Strong Completeness for Iteration-Free Coalgebraic Dynamic Logics

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Abstract. We present a (co)algebraic treatment of iteration-free dynamic modal logics such as Propositional Dynamic Logic (PDL) and Game Logic (GL), both without star. The main observation is that the program/game constructs of PDL/GL arise from monad structure, and the axioms of these logics correspond to certain compatibilty requirements between the modalities and this monad structure. Our main contribution is a general soundness and strong completeness result for PDL-like logics for T-coalgebras where T is a monad and the "program" constructs are given by sequential composition, test, and pointwise extensions of operations of T.

1 Introduction

Modal logics are a much used formalism in automated verification thanks to the good balance between their expressive power and their computational properties. Recently, it has been shown that modal logics can be developed in the general framework of coalgebra [4,18], and that the expressiveness and complexity results for Kripke semantics hold more generally across many types of structures [29,30].

In this paper, we aim to develop a coalgebraic framework for dynamic modal logics such as Propositional Dynamic Logic (PDL) [5,11] and Game Logic (GL) [24,25]. In PDL, modalities are indexed by programs whose semantics is given by relations, and program constructs are interpreted by relation algebra. Similarly, in GL, modalities are indexed by games whose semantics is given by monotonic neighbourhood functions. The common feature of these two logics is that programs/games are an explicit part of the syntax. Such logics are called *exogenous* (cf. [27]). In contrast, *endogenous* logics such as LTL and CTL take an inner perspective where programs are viewed as a single, monolithic structure.

Our framework for coalgebraic dynamic modal logic builds on the basic observation that in PDL and GL programs/games are interpreted as maps of the form $X \to TX$ where T is a monad. For PDL, T is the covariant powerset monad, and for GL, T is the monotonic neighbourhood monad (both are described in

⋆ Supported by NWO-Veni grant 639.021.231.
⋆⋆ Supported by University of Strathclyde starter grant.

detail later). Such maps can be viewed as arrows in the Kleisli category of the monad T which yields semantics of sequential composition as Kleisli composition. Alternatively, a map $X \to TX$ can be viewed as a T-coalgebra which leads to a (coalgebraic) modal logic of T-computations. Other constructs, such as choice ($\cup$) and dual (^{d}) in GL, are interpreted by algebraic structure on the set $(TX)^X = \{X \to TX\}$ which arises pointwise from algebraic structure on TX. We formalise such constructs using natural operations on functors. We also note that PDL and GL are usually interpreted over so-called standard models, in which the program/game constructs have a certain intended meaning. In our general framework this leads to the notion of a standard model relative to some algebraic structure θ on T. In the current paper, we include tests, but not iteration which will require more assumptions on the monad.

Our main contributions are: (i) a method for associating rank-1 axioms to natural operations, (ii) a method for axiomatising tests, and (iii) strong completeness for the ensuing dynamic modal logic.

The rest of the paper is organised as follows. In Section 2 we recall the basics of PDL and GL, and of coalgebraic modal logic and monads. In Section 3, we introduce our general framework for coalgebraic dynamic modal logic. In Section 4, we show how to obtain axioms for sequential composition and natural operations, and provide sufficient conditions for their soundness. In Sections 5 and 6, we prove our strong completeness result which builds on the generic strong completeness result in [31] by showing that a quasi-canonical model can be modified to validate also the non-rank-1 sequential composition axioms. Finally, in Section 7 we conclude and discuss related work.

Acknowledgements We thank Bart Jacobs, Alexander Kurz and Yde Venema for helpful discussions, and the anonymous referees for their useful comments.

2 Preliminaries

2.1 PDL and GL

We briefly recall the basics of the two dynamic modal logics that form our guiding examples. See the references given for more detail and background information.

Propositional Dynamic Logic (PDL) [5,11] is a modal logic for reasoning about program correctness. Modalities are indexed by programs, and a formula $[\alpha]\varphi$ should be read as "after all halting executions of program α, φ holds". PDL programs are built inductively from a set $Prog_0$ of atomic programs using the operations of *sequential composition* (;), *choice* ($\cup$) and *iteration* (*). Moreover, a formula φ can be turned into a program φ? by the *test* operation ?. The semantics of PDL is given by multi-modal Kripke models that contain a relation R_α for each program α. These models are generally assumed to be *standard* which means that relations for complex programs are defined inductively via composition, union and reflexive, transitive closure of relations over some given interpretation of atomic programs, and a test program φ? is interpreted by restricting the identity relation to the states that satisfy φ. As a deductive system, PDL is the least normal multi-modal logic that contains the axioms:

$$\begin{array}{lll} [\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi & [\alpha\cup\beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi & [\psi?]\varphi \leftrightarrow (\psi \rightarrow \varphi) \\ \varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi & \varphi \wedge [\alpha^*](\varphi \rightarrow [\alpha]\varphi) \rightarrow [\alpha^*]\varphi & \end{array} \tag{1}$$

for all programs α, β and all formulas ψ, φ. It is well known that PDL is (weakly) complete with respect to the class of standard PDL models [17]. Strong completeness fails due to the presence of * which makes PDL non-compact.

Game Logic (GL) [24,25] is a modal logic for reasoning about strategic ability in determined 2-player games. Here, a modal formula $[\gamma]\varphi$ should be read as "player 1 has a strategy in the game γ to ensure an outcome where φ holds". The modal language of GL is obtained by extending the program operations of PDL with the game operation *dual* (^{d}) which corresponds to a role switch of the two players. Game Logic semantics is given by multi-modal monotone neighbourhood models [3,8,9] in which each game γ is interpreted as a monotone neighbourhood function $E_\gamma \colon X \to \mathcal{M}(X)$ (we formally define $\mathcal{M}$ later in Example 1(iii)) which assigns to each state $x \in X$ the collection of all subsets $U \subseteq X$ for which player 1 has a strategy in γ starting in x to ensure an outcome in U. As a deductive system, GL is defined to be the least monotone multi-modal logic containing the following axioms and rule:

$$\begin{array}{lll} [\gamma;\delta]\varphi \leftrightarrow [\gamma][\delta]\varphi & [\gamma\cup\delta]\varphi \leftrightarrow [\gamma]\varphi \vee [\delta]\varphi & [\psi?]\varphi \leftrightarrow (\psi \wedge \varphi) \\ \varphi \vee [\gamma][\gamma^*]\varphi \rightarrow [\gamma^*]\varphi & \dfrac{\varphi \vee [\gamma]\varphi \rightarrow \psi}{[\gamma^*]\varphi \rightarrow \psi} & [\gamma^d]\varphi \leftrightarrow \neg[\gamma]\neg\varphi \end{array} \tag{2}$$

Both iteration-free GL and dual-free GL are known to be complete for standard GL models (restricted to the appropriate fragment), however, completeness of GL with both * and ^{d} remains an open question. One indication of why completeness for full GL is difficult is that GL can be viewed as a fragment of the modal μ-calculus that spans all levels of the alternation hierarchy [2,25].

2.2 Coalgebraic Modal Logic

Coalgebraic modal logic [4,18] is a general framework which encompasses many known modal logics such as normal, classical, graded and probabilistic modal logic. The uniform treatment of these is achieved by viewing the corresponding semantic structures as coalgebras for a functor T [28]. In the present paper, we only consider coalgebras for functors on Set, the category of sets and functions. Let T be a Set-(endo)functor. A *T-coalgebra* is a map $\xi \colon X \to TX$, and a *T-coalgebra morphism* from $\xi \colon X \to TX$ to $\xi' \colon X' \to TX'$ is a map $f \colon X \to X'$ such that $\xi' \circ f = Tf \circ \xi$. T-coalgebras and their morphisms form a category $\mathsf{Coalg}(T)$.

We follow the notation from [31] in defining syntax and semantics of coalgebraic modal logic. A modal signature Λ consists of a collection of modal operators with associated arities. Given a modal signature Λ and a countable set P

of atomic propositions, the set $\mathcal{F}(\Lambda)$ of Λ-formulas is generated by the following grammar:

$$\varphi \quad ::= \quad p \in P \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_\lambda(\varphi_1, \ldots, \varphi_n)$$

where $\Box_\lambda \in \Lambda$ is n-ary. For any set X, $\mathsf{Prop}(X)$ denotes the set of all propositional formulas over X, and $\Lambda(X) = \{\Box_\lambda(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in X, \Box_\lambda \in \Lambda \text{ is } n\text{-ary}\}$.

Modal formulas will be interpreted in coalgebras. We use the approach to coalgebraic modal logic in which modalities are interpreted via predicate liftings. First, we denote by $\mathcal{Q} : \mathsf{Set} \to \mathsf{Set}^{\mathrm{op}}$ the *contravariant powerset functor* which maps a set X to its powerset, and a function f to its inverse image map. An *n-ary predicate lifting for T* is a natural transformation $\lambda \colon \mathcal{Q}^n \Rightarrow \mathcal{Q} \circ T$. A *$(\Lambda, T)$-model* $\mathfrak{M}$ consists of a T-coalgebra $\xi \colon X \to TX$, a valuation $V \colon P \to \mathcal{P}(X)$ of atomic propositions, and an n-ary predicate lifting for each n-ary modality in Λ. For formulas $\varphi \in \mathcal{F}(\Lambda)$ the truth set $[\![\varphi]\!]^{\mathfrak{M}}$ is defined in the expected manner for the atomic propositions and Boolean connectives, and for modal formulas, $[\![\Box_\lambda(\varphi_1, \ldots, \varphi_n)]\!]^{\mathfrak{M}} = \xi^{-1}(\lambda_X([\![\varphi_1]\!]^{\mathfrak{M}}, \ldots, [\![\varphi_n]\!]^{\mathfrak{M}}))$. The map $\xi^{-1} \circ \lambda_X \colon \mathcal{P}(X)^n \to \mathcal{P}(X)$ is the *n-ary predicate transformer associated with ξ and* λ. In the remainder of this paper, we will only consider unary modalities and unary predicate liftings.

Example 1. The following well known instances of coalgebraic modal logic will be of central interest to the paper. See e.g. [28,29,31] for many other examples.

(i) Coalgebras for the *covariant powerset functor* $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ are Kripke frames, and $\mathcal{P}$-coalgebra morphisms are bounded morphisms. The Kripke box modality is interpreted via the predicate lifting $\lambda^{\Box}_X(U) = \{V \in \mathcal{P}(X) \mid V \subseteq U\}$.

(ii) The *neighbourhood functor* $\mathcal{N} = \mathcal{Q}^{\mathrm{op}}\mathcal{Q} \colon \mathsf{Set} \to \mathsf{Set}$ is the composition of $\mathcal{Q}$ with its dual $\mathcal{Q}^{\mathrm{op}}$. $\mathcal{N}$ maps a set X to $\mathcal{P}(\mathcal{P}(X))$, and function f to the double-inverse-image map $\mathcal{N}(f) = (f^{-1})^{-1}$. An $\mathcal{N}$-coalgebra $\nu \colon X \to \mathcal{N}(X)$ is known in modal logic as a neighbourhood frame, and $\mathcal{N}$-coalgebra morphisms as bounded neighbourhood morphisms [3,10]. The neighbourhood modality is interpreted via the predicate lifting given by $\lambda_X(U) = \{N \in \mathcal{N}(X) \mid U \in N\}$. In this paper we will refer to $\mathcal{N}$-coalgebras as *neighbourhood functions.*

(iii) The *monotone neighborhood functor* $\mathcal{M} \colon \mathsf{Set} \to \mathsf{Set}$ is the subfunctor of $\mathcal{N}$ which maps a set X to the set of upwards closed neighbourhood collections $H \subseteq \mathcal{P}(X)$, i.e., $\mathcal{M}(X) = \{H \in \mathcal{P}(\mathcal{P}(X)) \mid \forall U \subseteq V \subseteq X : U \in H \Rightarrow V \in H\}$, and for a function f, $\mathcal{M}(f)$ is obtained by restricting $\mathcal{N}(f)$ to upwards closed neighbourhood collections. Similarly, for the predicate lifting that interprets the monotonic neighbourhood modality. $\mathcal{M}$-coalgebras are known in modal logic as monotonic neighbourhood frames [3,8,9]. We will refer to $\mathcal{M}$-coalgebras as *monotonic neighbourhood functions.* The name "monotonic" refers to the upwards closure, and will be explained further in the next remark.

Remark 2. Neighbourhood functions and (unary) predicate transformers are essentially the same mathematical objects. This basic correspondence arises from the adjunction of the contravariant powerset functor $\mathcal{Q} \colon \mathsf{Set} \to \mathsf{Set}^{\mathrm{op}}$ with its dual:

$$\mathsf{Set} \underset{\mathcal{Q}^{\mathrm{op}}}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathsf{Set}^{\mathrm{op}} \quad (\bot) \qquad\qquad \frac{\mathcal{Q}X \to Y \quad \text{in } \mathsf{Set}^{\mathrm{op}}}{X \to \mathcal{Q}^{\mathrm{op}}Y \quad \text{in } \mathsf{Set}}$$

Hence, for all sets X and Y there is a bijection $\mathsf{Set}(X, \mathcal{Q}^{\mathrm{op}}Y) \cong \mathsf{Set}(Y, \mathcal{Q}X)$ given by exponential transpose $f(x)(y) = \widehat{f}(y)(x)$. Taking $Y = \mathcal{Q}X$, we get a bijection $\mathsf{Set}(X, \mathcal{Q}^{\mathrm{op}}\mathcal{Q}X) \cong \mathsf{Set}(\mathcal{Q}X, \mathcal{Q}X)$ between neighbourhood functions and predicate transformers given by $U \in \nu(x)$ iff $x \in \widehat{\nu}(U)$ for all $x \in X$ and $U \subseteq Y$. Note that $\widehat{\nu}\colon \mathcal{Q}(X) \to \mathcal{Q}(X)$ is a monotonic map (w.r.t. set-inclusion) if and only if $\nu\colon X \to \mathcal{M}(X) \subseteq \mathcal{Q}^{\mathrm{op}}\mathcal{Q}(X)$ is a monotonic neighbourhood function.

The Set-monad arising from the above adjunction is the neighbourhood monad $\mathcal{N} = \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ (cf. Example 3(2) below) and it will play a central role in what follows.

2.3 Monads

Monads will be used in two different ways. One is related to the view that monads model computational effects [23]. The other is related to their role as abstract algebraic theories [21].

We briefly recall the basic definitions. A *monad on* Set is a triple $\mathbb{T} = (T, \eta, \mu)$ where T is an Set-functor, and $\eta\colon \mathrm{Id} \Rightarrow T$ (unit) and $\mu\colon T^2 \Rightarrow T$ (multiplication) are natural transformations that satisfy the following coherence laws: $\mu \circ \eta_T = \mu \circ T\eta = id_T$ and $\mu \circ \mu_T = \mu \circ T\mu$.

For a Set-monad $\mathbb{T} = (T, \eta, \mu)$, the *Kleisli category* $\mathcal{K}\ell(\mathbb{T})$ has sets as objects and functions $X \to TY$ as arrows. Such a Kleisli map $X \to TY$ denotes a program with input in X and structured output in Y. Program composition is obtained as Kleisli composition which we denote by $*$, i.e., for maps $f\colon X \to TY$ and $g\colon Y \to TZ$, their composition in $\mathcal{K}\ell(\mathbb{T})$ is the map $g * f = \mu_Z \circ Tg \circ f\colon X \to TZ$.

A *(finitary) algebraic signature* Σ consists of a set of operation symbols $\{\sigma_i \mid i \in I\}$ where each σ_i has an arity $n_i \in \mathbb{N}$, $i \in I$. As is standard, we identify Σ with the Set-functor defined by $\Sigma X = \coprod_{i \in I} X^{n_i}$. The (classical) notion of an algebra for the signature Σ now coincides with the categorical notion of an algebra for the functor Σ, i.e., a function $\Sigma X \to X$.

Given a signature functor Σ, the *free monad generated by* Σ is the triple (T_Σ, η, μ) where for a set X, $T_\Sigma(X)$ is the set of Σ-terms over X, $\eta_X\colon X \to T_\Sigma(X)$ is the inclusion of variables as terms, and $\mu_X\colon T_\Sigma T_\Sigma(X) \to T_\Sigma(X)$ is the "flattening" of terms of terms into terms.

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad, an *Eilenberg-Moore algebra for* $\mathbb{T}$ (or just *EM*-algebra), is a map $\alpha\colon TX \to X$ such that $\alpha \circ \eta_X = id_X$ and $\alpha \circ \mu_X = \alpha \circ T\alpha$. A morphism of $\mathbb{T}$-algebras, written $f\colon (X, \alpha) \to (Y, \beta)$, is a map $f\colon X \to Y$ such that $\alpha \circ f = \beta \circ T(f)$. The free $\mathbb{T}$-algebra over X is given by multiplication (TX, μ_X). Every Σ-algebra $\alpha\colon \Sigma X \to X$ induces an Eilenberg-Moore algebra $\alpha^\sharp\colon T_\Sigma(X) \to X$ for (T_Σ, η, μ); this can be shown by structural induction on terms.

A *monad morphism* from (T, η, μ) to (T', η', μ') is a natural transformation $\rho\colon T \Rightarrow T'$ which respects the monad structure meaning that: $\rho \circ \eta = \eta'$ and $\rho \circ \mu = \mu' \circ \rho_{T'} \circ T\rho$. Since ρ is natural the last equation is equivalent to $\rho \circ \mu = \mu' \circ T'\rho \circ \rho_T$. Monads and monad morphisms together form a category.

Monads are used to capture computational effects such as I/O and state by viewing (functional) programs as arrows in the Kleisli category [23]. Here, we consider state-based computing rather than functional programming. This means that we generally view programs as functions $X \to TX$ where X is the state-space of the computation. However, the fact that such functions are also Kleisli maps is, of course, essential for the definition of sequential composition. We write $*$ for composition in $\mathcal{K}\ell(T)$.

In order to give semantics to test operations, we need TX to contain an element that represents an aborted computation. We will say that a monad T *is pointed*[4] if for each set X, TX contains a distinguished element $\bot_{TX}$ (or just $\bot$), and for all maps $f\colon X \to Y$, $Tf(\bot) = \bot$.

Example 3. Let X be an arbitrary set. For $U \subseteq X$, we denote by $\uparrow\{U\}$ the up-set of $\{U\}$ in the poset $\mathcal{N}(X)$, i.e., $\uparrow\{U\} = \{N \in \mathcal{N}(X) \mid U \in N\}$.

1. The covariant power set functor $\mathcal{P}$ is a monad with unit $\eta_X(x) = \{x\}$ and multiplication $\mu_X(\{U_i \mid i \in I\}) = \bigcup_{i\in I} U_i$. Arrows in $\mathcal{K}\ell(\mathcal{P})$ are relations, and $*$ is just relation composition. For a set X, $\mathcal{P}(X)$ is the free join-semilattice with bottom on X, and $\mathcal{P}$ is pointed by taking $\bot = \emptyset \in \mathcal{P}(X)$.
2. The neighbourhood functor $\mathcal{N}$ is a monad with

$$\eta_X(x) = \{U \subseteq X \mid x \in U\} \qquad \mu_X(W) = \{U \subseteq X \mid \uparrow\{U\} \in W\}.$$

   Also $\mathcal{N}$ is pointed by taking $\bot = \emptyset \in \mathcal{N}(X)$. An arrow $X \to \mathcal{N}Y$ in $\mathcal{K}\ell(\mathcal{N})$ is essentially a predicate transformer $\mathcal{Q}Y \to \mathcal{Q}X$ using the isomorphism via transpose (cf. Remark 2) which translates Kleisli composition of $\mathcal{N}$ into (function) composition of predicate transformers. In particular, for all $\nu_1, \nu_2\colon X \to \mathcal{N}X$, all $x \in X$ and $U \subseteq X$,

$$U \in (\nu_2 * \nu_1)(x) \iff x \in \widehat{\nu_1}(\widehat{\nu_2}(U)) \tag{3}$$

3. The functor $\mathcal{M}$ is also a pointed monad. The unit η and multiplication μ are obtained by restricting the ones for $\mathcal{N}$, and $\bot = \emptyset$. For a set X, $\mathcal{M}(X)$ is the free completely distributive lattice on X, cf. [22] (see also [15, 3.8,4.8]).
4. The functor $L = 1 + \mathrm{Id}$ is the "lift monad" (where $1 = \{*\}$). The unit $\eta_X : X \to 1 + X$ is inclusion. The multiplication μ_X maps $x \in 1 + (1 + X)$ to x iff $x \in X$, and otherwise to $*$. L is pointed by taking $\bot = * \in LX$.

3 Dynamic Coalgebraic Modal Logic

Our goal is to generalise the situation of PDL and GL to dynamic modal logics for other monads T. For the pointwise operations it seems at first that the natural

[4] Our notion of pointed monad is equivalent to requiring a natural transformation $1 \Rightarrow T$ where 1 is the constant Set-functor that maps every set to the singleton set.

operations are those coming from EM-algebras of $\mathbb{T}$. For example, PDL *choice* $\cup$ is interpreted via the join-semilattice structure on $\mathcal{P}(X)$. Similarly, the game operations *choice* and *dual* are interpreted via the lattice structure on $\mathcal{M}(X)$. While it is known that all set monads have a presentation in terms of operations and equations (cf. [20]), such a canonical presentation might be a proper class — a property that is not desirable for the design of a clear and concise programming language. As no "small" canonical choice of pointwise operations seems to be given, we generalise pointwise operations such as *choice* and *dual* using the notion of a natural operation and natural algebra.

Definition 4. *Let $T\colon \mathsf{C} \to \mathsf{Set}$ be a functor, a* natural n-ary operation *on T is a natural transformation $\theta\colon T^n \Rightarrow T$. More generally, given a signature functor $\Sigma\colon \mathsf{Set} \to \mathsf{Set}$, a* natural Σ-algebra *on T is a natural transformation $\theta\colon \Sigma T \Rightarrow T$.*

Example 5. 1. All Boolean operations are natural on $\mathcal{Q}^{\mathrm{op}}$. The reason is that the inverse-image map of a function preserves all of those.
2. For similar reasons, all Boolean operations on neighbourhood collections, such as e.g. $N \cup K$ for $N, K \in \mathcal{N}X$, are natural on $\mathcal{N}$. The neighbourhood-wise Boolean operations such as e.g. $N \Cap K = \{U \cap V \mid U \in N, V \in K\}$ are not natural on $\mathcal{N}$.
3. Union and intersection are natural on $\mathcal{M}$ (complement does not preserve monotonicity).
4. The *dual* operation defined for all $N \in \mathcal{N}X$ and $U \subseteq X$ by $U \in N^d$ iff $X \setminus U \notin N$ is natural on $\mathcal{N}$ (and $\mathcal{M}$).
5. The only Boolean operation that is natural on $\mathcal{P}$ is union, because the direct image of a function preserves unions, but not intersections or complements.
6. Apart from identity, the lift monad has only one (rather boring) operation *nil* where $nil_X(t) = *$ for all $t \in LX$.

A natural n-ary operation $\theta\colon T^n \Rightarrow T$ induces for each set X a pointwise operation θ^X_X on $\mathsf{Set}(X, TX) = (TX)^X$ in the expected manner. By cotupling, a natural Σ-algebra $\theta\colon \Sigma T \Rightarrow T$ induces a pointwise Σ-algebra θ^X_X on $(TX)^X$. For n-ary $\sigma \in \Sigma$, we denote the σ-component of θ^X_X by $(\theta_\sigma)^X_X\colon (TX)^n \to TX$.

Just as the syntax and semantics of PDL and GL is defined relative to a particular set of program/game operations, so is our notion of dynamic syntax and semantics. For the syntax, however, one only needs to fix a signature.

Definition 6 (Dynamic syntax). *Given a signature functor Σ, a set of atomic actions A_0 and a countable set P of atomic propositions, we define the set $\mathcal{F}(P, A_0, \Sigma)$ of dynamic formulas and the set $A = A(P, A_0, \Sigma)$ of complex actions by mutual induction:*

$$\begin{aligned} \mathcal{F}(P, A_0, \Sigma) &\ni \varphi ::= p \in P \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\alpha]\varphi \\ A(P, A_0, \Sigma) &\ni \alpha ::= a \in A_0 \mid \alpha;\alpha \mid \sigma(\alpha_1, \ldots, \alpha_n) \mid \varphi? \end{aligned}$$

where $\sigma \in \Sigma$ is n-ary.

For the semantics, we first note that the multi-modal structures of PDL and GL easily generalise to a coalgebra $X \to (TX)^A$ for the "labelled functor" T^A. A T^A-coalgebra will be called standard relative to some choice of natural algebra on T.

Definition 7 (Standard). *Let $\theta\colon \Sigma T \Rightarrow T$ be a natural Σ-algebra on a monad $\mathbb{T}$, and let $\delta\colon \Sigma A \to A$ be given by restricting action formation to Σ-operations. A coalgebra $\xi\colon X \to (TX)^A$ is called θ-*standard *if the transpose $\widehat{\xi}\colon A \to (TX)^X$ is a Σ-algebra morphism, i.e.,*

$$\widehat{\xi} \circ \delta = \theta_X^X \circ \Sigma\widehat{\xi} \tag{4}$$

We say that ξ is ;-standard *if for all $\alpha, \beta \in A$, $\widehat{\xi}(\alpha;\beta) = \widehat{\xi}(\alpha) * \widehat{\xi}(\beta)$.*

We now define the notion of a dynamic model relative to a choice of natural algebra $\theta\colon \Sigma T \Rightarrow T$.

Definition 8 (Dynamic semantics). *Let $\mathbb{T} = (T, \eta, \mu)$ be a pointed monad, and $\theta\colon \Sigma T \Rightarrow T$ a natural Σ-algebra on T. A (P, A_0, θ)-*dynamic $\mathbb{T}$-model *is a triple $\mathfrak{M} = (\xi_0, \lambda, V)$ where $\widehat{\xi}_0\colon A_0 \to (TX)^X$ is an interpretation of atomic actions in $(TX)^X$, $\lambda\colon \mathcal{Q} \Rightarrow \mathcal{Q} \circ T$ is a unary predicate lifting for T, and $V\colon P \to \mathcal{P}(X)$ is a valuation. We define the truth set $[\![\varphi]\!]^{\mathfrak{M}}$ of dynamic formulas and the semantics $\widehat{\xi}\colon A \to (TX)^X$ of complex actions in $\mathfrak{M}$ by mutual induction:*

$$\begin{array}{ll}
[\![p]\!]^{\mathfrak{M}} = V(p), \quad [\![\varphi \wedge \psi]\!]^{\mathfrak{M}} = [\![\varphi]\!]^{\mathfrak{M}} \cap [\![\psi]\!]^{\mathfrak{M}}, & [\![\neg\varphi]\!]^{\mathfrak{M}} = X \setminus [\![\varphi]\!]^{\mathfrak{M}}, \\
[\![[\alpha]\varphi]\!]^{\mathfrak{M}} = (\widehat{\xi}(\alpha)^{-1} \circ \lambda_X)([\![\varphi]\!]^{\mathfrak{M}}), & \\
\widehat{\xi}(\sigma(\alpha_1, \ldots, \sigma_n)) = (\theta_\sigma)_X^X(\widehat{\xi}(\alpha_1), \ldots, \widehat{\xi}(\alpha_n)) & \text{where } \sigma \in \Sigma \text{ is } n\text{-ary}, \\
\widehat{\xi}(\alpha;\beta) = \widehat{\xi}(\alpha) * \widehat{\xi}(\beta) & \text{(Kleisli composition)}, \\
\widehat{\xi}(\varphi?)(x) = \eta_X(x) \text{ if } x \in [\![\varphi]\!]^{\mathfrak{M}},\ \bot_{TX} \text{ otherwise.} &
\end{array}$$

We will sometimes refer to the induced $\widehat{\xi}$, and its transpose $\xi\colon X \to (TX)^A$, simply as a θ-dynamic $\mathbb{T}$-model.

Note that, by definition, a θ-dynamic $\mathbb{T}$-model $\xi\colon X \to (TX)^A$ is both θ-standard and ;-standard.

Remark 9. If we would not include tests, then we could drop the requirement of $\mathbb{T}$ being pointed, and define a $\mathbb{T}$-dynamic (P, A_0, θ)-structure to be a coalgebra $\xi\colon X \to (TX)^A$ whose transpose is the unique $\Sigma \cup \{;\}$-algebra morphism induced by $\widehat{\xi}_0\colon A_0 \to (TX)^X$ and the $\Sigma \cup \{;\}$-algebra structure on $(TX)^X$ given by θ and Kleisli composition.

4 Soundness

In this section we give a general method for finding axioms for (P, A_0, θ)-dynamic $\mathbb{T}$-models. In order for these axioms to be sound, it will be necessary to require the predicate lifting $\lambda\colon \mathcal{Q} \Rightarrow \mathcal{Q} \circ T$ to interact well with monad structure and pointwise structure.

We start with sequential composition. Not surprisingly, ;-standard models are captured by the axiom $[\alpha;\beta]p \leftrightarrow [\alpha][\beta]p$, for all $\alpha, \beta \in A$.

Lemma 10. *Let $\xi\colon X \to (TX)^A$ be ;-standard. If $\widehat{\lambda}\colon \mathbb{T} \Rightarrow \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ is a monad morphism, then the axiom $[\alpha;\beta]p \leftrightarrow [\alpha][\beta]p$ is valid in ξ.*

Proof. By the assumption that $\widehat{\lambda}\colon \mathbb{T} \Rightarrow \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ is a monad morphism, we have a functor (cf. [21]) $\widehat{\lambda} \circ - : \mathcal{K}\ell(T) \to \mathcal{K}\ell(\mathcal{Q}^{\mathrm{op}}\mathcal{Q})$ which acts as follows

$$X \mapsto X; \quad (f : X \to TY) \mapsto (\widehat{\lambda}_Y \circ f : X \to \mathcal{Q}^{\mathrm{op}}\mathcal{Q}Y).$$

We will also use the fact that for all $\alpha \in A$, the transpose of the predicate transformer $m_\alpha = \widehat{\xi}(\alpha)^{-1} \circ \lambda_X$ is $\widehat{m_\alpha} = \widehat{\lambda}_X \circ \widehat{\xi}(\alpha)$. We now have, for all sets X, all $x \in X$ and $U \subseteq X$,

$$\begin{array}{ll}
x \in (\widehat{\xi}(\alpha;\beta)^{-1} \circ \lambda_X)(U) & \iff \\
x \in ((\widehat{\xi}(\alpha) *_T \widehat{\xi}(\beta))^{-1} \circ \lambda_X)(U) & \iff \\
U \in (\widehat{\lambda_X} \circ (\widehat{\xi}(\alpha) *_T \widehat{\xi}(\beta)))(x) & \iff (\widehat{\lambda_X} \circ - \text{ is a functor}) \\
U \in ((\widehat{\lambda_X} \circ \widehat{\xi}(\alpha)) *_{\mathcal{Q}^{\mathrm{op}}\mathcal{Q}} (\widehat{\lambda_X} \circ \widehat{\xi}(\beta)))(x) & \iff (\text{cf. } (3)) \\
x \in (\widehat{\xi}(\alpha)^{-1} \circ \lambda_X \circ \widehat{\xi}(\beta)^{-1} \circ \lambda_X)(U). &
\end{array}$$

Remark 11. As noted in e.g. [16], giving a monad morphism $\mathbb{T} \Rightarrow \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ is the same as giving an Eilenberg-Moore algebra $T2 \to 2$. The view of modalities as EM-algebras for $\mathbb{T}$ was already suggested in [23], and more recently in [12]. The correspondence (via the Yoneda lemma) between unary predicate liftings and subsets of $T2$ was observed in [29]: Given $\lambda\colon \mathcal{Q} \Rightarrow \mathcal{Q}T$, define $C_\lambda \subseteq T2$ by $\lambda_2(\{1\})$. Conversely, given $C \subseteq T2$, define $\lambda_{C,X}(U) = (T\chi_U)^{-1}(C)$ where $\chi_U : X \to 2$ is the characteristic function of U. Moreover, it is easy to verify that λ corresponds to an EM-algebra iff its Boolean dual $\neg\lambda\neg$ does.

Example 12. (i) The Kripke diamond $\lambda^{\Diamond}_X(U) = \{V \in \mathcal{P}(X) \mid U \cap V \neq \emptyset\}$ corresponds (via Yoneda) to the free $\mathcal{P}$-algebra $\mathcal{P}\mathcal{P}(1) \to \mathcal{P}(1)$, hence the transpose of $\lambda^{\Diamond}$ and of its dual, the Kripke box $\lambda^{\Box}$, are both monad morphisms.

(ii) The transpose of the monotonic λ is the natural inclusion $\widehat{\lambda}\colon \mathcal{M} \Rightarrow \mathcal{N}$ and hence a monad morphism.

(iii) In [12], the EM-algebras $L2 \to 2$ for the lift monad were shown to correspond to λ^{tl} (total correctness) and λ^{pl} (partial correctness) where $t \in \lambda^{\mathsf{tl}}_X(U)$ iff $t \in U$ and $t \in \lambda^{\mathsf{pl}}_X(U)$ iff $t = *$ or $t \in U$.

Finding axioms for pointwise operations from natural algebras requires a bit more work. We will use the observation that an operation $\sigma\colon ((\mathcal{N}X)^X)^n \to (\mathcal{N}X)^X$ on neighbourhood functions is isomorphic to an operation $\overline{\sigma} = \psi^{-1} \circ \sigma \circ$

ψ^n on predicate transformers via the bijection $\psi\colon \mathcal{Q}X^{\mathcal{Q}X} \to (\mathcal{Q}^{\mathrm{op}}\mathcal{Q}X)^X$ given in Remark 2. That is, the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{Q}X^{\mathcal{Q}X})^n & \xrightarrow{\psi^n} & ((\mathcal{Q}^{\mathrm{op}}\mathcal{Q}X)^X)^n \\ \downarrow{\scriptstyle \psi^{-1}\circ\sigma\circ\psi^n} & & \downarrow{\scriptstyle \sigma} \\ \mathcal{Q}X^{\mathcal{Q}X} & \xrightarrow{\psi} & (\mathcal{Q}^{\mathrm{op}}\mathcal{Q}X)^X \end{array} \tag{5}$$

In particular, if $\chi\colon \mathcal{N}^n \Rightarrow \mathcal{N}$ is a natural operation on $\mathcal{N}$ with pointwise lifting $\chi^X_X\colon ((\mathcal{N}X)^X)^n \Rightarrow (\mathcal{N}X)^X$ to neighbourhood functions for any set X, then the corresponding $\overline{\chi}^X_X = \psi^{-1} \circ \chi^X_X \circ \psi^n$ is concretely given by

$$x \in \overline{\chi}^X_X(m_1,\ldots,m_n)(U) \iff U \in \chi^X_X(\psi(m_1),\ldots,\psi(m_n))(x) \tag{6}$$

for all $m_1,\ldots,m_n \in \mathcal{Q}X^{\mathcal{Q}X}$, $x \in X$ and $U \subseteq X$.

Example 13. The operation on predicate transformers corresponding to the *dual* operation $d\colon \mathcal{N} \Rightarrow \mathcal{N}$ is $\overline{d}(m)(U) = X \setminus m(X \setminus U)$. The operations on predicate transformers corresponding to Boolean operations on $\mathcal{N}$ are $(m_1 \overline{\cup} m_2)(U) = m_1(U) \cup m_2(U)$, $(\overline{\neg} m)(U) = X \setminus m(U)$ and so on.

The axioms for pointwise operations turn operations on labels into operations on predicate transformers. Using the above correspondence, we find the axioms via representations of natural operations on $\mathcal{N}$. For all $\chi\colon \mathcal{N}^n \Rightarrow \mathcal{N}$ and all $\alpha_1,\ldots,\alpha_n \in A$, we will define a rank-1 formula $\varphi(\chi,,\alpha_1,\ldots,\alpha_n,p)$. We start by showing how to do so for unary operations. Let $\chi\colon \mathcal{N} \Rightarrow \mathcal{N}$ be a unary natural operation on $\mathcal{N} = \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$. We have the following correspondence via the adjunction $\mathcal{Q} \dashv \mathcal{Q}^{\mathrm{op}}$ from Remark 2:

$$\frac{\chi_X\colon \mathcal{Q}^{\mathrm{op}}\mathcal{Q}X \to \mathcal{Q}^{\mathrm{op}}\mathcal{Q}X \in \mathsf{Set}}{\widehat{\chi}_X\colon \mathcal{Q}\mathcal{Q}^{\mathrm{op}}\mathcal{Q}X \to \mathcal{Q}X \in \mathsf{Set}^{\mathrm{op}}}$$

Therefore χ corresponds uniquely to (a predicate lifting) $\widehat{\chi} = \lambda^\chi\colon \mathcal{Q} \Rightarrow \mathcal{Q}\mathcal{Q}^{\mathrm{op}}\mathcal{Q}$, and by the Yoneda lemma to an element $\check{\chi} = \widehat{\chi}_2(id_2) = \widehat{\chi}_2(\{1\}) \in \mathcal{Q}\mathcal{Q}^{\mathrm{op}}\mathcal{Q}(2)$. Note that $\mathcal{Q}\mathcal{Q}^{\mathrm{op}}\mathcal{Q}(2)$ is the free Boolean algebra on four generators that can be identified with the elements of $\mathcal{Q}(2) = \{\emptyset, \{0\}, \{1\}, 2\}$. Consider the following four natural operations on $\mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ and their Yoneda correspondents:

operation	$\chi_X\colon \mathcal{N}X \to \mathcal{N}X, N \in \mathcal{N}X, U \subseteq X$	$\check{\chi}\colon \mathcal{Q}\mathcal{Q}^{\mathrm{op}}\mathcal{Q}(2), N \in \mathcal{Q}^{\mathrm{op}}\mathcal{Q}(2)$
id	$U \in i_X(N) \iff U \in N$	$N \in \check{\imath} \iff \{1\} \in N$
compl.	$U \in c_X(N) \iff X \setminus U \in N$	$N \in \check{c} \iff \{0\} \in N$
zero	$U \in z_X(N) \iff \emptyset \in N$	$N \in \check{z} \iff \emptyset \in N$
top	$U \in t_X(N) \iff X \in N$	$N \in \check{t} \iff 2 \in N$

Since $\check{\imath}, \check{c}, \check{z}, \check{t}$ generate all of $\mathcal{Q}\mathcal{Q}^{\mathrm{op}}\mathcal{Q}(2)$ it follows that for every unary natural operation $\chi\colon \mathcal{Q}^{\mathrm{op}}\mathcal{Q} \Rightarrow \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ the correspondent $\check{\chi}$ is a Boolean combination over $\check{\imath}, \check{c}, \check{z}, \check{t}$.

For an n-ary $\chi\colon \mathcal{N}^n \Rightarrow \mathcal{N}$, we get as Yoneda correspondent $\breve{\chi} \in \mathcal{Q}(\mathcal{N}(2)^n) \cong \mathcal{N}(n \cdot \mathcal{Q}(2))$ where $n \cdot \mathcal{Q}(2)$ is the n-fold coproduct of $\mathcal{Q}(2)$. $\mathcal{N}(n \cdot \mathcal{Q}(2))$ is the free Boolean algebra over $n \cdot \mathcal{Q}(2)$, and hence any n-ary natural operation on $\mathcal{N}$ corresponds to a Boolean expression over n copies of the generators $\breve{\imath}, \breve{c}, \breve{z}, \breve{t}$. For example, the binary union $\chi = \cup$ has correspondent $\breve{\chi} = \breve{\imath}_1 \vee \breve{\imath}_2$, i.e., $(N,K) \in \breve{\chi}$ iff $N \in \breve{\imath}_1$ or $K \in \breve{\imath}_2$. This leads to the following definition.

Definition 14. *Let $\{\breve{\imath}_j, \breve{c}_j, \breve{z}_j, \breve{t}_j \mid j = 1, \ldots, n\}$ be the generators of $\mathcal{N}(n \cdot \mathcal{Q}(2))$. For $\chi\colon \mathcal{N}^n \Rightarrow \mathcal{N}$ we define $\varphi(\breve{\chi}, \alpha_1, \ldots, \alpha_n, p)$ inductively as follows:*

- $\varphi(\breve{\imath}_j, \alpha_1, \ldots, \alpha_n, p) = [\alpha_j]p$ *for all* $j = 1, \ldots, n$.
- $\varphi(\breve{c}_j, \alpha_1, \ldots, \alpha_n, p) = [\alpha_j]\neg p$ *for all* $j = 1, \ldots, n$.
- $\varphi(\breve{z}_j, \alpha_1, \ldots, \alpha_n, p) = [\alpha_j]\bot$ *for all* $j = 1, \ldots, n$.
- $\varphi(\breve{t}_j, \alpha_1, \ldots, \alpha_n, p) = [\alpha_j]\top$ *for all* $j = 1, \ldots, n$.
- $\varphi(\neg\breve{\chi}, \alpha_1, \ldots, \alpha_n, p) = \neg\varphi(\breve{\chi}, \alpha_1, \ldots, \alpha_n, p)$.
- $\varphi(\breve{\chi} \wedge \breve{\delta}, \alpha_1, \ldots, \alpha_n, p) = \varphi(\breve{\chi}, \alpha_1, \ldots, \alpha_n, p) \wedge \varphi(\breve{\delta}, \alpha_1, \ldots, \alpha_n, p)$.

For example, the dual operation on $\mathcal{N}$ is $\breve{d} = \neg\breve{c}$ and we have $\varphi(\breve{d}, \alpha, p) = \neg\varphi(\breve{c}, \alpha, p) = \neg[\alpha]\neg p$. Similarly, for $\breve{d}_1 \wedge \neg\breve{t}_2$, we get $\varphi(\breve{d}_1 \wedge \neg\breve{t}_2, \alpha_1, \alpha_2, p) = (\neg[\alpha_1]\neg p) \wedge \neg([\alpha_2]\top)$.

The following theorem says that whenever λ transforms θ-structure on T-coalgebras into χ-structure on neighbourhood functions, for some natural χ, then we can associate with θ and χ a rank-1 axiom which is sound on θ-standard coalgebras.

Theorem 15. *Let $\theta\colon T^n \Rightarrow T$ be a natural operation on T, and let $\xi\colon X \to (TX)^A$ be a θ-standard T^A-coalgebra. Let $\lambda\colon \mathcal{Q} \Rightarrow \mathcal{Q} \circ T$ be a predicate lifting for T. If there is a natural operation $\chi\colon \mathcal{N}^n \Rightarrow \mathcal{N}$ such that*

$$\widehat{\lambda} \circ \theta = \chi \circ \widehat{\lambda}^n \tag{7}$$

then for all $\alpha_1, \ldots, \alpha_n \in A$, the θ-axiom $[\underline{\theta}(\alpha_1, \ldots, \alpha_n)]p \leftrightarrow \varphi(\breve{\chi}, \alpha_1, \ldots, \alpha_n, p)$ is valid in ξ (where $\underline{\theta}$ denotes the syntax/term constructor associated with θ). The above statement generalises to natural Σ-algebras $\theta\colon \Sigma T \Rightarrow T$ by considering the axioms for the components θ_σ and χ_σ for $\sigma \in \Sigma$.

Proof. For all $\alpha \in A$, let $m_\alpha = \xi(\alpha)^{-1} \circ \lambda$ be the predicate transformer for $[\alpha]$ in ξ. First, we prove that for all $x \in X$,

$$(\xi, V), x \models \varphi(\breve{\chi}, \alpha_1, \ldots, \alpha_n, p) \iff x \in \overline{\chi}_X^X(m_{\alpha_1}, \ldots, m_{\alpha_n})([\![p]\!]) \tag{8}$$

The proof is by straightforward induction on $\breve{\chi}$.

$$\text{Case } \breve{\imath}_j: \quad (\xi, V), x \models \varphi(\breve{\imath}_j, \alpha_1, \ldots, \alpha_n, p) \iff (\xi, V), x \models [\alpha_j]p \iff x \in m_{\alpha_j}([\![p]\!]) \iff x \in \bar{\imath}_X^X(m_{\alpha_j})([\![p]\!])$$

$$\text{Case } \breve{c}_j: \quad (\xi, V), x \models \varphi(\breve{c}_j, \alpha_1, \ldots, \alpha_n, p) \iff (\xi, V), x \models [\alpha_j]\neg p \iff x \in m_{\alpha_j}(X \setminus [\![p]\!]) \iff x \in \bar{c}_X^X(m_{\alpha_j})([\![p]\!])$$

$$\text{Case } \breve{z}_j: \quad (\xi, V), x \models \varphi(\breve{z}_j, \alpha_1, \ldots, \alpha_n, p) \iff (\xi, V), x \models [\alpha_j]\bot \iff x \in m_{\alpha_j}(\emptyset) \iff x \in \bar{z}_X^X(m_{\alpha_j})([\![p]\!])$$

$$\text{Case } \breve{t}_j: \quad (\xi, V), x \models \varphi(\breve{t}_j, \alpha_1, \ldots, \alpha_n, p) \iff (\xi, V), x \models [\alpha_j]\top \iff x \in m_{\alpha_j}(X) \iff x \in \bar{t}_X^X(m_{\alpha_j})([\![p]\!])$$

$$\text{Case } \neg: \quad (\xi, V), x \models \varphi(\neg\breve{\chi}, \alpha_1, \ldots, \alpha_n, p) \iff (\xi, V), x \models \neg\varphi(\chi, \alpha_1, \ldots, \alpha_n, p) \overset{\text{(IH)}}{\iff} x \notin \overline{\chi}_X^X(m_{\alpha_1}, \ldots, m_{\alpha_n})([\![p]\!]) \iff x \in \overline{\neg\chi}_X^X(m_{\alpha_1}, \ldots, m_{\alpha_n})([\![p]\!])$$

$$\begin{aligned} \text{Case } \wedge: \quad & (\xi, V), x \models \varphi(\neg\breve{\chi} \wedge \breve{\delta}, \alpha_1, \ldots, \alpha_n, p) \\ \iff & (\xi, V), x \models \varphi(\breve{\chi}, \alpha_1, \ldots, \alpha_n, p) \wedge \varphi(\breve{\delta}, \alpha_1, \ldots, \alpha_n, p) \\ \text{(IH)} \iff & x \in \overline{\chi}_X^X(m_{\alpha_1}, \ldots, m_{\alpha_n})([\![p]\!]) \cap \overline{\delta}_X^X(m_{\alpha_1}, \ldots, m_{\alpha_n})([\![p]\!]) \\ \iff & x \in \overline{(\chi \wedge \delta)}_X^X(m_{\alpha_1}, \ldots, m_{\alpha_n})([\![p]\!]) \end{aligned}$$

We can now prove the theorem:

$$\begin{aligned} (\xi, V), x \models [\underline{\theta}(\alpha_1, \ldots, \alpha_n)]p \iff & \xi(x)(\underline{\theta}(\alpha_1, \ldots, \alpha_n) \in \lambda_X [\![p]\!] \\ (\xi \text{ is } \theta\text{-standard}) \iff & \theta_X^X(\widehat{\xi}(\alpha_1), \ldots, \widehat{\xi}(\alpha_n))(x) \in \lambda_X([\![p]\!]) \\ (\text{transpose}) \iff & [\![p]\!] \in \widehat{\lambda}_X(\theta_X(\widehat{\xi}(\alpha_1)(x), \ldots, \widehat{\xi}(\alpha_n)(x))) \\ (\text{assumption (7)}) \iff & [\![p]\!] \in \chi_X(\widehat{\lambda}_X(\widehat{\xi}(\alpha_1)(x)), \ldots, \widehat{\lambda}(\widehat{\xi}(\alpha_n)(x))) \\ (\dagger) \iff & x \in \overline{\chi}_X^X(m_{\alpha_1}, \ldots, m_{\alpha_n})([\![p]\!]) \\ (\text{equation (8)}) \iff & (\xi, V), x \models \varphi(\breve{\chi}, \alpha_1, \ldots, \alpha_n, p) \end{aligned}$$

where (†) follows from the fact that $\psi(m_{\alpha_j}) = \psi(\xi(\alpha_j)^{-1} \circ \lambda_X) = \widehat{\lambda} \circ \widehat{\xi}(\alpha_j)$ together with equation (6).

Example 16. Using Theorem 15 we find that

(i) The PDL axiom $[\alpha \cup \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$ is valid because $\widehat{\lambda^{\Box}}: \mathcal{P} \Rightarrow \mathcal{N}$ transforms unions into intersections, i.e., $\widehat{\lambda}_X^{\Box}(U_1 \cup U_2) = \widehat{\lambda}_X^{\Box}(U_1) \cap \widehat{\lambda}_X^{\Box}(U_2)$. That is, we can apply Theorem 15 with $\theta = \cup: \mathcal{P}^2 \Rightarrow \mathcal{P}$ and $\chi = \cap: \mathcal{N}^2 \Rightarrow \mathcal{N}$.

(ii) The axiom $[\alpha \cup \beta]p \leftrightarrow [\alpha]p \vee [\beta]p$ is valid in standard GL-models because the transpose of the predicate lifting $\lambda_X(U) = \{N \in \mathcal{M}X \mid U \in N\}$ for the monotonic modality is the natural inclusion $\widehat{\lambda}: \mathcal{M} \Rightarrow \mathcal{N}$, i.e., $\theta = \chi = \cup$. Similarly, for the dual axiom $[\alpha^d]p \leftrightarrow \neg[\alpha]\neg p$.

(iii) For the lift monad we find that λ^{tl} turns *nil* into χ_{nil} where $\chi_{nil,X}(N) = \emptyset$ for all $N \in \mathcal{N}X$. Hence, we have the axiom $[nil]p \leftrightarrow \bot$. Dually, λ^{pl} turns *nil* into χ_{all} where $\chi_{all,X}(N) = \mathcal{P}(X)$ and we get the axiom $[nil]p \leftrightarrow \top$.

5 Completeness

In this section we will prove a generic strong completeness result for our family of coalgebraic dynamic logics.

Our completeness proof makes use of results from coalgebraic modal logic. Therefore we need to recall some terminology: A modal logic $\mathcal{L} = (\Lambda, \mathrm{Ax}, \mathrm{Fr})$ consists of a modal signature Λ, a collection $\mathrm{Ax} \subseteq \mathsf{Prop}(\Lambda(\mathsf{Prop}(P)))$ of rank-1 axioms, and a collection $\mathrm{Fr} \subseteq \mathcal{F}(\Lambda)$ of frame conditions. For a formula $\varphi \in \mathcal{F}(\Lambda)$, we write $\vdash_{\mathcal{L}} \varphi$ if φ can be derived from $\mathrm{Ax} \cup \mathrm{Fr}$ using propositional reasoning, uniform substitution and the congruence rule: from $\varphi_1 \leftrightarrow \varphi_1, \ldots, \varphi_n \leftrightarrow \psi_n$ infer $\Box_\lambda(\varphi_1, \ldots, \varphi_n) \leftrightarrow \Box_\lambda(\psi_1, \ldots, \psi_n)$ for any n-ary $\Box_\lambda \in \Lambda$.

- A formula $\varphi \in \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ is *one-step $\mathcal{L}$-derivable*, denoted $\vdash^1_{\mathcal{L}} \varphi$, if φ is propositionally entailed by the set $\{\psi\tau \mid \tau : P \to \mathcal{P}(X), \psi \in \mathrm{Ax}\}$.
- A set $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ is called *one-step $\mathcal{L}$-consistent* if there are no formulas $\varphi_1, \ldots, \varphi_n \in \Phi$ such that $\vdash^1_{\mathcal{L}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$.
- Let T be a Set-functor and assume a predicate lifting λ is given for each $\Box_\lambda \in \Lambda$. For a formula $\varphi \in \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ the *one-step semantics* $[\![\varphi]\!]_1 \subseteq TX$ is defined by putting $[\![\Box_\lambda(U_1, \ldots, U_n)]\!]_1 = \lambda_X(U_1, \ldots, U_n)$ and by inductively extending this definition to Boolean combinations of boxed formulas.
- For a set $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ of formulas, we let $[\![\Phi]\!]_1 = \bigcap_{\varphi \in \Phi} [\![\varphi]\!]_1$, and we say that Φ is *one-step satisfiable* if $[\![\Phi]\!]_1 \neq \emptyset$.
- $\mathcal{L}$ is called *one-step sound* if for any one-step derivable formula $\varphi \in \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ we have $[\![\varphi]\!]_1 = TX$, i.e., if any such formula φ is *one-step valid*.
- $\mathcal{L}$ is called *strongly one-step complete* if for every set X and every one-step consistent set $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ is one-step satisfiable.
- $\mathcal{L}$ is *separating* if $t \in TX$ is uniquely determined by the set $\{\Phi \in \Lambda(\mathcal{P}(X)) \mid t \in [\![\Phi]\!]_1\}$.

Throughout the section we assume the following are given:

- a pointed monad $\mathbb{T}$ on Set,
- a single, unary predicate lifting $\lambda \colon \mathcal{Q} \Rightarrow \mathcal{Q} \circ T$ for T whose transpose $\widehat{\lambda}$ is a monad morphism,
- a countable set P of atomic propositions,
- a countable set A_0 of atomic actions, and $\theta \colon \Sigma T \Rightarrow T$, a natural Σ-algebra on T.
- To ensure soundness, we also assume that there is a natural algebra $\chi \colon \Sigma\mathcal{N} \Rightarrow \mathcal{N}$ such that $\widehat{\lambda} \circ \theta = \chi \circ \widehat{\lambda}^n$ (cf. Theorem 15).
- We let $\Lambda = \{[\alpha] \mid \alpha \in A(P, A_0, \Sigma)\}$.

Let us now clarify which logics we are considering. Firstly, we assume we have a separating, one-step sound and strongly one-step complete rank-1 axiomatisation $\mathrm{Ax}(T, \Box)$ over T-coalgebras in the basic modal language $\mathcal{F}(\{\Box\})$. The "underlying" logic $(\{\lambda\}, \mathrm{Ax}(T, \Box), \emptyset)$ will be denoted by $\mathcal{L}_b$. Given an action $\alpha \in A$, we denote by $\mathrm{Ax}(T, \Box)_\alpha$ the set of rank-1 axioms over the labelled modal language $\mathcal{F}(\{[\alpha] \mid \alpha \in A\})$ obtained by replacing all ocurrences of $\Box$ by $[\alpha]$, and we let $\mathrm{Ax}(T, \Box)_A = \bigcup_{\alpha \in A} \mathrm{Ax}(T, \Box)_\alpha$ be all labelled instances of rank-1 axioms in $\mathrm{Ax}(T, \Box)$.

Definition 17 (Dynamic logic). *We define*

$$\begin{aligned} \mathrm{Ax} &= \mathrm{Ax}(T, \Box)_A \cup \{\varphi(\sigma, \alpha_1, \dots, \alpha_n, p) \mid \sigma \in \Sigma, \alpha_i \in A\} \\ \mathrm{Fr} &= \{[\alpha; \beta]p \leftrightarrow [\alpha][\beta]p \mid \alpha, \beta \in A(P, A_0, \Sigma), p \in P\} \\ \mathcal{L}(\theta) &= (\Lambda, \mathrm{Ax}, \emptyset), \\ \mathcal{L}(\theta, ;) &= (\Lambda, \mathrm{Ax}, \mathrm{Fr}). \end{aligned}$$

We refer to $\mathcal{L}(\theta)$ and $\mathcal{L}(\theta, ;)$ as (P, A_0, θ)-dynamic logics.

We are now going to prove completeness of both $\mathcal{L}(\theta)$ and $\mathcal{L}(\theta, ;)$ with respect to θ-standard and $\theta, ;$-standard models, respectively. This will be done in the following steps: we first prove that $\mathcal{L}(\theta)$ is sound and strongly complete with respect to θ-standard models. This will be achieved by proving the existence of a θ-standard "quasi-canonical" model for $\mathcal{L}(\theta)$ using results from [31]. After that we prove that the θ-standard quasi-canonical model can be turned into a $\theta, ;$-standard one. This will prove that $\mathcal{L}(\theta, ;)$ is sound and strongly complete with respect to $\theta, ;$-standard models.

We now turn to the details of the proof. In order to be able to use the existence result for quasi-canonical models from [31] we will first show that θ-standard models can be characterised as those models that are based on T^A_{st}-coalgebras for a suitable subfunctor T^A_{st} of T^A. This is done using the following definition of θ-standard that can be seen as a "point-wise" version of Definition 7.

Definition 18. *We say a function $f : A \to TX$ is θ-standard if f is a Σ-algebra morphism (from $\delta \colon \Sigma A \to A$ to $\theta_X \colon \Sigma TX \to TX$ where δ is as in Def. 7):*

$$f \circ \delta = \theta_X \circ \Sigma f \tag{9}$$

Furthermore we let $T^A_{\mathrm{st}}X = \{f : A \to TX \mid f \text{ is } \theta\text{-standard}\}$. It is easy to check that T^A_{st} can be extended to a subfunctor of T^A.

The following lemma establishes the correspondence between θ-standard coalgebras and coalgebras for the functor T^A_{st}:

Lemma 19. *A coalgebra $\xi : X \to (TX)^A$ is θ-standard iff ξ is a T^A_{st}-coalgebra.*

Proof. For simplicity we assume that $\Sigma = (\_)^2$, i.e., the signature consists of one binary operation. Then ξ is a T^A_{st}-coalgebra iff for all $x \in X$ and all $a_1, a_2 \in A$ we have $\theta_X(\xi(a_1)(x), \xi(a_2)(x)) = \xi(\delta(a_1, a_2))(x)$ iff for all $a_1, a_2 \in A$ we have

$\lambda x.(\theta_X(\xi(a_1)(x), \xi(a_2)(x))) = \lambda x.(\xi(\delta(a_1, a_2))(x))$. Now the claim follows from the observation that $\lambda x.(\xi(\delta(a_1, a_2))(x)) = \hat{\xi}(\delta(a_1, a_2))$ and that

$$\lambda x.(\theta_X(\xi(a_1)(x), \xi(a_2)(x))) = \theta_X^X(\lambda x.\xi(a_1)(x), \lambda x.\xi(a_2)(x)) = \theta_X^X(\hat{\xi}(a_1), \hat{\xi}(a_2))$$

Let us now start with our completeness proof. We are first going to check that $\mathcal{L}(\theta)$ is *one-step* sound over θ-standard models.

Proposition 20. *The logic $\mathcal{L}(\theta)$ is one-step sound for T_{st}^A.*

Proof. Consider an arbitrary one-step derivable formula $\varphi \in \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ for some set X. By the definition of one-step derivability we have that φ is propositionally entailed by the set $\{\psi\tau \mid \tau : P \to \mathcal{P}(X), \psi \in \text{Ax}\}$. Clearly the claim that $[\![\varphi]\!]_1 = T_{\text{st}}^A X$ follows if we can show that $[\![\psi\tau]\!]_1 = T_{\text{st}}^A X$ for all $\psi \in \text{Ax}$ and all $\tau : P \to \mathcal{P}(X)$.

Consider some $\psi \in \text{Ax}$ and recall from Section 4 (and our assumptions) that the axioms in Ax are sound wrt T_{st}^A-coalgebras. Suppose for a contradiction that there is a $\tau : P \to \mathcal{P}(X)$ such that $[\![\psi\tau]\!]_1 \neq T_{\text{st}}^A X$, i.e., there exists some $t \in T_{\text{st}}^A X$ such that $t \notin [\![\psi\tau]\!]_1$. Define the model $\mathfrak{M} = (X, \xi, V)$ by putting $\xi(x) = t$ for all $x \in X$ and by putting $V(p) = \tau(p)$ for all $p \in P$. With these definitions in place it is now a matter of routine checking to see that ψ is not satisfied in any state $x \in X$ in the model $\mathfrak{M}$. This contradicts our assumption of soundness of Ax and we conclude that $[\![\psi\tau]\!]_1 = T_{\text{st}}^A X$ for an arbitrary $\psi \in \text{Ax}$ and a valuation τ. This implies one-step soundness of $\mathcal{L}(\theta)$.

One-step soundness will be enough to prove soundness of our logic. In order to prove completeness we are now going to show that the logic is strongly one-step complete.

Proposition 21. *The logic $\mathcal{L}(\theta)$ is strongly one-step complete for T_{st}^A, that is, for all sets X, every one-step $\mathcal{L}(\theta)$-consistent set of formulas $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ is one-step satisfiable in $T_{\text{st}}^A X$.*

Proof. By the Lindenbaum lemma, it suffices to prove that for all sets X, every maximally one-step $\mathcal{L}(\theta)$-consistent set of formulas $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ is one-step satisfiable in $T_{\text{st}}^A X$. For any set $\Psi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ we define for each $\alpha \in A$ the following sets

$$\Psi_\alpha = \{[\alpha]Y \mid [\alpha]Y \in \Psi\} \cup \{\neg[\alpha]Y \mid \neg[\alpha]Y \in \Psi\}.$$

and we let $\text{Lit}(\Psi) = \bigcup_{\alpha \in L} \Psi_\alpha$ the set of "boxed literals" in Ψ.

Our first observation is that any maximally one-step $\mathcal{L}(\theta)$-consistent Ψ is one-step satisfiable iff $\text{Lit}(\Psi)$ is. The implication from left to right is clear. To see why the other implication holds, assume that $t \in [\![\text{Lit}(\Psi)]\!]_1$. We show that for all $\psi \in \Psi$, $t \in [\![\psi]\!]_1$ by induction on ψ. We may assume that ψ is in negation normal form, i.e., that all negations are in front of modalities. The cases where ψ is of the form $[\alpha]Y$ or $\neg[\alpha]Y$ are immediate. The cases where ψ is a disjunction or a conjunction follow from maximality of Ψ.

Let $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ be maximally one-step $\mathcal{L}(\theta)$-consistent. We are going to define an element $t \in T^A_{\mathrm{st}}X$ that (one-step) satisfies Φ, i.e., with the property that $t \in [\![\Phi]\!]_1^{T^A_{\mathrm{st}}X}$. For each action α we define the *term height* $\mathrm{th}(\alpha)$ inductively as follows:

$$\begin{array}{ll} \mathrm{th}(a) \quad = 0 & \mathrm{th}(\varphi?) \qquad = 0 \\ \mathrm{th}(\alpha;\beta) = 0 & \mathrm{th}(\underline{\theta}(\alpha_1,\dots,\alpha_n)) = \max\{\mathrm{th}(\alpha_1),\dots,\mathrm{th}(\alpha_n)\} + 1 \end{array}$$

where a denotes an arbitrary atomic action, $\alpha_1,\dots,\alpha_n$ denote arbitrary actions and φ denotes an arbitrary formula of $\mathcal{L}(\theta)$. Finally we denote by $\mathrm{Lit}_i(\Phi)$ the set of boxed literals in Φ in which all occurring modalities $[\alpha]$ satisfy $\mathrm{th}(\alpha) \le i$.

Since Φ is (maximally) one-step $\mathcal{L}(\theta)$-consistent, the sets $\Phi^{\Box}_\alpha = \{\Box Y \mid [\alpha]Y \in \Phi\} \cup \{\neg\Box Y \mid [\alpha]Y \in \Phi\}$ are one-step $\mathcal{L}_b$-consistent and hence, by assumption of one-step completeness of $\mathcal{L}_b$, we know that there is an element $t_\alpha \in TX$ with $t_\alpha \in [\![\Phi^{\Box}_\alpha]\!]_1^{TX}$ for all $\alpha \in A$. This can be used in order to define an element $t\colon A \to TX$ of $T^A_{\mathrm{st}}X$ as follows:

$$\begin{array}{rl} t(\alpha) = t_\alpha & \text{for actions } \alpha \text{ with } \mathrm{th}(\alpha) = 0 \\ t(\underline{\theta}(\alpha_1,\dots,\alpha_n)) = \theta_X(t(\alpha_1),\dots,t(\alpha_n)) & \text{for pointwise operations } \underline{\theta} \end{array}$$

We now claim that $t \in [\![\Phi]\!]_1^{T^A_{\mathrm{st}}X}$ as required. As discussed previously it suffices to prove the claim for boxed literals in Φ, i.e., that $t \in [\![\mathrm{Lit}(\Phi)]\!]_1^{T^A_{\mathrm{st}}X}$. The proof proceeds by induction on the height of the action α of a given boxed literal $[\alpha]Y$ or $\neg[\alpha]Y$, for presentational reasons we restrict ourselves to positive boxed atoms.

In case $\mathrm{th}(\alpha) = 0$ the claim that $t \in [\![[\alpha]Y]\!]_1^{T^A_{\mathrm{st}}X}$ is equivalent to $t(\alpha) \in [\![\Box Y]\!]_1^{T^A_{\mathrm{st}}X}$ and the latter holds by definition of t_α, because $t(\alpha) = t_\alpha$.

In case $\alpha = \underline{\theta}(\alpha_1,\dots,\alpha_n)$ is of term height $i+1$ the axiomatisation Ax contains the axiom $[\underline{\theta}(\alpha_1,\dots,\alpha_n)]p \leftrightarrow \varphi(\check{\chi},\alpha_1,\dots,\alpha_n,p)$ where $\varphi(\check{\chi},\alpha_1,\dots,\alpha_n,p)$ contains only actions of height strictly less than $i+1$. By one-step soundness of Ax on T^A_{st} we obtain

$$t \in [\![[\alpha]Y]\!]_1^{T^A_{\mathrm{st}}X} \qquad \text{iff} \qquad t \in [\![\varphi(\check{\chi},\alpha_1,\dots,\alpha_n,Y)]\!]_1^{T^A_{\mathrm{st}}X} \tag{10}$$

By maximality of Φ we have $\varphi(\check{\chi},\alpha_1,\dots,\alpha_n,Y) \in \Phi$. By I.H. we know that $\mathrm{Lit}_i(\Phi)$ is one-step satisfied at t which in turn implies $\varphi(\check{\chi},\alpha_1,\dots,\alpha_n,Y)$ is one-step satisfied at t (truth of $\varphi(\check{\chi},\alpha_1,\dots,\alpha_n,Y)$ only depends on literals with actions β such that $\mathrm{th}(\beta) < i$). This shows, using (10) that $t \in [\![[\alpha]Y]\!]_1^{T^A_{\mathrm{st}}X}$

The property of a functor preserving inverse limits of surjective ω-cochains from [31] is one of the main conditions for the existence of quasi-canonical models in Proposition 25 below.[5] Please consult *loc.cit.* for the definition.

Proposition 22. *If T weakly preserves inverse limits of surjective ω-cochains, then so does the functor T^A_{st}.*

[5] The condition in [31] is motivated by a stronger condition used in [19, Thm. 9.4].

Proof. Consider a surjective ω-cochain $\{X_i\}_{i\in\omega}$. As T preserves the inverse limit of the cochain the map $e : T(\varprojlim X_i) \to \varprojlim TX_i$ that exists by the universal property of $\varprojlim TX_i$ is surjective. It is easy to see that the corresponding map $e' : (T^A_{\mathrm{st}} \varprojlim X_i) \to \varprojlim T^A_{\mathrm{st}} X_i$ for the functor T^A_{st} is given by restricting $e^A : T^A(\varprojlim X_i) \to \varprojlim T{X_i}^A$ to $T^A_{\mathrm{st}}(\varprojlim X_i)$: In order to verify this, it suffices to show that the following diagram commutes where we denote by $i_X : T^A_{\mathrm{st}} X \to T^A X$ the natural inclusion map.

$$\begin{array}{ccc} T^A_{\mathrm{st}}(\varprojlim X_i) & \xrightarrow{e'} & \varprojlim T^A_{\mathrm{st}} X_i \\ {\scriptstyle i_{\varprojlim X_i}}\downarrow & & \downarrow{\scriptstyle \varprojlim i} \\ T^A(\varprojlim X_i) & \xrightarrow{e^A} & \varprojlim T{X_i}^A \end{array}$$

This can be proven by showing that for all projections $p_i : \varprojlim T^A X_i \to T^A X_i$ we have the following:

$$\begin{aligned} p_i \circ \varprojlim i \circ e' &= i_{X_i} \circ p_i^{T^A_{\mathrm{st}}} \circ e' \qquad (p_i^{T^A_{\mathrm{st}}} \text{ denotes } i\text{th projection from } \varprojlim T^A_{\mathrm{st}} X_i) \\ &= i_{X_i} \circ T^A_{\mathrm{st}} \pi_i \qquad (\text{by definition of } e') \\ &= T^A \pi_i \circ i_{\varprojlim X_i} \qquad (\text{naturality of } i) \\ &= p_i \circ e^A \circ i_{\varprojlim X_i} \qquad (\text{by definition of } e) \end{aligned}$$

Furthermore for all $j \in \omega$ we have a simple naturality diagram that shows that the projections $T\pi_j : T \varprojlim X_i \to TX_j$ are Σ-algebra morphisms:

$$\begin{array}{ccc} \Sigma T \varprojlim X_i & \xrightarrow{\Sigma T \pi_j} & \Sigma T X_j \\ {\scriptstyle \theta_{\varprojlim X_i}}\downarrow & & \downarrow{\scriptstyle \theta_{X_j}} \\ T \varprojlim X_i & \xrightarrow[T\pi_j]{} & TX_j \end{array} \tag{11}$$

The claim of the proposition can now be obtained as follows: Let $\{f_i : L \to TX_i \mid f_i \in T^A_{\mathrm{st}} X_i\}$ be an element of $\varprojlim T^A_{\mathrm{st}} X_i$. By assumption on T^A_{st} this means that all f_i's are Σ-algebra morphisms. We define a function f_Σ from the set A_Σ of Σ-atomic actions to $T(\varprojlim X_i)$ by putting

$$f_\Sigma(\alpha) = t \qquad \text{for some } t \in T(\varprojlim X_i) \text{ with } e(t) = \{f_i(\alpha) \mid i \in \omega\} \in \varprojlim TX_i$$

where $\alpha \in A_\Sigma$. The necessary t's exist by surjectivity of e. Because A can be seen as the free Σ-algebra over A_Σ, the function f_Σ uniquely determines a Σ-algebra morphism $\hat{f}_\Sigma : A \to T(\varprojlim X_i)$ where the algebra structure on the codomain is given by the natural algebra $\theta_{\varprojlim X_i}$. Obviously we have $\hat{f}_\Sigma \in T^A_{\mathrm{st}}(\varprojlim X_i)$ because $\hat{f}_\Sigma$ is a Σ-algebra morphism by definition. We claim that $e'(\hat{f}_\Sigma) = \{f_i \in T^A_{\mathrm{st}} X_i \mid i \in \omega\}$ as required. Consider an arbitrary $i \in \omega$. By the fact that e^A restricts to e' we have $\pi_i \circ e'(\hat{f}_\Sigma) = \pi_i \circ e \circ \hat{f}_\Sigma$. The latter is equal to $T\pi_i \circ \hat{f}_\Sigma$. Because $T\pi_i \circ \hat{f}_\Sigma$ is the unique Σ-algebra morphism such that its restriction to A_Σ yields f_Σ and because we have $f_i(\alpha) = f_\Sigma(\alpha)$ for all $\alpha \in A_\Sigma$ by definition, we obtain $T\pi_i \circ \hat{f}_\Sigma = f_i$. This implies $\pi_i \circ e'(\hat{f}_\Sigma) = f_i$ for all $i \in \omega$ and hence $e'(\hat{f}_\Sigma) = f$ as required.

Using the results from [31], Propositions 21 and 22 imply that $\mathcal{L}(\theta)$ is sound and strongly complete with respect to T^A_{st}-coalgebras. This is achieved by proving the existence of so-called quasi-canonical models.

Definition 23. *Let F be any* Set*-functor. A quasi-canoncial F-model for a dynamic modal logic $\mathcal{L} = (\Lambda, \text{Ax}, \text{Fr})$ is a T-model $(S, \xi \colon S \to F(S), V)$ that satisfies all axioms* Ax *and frame conditions* Fr *and such that*

- *S is the set of maximal $\mathcal{L}$-consistent sets of formulas,*
- *$V(p) = \{\Gamma \in S \mid p \in \Gamma\}$ and*
- *for all $\Gamma \in S$, $\alpha \in A$ and all formulas φ we have:*

$$\xi(\Gamma)(\alpha) \in \lambda(\hat{\varphi}) \qquad \textit{iff} \qquad [\alpha]\varphi \in \Gamma, \tag{12}$$

where $\hat{\varphi} = \{\Gamma \in S \mid \varphi \in \Gamma\}$.

In other words, the states of a quasi-canonical model for some logic $\mathcal{L}$ are precisely all maximally $\mathcal{L}$-consistent sets of formulae. The crucial property of quasi-canonical models is that they have the "truth is membership" property, often referred to as the truth lemma.

Lemma 24 (Truth Lemma). *If $(S, \xi : S \to F(S), V)$ is a quasi-canonical F-model for $\mathcal{L} = (\Lambda, \text{Ax}, \text{Fr})$, then for all $\Gamma \in S$, and all formulas $\varphi \in \mathcal{F}(\Lambda)$,*

$$\Gamma \in [\![\varphi]\!] \quad \textit{iff} \quad \varphi \in \Gamma. \tag{13}$$

Proof. The proof is by induction on φ. The base case follows from the condition on the valuation V, induction steps for Boolean connectives follow from Γ being maximally $\mathcal{L}$-consistent, and the modal case follows from condition (12).

The existence of a quasi-canonical F-model for a logic $\mathcal{L}$ ensures that all maximally $\mathcal{L}$-consistent sets of formulae are satisfied in an F-model that validates all axioms and frame conditions, and hence that the logic is strongly complete with respect this class of F-models. The following proposition summarises our findings for the logic $\mathcal{L}(\theta)$:

Proposition 25. *The logic $\mathcal{L}(\theta)$ has a θ-standard quasi-canonical T^A-model. Consequently, $\mathcal{L}(\theta)$ is sound and strongly complete with respect to the class of all θ-standard T^A-models.*

Proof. This follows from the results in [31] and the facts that $\mathcal{L}(\theta)$ is one-step sound and strongly one-step complete for T^A_{st} and that T^A_{st} preserves inverse limits of surjective ω-cochains. Hence $\mathcal{L}(\theta)$ has a quasi-canonical T^A_{st}-model, which by Lemma 19 is a quasi-canonical θ-standard T^A-model, as required.

Next, we prove that $\mathcal{L}(\theta, ;)$ is complete with respect to the class of $\theta, ;$-standard dynamic T-models. To this end, we show the existence of a $\theta, ;$-standard quasi-canonical T^A-model. (Tests will be discussed in section 6.) In other words, we want to ensure the *validity* of the sequential composition axioms on the quasi-canonical frame $\xi \colon X \to (TX)^A$. From a coalgebraic perspective this is a non-trivial task as one cannot deal with axioms of rank greater than 1 in a generic coalgebraic way. In particular, we cannot assume that a quasi-canonical T^A_{st}-model of $\mathcal{L}(\theta)$ is $;$-standard, but show that we can modify a quasi-canonical T^A-model into $\theta, ;$-standard T^A-model, which is again quasi-canonical.

Theorem 26. *The logic $\mathcal{L}(\theta, ;)$ is sound and strongly complete with respect to all $\theta, ;$-standard models.*

Proof. Soundness follows easily from the results in Section 4. Let $(S, \gamma \colon S \to (TS)^A, V)$ be a quasi-canonical T^A-model which exists by Prop. 25). Note that we do not use that $(S, \gamma \colon S \to (TS)^A, V)$ is θ-standard, as we will modify its structure, and in doing so, we make sure that the resulting structure is θ-standard. Define the T^A-coalgebra $\xi \colon S \to (TS)^A$ by taking $\widehat{\xi}(a) = \widehat{\gamma}(a)$ for all atomic actions $a \in A_0$, and extend inductively to complex programs such that ξ is the unique $\theta, ;$-standard T^A-coalgebra that agrees with γ on atomic actions. From Lemma 27 below, it follows that ξ also satisfies the coherence condition (12), and hence that (S, ξ, V) is a $\theta, ;$-standard quasi-canonical T^A-model.

Lemma 27. *For all dynamic (test-free) formulas $[\alpha]\varphi \in \mathcal{F}(\Lambda)$ and all $\Gamma \in S$,*

$$\widehat{\gamma}(\alpha)(\Gamma) \in \lambda_S(\hat{\varphi}) \quad \textit{iff} \quad \widehat{\xi}(\alpha)(\Gamma) \in \lambda_S(\hat{\varphi}) \tag{14}$$

In the proof of Lemma 27, the following lemma will be useful.

Lemma 28. *For all $\zeta \colon S \to (TS)^A$ and all $\beta \in A$, if ζ is coherent at β, meaning that for all $\Gamma \in S$ and all dynamic formulas φ, $\widehat{\zeta}(\beta)(\Gamma) \in \lambda_S(\hat{\varphi}) \iff [\beta]\varphi \in \Gamma$, then for all $f \colon S \to T(S)$, all $\Gamma \in S$ and all dynamic formulas φ, we have that*

$$(f * \widehat{\zeta}(\beta))(\Gamma) \in \lambda_S(\hat{\varphi}) \iff f(\Gamma) \in \lambda_S(\widehat{[\beta]\varphi})$$

Proof. We have:

$$\begin{aligned}
(f * \widehat{\zeta}(\beta))(\Gamma) \in \lambda_S(\hat{\varphi}) \quad & \text{iff} \quad \mu_S\left(T\widehat{\zeta}(\beta)\,(f(\Gamma))\right) \in \lambda_S(\hat{\varphi}) \\
\text{(def. of } \hat{\lambda}) \quad & \text{iff} \quad \hat{\varphi} \in \hat{\lambda}_S(\mu_S(T\widehat{\zeta}(\beta)(\zeta(\Gamma)(\alpha)))) \\
(\hat{\lambda} \text{ monad morph.}) \quad & \text{iff} \quad \hat{\varphi} \in \mu^{\mathcal{N}}_S\left(\mathcal{N}\hat{\lambda}_S(\hat{\lambda}_{TS}(T\widehat{\zeta}(\beta)(f(\Gamma))))\right) \\
\text{(def. of } \mu^N) \quad & \text{iff} \quad \eta_{\mathcal{P}(S)}(\hat{\varphi}) \in \mathcal{N}\hat{\lambda}_S\left(\hat{\lambda}_{TS}(T\widehat{\zeta}(\beta)(f(\Gamma)))\right) \\
\text{(def. of } \mathcal{N}) \quad & \text{iff} \quad \hat{\lambda}_S^{-1}\left(\eta_{\mathcal{P}(S)}(\hat{\varphi})\right) \in \hat{\lambda}_{TS}(T\widehat{\zeta}(\beta)(f(\Gamma))) \\
\text{(def. of } \eta) \quad & \text{iff} \quad \{t \in TS \mid \hat{\varphi} \in \widehat{\lambda}_S(t)\} \in \hat{\lambda}_{TS}(T\widehat{\zeta}(\beta)(f(\Gamma))) \\
\text{(def. of } \hat{\lambda}) \quad & \text{iff} \quad \{t \in TS \mid t \in \lambda_S(\hat{\varphi})\} \in \hat{\lambda}_{TS}(T\widehat{\zeta}(\beta)(f(\Gamma))) \\
\text{(naturality of } \hat{\lambda}) \quad & \text{iff} \quad \{t \in TS \mid t \in \lambda_S(\hat{\varphi})\} \in \mathcal{N}(\widehat{\zeta}(\beta))(\hat{\lambda}_S(f(\Gamma))) \\
\text{(def. of } \mathcal{N}) \quad & \text{iff} \quad \widehat{\zeta}(\beta)^{-1}\left(\{t \in TS \mid t \in \lambda_S(\hat{\varphi})\}\right) \in \hat{\lambda}_S(f(\Gamma)) \\
(\widehat{\zeta}(\beta) \text{ coherent}) \quad & \text{iff} \quad \{\Delta \in S \mid [\beta]\varphi \in \Delta\} \in \hat{\lambda}_S(f(\Gamma)) \\
\text{(def. of } \hat{\lambda}) \quad & \text{iff} \quad f(\Gamma) \in \lambda_S(\widehat{[\beta]\varphi})
\end{aligned}$$

Proof of Lemma 27. The proof is by induction on the structure of α. The base case for atomic actions ($\alpha = a \in A_0$) holds by definition of ξ. For the induction step for pointwise operations, assume $\alpha = \sigma(\alpha_1, \ldots, \alpha_n)$ where $\sigma \colon T^n \Rightarrow T$ and $\chi \colon \mathcal{N}^n \Rightarrow \mathcal{N}$ are natural operations such that $\widehat{\lambda} \circ \sigma = \chi \circ \widehat{\lambda}^n$. We write $\varphi(\chi, , \alpha_1, \ldots, \alpha_n, \varphi)$ for the formula obtained by uniformly substituting φ for p in $\varphi(\chi, , \alpha_1, \ldots, \alpha_n, p)$. We have:

$$\begin{aligned}
\widehat{\gamma}(\sigma(\alpha_1, \ldots, \alpha_n))(\Gamma) \in \lambda_S(\hat{\varphi}) &\iff [\sigma(\alpha_1, \ldots, \alpha_n)]\varphi \in \Gamma && \text{((12) for } \gamma) \\
&\iff \varphi(\chi, , \alpha_1, \ldots, \alpha_n, \varphi) \in \Gamma && \text{(by } \sigma\text{-axiom)}
\end{aligned}$$

and since ξ is σ-standard,

$$\begin{aligned}
\widehat{\xi}(\sigma(\alpha_1, \ldots, \alpha_n)(\Gamma) \in \lambda_S(\hat{\varphi}) &\iff \sigma_S^S(\widehat{\xi}(\alpha_1), \ldots, \widehat{\xi}(\alpha_n))(\Gamma) \in \lambda_S(\hat{\varphi}) \\
&\iff \hat{\varphi} \in \widehat{\lambda}_S(\sigma_S(\widehat{\xi}(\alpha_1)(\Gamma), \ldots, \widehat{\xi}(\alpha_n)(\Gamma)) \\
&\iff \hat{\varphi} \in \chi_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma)))
\end{aligned}$$

Hence it suffices to show that

$$\varphi(\chi, , \alpha_1, \ldots, \alpha_n, \varphi) \in \Gamma \iff \hat{\varphi} \in \chi_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) \quad (15)$$

We prove (15) by induction in $\breve{\chi}$.
Base case $\breve{\chi} = \breve{i}_j$:

$$\begin{aligned}
[\alpha_j]\varphi \in \Gamma &\iff \widehat{\gamma}(\alpha_j)(\Gamma) \in \lambda_S(\hat{\varphi}) && (\gamma\text{-coh.}) \\
&\iff \widehat{\xi}(\alpha_j)(\Gamma) \in \lambda_S(\hat{\varphi}) && \text{(IH for (14))} \\
&\iff \hat{\varphi} \in \widehat{\lambda}_S(\widehat{\xi}(\alpha_j)(\Gamma)) \\
&\iff \hat{\varphi} \in (i_j)_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) && \text{(def. of } i_j)
\end{aligned}$$

Base case $\breve{\chi} = \breve{c}_j$:

$$\begin{aligned}
[\alpha_j]\neg\varphi \in \Gamma &\iff \widehat{\gamma}(\alpha_j)(\Gamma) \in \lambda_S(\widehat{\neg\varphi}) && (\gamma\text{-coh.}) \\
&\iff \widehat{\xi}(\alpha_j)(\Gamma) \in \lambda_S(\widehat{\neg\varphi}) && \text{(IH for (14))} \\
&\iff \widehat{\neg\varphi} \in \widehat{\lambda}_S(\widehat{\xi}(\alpha_j)(\Gamma)) \\
&\iff \hat{\varphi} \in (c_j)_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) && \text{(def. of } c_j)
\end{aligned}$$

Base case $\breve{\chi} = \breve{z}_j$:

$$\begin{aligned}
[\alpha_j]\bot \in \Gamma &\iff \widehat{\gamma}(\alpha_j)(\Gamma) \in \lambda_S(\emptyset) && (\gamma\text{-coh.}) \\
&\iff \widehat{\xi}(\alpha_j)(\Gamma) \in \lambda_S(\emptyset) && \text{(IH for (14))} \\
&\iff \emptyset \in \widehat{\lambda}_S(\widehat{\xi}(\alpha_j)(\Gamma)) \\
&\iff \mathcal{P}(S) = (z_j)_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) && \text{(def. of } z_j) \\
&\iff \hat{\varphi} \in (z_j)_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) && \text{(def. of } z_j)
\end{aligned}$$

Base case $\breve{\chi} = \breve{t}_j$:

$$\begin{aligned}
[\alpha_j]\top \in \Gamma &\iff \widehat{\gamma}(\alpha_j)(\Gamma) \in \lambda_S(S) && (\gamma\text{-coh.}) \\
&\iff \widehat{\xi}(\alpha_j)(\Gamma) \in \lambda_S(S) && \text{(IH for (14))} \\
&\iff S \in \widehat{\lambda}_S(\widehat{\xi}(\alpha_j)(\Gamma)) \\
&\iff \mathcal{P}(S) = (t_j)_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) && \text{(def. of } t_j) \\
&\iff \hat{\varphi} \in (t_j)_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) && \text{(def. of } t_j)
\end{aligned}$$

Induction step for ¬:

$$\begin{aligned}
& \varphi(\neg\breve{\chi}, \alpha_1, \ldots, \alpha_n, \varphi) \in \Gamma \\
\iff & \neg\varphi(\breve{\chi}, \alpha_1, \ldots, \alpha_n, \varphi) \in \Gamma \\
\iff & \hat{\varphi} \notin \chi_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) \qquad \text{(IH and } \Gamma \text{ max.cons.)} \\
\iff & \hat{\varphi} \in (\neg\chi)_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma)))
\end{aligned}$$

Induction step for ∨:

$$\begin{aligned}
& \varphi(\breve{\chi} \vee \breve{\delta}, \alpha_1, \ldots, \alpha_n, \varphi) \in \Gamma \\
\iff & \varphi(\breve{\chi}, \alpha_1, \ldots, \alpha_n, \varphi) \in \Gamma \text{ or } \varphi(\breve{\delta}, \alpha_1, \ldots, \alpha_n, \varphi) \in \Gamma \\
\iff & \hat{\varphi} \in \chi_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) \text{ or} \\
& \hat{\varphi} \in \delta_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma))) \qquad \text{(IH and } \Gamma \text{ max.cons.)} \\
\iff & \hat{\varphi} \in (\chi \cup \delta)_S(\widehat{\lambda}_S(\widehat{\xi}(\alpha_1)(\Gamma)), \ldots, \widehat{\lambda}_S(\widehat{\xi}(\alpha_n)(\Gamma)))
\end{aligned}$$

We now return to the main induction for (14), and the induction step for sequential composition. We first prove that

$$\widehat{\gamma}(\alpha;\beta)(\Gamma) \in \lambda_S(\hat{\varphi}) \iff (\widehat{\gamma}(\alpha) * \widehat{\gamma}(\beta))(\Gamma) \in \lambda_S(\hat{\varphi}) \tag{16}$$

We have,

$$\begin{aligned}
& (\widehat{\gamma}(\alpha) * \widehat{\gamma}(\beta))(\Gamma) \in \lambda_S(\hat{\varphi}) \\
(\gamma \text{ coherent and Lemma 28}) \text{ iff } & \widehat{\gamma}(\alpha)(\Gamma) \in \lambda_S(\widehat{[\beta]\varphi}) \\
(\gamma \text{ coherent}) \text{ iff } & [\alpha][\beta]\varphi \in \Gamma \\
(\text{Axiom}) \text{ iff } & [\alpha;\beta]\varphi \in \Gamma \\
(\gamma \text{ coherent}) \text{ iff } & \widehat{\gamma}(\alpha;\beta)(\Gamma) \in \lambda_S(\hat{\varphi})
\end{aligned}$$

By the induction hypothesis,

$$\widehat{\gamma}(\beta)(\Gamma) \in \lambda_S(\hat{\varphi}) \iff \widehat{\xi}(\beta)(\Gamma) \in \lambda_S(\hat{\varphi}) \tag{17}$$

$$\widehat{\gamma}(\alpha)(\Gamma) \in \lambda_S([\hat{\beta}]\varphi) \iff \widehat{\xi}(\alpha)(\Gamma) \in \lambda_S([\hat{\beta}]\varphi) \tag{18}$$

Lemma 28 can clearly be applied on the left-hand side of (18), but also on the right-hand side, since ξ is coherent at β due to (17). We therefore have,

$$(\widehat{\gamma}(\alpha) * \widehat{\gamma}(\beta))(\Gamma) \in \lambda_S(\hat{\varphi}) \iff (\widehat{\xi}(\alpha) * \widehat{\xi}(\beta))(\Gamma) \in \lambda_S(\hat{\varphi})$$

which together with (16) proves the induction step for sequential composition.
QED

6 Tests

We will now incorporate axioms for tests into our axiomatisation of $\mathcal{L}(\theta, ;)$ and prove soundness and completeness with respect to dynamic models. Note that by including tests, our languages of formulas and programs are defined by mutual induction, cf. Definition 6, as is their semantics, cf. Definition 8.

When choosing the axioms for tests there are two obvious choices, depending on our choice of underlying modality. This can best be seen at the example $T = \mathcal{P}$: Taking the Kripke $\Diamond$ as basic modality, the axiom for tests will be $[\psi?]\varphi \leftrightarrow \psi \wedge \varphi$. Taking the Kripke $\Box$, the axiom for tests will be $[\psi?]\varphi \leftrightarrow \psi \to \varphi$.

In order to obtain an axiomatisation that is generic in the functor and chosen modality, we need a definition for when a modal operator is "box-like" or "diamond"-like. Apart from Def. 8 (semantics of tests), this is the only time we need that the monad is pointed, cf. Remark 9.

Definition 29. *Let λ be a predicate lifting for a pointed set monad $\mathbb{T}$. We say λ is* box-like *if for all sets X and all $U \subseteq X$ we have that the distinguished element $\bot_{TX} \in TX$ is in the λ-lifting of U, i.e., $\bot_{TX} \in \lambda_X(U)$. Likewise we call λ* diamond-like *if for all sets X and all $U \subseteq X$ we have $\bot_{TX} \notin \lambda_X(U)$.*

Any modality for a pointed monad falls into one of the above categories. For example, the (monotonic) neighbourhood modality is diamond-like.

Lemma 30. *Let λ be a predicate lifting for a pointed set monad $\mathbb{T}$. Then λ is either box-like or diamond-like.*

Proof. Let X be a set, let $U \subseteq X$ and let λ be a modality. Consider the characteristic function $\chi_U : X \to 2$ and the function $\Box_\lambda : T2 \to 2$ that corresponds to λ. By an observation in [7] we know that $t \in \lambda_X(U)$ iff $(\Box_\lambda \circ T\chi_U)(t) = 1$. The claim of the lemma follows now from the observation that $\Box_\lambda(\bot_{T2})$ is either equal to 0 or 1 and that $T\chi_U(\bot_{TX}) = \bot_{T2}$, since T is pointed. Therefore we get λ is box-like iff $\Box_\lambda(\bot_{T2}) = 1$ and diamond-like otherwise.

This allows us to add test axioms to $\mathcal{L}(\theta, ;)$.

Definition 31. *If λ is box-like, then we define the dynamic logic $\mathcal{L}(\theta, ;, ?)$ by adding the frame condition $[?\psi]p \leftrightarrow (\psi \to p)$ to* Fr *in $\mathcal{L}(\theta, ;)$. If λ is diamond-like, then we define $\mathcal{L}(\theta, ;, ?)$ by adding the frame condition $[?\psi]p \leftrightarrow (\psi \wedge p)$ to* Fr *in $\mathcal{L}(\theta, ;)$.*

Our soundness and completeness results relative to $\theta, ;$-standard models can now be extended to $\mathcal{L}(\theta, ;, ?)$ relative to the dynamic semantics.

Theorem 32. *The logic $\mathcal{L}(\theta, ;, ?)$ is sound and strongly complete with respect to the dynamic semantics (cf. Def. 8).*

Proof. Soundness is an easy exercise. For completeness, we again modify the quasi-canonical T^A-model (S, γ, V). First, we modify it for tests, and then we extend inductively from atomic actions, as in the proof of Theorem 26. The modification for tests goes as follows. Let $\xi \colon S \to (TS)^A$ be defined on all actions of the form $\alpha = \psi?$ by

$$\xi(\Gamma)(\psi?) := \begin{cases} \eta_S(\Gamma) & \text{if } \psi \in \Gamma \\ \bot_{TS} & \text{if } \psi \notin \Gamma \end{cases} \tag{19}$$

On non-test actions α, $\xi(\alpha) = \gamma(\alpha)$. We prove that for all dynamic formulas $[\psi?]\varphi \in \mathcal{F}(\Lambda)$ and all $\Gamma \in S$,

$$\widehat{\gamma}(\psi?)(\Gamma) \in \lambda_S(\hat{\varphi}) \quad \text{iff} \quad \widehat{\xi}(\psi?)(\Gamma) \in \lambda_S(\hat{\varphi}) \tag{20}$$

First, note that we have:

$$\begin{aligned} \eta_S(\Gamma) \in \lambda_S(\hat{\varphi}) \ &\text{iff} \ \hat{\varphi} \in \hat{\lambda_S}(\eta_S(\Gamma)) \\ &\text{iff} \ \hat{\varphi} \in \eta_S^{\mathcal{N}}(\Gamma) = \{U \subseteq S \mid \Gamma \in U\} \qquad (\hat{\lambda} \text{ is monad morph.}) \\ &\text{iff} \ \Gamma \in \hat{\varphi} \\ &\text{iff} \ \varphi \in \Gamma \end{aligned}$$

Hence,

$$\widehat{\xi}(\psi?)(\Gamma) \in \lambda_S(\hat{\varphi}) \quad \Longleftrightarrow \quad \begin{cases} \psi \in \Gamma \text{ and } \varphi \in \Gamma, & \text{or} \\ \psi \notin \Gamma \text{ and } \perp_{TS} \in \lambda_S(\hat{\varphi}) \end{cases}$$

Since (S, γ, V) is quasi-canonical, we have that $\widehat{\gamma}(\psi?)(\Gamma) \in \lambda_S(\hat{\varphi})$ iff $[\psi?]\varphi \in \Gamma$. If λ is diamond-like, then using the test axiom, we have that $\widehat{\gamma}(\psi?)(\Gamma) \in \lambda_S(\hat{\varphi})$ iff $\psi \wedge \varphi \in \Gamma$ iff $\psi \in \Gamma$ and $\varphi \in \Gamma$ iff $\widehat{\xi}(\psi?)(\Gamma) \in \lambda_S(\hat{\varphi})$.

If λ is box-like, then the test axiom gives us that $\widehat{\gamma}(\psi?)(\Gamma)$ iff $\psi \to \varphi \in \Gamma$. We again consider two cases: If $\psi \in \Gamma$, then $\psi \to \varphi \in \Gamma$ iff $\varphi \in \Gamma$ iff $\widehat{\xi}(\psi?)(\Gamma) \in \lambda_S(\hat{\varphi})$. If $\psi \notin \Gamma$, then $\psi \to \varphi \in \Gamma$ iff $\perp_{TS} \in \lambda_S(\hat{\varphi})$ (since λ is box-like) iff $\widehat{\xi}(\psi?)(\Gamma) \in \lambda_S(\hat{\varphi})$.

We have now proved (20). We now modify ξ on actions that are sequential compositions or pointwise operations of other actions, as described in the proof of Theorem 26. The resulting $\xi \colon S \to (TS)^A$ will satisfy the condition in Lemma 27 for *all* programs α, and consequently, the truth lemma for (S, ξ, V) follows, i.e., that $[\![\varphi]\!] = \hat{\varphi}$ for all formulas φ. This in turn implies that (S, ξ, V) is a dynamic $\mathbb{T}$-model, as the $\theta, ;$-standardness is immediate by definition of ξ, and the semantics of tests is as in Definition 8 due to the truth lemma and the definition of ξ on tests.

As special instances we obtain the following results of which (i) and (ii) were already known, but to our knowledge item (iii) is a modest new addition.

Corollary 33. *(i) Iteration-free PDL is sound and strongly complete with respect to $\cup$-dynamic $\mathcal{P}$-models. (ii) Iteration-free Game Logic is sound and strongly complete with respect to $\cup,^d$-dynamic $\mathcal{M}$-models. (iii) Let $\mathcal{L}_l = (\lambda^{\mathsf{tl}}, \mathrm{Ax}, \emptyset)$ be the "underlying logic" for the lift monad L where $\mathrm{Ax} = \{\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q, \Box(\neg p) \leftrightarrow \Box\top \wedge \neg\Box p\}$. Then the dynamic logic $\mathcal{L}_l(\emptyset, ;, ?)$ (over $\mathcal{L}_l$) is sound and strongly complete with respect to dynamic L-models.*

Proof. Item (i): Strong completeness of iteration-free PDL is well known. We show here how to obtain this result from our framework. Taking normal modal logic K as base logic with $\lambda^{\Box}$, and $\theta = \cup$, we obtain iteration-free PDL as our dynamic logic $\mathcal{L}(\cup, ;, ?)$ (using that $\lambda^{\Box}$ is box-like for the test axioms). In [31] it was shown that $\mathcal{P}$ weakly preserves inverse limits of surjective ω-cochains, and that K is strongly one-step complete for $\mathcal{P}$. It is also well known (and easy to check) that K is separating. In Examples 12 and 16 we verified that $\widehat{\lambda}^{\Box} : \mathcal{P} \Rightarrow \mathcal{N}$ is a monad morphism and that the pointwise $\cup$-axiom is valid in all $\cup$-standard $\mathcal{P}$-models. It now follows from our results that iteration-free PDL is sound and strongly complete with respect to $\cup$-dynamic $\mathcal{P}$-models.

Item (ii): Again, strong completeness of iteration-free GL is known [25]. We derive it here as follows. We take as base logic monotonic modal logic M with the monotonic predicate lifting λ. A rank-1 axiomation can be derived from the well known ones [3] - take e.g. $\mathrm{Ax} = \{\Box(p \wedge q) \rightarrow \Box p\}$ (expressing that $\Box$ is monotone) - and strong one-step completeness can also be verified by noting that any maximal one-step consistent set $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ can be satisfied by

$$\Gamma = \{U \subseteq X \mid \Box U \in \Phi\} \in \mathcal{M}X.$$

Here the axiom is crucial in order to show that Γ is indeed a *monotone*, i.e., an upwards closed neighbourhood collection. The weak limit preservation property is proved in Lemma 34 below. It is well known that M is separating. In Examples 12 and 16 we verified that $\widehat{\lambda}$: $\mathcal{M} \Rightarrow \mathcal{N}$ is a monad morphism and that the pointwise axioms for $\cup$ and ^{d} are valid in all $\cup,^d$-standard $\mathcal{M}$-models. Recall that the monotonic λ is diamond-like. It now follows from our results that iteration-free GL is sound and strongly complete with respect to $\cup,^d$-dynamic $\mathcal{M}$-models.

Item (iii): Inverse limit preservation of L and strong one-step completeness of $\mathcal{L}_l$ with respect to L is proved in Lemma 34 and Lemma 35 below, respectively. To see that $\mathcal{L}_l$ is separating, suppose $t_1 \neq t_2 \in LX$. If $t_1, t_2 \in X$ then they are separated by $\lambda^{\mathsf{tl}}(\{t_1\})$ since t_1 is in $\lambda^{\mathsf{tl}}(\{t_1\})$, but t_2 is not. If one of them equals $*$, then they are separated by $\lambda^{\mathsf{tl}}(X)$. From Example 12 we know that $\widehat{\lambda}^{\mathsf{tl}}$ is a monad map. We take sequential composition as our only program construct, i.e., no further pointwise program operations ($\theta = \emptyset$). Since $* \notin \lambda^{\mathsf{tl}}_X(U)$ for all $U \subseteq X$, λ^{tl} is box-like, and we take the corresponding test axioms, and consider the dynamic logic $\mathcal{L}_l(\emptyset, ;, ?)$. By our results, this dynamic logic is sound and strongly complete for dynamic L-models.

Lemma 34. *The monotonic neighbourhood functor $\mathcal{M}$ and the lift monad L weakly preserve inverse limits of surjective ω-cochains.*

Proof. Consider a surjective cochain $\{X_j\}_{j\in\omega}$ with morphisms $p_i : X_{i+1} \to X_i$ and projections $\pi_i : \varprojlim X_j \to X_i$ for all $i \in \omega$.

To prove the property for $\mathcal{M}$, let $\{\Gamma_i\}_{i\in\omega} \in \varprojlim \mathcal{M}X_j$. We define

$$\Gamma \;=\; \bigcup_{i\in\omega} \{\pi_i^{-1}[U] \mid U \in \Gamma_i\}$$

We claim that $g(\Gamma) = \{\Gamma_i\}_{i\in\omega}$. For this it suffices to show that for all $U \subseteq X_i$: $\pi_i^{-1}[U] \in \Gamma \iff U \in \Gamma_i$. This will follow if we can show that for $j < i$ and $U \subseteq X_j, V \subseteq X_i$ the following holds:

$$\pi_j^{-1}[U] = \pi_i^{-1}[V] \quad \text{implies} \quad U \in \Gamma_j \iff V \in \Gamma_i$$

This follows from the fact that

$$\pi_j^{-1}[U] = \pi_i^{-1}[(p_j^i)^{-1}[U]]$$

where $p_j^i : X_i \to X_j$, and hence (by surjectivity of all p_i's and thus of all π_i's) we have $(p_j^i)^{-1}[U] = V$. By the fact that the Γ_i's form a coherent family, the latter implies $U \in \Gamma_j \iff V \in \Gamma_i$ as required.

Now, to prove the property for L, let $\{\Gamma_i\}_{i\in\omega} \in \varprojlim LX_j$. We must prove that there is a $\Gamma \in L(\varprojlim X_j)$ such that $g(\Gamma) = \{\Gamma_i\}_{i\in\omega}$. Note that from the definition of Lp_n it follows that $\Gamma_0 = *$ iff $\Gamma_i = *$ for all $i \in \omega$. Hence either $\{\Gamma_i\}_{i\in\omega} = \{*\}_{i\in\omega}$ in which case we can take $\Gamma = *$; or $\{\Gamma_i\}_{i\in\omega} = \{x_i\}_{i\in\omega} \in \varprojlim X_j$ in which case we can take $\Gamma = \{x_i\}_{i\in\omega}$.

Lemma 35. *The logic $\mathcal{L}_l$ consisting of the axioms $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$ and $\Box(\neg p) \leftrightarrow \Box\top \wedge \neg\Box p$ is strongly one-step complete over finite sets with respect to coalgebras for the lift monad.*

Proof. Let X be a set and let $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ be a maximally one-step consistent set of formulas. We need to show that Φ is one-step satisfiable in $TX = 1 + X$.

Suppose first that $\neg\Box\top \in \Phi$. Then the second axiom implies that $\neg\Box U \in \Phi$ for all formulas $U \subseteq X$. This implies that $* \in \bigwedge_{\varphi\in\Phi} [\![\varphi]\!]_1$ for $* \in 1$ because clearly $*$ satisfies all boxed literals in Φ.

Suppose now that $\Box\top \in \Phi$. Using the first axiom one can show that $U \subseteq V$ entails that $\Box U \in \Phi$ implies $\Box V \in \Phi$ ("monotonicity"). Then by the second axiom we have for all $U \subseteq X$ that $\Box(X \setminus U) \in \Phi$ iff $\neg\Box U \in \Phi$. In particular we have $\Box\emptyset \in \Phi$ implies $\neg\Box X \in \Phi$. Because of monotonicity this means that we must have $\Box\emptyset \notin \Phi$.

We now claim that there is a unique $x \in X$ such that $\Box\{x\} \in \Phi$. Suppose first there is no such x. Then for all $x \in X$ we have $\Box X \setminus \{x\} \in \Phi$. Hence $\bigwedge_{x\in X} \Box X \setminus \{x\} \in \Phi$ and thus, using the first axiom, $\Box\left(\bigcap_{x\in X} X \setminus \{x\}\right) = \Box\emptyset \in \Phi$ which is a contradiction to what we discussed at the beginning of the proof. Similary, in case we have $\Box\{x_1\}, \Box\{x_2\} \in \Phi$ for $x_1 \neq x_2$ then $\Box\emptyset \in \Phi$ follows using the first axiom which again yields a contradiction.

It is now easy to check that the unique $x \in X$ with $\Box\{x\} \in \Phi$ satisfies all boxed literals in Φ and therefore x satisfies all of Φ as required.

7 Discussion and Conclusion

We have presented a framework for iteration-free coalgebraic dynamic modal logic where programs are modelled as T-coalgebras for a monad T, and program constructs are modelled via natural operations on T. We have proved a generic strong completeness result relative to a chosen set θ of natural operations. We note that our notion of natural operation is more general than the notion of *algebraic operation* [26] which is used in the context of computational effects. For example, it can be checked that *dual* is not an algebraic operation for $\mathcal{M}$.

We also note that the fact that intersection is not natural on $\mathcal{P}$ can be seen as an explanation of why PDL with intersection is difficult to axiomatise [1].

We leave it as future work to incorporate iteration into our framework. From PDL we know that dynamic modal logics with iteration cannot be strongly complete (due to non-compactness). Moreover, the fact that the completeness of GL remains an open problem tells us that a general weak completeness theorem is highly non-trivial. In any case, we will need to assume that the monad in question is order-enriched, perhaps along the lines of [12,6].

We note that our notion of pointed monad is weaker than requiring that the Kleisli category is enriched over the categoy of pointed sets, or over pointed CPOs. For example, it can be checked that the Kleisli category of the pointed monad $\mathcal{M}$ has neither form of enrichment.

A limitation of our framework is that it is unsuitable for designing dynamic modal logics for probabilistic or weighted systems. For probabilistic systems that are coalgebras for the (sub)distribution monad $\mathcal{D}_\omega$ there seem to be no interesting EM-algebras on 2. Similarly, for the weighted semiring monad $S_\omega(X) = \{f\colon X \to S \mid f \text{ has finite support }\}$ (where S is a semiring). Dynamic logics for such quantitative systems seem to require a multi-valued setting where the truth object is $T(1)$ (instead of 2).

Such a multi-valued approach to weakest preconditions for non-deterministic, probabilistic and quantum computation has recently been investigated in a categorical setting via so-called state-and-effect-triangles [13,14], see also [12,6]. Weakest preconditions are closely related to dynamic modal logic, e.g., the weakest precondition for φ with respect to program α is expressed in PDL as $[\alpha]\varphi$. Also in [12,14], as in our Lemma 10, it is noted that weakest preconditions/predicate liftings must be monad morphisms in order to obtain compositionality for sequential composition. Moreover, it is noted in [14] that algebraic structure on programs can be described in terms of enrichment. This works out nicely for $T = \mathcal{P}$, where also the predicate functor is enriched which explains the PDL axiom for choice. However, in the general picture discussed in [14, section 6] the enrichment is not captured. Interestingly, the neighbourhood monad occurs in this context as Lawvere's dual monad. An important difference with our work is that [12,14] focus on semantics, and no syntax or axiomatisation is investigated. We would like to investigate further the connections between our work and the multi-valued predicate transformer approach of [12,14].

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