Note on Hypergraphs and Sphere Orders

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ABSTRACT

We show that each partial order $\leq$ of height 2 can be represented by spheres in Euclidean space, where inclusion represents $\leq$. If each element has at most k elements under it, we can do this in $2k - 1$-dimensional space. This extends a result (and a method) of Scheinerman for the case $k = 2$. © 1993 John Wiley & Sons, Inc.

A partial order $\leq$ on a set P is called a *sphere order* (in dimension n) if for each $u \in P$ there exists a ball B_u in $\mathbb{R}^n$ so that for all $u, v \in P$ one has $u < v$ if and only if $B_u \subset B_v$. Sphere orders were introduced by Brightwell and Winkler [1], who posed the intriguing question of whether *each* partially order is a sphere order. They conjectured that the answer is negative.

In [3], Scheinerman showed that each partial order on the vertices and edges of a graph (ordered by inclusion) is a sphere order in dimension 3. Here we extend Scheinerman's result (and his construction) to hypergraphs:

Theorem. For any hypergraph $H = (V, E)$, the partial order on $V \cup E$, given by

$$x < y \Leftrightarrow x \in V, y \in E, x \in y, \tag{1}$$

is a sphere order in dimension $2k - 1$, where k is the maximum edge size of H.

Since the reverse order to a sphere order is a sphere order again, in the same dimension, we could equally take for k the maximum degree of H.

Another formulation of the theorem is that each partial order P of height 2 is a sphere order in dimension $2k - 1$, where $k := \max_{u \in P} |\{v \in P \mid v < u\}|$.

Journal of Graph Theory, Vol. 17, No. 2, 173–176 (1993)
 CCC 0364-9024/93/020173-04

The theorem follows directly from the following lemma (extending the lemma in [3]). Let C be the following curve in $\mathbb{R}^{2k}$:

$$C := \{(1, x, x^2, x^3, \ldots, x^{2k-1}) | x \in \mathbb{R}\}. \tag{2}$$

Lemma. For each subset A of C with $|A| = k$ there exists a ball B with $B \cap C = A$.

Proof. Let A consist of the points

$$(1, a_i, a_i^2, a_i^3, \ldots, a_i^{2k-1}) \tag{3}$$

on C, for $i = 1, \ldots, k$. Let the polynomials $p(x)$ and $q(x)$ be given by

$$p(x) := 1 + x^2 + x^4 + \cdots + x^{4k-2},$$
$$q(x) := \prod_{i=1}^{k} (x - a_i)^2. \tag{4}$$

Since $q(x)$ has degree $2k$, there exists a polynomial $f(x)$ so that the polynomial

$$r(x) := p(x) - f(x) \cdot q(x) \tag{5}$$

has degree at most $2k - 1$ (as we can reduce $p(x)$ modulo $q(x)$ to a polynomial of degree at most $2k - 1$).

Write $r(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{2k-1} x^{2k-1}$, and let $g := \frac{1}{2}(r_0, r_1, r_2, \ldots, r_{2k-1})$. Then the ball $B(g, \|g\|)$ with center g and radius $\|g\|$ intersects C exactly in the set A. This can be seen as follows.

Let $z = (1, x, x^2, \ldots, x^{2k-1})$ be a point on C. Then

$$\|g - z\|^2 = \|g\|^2 + \|z\|^2 - 2g^T z = \|g\|^2 + p(x) - r(x)$$
$$= \|g\|^2 + f(x) \cdot q(x). \tag{6}$$

Now the polynomial $f(x)$ has no real zeros, since the polynomial $h(x) := f(x) \cdot q(x)$ has at most $2k$ real zeros (counting multiplicities). This follows from the fact that the $2k$th derivative $h^{(2k)}(x)$ of $h(x)$ has no real zeros, as it satisfies

$$h^{(2k)}(x) = (2k)! + \frac{(2k+2)!}{2!} x^2 + \frac{(2k+4)!}{4!} x^4$$
$$+ \cdots \frac{(4k-2)!}{(2k-2)!} x^{2k-2} \tag{7}$$

(since $h(x) = p(x) - r(x) = \cdots + x^{2k} + \cdots + x^{4k-4} + x^{4k-2}$).

As the main coefficient of $f(x)$ is 1, we know that $f(x) > 0$ for all $x \in \mathbb{R}$. So $\|g - z\|^2 = \|g\|^2$ if $z \in A$ and $\|g - z\|^2 > \|g\|^2$ if $z \notin A$. ∎

The theorem now follows by first observing that we may assume that each edge of H contains exactly k vertices (by adding dummy vertices). We take $|V|$ arbitrary points on C, to be considered as balls of radius 0, representing the vertices of H. For each edge e of H we take the ball intersecting C exactly in the points representing the vertices in e. Since C is in a $2k - 1$-dimensional subspace of $\mathbb{R}$, we obtain a sphere order in dimension $2k - 1$.

We remark that our construction is related to the construction of cyclic polytopes (Gale [2]).

Now one may ask:

$$\text{Is } 2k - 1 \text{ best possible in the theorem (for fixed } k\text{)?} \tag{8}$$

We do not know the answer to this question. However, if the balls associated with the vertices of the hypergraph have radius 0 (as is the case in our construction above) then $2k - 1$ is best possible, as follows from the following proposition.

Proposition. There is no subset V of $\mathbb{R}^{2k-2}$ such that $|V| = 2k + 1$ and such that for each subset X of V with $|X| = k$ there exists a ball B_X satisfying $B_X \cap V = X$.

Proof. Suppose such a set V exists. Then for any two disjoint subsets X, Y, of V with $|X| = |Y| = k$ one has that conv $X \cap$ conv $Y = \varnothing$, since $\operatorname{conv}(B_X \setminus B_Y) \cap \operatorname{conv}(B_Y \setminus B_X) = \varnothing$.

Let $V = \{v_1, \ldots, v_{2k+1}\}$. Let W be the linear subspace of $\mathbb{R}^{2k+1}$ consisting of all vectors $w = (w_1, \ldots, w_{2k+1})$ satisfying

$$\begin{aligned} w_1 v_1 + \cdots + w_{2k+1} v_{2k+1} &= 0, \\ w_1 + \cdots + w_{2k+1} &= 0. \end{aligned} \tag{9}$$

Note that dim $W \geq 2$.

For any vector $w = (w_1, \ldots, w_{2k+1})$, let $p_+(w)$ be the number of $i \in \{1, \ldots, 2k + 1\}$ satisfying $w_i > 0$, and let $p_-(w)$ be the number of $i \in \{1, \ldots, 2k + 1\}$ satisfying $w_i < 0$. Now W contains a nonzero vector w satisfying $p_+(w) \leq k$ and $p_-(w) \leq k$. This can be seen as follows.

Let $W_+ := \{v \in W | p_+(v) \geq k + 1\}$ and $W_- := \{v \in W | p_-(v) \geq k + 1\}$. So W_+ and W_- are two disjoint open subsets of $W \backslash \{0\}$. Moreover, $W_+ \neq \backslash W \backslash \{0\}$ and $W_- \neq W \backslash \{0\}$, since $W_- = -W_+$. Hence by the connectedness of $W \backslash \{0\}$, $W \backslash \{0\} \neq W_+ \cup W_-$, implying that $W \backslash \{0\}$ contains a vector w satisfying $p_+(w) \leq k$ and $p_-(w) \leq k$.

We may assume that $w = (w_1, \ldots, w_{2k+1})$ satisfies $w_1, \ldots, w_k \geq 0$, $w_{k+1}, \ldots, w_{2k} \leq 0$, $w_{2k+1} = 0$ and $w_1 + \cdots + w_k = 1$. Hence $(-w_{k+1}) + \cdots + (-w_{2k}) = 1$. In particular, both $\mathrm{conv}\{v_1, \ldots v_k\}$ and $\mathrm{conv}\{v_{k+1}, \ldots, v_{2k}\}$ contain the vector

$$w_1 v_1 + \cdots + w_k v_k = (-w_{k+1}) v_{k+1} + \cdots + (-w_{2k}) v_{2k}. \qquad (10)$$

This contradicts the fact that $\mathrm{conv}\{v_1, \ldots v_k\} \cap \mathrm{conv}\{v_{k+1}, \ldots, v_{2k}\} = \emptyset$. ∎

Thus if $|V| = 2k + 1$ and E consists of all subsets of V of size k, then $2k - 1$ is best possible in the theorem if each ball associated with a vertex in V has radius 0.

ACKNOWLEDGMENTS

I am grateful to the referees for helpful comments and suggestions, in particular for suggesting question (8).

References

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