Note on Hypergraphs and Sphere Orders

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ABSTRACT

We show that each partial order $\leq$ of height 2 can be represented by spheres in Euclidean space, where inclusion represents $\leq$. If each element has at most $k$ elements under it, we can do this in $2k - 1$-dimensional space. This extends a result (and a method) of Scheinerman for the case $k = 2$. © 1993 John Wiley & Sons, Inc.

A partial order $\leq$ on a set $P$ is called a sphere order (in dimension $n$) if for each $u \in P$ there exists a ball $B_u$ in $\mathbb{R}^n$ so that for all $u, v \in P$ one has $u < v$ if and only if $B_u \subset B_v$. Sphere orders were introduced by Brightwell and Winkler [1], who posed the intriguing question of whether each partially order is a sphere order. They conjectured that the answer is negative.

In [3], Scheinerman showed that each partial order on the vertices and edges of a graph (ordered by inclusion) is a sphere order in dimension 3. Here we extend Scheinerman’s result (and his construction) to hypergraphs:

**Theorem.** For any hypergraph $H = (V, E)$, the partial order on $V \cup E$, given by

$$x < y \iff x \in V, y \in E, x \in y,$$

is a sphere order in dimension $2k - 1$, where $k$ is the maximum edge size of $H$.

Since the reverse order to a sphere order is a sphere order again, in the same dimension, we could equally take for $k$ the maximum degree of $H$.

Another formulation of the theorem is that each partial order $P$ of height 2 is a sphere order in dimension $2k - 1$, where $k := \max_{u \in P} |\{v \in P | v < u\}|$.

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The theorem follows directly from the following lemma (extending the lemma in [3]). Let \( C \) be the following curve in \( \mathbb{R}^{2k} \):

\[
C := \{(1, x, x^2, x^3, \ldots, x^{2k-1})| x \in \mathbb{R}\}. \tag{2}
\]

**Lemma.** For each subset \( A \) of \( C \) with \( |A| = k \) there exists a ball \( B \) with \( B \cap C = A \).

**Proof.** Let \( A \) consist of the points

\[
(1, a_i, a_i^2, a_i^3, \ldots, a_i^{2k-1}) \tag{3}
\]
on \( C \), for \( i = 1, \ldots, k \). Let the polynomials \( p(x) \) and \( q(x) \) be given by

\[
p(x) := 1 + x^2 + x^4 + \cdots + x^{4k-2},
\]

\[
q(x) := \prod_{i=1}^{k} (x - a_i)^2. \tag{4}
\]

Since \( q(x) \) has degree \( 2k \), there exists a polynomial \( f(x) \) so that the polynomial

\[
r(x) := p(x) - f(x) \cdot q(x) \tag{5}
\]

has degree at most \( 2k - 1 \) (as we can reduce \( p(x) \) modulo \( q(x) \) to a polynomial of degree at most \( 2k - 1 \)).

Write \( r(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{2k-1}x^{2k-1} \), and let \( g := \frac{1}{2}(r_0, r_1, r_2, \ldots, r_{2k-1}) \). Then the ball \( B(g, ||g||) \) with center \( g \) and radius \( ||g|| \) intersects \( C \) exactly in the set \( A \). This can be seen as follows.

Let \( z = (1, x, x^2, \ldots, x^{2k-1}) \) be a point on \( C \). Then

\[
||g - z||^2 = ||g||^2 + ||z||^2 - 2g^T z = ||g||^2 + p(x) - r(x)
\]

\[
= ||g||^2 + f(x) \cdot q(x). \tag{6}
\]

Now the polynomial \( f(x) \) has no real zeros, since the polynomial \( h(x) := f(x) \cdot q(x) \) has at most \( 2k \) real zeros (counting multiplicities). This follows from the fact that the \( 2k \)th derivative \( h^{(2k)}(x) \) of \( h(x) \) has no real zeros, as it satisfies

\[
h^{(2k)}(x) = (2k)! + \frac{(2k + 2)!}{2!} x^2 + \frac{(2k + 4)!}{4!} x^4
\]

\[
+ \cdots \frac{(4k - 2)!}{(2k - 2)!} x^{2k-2} \tag{7}
\]

(since \( h(x) = p(x) - r(x) = \cdots + x^{2k} + \cdots + x^{4k-4} + x^{4k-2} \)).
As the main coefficient of $f(x)$ is 1, we know that $f(x) > 0$ for all $x \in \mathbb{R}$. So $\|g - z\|^2 = \|g\|^2$ if $z \in A$ and $\|g - z\|^2 > \|g\|^2$ if $z \notin A$.

The theorem now follows by first observing that we may assume that each edge of $H$ contains exactly $k$ vertices (by adding dummy vertices). We take $|V|$ arbitrary points on $C$, to be considered as balls of radius 0, representing the vertices of $H$. For each edge $e$ of $H$ we take the ball intersecting $C$ exactly in the points representing the vertices in $e$. Since $C$ is in a $2k - 1$-dimensional subspace of $\mathbb{R}$, we obtain a sphere order in dimension $2k - 1$.

We remark that our construction is related to the construction of cyclic polytopes (Gale [2]).

Now one may ask:

Is $2k - 1$ best possible in the theorem (for fixed $k$)? \(8\)

We do not know the answer to this question. However, if the balls associated with the vertices of the hypergraph have radius 0 (as is the case in our construction above) then $2k - 1$ is best possible, as follows from the following proposition.

**Proposition.** There is no subset $V$ of $\mathbb{R}^{2k-2}$ such that $|V| = 2k + 1$ and such that for each subset $X$ of $V$ with $|X| = k$ there exists a ball $B_X$ satisfying $B_X \cap V = X$.

**Proof.** Suppose such a set $V$ exists. Then for any two disjoint subsets $X, Y$ of $V$ with $|X| = |Y| = k$ one has that $\text{conv } X \cap \text{conv } Y = \emptyset$, since $\text{conv}(B_X \setminus B_Y) \cap \text{conv}(B_Y \setminus B_X) = \emptyset$.

Let $V = \{v_1, \ldots, v_{2k+1}\}$. Let $W$ be the linear subspace of $\mathbb{R}^{2k+1}$ consisting of all vectors $w = (w_1, \ldots, w_{2k+1})$ satisfying

$$w_1 v_1 + \cdots + w_{2k+1} v_{2k+1} = 0,$$

$$w_1 + \cdots + w_{2k+1} = 0.$$ \(9\)

Note that $\dim W \geq 2$.

For any vector $w = (w_1, \ldots, w_{2k+1})$, let $p_+(w)$ be the number of $i \in \{1, \ldots, 2k + 1\}$ satisfying $w_i > 0$, and let $p_-(w)$ be the number of $i \in \{1, \ldots, 2k + 1\}$ satisfying $w_i < 0$. Now $W$ contains a nonzero vector $w$ satisfying $p_+(w) \leq k$ and $p_-(w) \leq k$. This can be seen as follows.

Let $W_+ := \{v \in W | p_+(v) \geq k + 1\}$ and $W_- := \{v \in W | p_-(v) \geq k + 1\}$. So $W_+$ and $W_-$ are two disjoint open subsets of $W \setminus \{0\}$. Moreover, $W_+ \neq W_+ \setminus \{0\}$ and $W_- \neq W_- \setminus \{0\}$, since $W_- = -W_+$. Hence by the connectedness of $W \setminus \{0\}$, $W \setminus \{0\} \neq W_+ \cup W_-$, implying that $W \setminus \{0\}$ contains a vector $w$ satisfying $p_+(w) \leq k$ and $p_-(w) \leq k$. 
We may assume that \( w = (w_1, \ldots, w_{2k+1}) \) satisfies \( w_1, \ldots, w_k \geq 0, \ w_{k+1}, \ldots, w_{2k} \leq 0, \ w_{2k+1} = 0 \) and \( w_1 + \cdots + w_k = 1 \). Hence \( (-w_{k+1}) + \cdots + (-w_{2k}) = 1 \). In particular, both \( \text{conv}\{v_1, \ldots, v_k\} \) and \( \text{conv}\{v_{k+1}, \ldots, v_{2k}\} \) contain the vector \( w_1 v_1 + \cdots + w_k v_k = (-w_{k+1})v_{k+1} + \cdots + (-w_{2k})v_{2k} \). (10)

This contradicts the fact that \( \text{conv}\{v_1, \ldots, v_k\} \cap \text{conv}\{v_{k+1}, \ldots, v_{2k}\} = \emptyset \).

Thus if \(|V| = 2k + 1\) and \( E \) consists of all subsets of \( V \) of size \( k \), then \( 2k - 1 \) is best possible in the theorem if each ball associated with a vertex in \( V \) has radius 0.

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**References**

