Computers Math. Applic. Vol. 25, No. 5, pp. 55–63, 1993
Printed in Great Britain.

0097-4943/93 $6.00 + 0.00

ASYMPTOTIC AND NUMERICAL ASPECTS OF THE NONCENTRAL CHI-SQUARE DISTRIBUTION

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(Received June 1992)

Abstract—The noncentral χ^2-distribution is related with the series

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}P(\mu+n,y)=1-e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}Q(\mu+n,y),$$

where $P(\alpha,z)$ and $Q(\alpha,z)$ are incomplete gamma functions (central χ^2-distributions). Another representation is in terms of

$$Q_\mu(x,y):=\int_y^\infty\left(\frac{z}{x}\right)^{1/2(\mu-1)}e^{-z-x}I_{\mu-1}(2\sqrt{xz})\,dz,$$

which is also known as the generalized Marcum Q-function; $I_\mu(z)$ is the modified Bessel function. $Q_\mu(x,y)$ plays a role in communication studies. From the integral representation recurrence relations for $Q_\mu(x,y)$ are derived. Next, it is shown that $Q_\mu(x,y)$ can be expressed in terms of the simpler integral

$$F_\mu(\xi,\sigma):=\int_\xi^\infty e^{-(\sigma+1)t}I_\mu(t)\,dt,$$

where

$$\xi=2\sqrt{xy}\quad\text{and}\quad\sigma=-1+\frac{1}{2}\left(\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}\right).$$

Two asymptotic expansions of $Q_\mu(x,y)$ are derived. In one form, the function $F_\mu(\xi,\sigma)$ is used with μ fixed and large ξ, giving an expansion which holds uniformly with respect to $\sigma\in(0,\infty)$. In a second expansion, both parameters ξ and μ may be large. In both asymptotic forms, an error function (the normal distribution function) is used to describe the behavior of $Q_\mu(x,y)$ as y crosses the value $x+\mu$. Series expansions in terms of incomplete gamma functions are discussed in connection with numerical evaluation of $Q_\mu(x,y)$ or $1-Q_\mu(x,y)$. It is also indicated when the asymptotic expansion can be used in order to obtain a certain relative accuracy.

1. INTRODUCTION

The noncentral χ^2-distribution is related with the Bessel function integral

$$Q_\mu(x,y):=\int_y^\infty\left(\frac{z}{x}\right)^{1/2(\mu-1)}e^{-z-x}I_{\mu-1}(2\sqrt{xz})\,dz,\qquad x,y>0. \tag{1.1}$$

In problems on radar communications, this function is known as the generalized Marcum Q-function, which for $\mu=1$ reduces to the ordinary Marcum function. See [1–5]. In this field, μ is the number of independent samples of the output of a square-law detector. In our analysis, μ is a not necessarily an integer number. We assume that $\mu>0$.

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The function $I_\mu(t)$ is the modified Bessel function [6, Section 9.6]

$$I_\mu(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{\mu+2n}}{n!\Gamma(\mu+n+1)}.$$

Substituting this expansion in (1.1) gives

$$Q_\mu(x,y) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} Q(\mu+n, y),$$

where $Q(\alpha, z)$ is the incomplete gamma function defined by [6, Chapters 6 and 26]

$$Q(\alpha,z) = \frac{\Gamma(\alpha,z)}{\Gamma(\alpha)}, \qquad \Gamma(\alpha,z) = \int_z^\infty t^{\alpha-1} e^{-t}\, dt. \tag{1.2}$$

By using the incomplete gamma function $P(\alpha, z) = 1 - Q(\alpha, z)$, we obtain

$$1 - Q_\mu(x,y) = \int_0^y \left(\frac{z}{x}\right)^{1/2(\mu-1)} e^{-z-x} I_{\mu-1}(2\sqrt{xz})\, dz = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} P(\mu+n, y). \tag{1.3}$$

This series defines the noncentral χ^2−distribution (in the notation of [6, 26.4.25]). The parameter μ is related with the degrees of freedom and y with the noncentrality.

In [7], recurrence relations for $Q_\mu(x,y)$ with respect to μ are derived. We rederive some of the results by using properties of the modified Bessel function. First, we write (1.1) in the form (by putting $z = r^2/(4x)$)

$$Q_\mu(x,y) = (2x)^{-\mu} e^{-x} \int_{2\sqrt{xy}}^\infty r^\mu e^{-r^2/(4x)} I_{\mu-1}(r)\, dr.$$

Next, we observe that (see [6, 9.6.28]) $r^\mu I_{\mu-1}(r) = \frac{d}{dr}[r^\mu I_\mu(r)]$. Integrating by parts, we obtain the inhomogeneous recursion

$$Q_{\mu+1}(x,y) = Q_\mu(x,y) + \left(\frac{y}{x}\right)^{\mu/2} e^{-x} I_\mu\left(2\sqrt{xy}\right). \tag{1.4}$$

In Section 5, this relation is discussed in connection with numerical algorithms. We can eliminate the Bessel function in (1.4) using (see [6, 9.6.26])

$$I_{\mu-1}(z) = I_{\mu+1}(z) + \frac{2\mu}{z} I_\mu(z). \tag{1.5}$$

This gives the homogeneous third order recurrence relation:

$$x\, Q_{\mu+2}(x,y) = (x-\mu) Q_{\mu+1}(x,y) + (y+\mu) Q_\mu(x,y) - y\, Q_{\mu-1}(x,y). \tag{1.6}$$

The purpose of the paper is to derive asymptotic expansions of the function defined in (1.1). When x and y are large, and $|x-y|$ is small compared to x and y, the integral (1.1) has a peculiar behavior. To see this, consider the well-known estimate

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \qquad \text{as } z \to \infty, \quad \nu \text{ fixed}.$$

We see that the term $\exp(-z-x) I_{\mu-1}(2\sqrt{xz})$ of (1.1) is exponentially small, except when $x \geq y$ and $z \sim x$. It follows that, when x and y are large (and μ small with respect to x and y), the behavior of $Q_\mu(x,y)$ significantly changes when y crosses the value x. It will appear that when μ is large too, this change in behavior occurs when y crosses the value $x+\mu$. In both cases, the asymptotic behavior can be described by using the error function, that is, the normal distribution function.

2. MORE INTEGRAL REPRESENTATIONS

First we show that (2.1) essentially reduces to a sum of two simpler functions. Moreover, we obtain symmetrical representations for the cases $x < y$ and $x > y$. The auxiliary function is defined by

$$F_\mu(\xi,\sigma) := \int_\xi^\infty e^{-(\sigma+1)t}\, I_\mu(t)\, dt, \qquad \sigma > 0. \tag{2.1}$$

To show that $Q_\mu(x,y)$ can be expressed in terms of this function, we use (see [6, 29.3.81])

$$\left(\frac{z}{x}\right)^{1/2(\mu-1)} I_{\mu-1}(2\sqrt{xz}) = \frac{1}{2\pi i}\int e^{sz+x/s} s^{-\mu}\, ds, \qquad \mu > 0, \tag{2.2}$$

where the path of integration may be any vertical line in the half plane $\Re s > 0$. The path may be deformed into a loop $\mathcal{L}$ starting and terminating at $-\infty$, initially with $\arg t = -\pi$, and finally with $\arg t = +\pi$. The loop integral encircles the origin in the positive sense. In this way an absolute (and fast) convergent integral is obtained. Substituting the loop integral into (1.1), we obtain

$$Q_\mu(x,y) = e^{-x}\int_y^\infty \frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(s-1)+x/s}\frac{ds}{s^\mu}\, dz.$$

Take $\mathcal{L}$ such that $\Re s < 1$ for any $s \in \mathcal{L}$. Then, by absolute convergence of the repeated integrals, we may interchange the order of integration. Deforming $\mathcal{L}$ back into a vertical with $0 < \Re s < 1$ we arrive at

$$Q_\mu(x,y) = \frac{e^{-x-y}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{x/s+ys}}{1-s}\frac{ds}{s^\mu}, \qquad 0 < c < 1. \tag{2.3}$$

When we move the vertical to the right, across the pole at $s = 1$, taking into account the residue, we obtain

$$1 - Q_\mu(x,y) = \frac{e^{-x-y}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{x/s+ys}}{s-1}\frac{ds}{s^\mu}, \qquad c > 1. \tag{2.4}$$

In (2.3), we substitute $s = t/\rho$ with $\rho = \sqrt{y/x}$. It follows that

$$Q_\mu(x,y) = e^{-x-y-2z\lambda}\rho^\mu \Phi(z),$$

$$\Phi(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{z(t+1/t+2\lambda)}}{\rho - t}\frac{dt}{t^\mu}, \qquad 0 < c < \rho, \tag{2.5}$$

where $z = \sqrt{xy}$. We now assume, for the time being, that $\rho > 1$. Taking $2\lambda = -(\rho + 1/\rho)$, and assuming (again, for the time being) that ρ does not depend on x, y, z, we obtain

$$\frac{d\Phi(z)}{dz} = -\frac{e^{2\lambda z}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z(t+1/t)}\left(t - \frac{1}{\rho}\right)\frac{dt}{t^{\mu+1}}.$$

Invoking (2.2), we derive

$$\frac{d\Phi(z)}{dz} = -e^{2\lambda z}\left[I_{\mu-1}(2z) - \frac{1}{\rho}I_\mu(2z)\right].$$

To integrate this, we use $\Phi(\infty) = 0$. This follows from standard techniques from asymptotics applied on (2.5), for instance the saddle point method, see [8]. Observe that the exponential function of the integrand in (2.5) has a saddle point at $s = 1$. We obtain

$$Q_\mu(x,y) = \rho^\mu\Phi(z) = \frac{1}{2}\rho^\mu\left[F_{\mu-1}(\xi,\sigma) - \frac{1}{\rho}F_\mu(\xi,\sigma)\right], \qquad y > x; \tag{2.6}$$

ρ has regained its original meaning and the F-function is defined in (2.1). Furthermore

$$\xi = 2\sqrt{xy}, \quad \sigma = -\lambda - 1 = \frac{(\sqrt{y}-\sqrt{x})^2}{\xi}, \quad \rho = \sqrt{\frac{y}{x}}. \tag{2.7}$$

Now let $\rho < 1$. Repeating the analysis that leads to (2.6), but now with starting point the integral in (2.4), we obtain for this case

$$Q_\mu(x,y) = 1 - \frac{1}{2}\rho^\mu \left[\frac{1}{\rho} F_\mu(\xi,\sigma) - F_{\mu-1}(\xi,\sigma)\right], \qquad y < x, \tag{2.8}$$

where the parameters are as in (2.7).

In the following sections the large ξ-behaviour of $Q_\mu(x,y)$ is discussed. We have, as $\xi \to \infty$ and ρ fixed,

$$Q_\mu(x,y) \sim \begin{cases} 1, & \text{if } \rho < 1; \\ \frac{1}{2}, & \text{if } \rho = 1; \\ 0, & \text{if } \rho > 1. \end{cases} \tag{2.9}$$

It will be shown that a smooth transition can be described in terms of the error function (the normal distribution function).

3. ASYMPTOTIC EXPANSION; μ FIXED, ξ LARGE

We concentrate on the function $F_\mu(\xi,\sigma)$ given in (2.1). We point out that this function with ξ and σ as in (2.7) is symmetric in x and y, and occurs in both (2.6) and (2.8). Hence, it is sufficient to assume $x < y$. The case $x = y$ follows from the asymptotic results when we let $x \to y$.

The asymptotic feature is that ξ is large, whereas σ tends to zero when $x \to y$. We give an asymptotic expansion that holds uniformly with respect to $\sigma \in [0,\infty)$. Note that the integral defining $F_\mu(\xi,\sigma)$ becomes undetermined when $\sigma = 0$. However, since we use a combination of two F-functions in (2.6), and ρ tends to unity as $x \to y$, the function $Q_\mu(x,y)$ is well defined in this limit.

We substitute the well-known expansion [6, Section 9.7]

$$e^{-t} I_\mu(t) \sim \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} (-1)^n \frac{A_n(\mu)}{t^n}, \qquad \text{as } t \to \infty, \tag{3.1}$$

in (2.1), where

$$A_n(\mu) = \frac{2^{-n}\Gamma(1/2+\mu+n)}{n!\,\Gamma(1/2+\mu-n)}, \qquad n = 0,1,2,\ldots,$$

with recursion

$$A_{n+1}(\mu) = -\frac{(2n+1)^2 - 4\mu^2}{8(n+1)} A_n(\mu), \qquad n \geq 0, \quad A_0(\mu) = 1.$$

This gives the formal expansion

$$F_\mu(\xi,\sigma) \sim \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n A_n(\mu) \phi_n, \tag{3.2}$$

where ϕ_n is an incomplete gamma function (see (1.2))

$$\phi_n = \int_\xi^\infty e^{-\sigma t}\, t^{-n-1/2}\, dt = \sigma^{n-1/2}\,\Gamma\left(\frac{1}{2} - n, \sigma\xi\right). \tag{3.3}$$

The function ϕ_0 is an error function:

$$\phi_0 = \sqrt{\pi/\sigma}\,\mathrm{erfc}\,\sqrt{\sigma\xi} = \sqrt{\pi/\sigma}\,\mathrm{erfc}\,c(\sqrt{y}-\sqrt{x}), \quad \mathrm{erfc}\,z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt. \tag{3.4}$$

Further terms can be obtained from the recursion

$$\left(n-\frac{1}{2}\right)\phi_n = -\sigma\phi_{n-1} + e^{-\sigma\xi}\xi^{-n+1/2}, \qquad n = 1,2,3,\ldots, \tag{3.5}$$

which follows from integrating by parts in (3.3).

Using (2.6) and (3.2), we obtain

$$Q_\mu(x,y) \sim \sum_{n=0}^{\infty}\psi_n, \qquad \psi_n = \frac{\rho^\mu}{2\sqrt{2\pi}}(-1)^n\left[A_n(\mu-1) - \frac{1}{\rho}A_n(\mu)\right]\phi_n. \tag{3.6}$$

Information on the asymptotic nature and error bounds of expansion (3.2) can be found in [9], in which also numerical aspects of recursion (3.5) are discussed. Expansion (3.2) holds for large values of ξ, uniformly with respect to $\sigma \in [0,\infty)$.

The first term approximation of the series in (3.6) reads, since $(\rho-1)/\sqrt{2\sigma} = \sqrt{\rho}$,

$$Q_\mu(x,y) \sim \psi_0 = \frac{1}{2}\rho^{\mu-1/2}\,\mathrm{erfc}\,(\sqrt{y}-\sqrt{x}). \tag{3.7}$$

We remark that the right-hand side reduces to $1/2$ when $x \uparrow y$.

REMARK. When $x > y$, that is $\rho < 1$, the expression $(\rho-1)/\sqrt{2\sigma}$ should be interpreted as $-\sqrt{\rho}$, and (2.8) gives

$$Q_\mu(x,y) \sim 1-\psi_0 = 1-\frac{1}{2}\rho^{\mu-1/2}\,\mathrm{erfc}(\sqrt{x}-\sqrt{y}).$$

Again, the right-hand side reduces to $1/2$ when $x \downarrow y$.

4. ASYMPTOTIC EXPANSION; ξ LARGE, μ ARBITRARY

In this case, we consider (2.3). We write

$$Q_\mu(x,y) = \rho^\mu\, e^{-x-y}\,\frac{1}{2\pi i}\int_{\mathcal{L}} e^{\xi\alpha(t)}\,\frac{dt}{\rho-t},$$

where ρ and ξ are as in (2.7), and

$$\alpha(t) = \frac{1}{2}\left(t+\frac{1}{t}\right) - \beta\ln t, \qquad \beta = \frac{\mu}{\xi}.$$

The path of integration $\mathcal{L}$ is a vertical $\Re t = c$, with $0 < c < \rho$. However, $\mathcal{L}$ may be deformed into a different contour, for instance into the path of steepest descent through a saddle point. The saddle points are solutions of the equation $\alpha'(t) = 0$. We select the positive saddle point

$$t_0 = \beta + \sqrt{\beta^2+1} = e^\gamma.$$

It is convenient to write

$$\beta = \sinh,\gamma, \quad \rho = e^\theta. \tag{4.1}$$

Then we have

$$t_0 = e^\gamma, \quad \alpha(t_0) = \cosh\gamma - \gamma\sinh\gamma.$$

Observe that when $\gamma \sim \theta$ the saddle point and the pole are close together. A well-known approach in asymptotics to handle this case is based on using an error function [8]. The case

$\gamma = \theta$ corresponds with $y = x + \mu$. When ξ is large and y crosses the value $x + \mu$, the function $Q_\mu(x, y)$ suddenly changes. We have (cf. (2.9))

$$Q_\mu(x,y) \sim \begin{cases} 1, & \text{if } x + \mu > y; \\ \frac{1}{2}, & \text{if } x + \mu = y; \\ 0, & \text{if } x + \mu < y. \end{cases}$$

In terms of μ and γ, these cases read $\theta < \gamma$, $\theta = \gamma$, $\theta > \gamma$, respectively. When $t_0 < \rho$, that is, when $\theta > \gamma$ or $y > x + \mu$, we can shift $\mathcal{L}$ through the saddle point, without passing the pole at $t = \rho$. We temporarily assume that $t_0 < \rho$.

The path of steepest descent $\mathcal{L}$ through t_0 follows from the equation $\Im\,\alpha(t) = 0$. Let $t = re^{i\phi}$. Then we can describe $\mathcal{L}$ by

$$r = \sinh\gamma \frac{\phi}{\sin\phi} + \sqrt{1 + \sinh^2\gamma \frac{\phi^2}{\sin^2\phi}}, \qquad -\pi < \phi < \pi.$$

We define a mapping $t \mapsto u(t)$ that maps $\mathcal{L}$ to $\mathbb{R}$ by writing

$$\frac{1}{2}u^2 = \alpha(t_0) - \alpha(t).$$

When t follows $\mathcal{L}$ we take $u \in \mathbb{R}$, with $\operatorname{sign}(u) = \operatorname{sign}(\Im t)$. The pole at $t = \rho$ is mapped to the point iu_0, where u_0 is defined by

$$\frac{1}{2}u_0^2 = \cosh\theta - \cosh\gamma + (\gamma - \theta)\sinh\gamma,$$

where θ and γ are introduced in (4.1). The sign of u_0 follows from the definition of the mapping $t \mapsto u(t)$: we have $\operatorname{sign}(u_0) = \operatorname{sign}(\gamma - \theta) = \operatorname{sign}(x + \mu - y)$. Integrating with respect to u, and splitting off the pole at $u = iu_0$, we obtain

$$Q_\mu(x,y) = \frac{1}{2}\operatorname{erfc}\left(-u_0\sqrt{\frac{\xi}{2}}\right) + \frac{e^{-1/2\xi u_0^2}}{2\pi i}\int_{-\infty}^{\infty} e^{-1/2\xi u^2} f(u)\, du, \tag{4.2}$$

where

$$f(u) = \frac{dt}{du}\frac{1}{\rho - t} + \frac{1}{u - iu_0}.$$

In deriving the term with the error function, we have used [6, 7.1.3 & 7.1.4]. The asymptotic expansion of $Q_\mu(x, y)$ now follows by expanding

$$f(u) = i\sum_{n=0}^{\infty} c_n u^n$$

and by substituting this in the above integral. This gives

$$Q_\mu(x,y) \sim \frac{1}{2}\operatorname{erfc}\left(-u_0\sqrt{\frac{\xi}{2}}\right) + \frac{e^{-1/2\xi u_0^2}}{\sqrt{2\pi\xi}}\sum_{n=0}^{\infty} c_{2n}\frac{\Gamma(n+1/2)}{\Gamma(1/2)}\left(\frac{2}{\xi}\right)^n, \qquad \xi \to \infty. \tag{4.3}$$

This expansion holds uniformly with respect to $\mu \in [0, \infty)$. The first coefficients are

$$c_0 = \frac{1}{\sqrt{\cosh\gamma}}\frac{1}{e^{\theta-\gamma} - 1} + \frac{1}{u_0},$$

$$c_2 = -\frac{e^{2\gamma} + e^{-2\gamma} - 8 + e^{\theta-\gamma}(10e^{2\gamma} - 2e^{-2\gamma} + 28) + e^{2\theta-2\gamma}(e^{2\gamma} + 13e^{-2\gamma} + 4)}{48\cosh^{7/2}\gamma\,(e^{\theta-\gamma} - 1)^3} - \frac{1}{u_0^3}.$$

REMARK. We have temporarily assumed $t_0 < \rho$, that is $y > x + \mu$. In (4.2), this condition can be dropped. The expansion in (4.3) also holds for $y \le x + \mu$. Note that a single error function describes the transition from $y > x + \mu$ to $y < x + \mu$, and we do not need different representations for $Q_\mu(x, y)$ as in the previous section; confer (2.6) and (2.8). The method of this section can also be used when μ is fixed. However, the method of the previous section gives very simple coefficients in expansion (2.6).

5. COMPUTATIONAL ASPECTS

In applications, it is of interest to have available algorithms for $Q_\mu(x,y)$ when $0 < Q_\mu(x,y) \le 1/2$ and for $1 - Q_\mu(x,y)$ when $1/2 < Q_\mu(x,y) < 1$. The first inequalities occur when (roughly speaking) $y \ge x+\mu$, the second ones when $y < x+\mu$ (this follows from the asymptotic expansion of the previous section).

Recurrence relation (1.4) is very useful for computing $Q_\mu(x,y)$. It is numerically stable in forward direction, since the right-hand side of (1.4) has positive terms. An algorithm for the modified Bessel function is needed. A point of warning: recurrence relation (1.5) should not be used in forward direction. In [10], a detailed discussion on this problem is given. Observe that the function $1 - Q_\mu(x,y)$ satisfies the recursion

$$1 - Q_\mu(x,y) = 1 - Q_{\mu+1}(x,y) + \left(\frac{y}{x}\right)^{\mu/2} e^{-x} I_\mu\left(2\sqrt{xy}\right),$$

which is stable in backward direction. In the homogeneous recurrence relation (1.6) Bessel functions do not occur. It is attractive to use this equation in order to avoid the forward recursion of the Bessel functions. However, one needs to investigate the stability of (1.6) in more detail, and for several combinations of the parameters, which is not a trivial problem. Observe that any constant function (with respect to μ) solves (1.6), and that, hence, $1-Q_\mu(x,y)$ satisfies the same recurrence.

For small and moderate values of x, y, μ the expansions in terms of the incomplete gamma functions of Section 1 can be used. We recall:

$$\begin{aligned} Q_\mu(x,y) &= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} Q(\mu+n, y), \\ 1 - Q_\mu(x,y) &= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} P(\mu+n, y). \end{aligned} \tag{5.1}$$

Both series have positive terms. In [5], the first series is rearranged into another series with positive terms, but this seems to give a more complicated expansion. Both series require the evaluation of one incomplete gamma function. An algorithm for computing these functions for large values of the parameters can be found in [11].

The first series in (5.1) requires the value $Q(\mu, y)$, and the remaining terms follow from the recursion

$$Q(\mu+n+1, y) = Q(\mu+n, y) + \frac{y^{n+\mu} e^{-y}}{\Gamma(\mu+n+1)}, \qquad n = 0, 1, 2, \ldots$$

The second series requires an initial P-value. The corresponding recursion should be used in the backward direction:

$$P(\mu+n, y) = P(\mu+n+1, y) + \frac{y^{n+\mu} e^{-y}}{\Gamma(\mu+n+1)},$$

since the forward form is not stable. Let n_0 be the (smallest) number such that

$$1 - Q_\mu(x,y) \simeq e^{-x} \sum_{n=0}^{n_0} \frac{x^n}{n!} P(\mu+n, y), \tag{5.2}$$

within the required relative accuracy. Then as starting value we need to compute $P(\mu+n_0, y)$, and the remaining values follow from the above recursion. To estimate n_0 we may use

$$P(\mu+n, y) \sim \frac{y^{n+\mu} e^{-y}}{\Gamma(\mu+n+1)}, \qquad \text{as } \mu+n \to \infty.$$

For obtaining *relative* accuracy, we need an estimate of $1 - Q_\mu(x,y)$ in the P-series of (5.1). We take the value of the integrand of (1.3) at $z = y$, that is,

$$1 - Q_\mu(x,y) \sim \rho^\mu e^{-x-y} I_\mu(\xi)$$

and we replace $I_\mu(\xi)$ by the dominant part of the uniform asymptotic expansion [6, 9.7.7]. That is, we replace $I_\mu(\xi)$ with

$$e^{\mu\eta}, \quad \eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}, \quad z = \frac{\xi}{\mu}.$$

Combining these estimates, and using the dominant terms in Stirling's approximation of the gamma function, we infer that the equation

$$\frac{e^{-x} x^n P(\mu+n, y)/n!}{1 - Q_\mu(x,y)} = \varepsilon$$

can be replaced by the equation

$$n \ln \frac{n}{ex} + (\mu+n) \ln \frac{\mu+n}{ey} + \mu(\ln \rho + \coth\gamma - \gamma) + \ln\varepsilon = 0.$$

The left-hand side assumes a minimal value at $n = 1/2\xi \exp(-\gamma)$, with γ as in the previous section. A Newton process (a safe start is $\xi \exp(-\gamma)$) gives the desired value of n, which is taken as the number n_0 in (5.2).

Table 5.1 shows the number of terms n_0 used in the series of (5.1), for several values of x. In all cases $\mu = 8192$, $y = 1.05\mu$, as in Table I of [5].

Table 5.1. n_0 is the number of terms used in the series of (5.1); $\mu = 8192$, $y = 1.05\mu$; the relative accuracy is 10^{-10}.

x/μ	n_0	$Q_\mu(x,y)$	$1 - Q_\mu(x,y)$
0.01	150	$1.984527803e-4$	$9.998015472e-1$
0.03	355	$4.000364970e-2$	$9.599963503e-1$
0.05	543	$4.985354536e-1$	$5.014645464e-1$
0.07	727	$9.556573418e-1$	$4.434265825e-2$
0.09	894	$9.996249724e-1$	$3.750276164e-4$
0.11	1054	$9.999997188e-1$	$2.811864384e-7$
0.13	1207	$1.000000000e+0$	$1.999694515e-11$

For large values of the parameters the computation can be based on the uniform expansion (4.3). Special care has to be taken when $y \sim x + \mu$, that is, $\theta \sim \gamma$. First it is convenient to have an expansion of u_0. We have

$$u_0 = (\gamma - \theta)\sqrt{\frac{2[\cosh\theta - \cosh\gamma - (\theta-\gamma)\sinh\gamma]}{(\theta-\gamma)^2}},$$

where the square root is should be taken positive. The expression inside the square root can easily be expanded in powers of $\theta - \gamma$.

The coefficient c_0 has the expansion

$$c_0 = \frac{1}{6\cosh^{3/2}\gamma}\left[\sinh\gamma - 3\cosh\gamma + \left(\frac{1}{2}\sinh^2\gamma + \frac{3}{4}\right)\zeta - \frac{\sinh\gamma(2\sinh^2\gamma + 27)}{360}\zeta^2 + \mathcal{O}(\zeta^3)\right],$$

as $\zeta \to 0$, where

$$\zeta = \frac{\theta - \gamma}{\cosh\gamma}.$$

For c_2 we have

$$c_2 = -\frac{e^{-3\gamma}(e^{6\gamma} + 6e^{4\gamma} + 309e^{2\gamma} - 46)}{4320\cosh^{9/2}\gamma} - \frac{e^{-4\gamma}(1 - 8e^{2\gamma} + e^{4\gamma})(1 + 16e^{2\gamma} + e^{4\gamma})}{4608\cosh^{9/2}\gamma}\zeta + \mathcal{O}(\zeta^2).$$

In our algorithm, we have used these approximations when $|\theta - \gamma| < (1.0)10^{-4}$ in c_0 and $|\theta-\gamma| < (0.8)10^{-3}$ in c_2. We have used (4.3) with these two coefficients under the condition $\sqrt{xy} + \mu > 1600$. Then the relative accuracy is about ten digits, unless $Q_\mu(x,y)$ or $1 - Q_\mu(x,y)$ is quite small, say smaller than 10^{-20}, in which case some digits may be lost.

References

1. J.I. Marcum, A statistical theory of target detection by pulsed radar, *IRE Trans. Inform. Theory* **6**, 59–268 (1960).
2. P.J. Nahin, An error analysis of an algorithm for Marcum's Q-function, *Comp & Maths. with Appls.* **2** (3/4), 207–210 (1976).
3. A.H. Nuttall, Some integrals involving the Q_M function, *IEEE Trans. Inform. Theory* **IT-21**, 95–96 (1975).
4. S.O. Rice, Uniform asymptotic expansions for saddle point integrals—Application to a probability distribution occurring in noise theory, *Bell System Tech. J.* **47**, 1971–2013 (1968).
5. G.H. Robertson, Computation of the noncentral chi-square distribution, *Bell System Tech. J.* **48**, 201–207 (1969).
6. M. Abramowitz and I.A. Stegun, Handbook of mathematical functions with formulas, graphs and mathematical tables, *Nat. Bur. Standards Appl. Series*, Vol. 55, U.S. Government Printing Office, Washington, D.C., (1964).
7. J.D. Cohen, Noncentral Chi-Square: Some observations on recurrence, *Am. Statist* **42**, 120–122 (1988).
8. R. Wong, *Asymptotic approximations of integrals*, Academic Press, (1989).
9. N.M. Temme, A double integral containing the modified Bessel function: Asymptotics and computation, *Math Comp.* **47**, 683–691 (1986).
10. W. Gautschi, Computational aspects of three-term recurrence relations, *SIAM Review* **9**, 24–82 (1967).
11. N.M. Temme, On the computation of the incomplete gamma functions for large values of the parameters, In *Algorithms For Approximation, Proceedings of the IMA-Conference*, Shrivenham July 15–19, 1985, (Edited by J.C. Mason and M.G. Cox), pp. 479–489, Clarendon, Oxford, (1987).