-semigroup on X as in Theorem 2.2. If follows by Prop. 1.3 (1) that $\{x_n^*\}_{n=1}^{\infty}$ is a nonshrinking basis of $[x_n^*] = X^{\odot}$ and hence by the same argument $X^{\odot\odot} = [x_n^{**}] \neq X^{\odot *}$, proving (3) $\Rightarrow$ (1).

Theorem A does not hold for arbitrary Banach spaces. For instance, let $X = L^{\infty}[0,1]$ or more generally any Grothendieck space with the Dunford-Pettis property. Since every C_0-semigroup on X has a bounded generator, it is obvious that the adjoint of such a semigroup is strongly continuous and has a bounded generator as well. Note that these spaces always are nonseparable [10]. Therefore one still may ask whether Theorem A holds for arbitrary *separable* Banach spaces X, since not every separable Banach space has a basis [6]. For instance, it is known [11] that c_0 and l^1 contain subspaces Y without a basis. In these two cases however the answer is easy, since Y contains a complemented subspace Z isomorphic to c_0 or l^1 respectively [11]. On Z we may construct a C_0-semigroup whose adjoint is not strongly continuous; this semigroup can be extended to Y by putting it identically 1 on the complement of Z. Hence, Theorem A holds for closed subspaces of c_0 and l^1.

By a theorem of A. Pelczynski [13] a Banach space is reflexive if and only if every closed subspace with a basis is. This, in combination with Theorem A, gives the following corollary.

COROLLARY 2.5. *A Banach space X is reflexive if and only if for every closed subspace Y of X, every C_0-semigroup $T(t)$ on Y has a strongly continuous adjoint $T^*(t)$ on Y^*.*

3. THE RADON-NIKODYM PROPERTY AND UNCONDITIONAL BASES

LEMMA 3.1. *Every weak*-continuous semigroup $T(t)$ on a dual Banach space X^* with the Radon-Nikodym property is strongly continuous for $t>0$.*

PROOF. Fix an arbitrary $x^* \in X^*$. By the uniform boundedness theorem, there is an $M<\infty$ such that $\|T(t)x^*\| \leq M$ for all $t \in [0,1]$. Define $S: L^1[0,1] \to X^*$ by

$$Sg = weak^* \int_0^1 g(t)T(t)x^* dt.$$

Since $\langle T(t)x^*, x\rangle$ is continuous for each $x \in X$, it follows that $\langle g(t)T(t)x^*, x\rangle \in L^1[0,1]$ for all $x \in X$, and the above integral is well-defined. S is bounded:

$$\|Sg\| = \sup_{\|x\|=1} \left|\int_0^1 \langle g(t)T(t)x^*, x\rangle dt\right| \leq \sup_{\|x\|=1} \int_0^1 |g(t)|\,|\langle T(t)x^*, x\rangle| dt \leq M\|g\|_1.$$

Since X^* has the Radon-Nikodym property, by Proposition 1.5 there is an $h \in L^\infty([0,1]; X^*)$ such that

$$Sg = \int_0^1 g(t)h(t)dt$$

for all $g \in L^1[0,1]$. For $0 \leq t < 1$ and $\varepsilon > 0$ small enough, let $E=[t, t+\varepsilon]$ and put $g=(1/\varepsilon)\chi_E$, where χ is the characteristic function. It follows that

$$weak^* \int_t^{t+\varepsilon} \frac{1}{\varepsilon} T(\tau)x^* d\tau = \int_t^{t+\varepsilon} \frac{1}{\varepsilon} h(\tau)d\tau.$$

By the Lebesgue differentiation theorem, for almost all $t \in [0,1)$ the right-hand side converges to $h(t)$ as $\varepsilon \to 0$. Hence, for such t we have

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \langle T(\tau)x^*, x\rangle d\tau \to \langle h(t), x\rangle \quad (\varepsilon \to 0)$$

for all $x \in X$. But the integrand on the left-hand side is continuous, and therefore the integral converges to $\langle T(t)x^*, x\rangle$. So $T(t)x^* = h(t)$ a.e. In particular, $T(t)x^*$ is measurable on $[0,1]$, hence on $[0,\infty)$. It follows from Prop. 1.2 (1) that $T(t)$ is strongly continuous for $t>0$.

If $T(t)$ in Lemma 3.1 is an *adjoint* semigroup, the above result is implicit in

W. Arendt [1], where it is obtained by an entirely different method of proof.

It is classical result of N. Dunford and B.J. Pettis [5] that separable duals have the Radon-Nikodym property. For such spaces the above lemma is much easier to prove. Indeed, by Pettis's measurability theorem [4, Cor. II.1.4], for each $x^* \in X^*$ the map $t \to T^*(t)x^*$ is strongly measurable. Now apply Prop. 1.2 (1).

Every nonreflexive Banach space X with a basis admits a C_0-semigroup whose adjoint is strongly continuous *precisely* for $t > 0$. In fact, the semigroup from the proof of Theorem A, (2) ⇒ (1), will do, as is easily verified. However, this is a rather non-constructive example. The following example is adapted from [1], where it is credited to H.P. Lotz.

EXAMPLE 3.2. Let J be the James space consisting of all sequences of scalars $x = (a_1, a_2, \dots)$ for which

$$\|x\| = \sup[(a_{p_1} - a_{p_2})^2 + (a_{p_2} - a_{p_3})^2 + \cdots + (a_{p_{m-1}} - a_{p_m})^2 + (a_{p_m} - a_{p_1})^2]^{1/2} < \infty$$

and

$$\lim_{n \to \infty} a_n = 0,$$

where the sup is taken over all possible choices of integers m and $p_1 < p_2 < \cdots < p_m$. Let x_n denote the nth unit vector. On J define a C_0-semigroup $T(t)$ by

$$T(t)x_n = e^{-nt}x_n.$$

Since $\{x_n\}_{n=1}^{\infty}$ is a shrinking basis for J, the unit vectors x_n^* of J^* form a basis for J^* and we have $J^{\odot} = J^*$. One can show that J^{**} is isomorphic to $J \oplus \mathbb{C}e$, where $e = (1, 1, \dots)$. Consequently J^{**} is separable and therefore has the Radon-Nikodym property. Hence $T^{**}(t)$ is strongly continuous for $t > 0$ by Lemma 3.1. It follows from Theorem 2.2 (b) (applied to the C_0-semigroup $T^*(t)$ on J^*) that $e \notin J^{*\odot}$. Therefore $T^{**}(t)$ is not strongly continuous at $t = 0$.

This example is interesting for another reason. There are many examples of C_0-semigroups on Banach spaces X such that $\dim X^*/X^{\odot} = \infty$. The above example shows that $X^{\odot}$ can also have any *finite* codimension in X^*:

COROLLARY 3.3. *For each* $n \in \mathbb{N}$ *there exists a Banach space* X *and a* C_0*-semigroup* $T(t)$ *on* X *such that* $\dim X^*/X^{\odot} = n$.

PROOF. If $n = 0$, let $T(t)$ be any C_0-semigroup on a reflexive space. Otherwise, consider the C_0-semigroup $T^*(t)$ on J^* from Example 3.2. Since $J^{**} = J \oplus \mathbb{C}e = J^{*\odot} \oplus \mathbb{C}e$ we see that $\dim J^{**}/J^{*\odot} = 1$. Let $X = J^* \times J^* \times \cdots \times J^*$, n times, together with the 'product' semigroup obtained from n copies of $T^*(t)$.

THEOREM B. *Let* X *be a Banach space with an unconditional basis* $\{x_n\}_{n=1}^{\infty}$. *The following statements are equivalent*:

(1) *X^* has the Radon-Nikodym property;*
(2) *Every adjoint semigroup on X^* is strongly continuous for $t>0$.*

PROOF. (1) ⇒ (2) follows from Lemma 3.1. It therefore remains to be shown that (2) ⇒ (1) holds. We already remarked that separable duals have the Radon-Nikodym property. Hence it suffices to show the unconditional basis $\{x_n\}_{n=1}^{\infty}$ of X is shrinking. Suppose the contrary is true. Then there are $x_0^* \in X^*$, $\|x_0^*\|=1$ and $0<\varepsilon<1$ such that

$$\lim_N \|x_0^*|_{[x_N, x_{N+1}, \dots]}\| > \varepsilon.$$

Choose inductively a sequence of integers $0=N_0<N_1<\cdots$ and a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ of norm-1 vectors as follows. Let $z_1=\sum_{n=1}^{\infty} \alpha_{1n} x_n$ be any norm-1 vector such that

$$|\langle x_0^*, z_1\rangle| > \varepsilon.$$

Choose N_1 sufficiently large such that

$$\left|\left\langle x_0^*, \sum_{n=1}^{N_1} \alpha_{1n} x_n\right\rangle\right| > \varepsilon.$$

Put $y_1=\sum_{n=1}^{N_1} \alpha_{1n} x_n$. We may, by choosing N_1 large enough, multiply y_1 with an appropriate scalar so as to make a norm-1 vector of it without affecting the above inequality. Choose $z_2=\sum_{n=N_1+1}^{\infty} \alpha_{2n} x_n \in [x_{N_1+1}, x_{N_1+2}, \dots]$ of norm 1 such that

$$|\langle x_0^*, z_2\rangle| > \varepsilon.$$

Choose N_2 such that

$$\left|\left\langle x_0^*, \sum_{n=N_1+1}^{N_2} \alpha_{2n} x_n\right\rangle\right| > \varepsilon.$$

Define $y_2=\sum_{n=N_1+1}^{N_2} \alpha_{2n} x_n$ and again assume without loss of generality that y_2 has norm 1. Continue in this way. By construction of the y_n we have for all n,

$$|\langle x_0^*, y_n\rangle| > \varepsilon.$$

For $N_{m-1}<n\le N_m$ define

$$T(t)x_n = e^{imt} x_n,$$

where x_n is the nth basis vector. By Prop. 1.3 (3), there is a $K>0$ such that $\|T(t)\| \le K$ for all $t \ge 0$. From this it is easy to see that $T(t)$ is a C_0-semigroup on X. Now let $t>0$ be arbitrary and fixed. We will show that $T^*(t)x_0^* \notin X^{\odot}$. Let $m \in \mathbb{N}$, $m \ge 1$. By the irrationality of the number π, we can find a positive integer k such that

$$|1-e^{i(k/m)}| > 2-\varepsilon.$$

We have the following estimates.

$$\left\| T^*\left(t+\frac{1}{m}\right)x_0^* - T^*(t)x_0^*\right\| \geq \left|\left\langle T^*\left(t+\frac{1}{m}\right)x_0^* - T^*(t)x_0^*, y_k\right\rangle\right| =$$

$$|e^{ik(t+1/m)} - e^{ikt}| \cdot |\langle x_0^*, y_k\rangle| \geq (2-\varepsilon)\cdot\varepsilon.$$

This proves Theorem B.

It is natural to ask whether an analogue of Corollary 2.5 holds for Banach spaces whose dual have the Radon-Nikodym property. H.P. Lotz's theorem on l^1 in Banach lattices [7] shows that for Banach *lattices* this is indeed the case: If the dual of a Banach lattice does not have the Radon-Nikodym property, then X contains a copy of l^1; on l^1 we have a C_0-semigroup whose adjoint is not strongly continuous for $t>0$ by Theorem B. For general Banach spaces we remark that J. Hagler [8] proved that a separable Banach space with a nonseparable dual has a subspace with a basis whose dual is nonseparable. Therefore it would be enough to prove Theorem B, $(2)\Rightarrow(1)$, without the assumption that the basis of X should be unconditional. (note that we made a rather crude step at this stage in just using that the basis of a space with nonseparable dual necessarily must be nonshrinking). The following theorem shows that in order to solve this problem, it suffices to construct a C_0-semigroup on X whose adjoint has a nonseparable orbit.

THEOREM 3.4. *Let $T(t)$ be a C_0-semigroup on a Banach space X. Let $x^*\in X^*$. The orbit $\{T^*(t)x^*\colon t\geq 0\}$ is separable if and only if $t\to T^*(t)x^*$ is strongly continuous for $t>0$ if and only if $t\to T^*(t)x^*$ is weakly continuous for $t>0$.*

PROOF. It is obvious that strong continuity implies weak continuity. If $t\to T^*(t)x^*$ is weakly continuous for $t>0$ then it is certainly weakly separable, which is the same as strongly separable. Suppose $\{T^*(t)x^*\colon t\geq 0\}$ is separable. The proof that the map $t\to T^*(t)x^*$ is strongly continuous for $t>0$ is a slight modification of the argument given in [9, Thm 10.3.2]. Choose numbers $0<\alpha<\tau<\beta<\xi$ and let η be so small that $\beta<\xi-\eta$. Now $T^*(\xi)x^*=T^*(\tau)T^*(\xi-\tau)x^*$ is independent of τ, hence certainly integrable on $[\alpha,\beta]$ wirth respect to τ. Therefore

$$(\beta-\alpha)[T^*(\xi\pm\eta)-T^*(\xi)]x^* = \int_\alpha^\beta T^*(\tau)[T^*(\xi\pm\eta-\tau)-T^*(\xi-\tau)]x^* d\tau.$$

The norm of the integrand is majorized by $2M\|x^*\|$, where M is such that $\|T^*(t)\| = \|T(t)\| \leq M$ on $[0,\xi+\eta]$. Since $\tau\to[T^*(\xi\pm\eta-\tau)-T^*(\xi-\tau)]x^*$

is measurable (by Pettis' measurability theorem), so is $\|[T^*(\xi \pm \eta - \tau) - T^*(\xi - \tau)]x^*\|$. This gives

$$(\beta - \alpha) \|[T^*(\xi \pm \eta) - T^*(\xi)]x^*\|$$

$$\leq M \int_{\xi - \beta}^{\xi - \alpha} \|[T^*(\sigma \pm \eta) - T^*(\sigma)]x^*\| \, d\sigma \to 0 \quad (\eta \to 0);$$

see [9, Thm 3.8.3].

THEOREM 3.5. *Let $T(t)$ be a C_0-semigroup on a Banach space X. Let $x^* \in X^*$. Then $t \to T^*(t)x^*$ is strongly continuous for $t \geq 0$ if and only if $t \to T^*(t)x^*$ is weakly continuous for $t \geq 0$.*

PROOF. We only have to prove the 'if' part. If $T^*(t)$ is an adjoint semigroup, then there is a positive M such that $\|T^*(t)\| \leq M$ in a neighbourhood of $t = 0$ (since such an estimate holds for its predual $T(t)$). Now the proof can be finished in exactly the same way as in [17, Ch. IX, 1].

These two theorems can be considered as the 'orbitwise' analogous for adjoint semigroups of Prop. 1.2. The point of their proofs is that we have bounds on $T^*(t)$ beforehand, since we are dealing with *adjoint* semigroups.

4. NONSHRINKING BASES IN c_0

Theorem A guarantees the existence of a C_0-semigroup without strongly continuous adjoint on the nonreflexive space c_0 (and, more generally, on every separable Banach space containing c_0, since by A. Sobczyk's theorem [11], c_0 is complemented in such spaces). The following theorem shows that it can be hard to give an explicit example of such a semigroup.

THEOREM 4.1. *Let $T(t)$ be a C_0-semigroup on c_0; $\|T(t)\| \leq Me^{\omega t}$. If $M < 2$, then $T^*(t)$ is strongly continuous on l^1.*

PROOF. Choose $\varepsilon > 0$ such that $M - 1 + \varepsilon < 1$. Let $x_0 = \sum_n \alpha_n e_n \in l^1$ be arbitrary (e_n denoting the nth unit vector of l^1); $\|x_0\| = 1$. Let N be such that $\|\sum_{n=N+1}^{\infty} \alpha_n e_n\| < \varepsilon/5$. Choose $t_1 > 0$ so small that $\|T^*(t_1)x_0\| \leq M + \varepsilon/5$ and $|(T^*(t_1)x_0 - x_0)_n| \leq \varepsilon/(5N)$ $(n = 1, 2, \ldots, N)$. Such t_1 exists by the weak*-continuity of the map $t \to T^*(t)x_0$ and by the estimate $\|T(t)\| \leq Me^{\omega t}$. We have

$$\sum_{n=1}^{N} |(T^*(t_1)x_0)_n| \geq \sum_{n=1}^{N} |(x_0)_n| - \sum_{n=1}^{N} |(T^*(t_1)x_0 - x_0)_n|$$

$$\geq 1 - \frac{\varepsilon}{5} - N \cdot \frac{\varepsilon}{5N} = 1 - \frac{2\varepsilon}{5}.$$

Therefore

$$\|x_0 - T^*(t_1)x_0\| = \sum_{n=1}^{N} |(T^*(t_1)x_0 - x_0)_n| + \sum_{n=N+1}^{\infty} |(T^*(t_1)x_0 - x_0)_n|$$

$$\le \frac{\varepsilon}{5} + \sum_{n=N+1}^{\infty} |(T^*(t_1)x_0)_n| + \sum_{n=N+1}^{\infty} |(x_0)_n|$$

$$\le \frac{\varepsilon}{5} + \left(\|T^*(t_1)x_0\| - \left(1 - \frac{2\varepsilon}{5}\right)\right) + \frac{\varepsilon}{5} \le M - 1 + \varepsilon.$$

Put $x_1 = x_0 - T^*(t_1)x_0$. In the same way, there is an $t_2 > 0$ such that

$$\|x_1 - T^*(t_2)x_1\| \le (M - 1 + \varepsilon)\, \|x_1\| \le (M - 1 + \varepsilon)^2.$$

Put $x_2 = x_1 - T^*(t_2)x_1$. Proceed with the construction inductively in the obvious way. After n steps, we have $t_1, t_2, \dots, t_n > 0$ and vectors $x_1, x_2, \dots, x_n$ such that $x_n = x_{n-1} - T^*(t_n)x_{n-1}$ and

$$\|x_0 - T^*(t_1)x_0 - T^*(t_2)x_1 - \cdots - T^*(t_n)x_{n-1}\|$$

$$= \|x_{n-1} - T^*(t_n)x_{n-1}\| \le (M - 1 + \varepsilon)^n$$

Since l^1 has the Radon-Nikodym property, by Lemma 3.1 we get that $T^*(t_i)x_{i-1} \in (c_0)^{\odot}$ for all $i = 1, 2, \dots$. Since $(M - 1 + \varepsilon)^n \to 0$ as $n \to \infty$ we have proved that x_0 is in the closure of $(c_0)^{\odot}$. By 1.1 (3), $(c_0)^{\odot}$ is closed and therefore $x_0 \in (c_0)^{\odot}$. Hence $(c_0)^* = l^1 = (c_0)^{\odot}$, as was to be shown.

We noted that the standard unit vector basis of c_0 is shrinking. Of course, this basis has basis constant $C = 1$. By M. Zippin's theorem we are told that there exists a nonshrinking basis for c_0, since c_0 is nonreflexive. What can be said of the basis constant of such a basis?

COROLLARY 4.2. *Every nonshrinking basis of c_0 has basis constant $C \ge 2$.*

PROOF. Let $\{x_n\}_{n=1}^{\infty}$ be nonshrinking basis of c_0 with basis constant C. Let $T(t)$ be the C_0-semigroup, defined with respect to $\{x_n\}_{n=1}^{\infty}$, as in Theorem A. Then $T^*(t)$ is not strongly continuous. By Theorem 2.2, $\|T(t)\| \le C$. Now by Theorem 4.1 we must have $C \ge 2$.

The results of Theorem 4.1 and Corollary 4.2 are optimal: let z_i denote the ith unit vector of c_0 and put $y_n = \sum_{i=1}^{n} z_i$, then the basis $\{y_n\}_{n=1}^{\infty}$ is nonshrinking and has basis constant 2. Moreover, the semigroup $T(t)$ as defined in Theorem 2.2 satisfies $\|T(t)\| \le 2$ and has an adjoint which is not strongly continuous. Using a very different approach, another example of such a semigroup on c_0 was constructed by A. Di Bucchianico and A.J. Stam [private communication].

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