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Tait's Flyping Conjecture for Well-Connected Links

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We call a diagram of a link *well-connected* if it is connected, has no 2-edge cuts, and the only 4-edge cuts are those made by a crossing. We prove Tait's flyping conjecture for well-connected diagrams, i.e., any two well-connected alternating diagrams represent equivalent (= ambient itotopic) links, if and only if these diagrams are the same up to trivial operations.

1. INTRODUCTION

A *knot* is a subset of $\mathbb{R}^3$ homeomorphic to the unit circle. A *link* is a disjoint union of a finite number of knots (cf. [1]).

We assume knots and links to be tame. Moreover, for the purpose of this paper we may assume that for each link K considered, the projection $\pi[K]$ of K to $\mathbb{R}^2$ is a 4-regular planar graph, with a finite set of vertices, edges, and faces. Here π denotes the projection from $\mathbb{R}^3$ onto $\mathbb{R}^2$ with $\pi(x_1, x_2, x_3) := (x_1, x_2)$. Throughout, by *projecting* we mean projecting by π.

We can associate with a link K the *diagram* of K that arises by projecting K to $\mathbb{R}^2$, indicating at each crossing which of the two curve segments goes over the other as in Fig. 1.

In this paper, by *the* diagram of a link we mean the diagram obtained under projecting by π. The diagram is called *alternating* if, when following each component of the link in its diagram, we go alternatingly over and under, like in Fig. 2.

Two links K and K' are *equivalent* if there exists an isotopy of $\mathbb{R}^3$ bringing K to K'. (An *isotopy* of a topological space X is a continuous function Φ: $[0, 1] \times X \to X$ such that $\Phi(0, u) = u$ for each $u \in X$, while for each fixed $t \in [0, 1]$ the function $\Phi(t, \cdot)$ is a homeomorphism of X. It *brings* Y to Y' if $\Phi(1, Y) = Y'$.)

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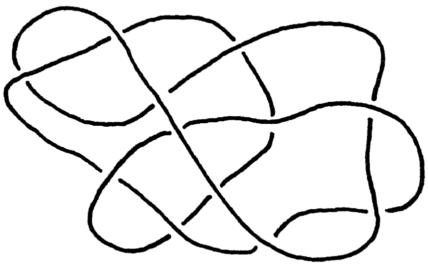

FIGURE 1

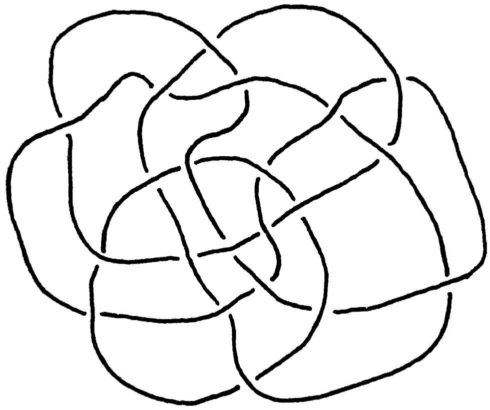

FIGURE 2

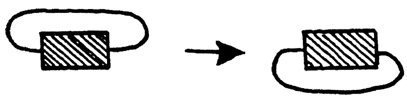

FIGURE 3

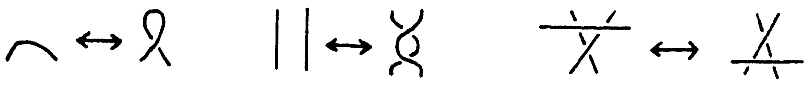

FIGURE 4

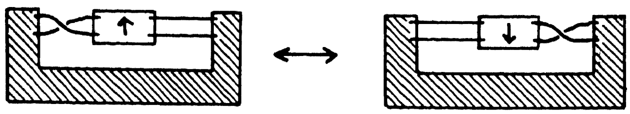

FIGURE 5

Two link diagrams are called *equivalent* if one arises from the other by a finite sequence of the following operations:

(i) reflecting the diagram in $\mathbb{R}^2$, e.g., with respect to the x_1 axis, and interchanging over and under;

(ii) rerouting one of the edges of the diagram through the unbounded face, as in Fig. 3. (1)

(Operation (i) corresponds to rotating the link (in $\mathbb{R}^3$), around the x_1-axis—this corresponds to an isotopy. The box in Fig. 3 denotes the rest of the diagram.)

Remark 1. It has been shown by Reidemeister [7] that if two diagrams represent equivalent links, then these diagrams can be obtained from each other by a finite sequence of the operations given in Fig. 4. These operations are called the *Reidemeister moves.*

Clearly, if two links have equivalent diagrams, they are equivalent. The converse need not hold in general. However, as we show in this paper, if the diagrams are well-connected and alternating the converse *does* hold. We call the diagram $\pi[K]$ of a link K *well-connected* if (as a graph) $\pi[K]$ is connected, has no 2-edge cut sets, and the only 4-edge cut sets are those determined by one vertex of $\pi[K]$ (that is, the four edges incident with a vertex (= crossing) of $\pi[K]$).

THEOREM. *Let K and K' be links with well-connected alternating diagrams. If K and K' are equivalent, then their diagrams are equivalent.*

This is a special case of the *Tait flying conjecture* [8], which does not require well-connectedness but the weaker *reducedness* instead (a diagram is *reduced* if the graph is connected and has no loops and no cutpoints), while the operations (1) should be extended by *flyping*—cf. Fig. 5 (it is also a flype if over and under in the two crossings given are interchanged).[1]

Note that flypes are not possible for well-connected diagrams. (Tait: "The deformation process is, in fact, simply one of *flyping*, an excellent word, very inadequately represented by the nearest equivalent English phrase turning outside in." [8]; "When we *flype* a glove (as in taking it off when very wet, or as we skin a hare), we perform an operation which changes its character from a right-hand glove to a left" [9].)

Remark 2. By an idea of Tait, the diagram $\pi[K]$ of any link K gives a planar graph H_K as follows. Color the faces of $\pi[K]$ black and white

[1] Meantime, W. W. Menasco and M. B. Thistlethwaite (The Tait flyping conjecture, *Bull. Amer. Math. Soc.* **25** (1991), 403–412) have announced a proof of the full Tait flyping conjecture.

such that adjacent faces have different colors, and such that the unbounded face is colored white. Put a vertex in each black face, and for each crossing, make an edge connecting the vertices in the two (possibly identical) black faces incident with the crossing (in such a way that the edge crosses the crossing).

Now a link K is well-connected if and only if the graph H_K is 3-vertex-connected (i.e., has no vertex cut of less than three vertices and has no parallel edges (except if it has only two vertices connected by at most three parallel edges)).

2. Proof of the Theorem

We will associate with any link K a compact bordered surface Σ_K in $\mathbb{R}^3$, with $\mathrm{bd}(\Sigma_K) = K$. (bd denotes boundary.) A pictorial impression of Σ_K is given in Fig. 6. Here any two black faces are connected at a crossing by a twisted band as in the Möbius strip (Fig. 7) (or the twist the other way around if over and under are interchanged).

More precisely, Σ_K is defined as follows. For any link K, let V_K denote the set of vertices of $\pi[K]$, and let

$$v(K) := |V_K|. \tag{2}$$

For each vertex v of the graph $\pi[K]$, let $p_v^{\uparrow}$ and $p_v^{\downarrow}$ be the two points in $K \cap \pi^{-1}(v)$, where $p_v^{\uparrow}$ is above $p_v^{\downarrow}$. (Here and below, *above* and *under* refer to larger and smaller x_3 coordinate.)

Moreover, let e_v be the open line segment in $\pi^{-1}(v)$ connecting $p_v^{\uparrow}$ and $p_v^{\downarrow}$. Define

$$T := K \cup \bigcup_{v \in V_K} e_v. \tag{3}$$

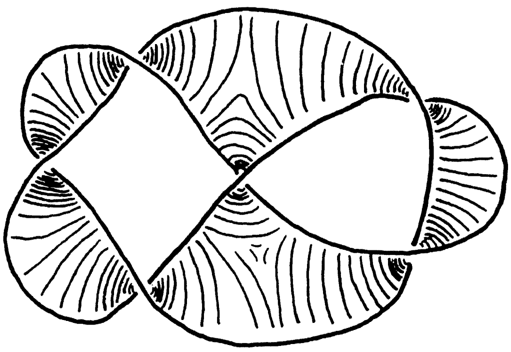

Figure 6

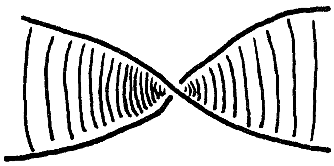

FIGURE 7

So T forms a 3-regular graph embedded in $\mathbb{R}^3$, with $2v(K)$ vertices and $3v(K)$ edges.

Let K be a link with alternating diagram $\pi[K]$. Call a face F of $\pi[K]$ *even* if at each vertex v incident with F one has a crossing as in Fig. 8. (So if F is bounded then when following the boundary of F in clockwise orientation, we follow the edges from up to down.) The other faces are called *odd*.

Note that of any two adjacent faces, one is even and the other is odd. So if the unbounded face is even, then the white faces are even, and the black faces are odd. If the unbounded face is odd, then the white faces are odd, and the black faces are even.

Note moreover that any link diagram can be transformed into one in which the unbounded face is even, by (possibly) rerouting through the unbounded face (operation (1)(ii)). So the condition that the unbounded face be even, is not a restriction.

Let K be a link with connected alternating diagram, such that the unbounded face of $\pi[K]$ is even. Let $\mathscr{B}$ denote the collection of odd faces. Consider an odd face F. The set $\pi^{-1}[\mathrm{bd}(F)] \cap T$ is a simple closed curve, consisting of parts of K and of the line segments e_v, for those vertices v of $\pi[K]$ that are incident with F. So it is the boundary of some open disk D_F such that π maps D_F one-to-one onto F. Fix for each odd face F one such open disk D_F. Then we define:

$$\Sigma_K := T \cup \bigcup_{F \in \mathscr{B}} D_F. \tag{4}$$

So Σ_K indeed is a compact bordered surface with boundary K.

Our proof is based on the following two theorems, which might be interesting in their own right:

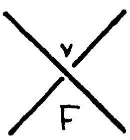

FIGURE 8

THEOREM A. *Let K and K' be links with well-connected alternating diagrams, such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If K and K' are equivalent, then there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$.*

(S^3 is the 3-dimensional sphere, considered as one-point compactification of $\mathbb{R}^3$.)

THEOREM B. *Let K and K' be links with well-connected alternating diagrams, such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$, then the diagrams of K and K' are equivalent.*

Theorems A and B clearly directly imply the theorem. Although Theorem A above holds in general, to avoid several technicalities, in this paper we prove Theorem A only under the condition that

> the unbounded face of $\pi[K]$ is bounded by at least four edges of $\pi[K]$. (5)

This is enough to derive the theorem, since we may assume that either $\pi[K]$ or $\pi[K']$ has at least one face that is bounded by at least four edges. (If all faces of $\pi[K]$ and of $\pi[K']$ are bounded by at most three edges, then, by the well-connectedness of K and K', $\pi[K]$ and $\pi[K']$ have either at most three vertices or both are the octahedron, and the theorem is easy to check under these assumptions.) Then by applying operations (1) and possibly mirroring K and K' in the x_1-x_2 plane we can obtain condition (5). (Mirroring in the x_1-x_2 plane by itself is not an isotopy, but it maintains equivalence of K and K'.)

Remark 3. In fact a more general statement than Theorem A holds:

> Let K and K' be links with reduced alternating diagrams such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If K and K' are equivalent, then there is an isotopy of S^3 bringing Σ_K to $\Sigma_{\tilde{K}'}$, where $\tilde{K}'$ is a link the diagram of which can be obtained from that of K' by a series of flypings. (6)

Remark 4. The following can be proved by methods similar to those used in this paper to show Theorem B:

> Let K and K' be links with reduced alternating diagrams, such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$, then the cycle spaces of H_K and $H_{K'}$ are isomorphic. (7)

Here the *cycle space* of a graph is the collection of its cycles. A *cycle* is an edge-disjoint union of circuits.

Statements (6) and (7) imply that if K and K' are equivalent links with reduced alternating diagrams such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even, then the cycle spaces of H_K and $H_{K'}$ are isomorphic. By a theorem of Whitney [14], (7) directly implies Theorem B.

3. Preliminaries on Links and Surfaces

We give some preliminaries on links and surfaces (see [4, Sects. I–IV] in which the information on links given below can be found).

Kauffman [3], Murasugi [5] and Thistlethwaite [10] (cf. Turaev [13]) showed that if K and K' are equivalent links with reduced alternating diagrams, then $v(K) = v(K')$. In fact they showed that any reduced alternating diagram of a link K attains the minimum number of crossings among all diagrams of links equivalent to K.

A second invariant is obtained as follows. Give each component of K some orientation. This way we obtain an *oriented link*. Then there are two types of crossings, positive and negative—see Fig. 9. The *writhe* $w(K)$ of K is the number of positive crossings minus the number of negative crossings. This number is not invariant under equivalence of links. However, Murasugi [6] and Thistlethwaite [11] showed that if K and K' are equivalent links with reduced alternating diagrams, then $w(K) = w(K')$. Similarly, Murasugi [6] and Thistlethwaite [12] showed that the number $b(K)$ of odd faces is an invariant for reduced alternating diagrams of equivalent links.

Let K_1 and K_2 be two disjoint oriented links. Consider the diagram made by $K_1 \cup K_2$. Define

$$\begin{aligned} \mathrm{lk}(K_1, K_2) := \tfrac{1}{2}((\#\text{ positive } K_1 - K_2 \text{ crossings}) \\ - (\#\text{ negative } K_1 - K_2 \text{ crossings})). \end{aligned} \tag{8}$$

(A $K_1 - K_2$ *crossing* is a crossing of K_1 with K_2. # means "number of." Here no condition is put on which of K_1 and K_2 is above the other at the crossing.) This number is invariant under isotopy of S^3: if (K_1', K_2')

Figure 9

is brought to (K_1, K_2) by some isotopy then $\mathrm{lk}(K_1', K_2') = \mathrm{lk}(K_1, K_2)$ (assuming that K_1' and K_2' are oriented as induced through the isotopy by the orientations of K_1 and K_2). This invariance of $\mathrm{lk}(\cdot, \cdot)$ follows directly by considering the Reidemeister moves.

Let K be an oriented link and let Σ be a disjoint union of a finite number of compact bordered surfaces embedded in $\mathbb{R}^3$ and containing K. We define a number $\tau(K, \Sigma)$ as follows.

If each component of K is an orientation-preserving curve on Σ, we take for each component κ of K a curve $\tilde{\kappa}$ parallel on Σ to κ. The union of these $\tilde{\kappa}$ forms a link $\tilde{K}$. Then $\tau(K, \Sigma) := 2\,\mathrm{lk}(K, \tilde{K})$, where we orient K and $\tilde{K}$ in the same direction.

If at least one component of K is orientation-reversing, we consider a link J homotopic on Σ to the set of closed curves that follow the components of K twice. So each component of J is orientation-preserving. We define $\tau(K, \Sigma) := \frac{1}{4}\tau(J, \Sigma)$.

Clearly, if K and K' are homotopic on Σ, then $\tau(K, \Sigma) = \tau(K', \Sigma)$. (This follows from the fact that there exists an isotopy fixing Σ bringing K to K'. Hence if each component of K is orientation-preserving, then there exists an isotopy fixing Σ bringing $(K, \tilde{K})$ to $(K', \tilde{K}')$, where $\tilde{K}$ and $\tilde{K}'$ are the shifted K and K', respectively. So $\mathrm{lk}(K, \tilde{K}) = \mathrm{lk}(K', \tilde{K}')$. Similarly for J if some component of K is orientation-reversing.)

More generally, if some isotopy of S^3 brings (K, Σ) to (K', Σ'), then $\tau(K, \Sigma) = \tau(K', \Sigma')$.

Direct calculation shows that for any oriented link K with alternating diagram for which the unbounded face of $\pi[K]$ is even one has

$$\tau(K, \Sigma_K) = 2(v(K) + w(K)) = 4(\#\text{ positive crossings of } K). \tag{9}$$

Indeed, observe that K is orientation-preserving, since it is a boundary component of Σ_K. Consider a positive crossing of K. Let K', K'' and $\tilde{K}'$, $\tilde{K}''$

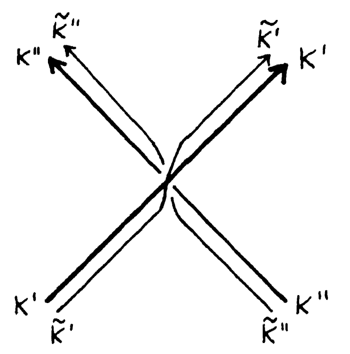

FIGURE 10

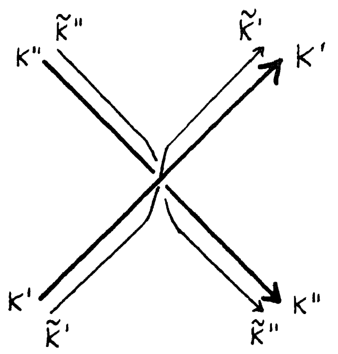

FIGURE 11

be the parts of K and $\tilde{K}$ as in Fig. 10. Then K' and $\tilde{K}'$ make a positive crossing, K' and $\tilde{K}''$ make a positive crossing, K'' and $\tilde{K}'$ make a positive crossing, and K'' and $\tilde{K}''$ make a positive crossing. So a positive crossing contributes 4 to $\tau(K, \Sigma_K)$.

Consider next a negative crossing of K (Fig. 11). Again K' and $\tilde{K}'$ make a positive crossing and K'' and $\tilde{K}''$ make a positive crossing. On the other hand, K'' and $\tilde{K}'$ make a negative crossing and also K' and $\tilde{K}''$ make a negative crossing. Hence a negative crossing contributes 0 to $\tau(K, \Sigma_K)$.

Finally, as is well-known, the *Euler characteristic* $\chi(\Sigma_K)$ of a surface Σ_K is equal to: number of faces, minus number of edges, plus number of vertices of any graph embedded on the surface (with all faces being an open disk). So

$$\chi(\Sigma_K) = b(K) - v(K), \tag{10}$$

where $b(K)$ denotes the number of odd faces of the diagram of K. (This follows from the facts that T has $2v(K)$ vertices and $3v(K)$ edges, and that $\Sigma_K \backslash T$ consists of $b(K)$ open disks.)

4. THEOREM A

In this section we consider:

THEOREM A. *Let K and K' be links with well-connected alternating diagrams such that the unbounded faces of $\pi[K]$ and of $\pi[K']$ are even. If K and K' are equivalent, then there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$.*

We show Theorem A under the condition that the unbounded face of $\pi[K]$ is bounded by at least four edges of $\pi[K]$.

Proof. It suffices to show:

LEMMA. *Let K be a link with well-connected alternating diagram such that the unbounded face of $\pi[K]$ is even. Let Σ be the disjoint union of compact bordered surfaces satisfying*:

$$\begin{aligned} &\text{(i)} \quad \mathrm{bd}(\Sigma)=K, \\ &\text{(ii)} \quad \chi(\Sigma)\geqslant b(K)-v(K), \\ &\text{(iii)} \quad \tau(K,\Sigma)=2(v(K)+w(K)). \end{aligned} \tag{11}$$

Then there exists an isotopy of S^3 bringing Σ to Σ_K.

("Disjoint union of compact bordered surfaces" implies that each component of Σ has a nonempty border (being a nonempty disjoint union of closed curves). Observe that condition (11)(iii) is independent of the orientations chosen for K (since $\tau(K,\Sigma)-2w(K)$ represents the total twist of annular neighbourhoods on Σ of the components of K). The conclusion in the lemma implies that Σ is connected and that equality holds in (11)(ii).)

We prove the lemma under the condition that the unbounded face of $\pi[K]$ is bounded by at least four edges.

Remark 5. The lemma also holds if this last condition is not satisfied. In fact, the lemma can be extended to links with reduced, not necessarily well-connected diagrams. In that case the conclusion is that there exists an isotopy of S^3 bringing Σ to $\Sigma_{\tilde{K}}$, where $\tilde{K}$ is some link the diagram of which is obtained from that of K by a series of flypings.

To derive Theorem A from the Lemma, let K and K' be equivalent links with well-connected alternating diagrams such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even, and such that the unbounded face of $\pi[K]$ is bounded by at least four edges.

Let Φ be an isotopy of S^3 bringing K' to K. Let $\psi(x):=\Phi(1,x)$ for all $x\in S^3$. So $K=\psi[K']$.

Applying the lemma to $\Sigma:=\psi[\Sigma_{K'}]$ gives Theorem A (since

$$\begin{aligned}\tau(K,\psi[\Sigma_{K'}])&=\tau(\psi[K'],\psi[\Sigma_{K'}])=\tau(K',\Sigma_{K'})\\ &=2(v(K')+w(K'))=2(v(K)+w(K))\end{aligned} \tag{12}$$

and

$$\chi(\psi[\Sigma_{K'}])=\chi(\Sigma_{K'})=b(K')-v(K')=b(K)-v(K)). \tag{13}$$

Proof of the Lemma. Let

$$G := \pi[K],$$
$$V := V_K, \tag{14}$$
$$P := \{p_v^{\uparrow} \mid v \in V\} \cup \{p_v^{\downarrow} \mid v \in V\}.$$

Throughout we identify an embedded graph with its image. We consider edges as *open* curves, and faces as *open* regions.

In proving the lemma, we make the assumption that Σ is tame and in general position with respect to the link K and the projection function π. In particular we assume that Σ has a simplicial decomposition into a finite number of vertices, edges, and faces, in such a way that each edge and each face projects one-to-one to $\mathbb{R}^2$. So the number

$$\omega(x) := |\Sigma \cap \pi^{-1}(x)| \tag{15}$$

is finite for each $x \in \mathbb{R}^2$.

Moreover, there exists a planar graph H in $\mathbb{R}^2$, with a finite number of vertices, edges, and faces, such that ω is constant on each edge and on each face of H. We may assume that ω takes the value 0 in the unbounded face of H. (So Σ does not contain the point in $S^3 \backslash \mathbb{R}^3$. This is no restriction as we can easily shift Σ slightly.)

The simplicial decomposition of Σ implies that there exists a finite set W of points on K that do not have a neighbourhood in Σ that projects one-to-one to $\mathbb{R}^2$. We may assume that the neighbourhood of any point in W is

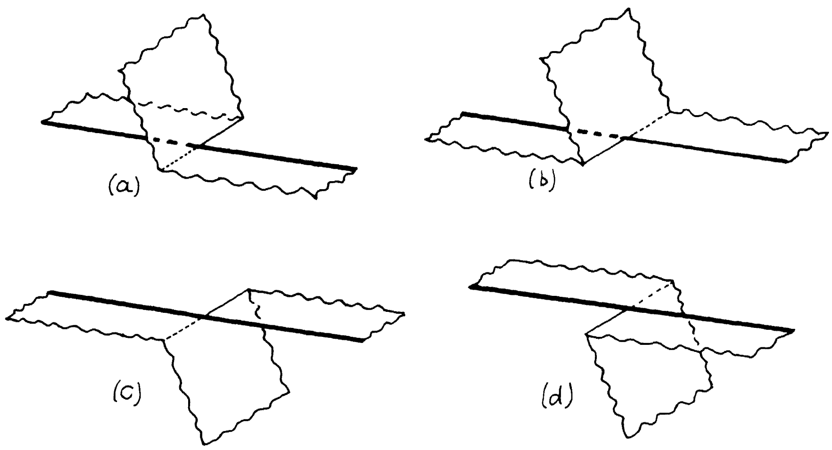

FIGURE 12

FIGURE 13

like one in Fig. 12. (In this and following figures, the bold lines indicate parts of K or of $\pi[K]$. The wriggled lines give the cuts through Σ bounding the neighbourhood.)

As an illustration, one could represent a twisted band as in Fig. 7 as follows. Take a slip of paper as in Fig. 13 and fold it as in Fig. 14. The two sides of the slip form a crossing at x, and locally the surface is isotopic to the part in Fig. 7. The points w_1 and w_2 are in W; the neighbourhood of w_1 is of type (a) in Fig. 12 and that of w_2 is of type (c) in Fig. 12. Doing this for each crossing we obtain a surface isotopic to Σ_K.

We may assume that $P \cap W = \emptyset$. Moreover, we may assume that the projection of any vertex of the simplicial decomposition of Σ is not contained in the projection of any edge of this decomposition.

Define

$$\Gamma := \Sigma \cap \pi^{-1}[G] \tag{16}$$

and

$$\Delta := \{x \in \Sigma \mid x \text{ has no neighbourhood on } \Sigma \text{ that is an open disk and that projects one-to-one to } \mathbb{R}^2\}. \tag{17}$$

By the tameness and general position assumption Γ and Δ are graphs (embedded in $\mathbb{R}^3$), with a finite number of vertices and edges.

The link K is contained both in Γ and in Δ. The graph Δ consists of K together with all "fold" edges of Σ. The set W is the set of vertices of Δ of degree 3, all other vertices of Δ having degree 2. Note that

$$H = \pi[\Delta]. \tag{18}$$

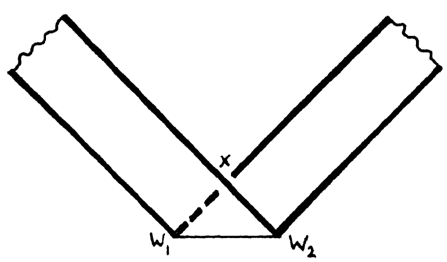

FIGURE 14

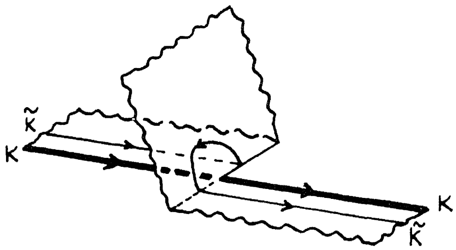

FIGURE 15

(It is not difficult to see that these assumptions can be satisfied. In fact, if we take $\Sigma = \psi[\Sigma_{K'}]$ as in the beginning of this section, these assumptions are easily fulfilled, as the isotopy can be described by Reidemeister moves.)

We introduce some further notation and terminology. Let $W^{\uparrow}$ denote the set of points of type (a) or (b) in Fig. 12, and let $W^{\downarrow}$ denote the set of points of type (c) or (d) in Fig. 12. Let W^{+} denote the set of points of type (a) or (c) in Fig. 12, and let W^{-} denote the set of points of type (b) or (d) in Fig. 12. This notation is motivated by the fact that

> a link $\tilde{K}$ on Σ parallel and close to K, makes a positive crossing with K near any point $w \in W^{+}$, and a negative crossing with K near any point $w \in W^{-}$. (19)

For instance, in (a) of Fig. 12, a positive $K - \tilde{K}$ crossing can be seen (Fig. 15).

Let U be the set of points in $\pi^{-1}[G]$ that are on fold edges of Δ. That is,

$$U := \Delta \cap \pi^{-1}[G] \setminus K. \tag{20}$$

So U is the set of points u that have in $\Sigma \cap \pi^{-1}[G]$ a neighbourhood as in Fig. 16. Moreover, define (for any X)

$$\begin{aligned} VX &:= \text{set of vertices of } X, \\ EX &:= \text{set of edges of } X, \\ FX &:= \text{set of faces of } X, \\ \mathscr{C} &:= \text{set of components of } \Sigma \setminus \pi^{-1}[G], \\ F_0 &:= \text{unbounded face of } G. \end{aligned} \tag{21}$$

Call a component of $K \setminus (P \cup W)$ (i.e., an edge of Γ on K) a *segment*.

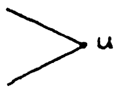

FIGURE 16

By extension, define for any $x \in \mathbb{R}^3 : \omega(x) := \omega(\pi(x))$. Call a point x in $\mathbb{R}^2$ or $\mathbb{R}^3$ *even* or *odd* if $\omega(x)$ is even or odd. For any set X, X_{even} denotes the set of even points in X, and X_{odd} denotes the set of odd points in X.

For any nonempty subset X of $\mathbb{R}^2$ or $\mathbb{R}^3$ let

$$\mu(X) := \min\{\omega(x) \mid x \in X\}. \tag{22}$$

Minimality of Σ. Suppose Σ is a counterexample to the lemma. We may assume that we have chosen Σ in such a way that:

$$\begin{aligned} &\text{(i)} && \chi(\Sigma) \text{ is as large as possible;} \\ &\text{(ii)} && \sum_{v \in VG \cap \mathrm{bd}(F_0)} \omega(v) \text{ is as small as possible;} \\ &\text{(iii)} && \sum_{v \in VG} \omega(v) \text{ is as small as possible;} \\ &\text{(iv)} && |W| \text{ is as small as possible;} \\ &\text{(v)} && \sum_{\sigma \text{ segment}} \mu(\sigma) \text{ is as small as possible;} \\ &\text{(vi)} && |U| \text{ is as small as possible.} \end{aligned} \tag{23}$$

(In this order: (ii) should hold under condition (i), and so on.)

Σ is determined by Γ. The surface Σ is determined by the graph Γ (up to inessential deformations). To see this, note that by our general position assumption, the boundary of any component $C \in \mathscr{C}$ is a disjoint union of simple closed curves. In fact it is only one closed curve:

CLAIM 1. *Each component $C \in \mathscr{C}$ is an open disk.*

Proof. Consider a face F of G. For any component $C \in \mathscr{C}$ contained in $\pi^{-1}[F]$, the boundary $\mathrm{bd}(C)$ of C is a union of pairwise disjoint simple closed curves on $\mathrm{bd}(\pi^{-1}[F])$.

Moreover, C is orientable, since we can extend C to a closed surface

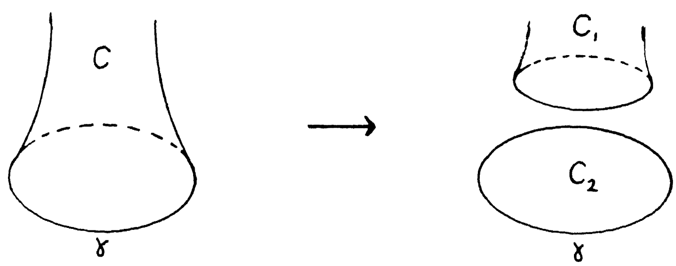

FIGURE 17

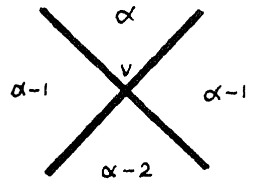

FIGURE 18

in $\mathbb{R}^3$ by adding disjoint closed disks to the boundary components of C (outside a finite section of $\pi^{-1}[F]$).

Suppose $\pi^{-1}[F]$ contains a component in $\mathscr{C}$ that is not an open disk. Then we can choose a component $C \in \mathscr{C}$ contained in $\pi^{-1}[F]$ such that C is not an open disk and such that for one of its boundary components γ, one of the components of $\pi^{-1}[\mathrm{bd}(F)] \backslash \gamma$ is minimal (inclusion-wise). (Minimal taken over all C that are not open disks, over all boundary components γ and over the two components of $\pi^{-1}[\mathrm{bd}(F)] \backslash \gamma$.)

By this minimality assumption, we know that there exists an open disk C_2 in $\pi^{-1}[F]$ with boundary γ, and disjoint from Σ. Near to C_2 we can do surgery on C so as to obtain a bounded surface C_1 in $\pi^{-1}[F]$ with boundary $\mathrm{bd}(C) \backslash \gamma$, and disjoint from $(\Sigma \backslash C) \cup C_2$ (cf. Fig. 17). Thus

$$\chi(C_1) + \chi(C_2) = \chi(C) + 2. \tag{24}$$

Let Σ' be the manifold obtained from Σ by replacing C by C_1 and C_2. So $\chi(\Sigma') = \chi(\Sigma) + 2$.

Let Σ'' be the union of those components of Σ' that have a nonempty border (i.e., are not closed). So $\mathrm{bd}(\Sigma'') = K$. Note that $\Sigma' \backslash \Sigma''$ has at most one component, because each component of Σ has a nonempty border. If $\chi(\Sigma'') > \chi(\Sigma)$, we would obtain a counterexample with larger Euler characteristic, contradicting our assumption (23)(i). (It is a counterexample, since clearly $\tau(K, \Sigma'') = \tau(K, \Sigma)$ and since $\chi(\Sigma'') > \chi(\Sigma) \geqslant b(K) - v(K) = \chi(\Sigma_K)$.)

So $\chi(\Sigma'') \leqslant \chi(\Sigma)$. Hence $\chi(\Sigma' \backslash \Sigma'') \geqslant 2$, and hence $\Sigma' \backslash \Sigma''$ is a 2-sphere S. Then K is either enclosed by S or is contained in the exterior of S. (Indeed, $\pi[K]$ attains the minimum number of crossings among all links equivalent

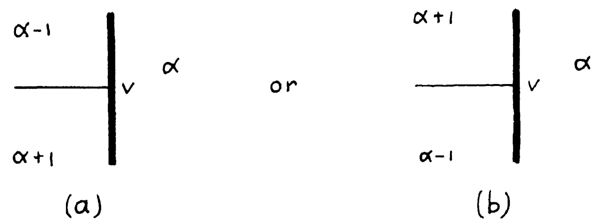

FIGURE 19

to K (cf. Section 3). Hence there cannot exist a 2-sphere separating two components of K.)

By (possibly) applying an isotopy of S^3 we may assume that K is contained in the exterior of S.

It follows that there is an isotopy bringing $(S\backslash(C_1 \cup C_2)) \cup C$ to a bordered surface contained in $\pi^{-1}[F]$, fixing $\Sigma\backslash((S\backslash(C_1 \cup C_2)) \cup C)$. Thereby we decrease $|U|$ or $\omega(v)$ for at least one $v \in V$, and we do not increase $\omega(v)$ for any $v \in VG$, or $|W|$, or $\mu(\sigma)$ for any segment σ. This contradicts the minimality assumption (23). ∎

It follows that, up to isotopy, we can reconstruct Σ from Γ, because up to isotopy there is a unique way to fill disjoint closed curves on a cylinder by disjoint disks inside the cylinder. (This follows inductively from the homotopic triviality of the solid cylinder.) Note that at edges e of Γ not on K, the surface Σ is attached at both sides of $\pi^{-1}[\pi[e]]$. At each segment σ on K ($=$edge of Γ on K), Σ is attached at only one side. We can determine this side, as it is at the "odd face side" if $\mu(\sigma)$ is odd, and at the "even face side" if $\mu(\sigma)$ is even. ($\mu(\sigma)$ is determined by Γ.)

The graphs G and H. Note that $G = \pi[K]$ is a subgraph of H, and if $x \notin H$, that $\omega(x)$ is odd if x belongs to some odd face of $\pi[K]$, and $\omega(x)$ is even if x belongs to some even face of $\pi[K]$.

Note moreover that if e is an edge of H, and F and F' are the two faces of H incident with e, then $|\mu(F) - \mu(F')| = 1$ if e is part of G, and $|\mu(F) - \mu(F')| = 2$ otherwise.

H has three types of vertices: vertices that are also vertices of G, vertices that are on an edge of G, and vertices that are in a face of G. Consider a vertex v of H, and let $\alpha := \omega(v)$.

If v is also a vertex of G, it has degree 4 both in G and in H. Its neighbourhood is like that in Fig. 18. (In Figs. 18–23, the numbers in the faces of H give their μ-values.)

If v is on an edge of G, it has degree 3 or 4. If it has degree 3, it is the projection $\pi(w)$ of some point w in W, and (see Fig. 12) its neighbourhood is as in Fig. 19.

If v has degree 4, it is the projection $\pi(u)$ of some point u in U, and its neighbourhood is as in Fig. 20.

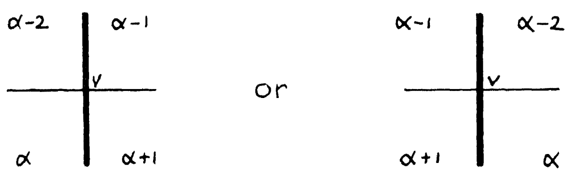

FIGURE 20

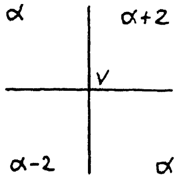

FIGURE 21

If v is in a face of G, it has degree 2 or 4 in H. If it has degree 4, its neighbourhood is as in Fig. 21.

Sometimes, we will indicate by a little arrow crossing any edge e of H which of the two faces incident with e has highest μ-value as in Fig. 22.

Moreover, we orient each edge e of H so that the face of H with highest μ-value is at the right hand side of e, cf. Fig. 23.

The set W. For any $w \in W$, let ε_w be the (unique) edge of H incident with $\pi(w)$ but not a part of G. Note that

> w belongs to W^+ if either $w \in W^\uparrow$ and ε_w is oriented towards $\pi(w)$, or $w \in W^\downarrow$ and ε_w is oriented away from $\pi(w)$; similarly, w belongs to W^- if either $w \in W^\uparrow$ and ε_w is oriented away from $\pi(w)$, or $w \in W^\downarrow$ and ε_w is oriented towards $\pi(w)$ (25)

(cf. Fig. 12). We show:

CLAIM 2. *Let w and w' be two points in W connected by a segment σ. Then one of w and w' belongs to $W^\uparrow$, the other to $W^\downarrow$.*

Proof. Suppose to the contrary that both w and w' belong to $W^\uparrow$, say. Thus we would have configurations (a) and (b) of Fig. 12 consecutively. (They can be pasted together in four ways.) For instance, we would obtain Fig. 24. This configuration can be replaced by Fig. 25. Note that Figs. 24 and 25 have a similar boundary (the wriggled curves and the part of the knot). So the rest of Σ can be attached to either of these figures. Moreover, locally Fig. 24 can be brought to Fig. 25. (One way of seeing this is that both Fig. 24 and Fig. 25 form an open disk with boundary the "same"

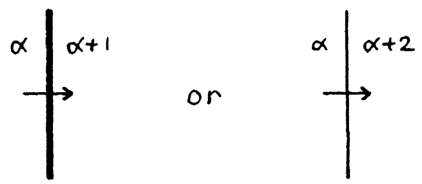

FIGURE 22

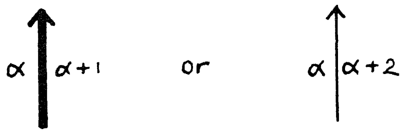

FIGURE 23

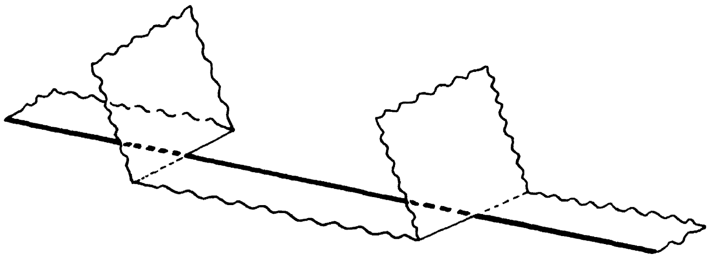

FIGURE 24

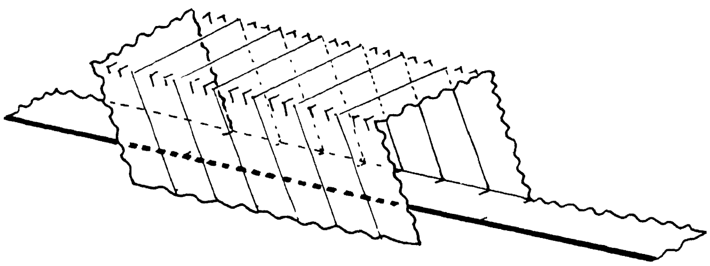

FIGURE 25

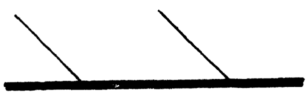

FIGURE 26

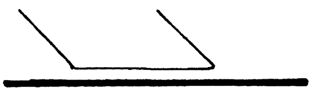

FIGURE 27

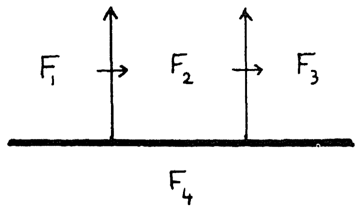

FIGURE 28

closed curve (being the union of the wriggled curve and the given part of the link).) Let e be the edge of G containing $\pi[\sigma]$.

In $\pi^{-1}[e]$, replacing Fig. 24 by Fig. 25 means replacing Fig. 26 by Fig. 27.

We thus do not change $\omega(v)$ for any $v \in VG$, but we do decrease $|W|$, contradicting the minimality assumption (23).

Similar arguments hold for the three other ways of pasting (a) and (b) of Fig. 12 together. ▮

Moreover:

CLAIM 3. *Let w and w' be two points in W connected by a segment. Then one of ε_{w}, $\varepsilon_{w'}$ is oriented towards $\pi(w)$ or $\pi(w')$, the other one away from $\pi(w)$ or $\pi(w')$.*

Proof. Suppose Fig. 28 would occur. Then $\mu(F_3) = \mu(F_2) + 2 = \mu(F_1) + 4$. However, $\mu(F_4)$ differs by at most one from both $\mu(F_1)$ and $\mu(F_3)$, a contradiction.

Similarly the configurations in Fig. 29 lead to a contradiction. ▮

As a direct corollary we have:

CLAIM 4. *For each edge e of G, either all points $w \in W$ with $\pi(w) \in e$ belong to W^+, or all belong to W^-.*

Proof. Directly from Claims 2 and 3 (cf. (25)). ▮

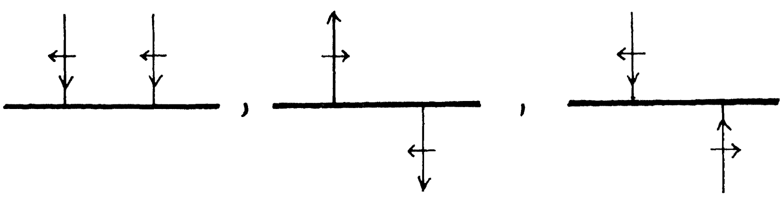

FIGURE 29

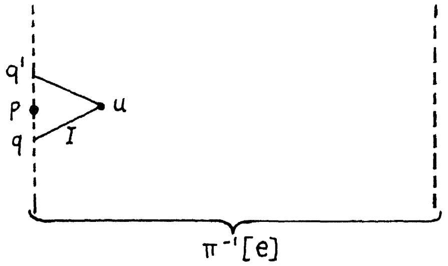

FIGURE 30

The set U. Consider an edge e of G, connecting vertices v and v' of G. Assume without loss of generality that e is a straight line segment in $\mathbb{R}^2$. Consider the intersection $J := \Sigma \cap \pi^{-1}[\bar{e}]$.

The set J forms a graph with vertices of degree 1 on the boundary of $\pi^{-1}[\bar{e}]$ and vertices of degree 3 in each point in $W \cap \pi^{-1}[\bar{e}]$. Moreover, one of $p_v^{\uparrow}$, $p_v^{\downarrow}$ and one of $p_{v'}^{\uparrow}$, $p_{v'}^{\downarrow}$ might be an isolated vertex of J. All other vertices of J have degree 2.

By the minimality assumption (23), we may assume that each component I of $J \backslash K$ is a straight line segment, or the union of two straight line segments "above each other," making an angle at a point u in U, as in Fig. 30. In the latter case, the straight line segment connecting the end points q and q' of I contains a point $p \in P$, which is an isolated point of J. Moreover, above or under I there is no point in W (i.e., $\pi[I] \cap \pi[W] = \emptyset$). So there is a segment σ of K such that $\pi[I] \subset \pi[\sigma]$ and such that σ is incident with at least one point in P.

The neighbourhood of $\pi^{-1}[v]$ for vertices v of G. Consider a vertex v of G and its neighbourhood, as in Fig. 31. Here F_1, F_2, F_3, F_4 denote the faces of G incident with v. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be the segments incident with $p_v^{\uparrow}$ and $p_v^{\downarrow}$ so that $\pi[\sigma_i]$ is incident with F_i and F_{i+1} ($i = 1, ..., 4$, taking indices mod 4).

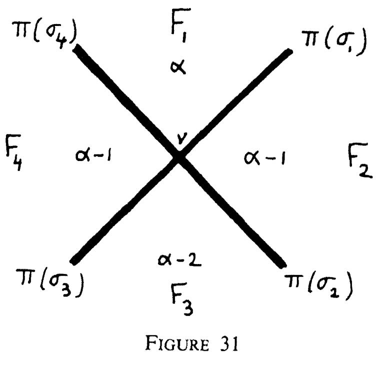

FIGURE 31

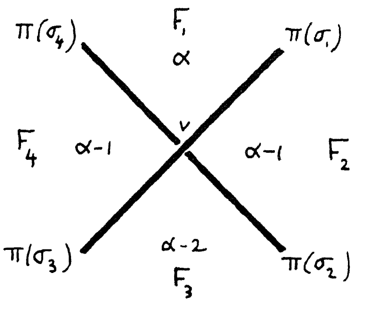

FIGURE 32

For each $i = 1, ..., 4$, choose some subinterval J_i of $\pi[\sigma_i] \cup \{v\} \cup \pi[\sigma_{i-1}]$ containing all points in $\pi[U] \cap (\pi[\sigma_i] \cup \pi[\sigma_{i-1}])$.

First consider the case where $\alpha := \omega(v)$ is odd. Then the diagram is locally as in Fig. 32. Consider now $\pi^{-1}[J_3]$ as "seen" from F_3. It is either as in Fig. 33 or as in Fig. 34.

The numbers $\beta_v^\uparrow, \beta_v^\downarrow, \varphi_v^\uparrow, \varphi_v^\downarrow, \zeta_v, \eta_v$ are the numbers of occurrences of the given type of curve.

We set $\eta_v := 0$ if Fig. 33 applies, and $\zeta_v := 0$ if Fig. 34 applies. Define

$$\varphi_v := \varphi_v^\uparrow + \varphi_v^\downarrow, \qquad \varphi := \sum_{v \in V} \varphi_v, \qquad \zeta := \sum_{v \in V} \zeta_v, \qquad \eta := \sum_{v \in V} \eta_v. \quad (26)$$

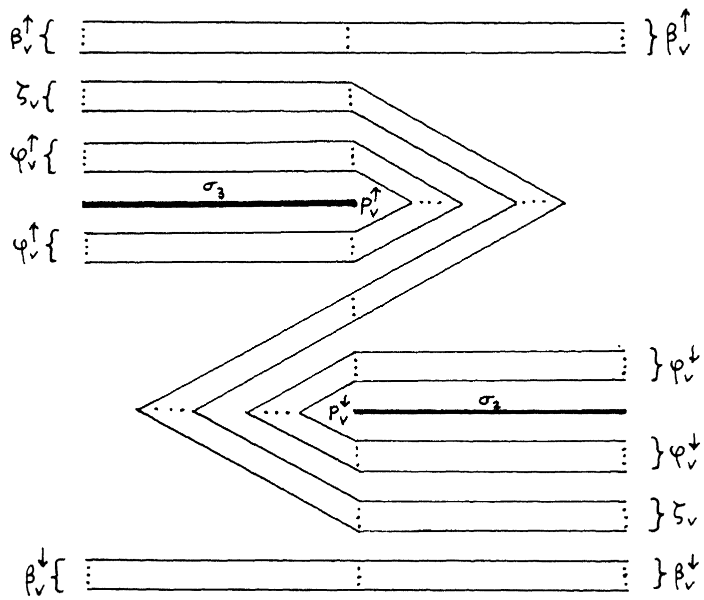

FIGURE 33

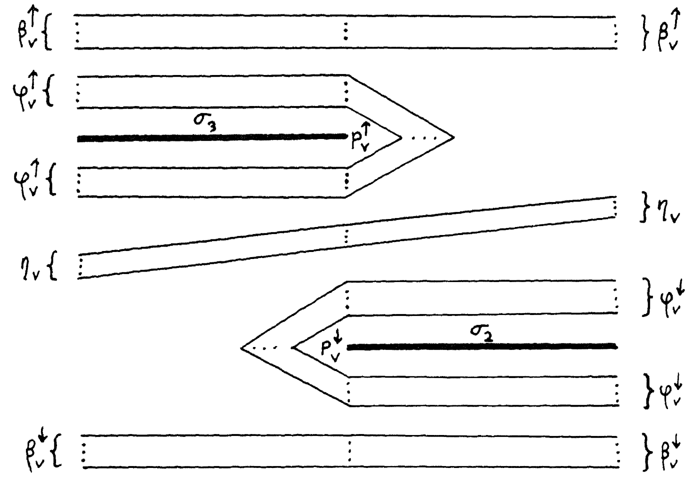

FIGURE 34

Note that

$$|U| = \varphi + 2\zeta. \tag{27}$$

$\pi^{-1}[J_2]$ seen from F_2 is as in Fig. 35. A symmetric picture applies to $\pi^{-1}[J_4]$ seen from F_4.

Finally, $\pi^{-1}[J_1]$ seen from F_1 is as in Fig. 36.

Symmetric pictures and notation apply in case $\omega(v)$ is even.

Segments connecting P and W. Consider a segment σ incident at one end with a point $p_v^\uparrow$ and at the other end with a point w in W. Let e be the edge of G containing $\pi[\sigma]$. Let I be the component of $(\pi^{-1}[e] \cap \Sigma)\backslash K$ incident with w. Then we have:

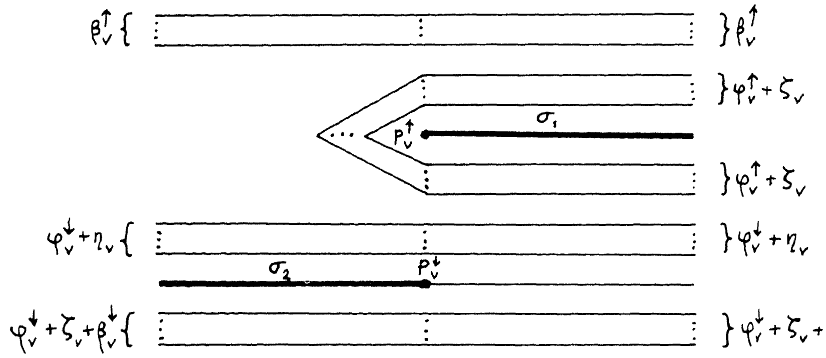

FIGURE 35

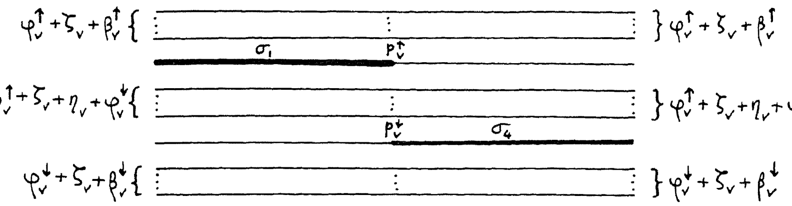

FIGURE 36

CLAIM 5. *Locally in* $\pi^{-1}[e]$, *the configuration is like one of those in Fig.* 37.

Proof. Indeed, the alternative would be that it is one of the configurations in Fig. 38. In both of these two cases there is an isotopy (moving w along K), reducing $\omega(v)$ (and not changing any $\omega(v')$ for any $v' \neq v$), contradicting the minimality assumptions (23)(ii) and (iii). ∎

Similar statements hold for segments connecting $p_v^{\downarrow}$ and a point in W.

The boundaries of components in $\mathscr{C}$. Consider a component $C \in \mathscr{C}$. Let $\pi[C]$ be contained in face F of G. Then either $\mathrm{bd}(C)$ is a homotopically trivial circuit on $\pi^{-1}[\mathrm{bd}(F)]$, or not. Let $\mathscr{C}_0$ be the collection of components of the first kind, and let $\mathscr{C}_1$ be the collection of components of the second kind. Note that if F is the unbounded face F_0 of G, then all components $C \in \mathscr{C}$ contained in $\pi^{-1}[F]$ belong to $\mathscr{C}_0$ (since $\Sigma \subseteq \mathbb{R}^3$).

In order to study $\mathscr{C}$, consider a segment σ. Let e and e' be two parts of edges of Γ above σ, in such a way that e and e' have the same projection as σ as in Fig. 39. (Here e might be incident with one of the end points of σ). Let e and e' be on the boundaries of components C and C' in $\mathscr{C}$, respectively. Then:

CLAIM 6. *C and C' are different.*

Proof. Suppose $C = C'$. Then we may assume that there is no other edge part of Γ in between e and e' with the same projection as σ. Otherwise there would be two such edges e'' and e''' in between being part of the boundary of the same C'' in $\mathscr{C}$. (This follows from the fact that if l is a line segment in $\pi^{-1}[\pi[\sigma]]$ connecting e and e', then l is contained in some

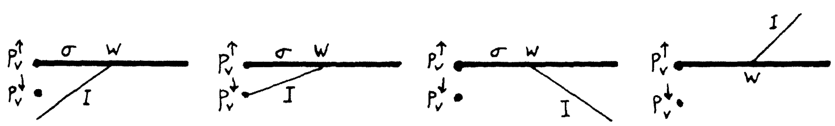

FIGURE 37

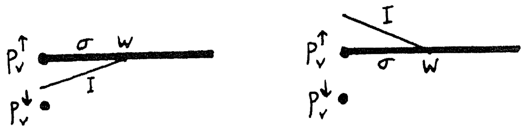

FIGURE 38

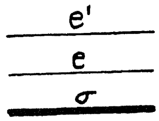

FIGURE 39

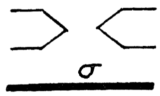

FIGURE 40

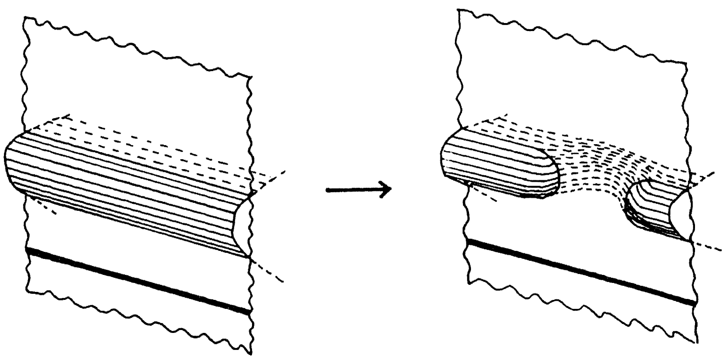

FIGURE 41

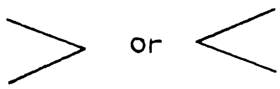

FIGURE 42

FIGURE 43

circuit in $l \cup \mathrm{bd}(C)$ that is homotopically trivial in $\pi^{-1}[\mathrm{bd}(F)]$, where F is the face of G containing $\pi[C]$. Hence for every component C'', $\mathrm{bd}(C'')$ crosses l an even number of times.)

By replacing e, e' by e'', e''' and repeating the argument, we obtain two "neighbouring" e, e'.

Now modify Γ by replacing the configuration in Fig. 39 by that in Fig. 40. In $\mathbb{R}^3$ this amounts to an isotopy with the effect as in Fig. 41. This way we reduce $\mu(\sigma)$ for at least one segment σ, and not changing any other $\mu(\sigma)$ or any $\omega(v)$ or $|W|$. This contradicts the minimality assumption (23)(v). ∎

It follows that for any $C \in \mathscr{C}$, with $\pi[C]$ contained in face F of G, and for any $x \in \mathrm{bd}(F)$, $\mathrm{bd}(C)$ has at most three intersections with $\pi^{-1}(x)$.

For any $C \in \mathscr{C}$, let $B(C)$ denote the set of all points in C for which no neighbourhood in C projects one-to-one to $\mathbb{R}^2$.

So

$$\Delta = K \cup \bigcup_{C \in \mathscr{C}} B(C), \tag{28}$$

and hence

$$H = \pi[\Delta] = G \cup \bigcup_{C \in \mathscr{C}} \pi[B(C)]. \tag{29}$$

Let $C \in \mathscr{C}$ and let F be the face of G containing $\pi[C]$. We call a point x of $\mathrm{bd}(C)$ a *turning point* of $\mathrm{bd}(C)$ if on $\pi^{-1}[\mathrm{bd}(F)]$ the neighbourhood of x in $\mathrm{bd}(C)$ is as in Fig. 42. Thus each turning point belongs to $W \cup U$.

If $C \in \mathscr{C}_0$, then Claim 6 implies that $\mathrm{bd}(C)$ has exactly two turning points, and $\pi[\mathrm{bd}(C)] \neq \mathrm{bd}(F)$. We may assume that $|\pi^{-1}(x) \cap C| \leqslant 2$ for all $x \in F$, and that $B(C)$ is a curve connecting the two turning points on $\mathrm{bd}(C)$. Moreover, we may assume that $C = D' \cup B(C) \cup D''$ for two open disks D' and D'' such that both D' and D'' project one-to-one to $\mathbb{R}^2$.

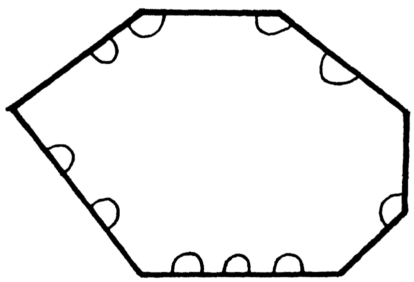

FIGURE 44

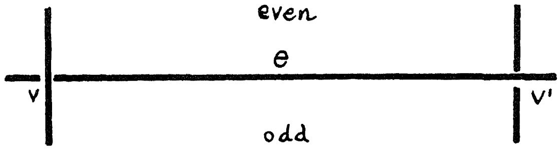

FIGURE 45

If $C \in \mathscr{C}_1$, then at any $x \in \text{bd}(F)$ one has $|\pi^{-1}(x) \cap \text{bd}(C)| = 1$ or 3, except if $\pi^{-1}(x)$ contains a turning point of bd(C). Call a curve of type ζ in Fig. 33 a *Z-type curve*. Then by Claim 6, if $|\pi^{-1}(x) \cap \text{bd}(C)| = 3$, the middle element of $\pi^{-1}(x) \cap \text{bd}(C)$ is either part of a Z-type curve or is on K. So bd(C) can have turning points only at Z-type curves or at segments σ on K that are incident with two points in W and that are locally as in Fig. 43, which we call *Z-type segments*.

We may assume that $|\pi^{-1}(x) \cap C| \leqslant 3$ for all $x \in F$. In fact, we may assume that the set $\{x \in F \mid |\pi^{-1}(x) \cap C| = 3\}$ forms a collection of pairwise disjoint open regions, each corresponding to one Z-type curve or Z-type segment. This follows from the fact that up to isotopy C is fully determined by bd(C). Since there exists an open disk $\tilde{C}$ with $\text{bd}(\tilde{C}) = \text{bd}(C)$ for which the set $\{x \in F \mid |\pi^{-1}(x) \cap \tilde{C}| = 3\}$ forms a collection of pairwise disjoint open regions, we may assume that C itself has this property. So "fold edges" going across from one Z-type curve or segment to another can be removed.

The set $B(C)$ forms a disjoint union of curves, each of them connecting two turning points on some Z-type curve or segment on the boundary of C. We may assume that $B(C)$ projects one-to-one to $\mathbb{R}^2$ (the curves do not touch each other), as in Fig. 44.

The graph H along an edge of G. Consider an edge e of G, let it go from v to v' as in Fig. 45. Following e from v to v', we first meet some (or none) points in $\pi[U]$, each having degree 4 in H. Next we meet some (or none) points in $\pi[W]$, each having degree 3 in H. Finally we meet again some (or none) points in $\pi[U]$, each of degree 4 in H (cf. the observations concerning Fig. 30).

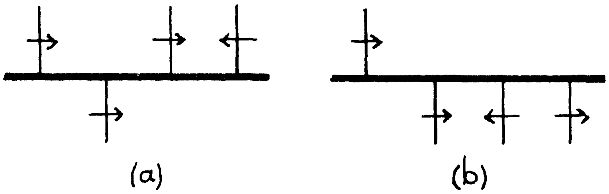

FIGURE 46

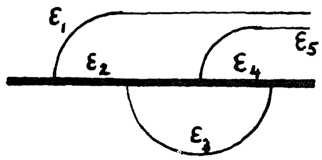

FIGURE 47

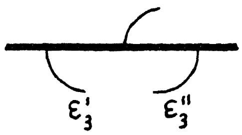

FIGURE 48

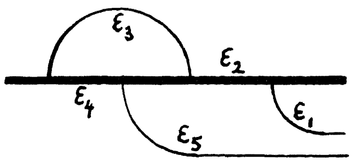

FIGURE 49

FIGURE 50

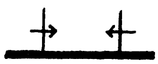

FIGURE 51

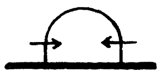

FIGURE 52

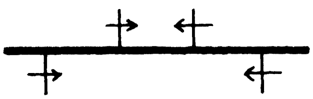

FIGURE 53

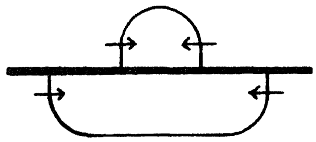

FIGURE 54

We first show:

CLAIM 7. *The configurations in Fig. 46 do not occur on any edge e of G. Similarly for those configurations which arise from exchanging up and down and left and right in this figure.*

Proof. By Claim 2, Fig. 46(a) gives Fig. 47 in $\pi^{-1}[e]$ (up to exchanging up and down). Curve ε_3 comes from the fact that ε_3' and ε_3'' in Fig. 48 should lead to each other (by Claim 6).

Then the boundary of some component $C \in \mathscr{C}$ contains $\varepsilon_1, \ldots, \varepsilon_5$ (at one side of $\pi^{-1}[e]$ or the other). So bd(C) contains both ε_1 and ε_5. This contradicts Claim 6.

Similarly, Fig. 46(b) gives Fig. 49 (up to exchanging up and down), again contradicting Claim 6. ∎

Partition the set $W \cap \pi^{-1}[e]$ into classes $W_1, W_2, \ldots, W_k$ in such a way that

(i) k is even;
(ii) $\pi[W_1], \ldots, \pi[W_k]$ occur consecutively along e, as in Fig. 50; (30)
(iii) $W_i \neq \varnothing$ for $i = 2, \ldots, k-1$;
(iv) the arrow crossing any edge ε_w with $w \in W_i$ goes from right to left if i is odd, and from left to right if i is even.

(Again, ε_w denotes the edge of H incident with $\pi(w)$ not being part of G.) This partition is trivially unique.

As Fig. 46(a) does not occur, by Claim 3 we know that if i is even and $i \leqslant k-2$ then $|W_i| \leqslant 2$. Similarly, if i is odd and $i \geqslant 3$ then $|W_i| \leqslant 2$. So $1 \leqslant |W_i| \leqslant 2$ for $i = 2, \ldots, k-1$.

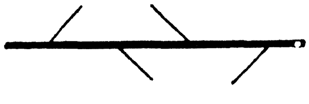

FIGURE 55

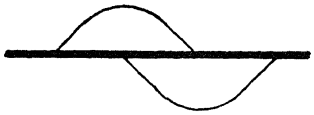

FIGURE 56

As Fig. 46(b) does not occur, we know that if i is even, $i \leqslant k-2$ and $W_{i+2} \neq \emptyset$ then $|W_i| \leqslant |W_{i+1}|$. Similarly, if i is odd, $i \geqslant 3$ and $W_{i-2} \neq \emptyset$ then $|W_i| \leqslant |W_{i-1}|$. Hence $|W_i| = |W_{i+1}|$ for i even, $4 \leqslant i \leqslant k-4$; moreover, $|W_2| = |W_3|$ if $W_1 \neq \emptyset$, and $|W_{k-2}| = |W_{k-1}|$ if $W_k \neq \emptyset$. So there are the following possibilities:

(i) $|W_i| = |W_{i+1}| \in \{1, 2\}$ for each even i with $2 \leqslant i \leqslant k-2$;

(ii) $k \geqslant 4$, $W_1 = \emptyset$, $|W_2| = 1$, $|W_3| = 2$, $|W_i| = |W_{i+1}| \in \{1, 2\}$ for each even i with $4 \leqslant i \leqslant k-2$;

(iii) $k \geqslant 4$, $|W_i| = |W_{i+1}| \in \{1, 2\}$ for each even i with $2 \leqslant i \leqslant k-4$, $|W_{k-2}| = 2$, $|W_{k-1}| = 1$, $W_k = \emptyset$; (31)

(iv) $k \geqslant 6$, $W_1 = \emptyset$, $|W_2| = 1$, $|W_3| = 2$, $|W_i| = |W_{i+1}| \in \{1, 2\}$ for each even i with $4 \leqslant i \leqslant k-4$, $|W_{k-2}| = 2$, $|W_{k-1} = 1$, $W_k = \emptyset$.

If we have two neighbouring edges ε_w and $\varepsilon_{w'}$ with arrows pointing towards each other as in Fig. 51 (up to exchanging up and down and left and right in this figure), then they are in fact one and the same edge as in Fig. 52. This follows from the fact that they are projections of some component of $B(C)$ for some $C \in \mathscr{C}_1$, as the segment on K in between is a Z-type segment.

If $|W_i| = |W_{i+1}| = 2$ with $2 \leqslant i \leqslant k-2$ and i even, then we have Fig. 53 (up to exchanging up and down in this figure). In that case they are part of Fig. 54, since in $\pi^{-1}[e]$ we have Fig. 55 (up to exchanging up and down), and hence we have Fig. 56.

We now consider what we see when following edge e from v to v' (cf. Fig. 45). First assume that alternative (31)(i) applies. We first meet a number $t \geqslant 0$ of points in $\pi[U]$, each having degree 4 in H as in Fig. 57. We say that these points of $\pi[U]$ (and their liftings in U) are *near to v*.

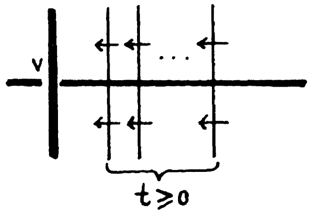

FIGURE 57

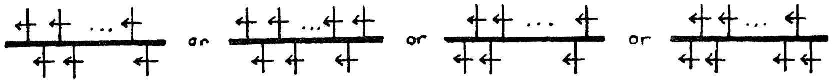

FIGURE 58

Next we meet a series of points in $\pi[W]$ of degree 3. First we meet the points in $\pi[W_1]$ (possibly none) as in Fig. 58. Again we say that these points of $\pi[W]$ (and their liftings in W) are *near to v*.

Next we meet a series of configurations given in Fig. 59 made by $W_2 \cup W_3$, $W_4 \cup W_5, \ldots, W_{k-2} \cup W_{k-1}$, in some amount and in some order. (In fact, Claim 3 gives conditions under which these configurations can succeed each other.)

After that we have points in $\pi[W_k]$ (possibly none) as in Fig. 60. These points (and their liftings) are called *near to v'*.

Finally, we meet again a number of points of degree 4 in $\pi[U]$ ($t' \geq 0$ say) as in Fig. 61. These points (and their liftings) are called *near to v'*.

Next assume that alternative (31)(ii) applies. Then again we first meet a number of points in $\pi[U]$ each having degree 4 in H as in Fig. 57. Again we call these points and their liftings *near to v*.

Next we meet a configuration made by $W_2 \cup W_3$ as in Fig. 62. We say that these three points, and their liftings, are *near to v*.

After that we meet a series of points as in Figs. 59, 60, and 61. Again we call the points in Figs. 60 and 61 and their liftings *near to v'*.

If alternative (31)(iii) applies we have a symmetric situation. Finally, if (31)(iv) applies, we obtain a sequence beginning as for (31)(ii) and ending as for (31)(iii).

We analyze a little further. Suppose that w belongs to W^+ whenever $\pi(w)$ is on e (cf. Claim 4). Let alternative (31)(i) apply. If $W_1 \neq \emptyset$ then by Claim 5 the first vertex w in W_1 should belong to $W^{\uparrow}$. Hence it is as shown in Fig. 12(a), and therefore it should be as in Fig. 63. Similarly, if $W_k \neq \emptyset$ and w' is the last point in W_k it should be in $W^{\downarrow}$ and hence of type Fig. 12(c). So it is as in Fig. 64. Hence using Claim 3 if $|W_1|$ is even and nonzero then $|W_k|$ is even, and we have Fig. 65 where the interrupted parts are optional. Similarly, if $|W_1|$ is odd then $|W_k|$ is odd or zero, and we have Fig. 66.

Next let alternative (31)(ii) apply. Then we start like in Fig. 67. This follows from the fact that if we would have alternatively Fig. 68, then seen

FIGURE 59

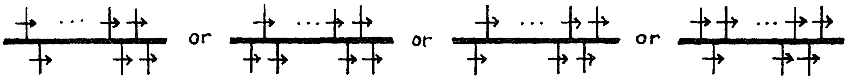

FIGURE 60

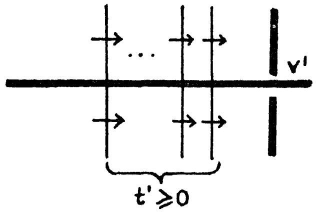

FIGURE 61

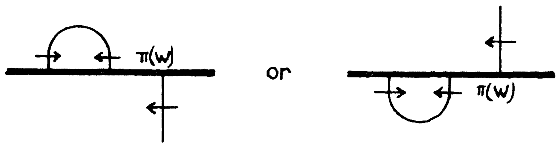

FIGURE 62

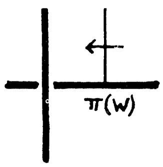

FIGURE 63

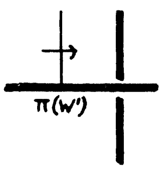

FIGURE 64

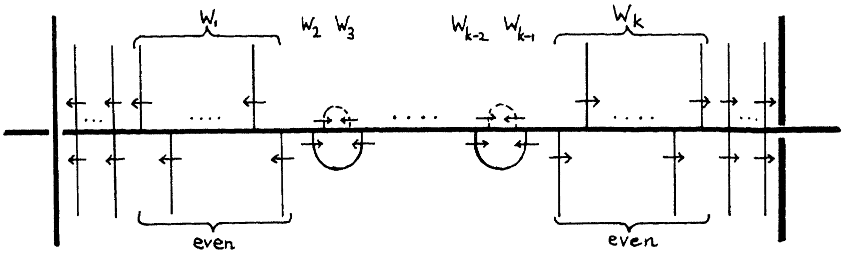

FIGURE 65

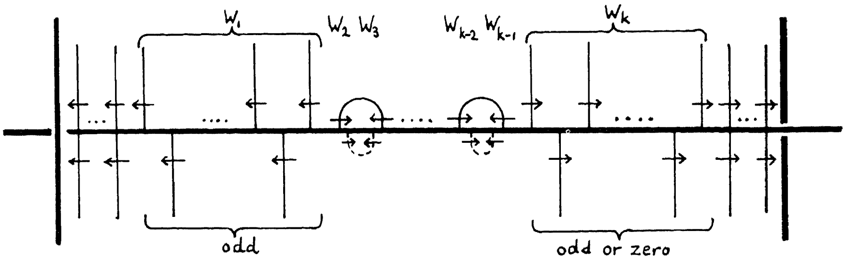

FIGURE 66

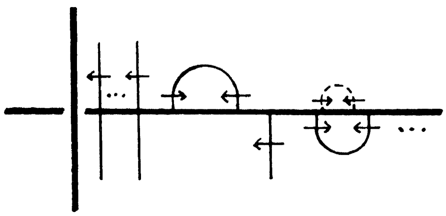

FIGURE 67

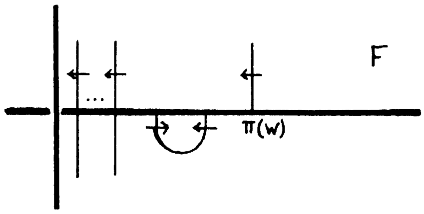

FIGURE 68

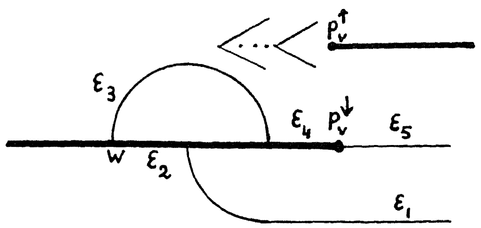

FIGURE 69

from F we have Fig. 69 (since w belongs to W^+ (using (25))): Then the boundary of some component $C \in \mathscr{C}$ contains $\varepsilon_1, \ldots, \varepsilon_5$. So bd($C$) contains both ε_1 and ε_5, contradicting Claim 6.

Similarly for the alternatives (31)(iii) and (iv).

Summarizing, we have the following five types of edges e with $w \in W^+$ when $\pi(w) \in e$: Fig. 70 with γ and δ even; Fig. 71 with γ odd or 0, and δ odd or 0; Fig. 72 with δ even; Fig. 73 with γ even; and Fig. 74.

If w belongs to W^- whenever $\pi(w) \in e$ we should reflect these figures with respect to e.

Note:

CLAIM 8. *All points in U near to a vertex v of G project to at most two edges of G incident with v.*

Proof. This follows directly from Figs. 33–36. ∎

The graph H in the faces of G. Let F be a face of G. Let C and C' be two components in $\mathscr{C}$ contained in $\pi^{-1}[F]$. Consider a component Q of $B(C)$ and a component Q' of $B(C')$. So Q and Q' are curves. Suppose that the projections $\pi[Q]$ and $\pi[Q']$ cross. Then $C \neq C'$ (by our analysis after Claim 6). If $C \in \mathscr{C}_0$ and F is even, then one turning point x of bd(C) is as one in Fig. 75 and the other turning point y is as in Fig. 76. So for each $z \in \pi[\text{bd}(C)]$ the closed vertical line segment connecting the (at most two) points in $\pi^{-1}(z) \cap \text{bd}(C)$ intersects K, except near the two turning points of bd(C).

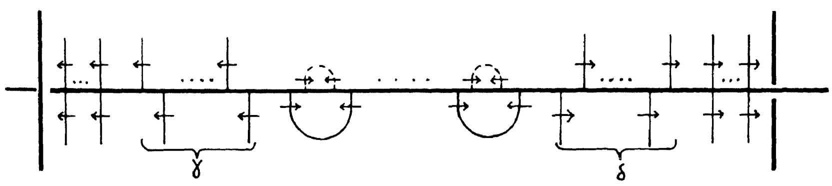

FIGURE 70

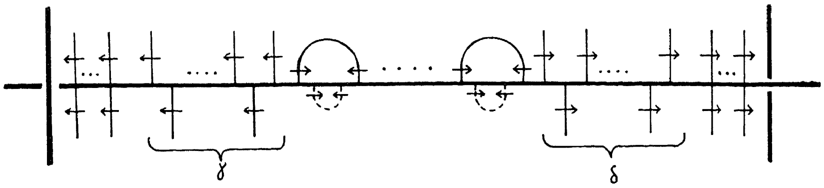

FIGURE 71

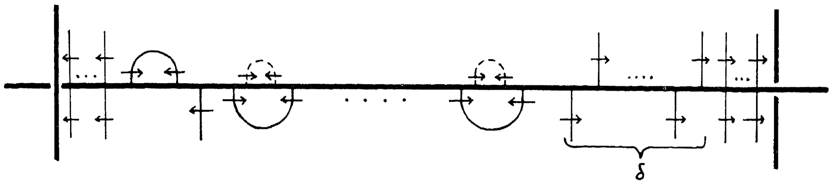

FIGURE 72

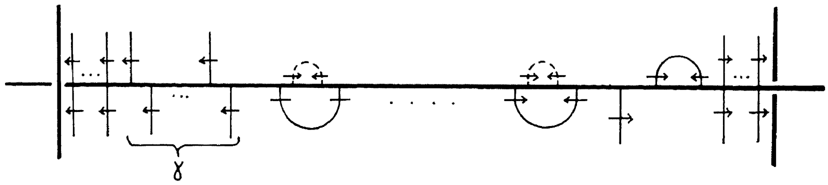

FIGURE 73

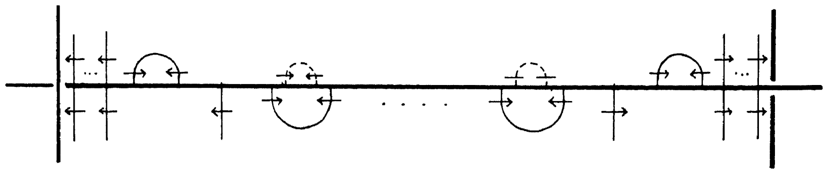

FIGURE 74

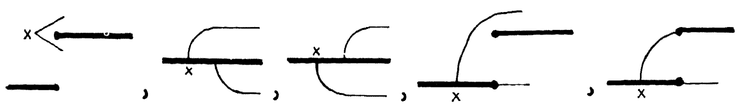

FIGURE 75

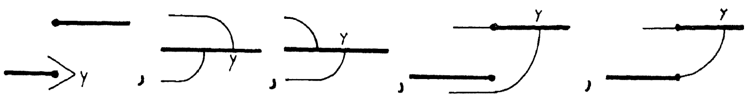

FIGURE 76

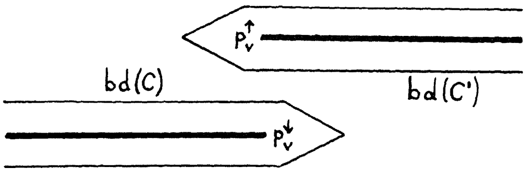

FIGURE 77

Suppose that also C' belongs to $\mathscr{C}_0$. As $\pi[Q]$ and $\pi[Q']$ cross we know $\pi[\mathrm{bd}(C)] \nsubseteq \pi[\mathrm{bd}(C')] \nsubseteq \pi[\mathrm{bd}(C)]$, $\pi[\mathrm{bd}(C)] \cap \pi[\mathrm{bd}(C')] \neq \emptyset$ and $\pi[\mathrm{bd}(C)] \cup \pi[\mathrm{bd}(C')] \neq \mathrm{bd}(F)$. So $\mathrm{bd}(C)$ and $\mathrm{bd}(C')$ do not enclose each other. Hence $\mathrm{bd}(C)$ and $\mathrm{bd}(C')$ should have turning points in U near to some vertex v as in Fig. 77. In $\mathbb{R}^2$ this gives Fig. 78. Suppose next that C' belongs to $\mathscr{C}_1$. Then by the observation on Fig. 44 we know that Q' must be a curve coming from a Z-type curve, say near vertex v of G. Then $\pi^{-1}(v)$ as seen from F is as in Fig. 79. In $\mathbb{R}^2$ this gives Fig. 80. Finally, assume that both C and C' belong to $\mathscr{C}_1$. Then again by the observation on Fig. 44 we know that both Q and Q' come from a Z-type curve, say near vertex v of G. Then $\pi^{-1}(v)$ as seen from F is as in Fig. 81. In $\mathbb{R}^2$ this gives Fig. 82.

Symmetric situations arise if F is odd. Therefore, we always have:

> If $\pi[B(C)]$ and $\pi[B(C')]$ have a crossing in F, then there exist points, $u, u' \in U$ such that $u \in \mathrm{bd}(C)$ and $u' \in \mathrm{bd}(C')$, such that u and u' are near to the same vertex v of G, and such that $\pi(u)$ and $\pi(u')$ are on different edges of G (incident with v). (32)

We say that such a crossing is *near to v*. So:

> Each vertex of H of degree 4 in some face of G is a crossing near to some vertex v of G; it can occur in only one of the four faces of G incident with v (viz. the one with smallest μ-value near to v). (33)

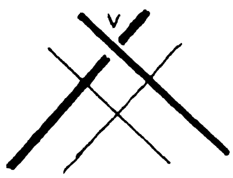

FIGURE 78

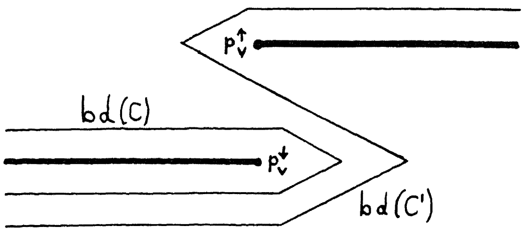

FIGURE 79

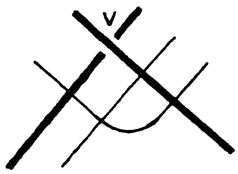

FIGURE 80

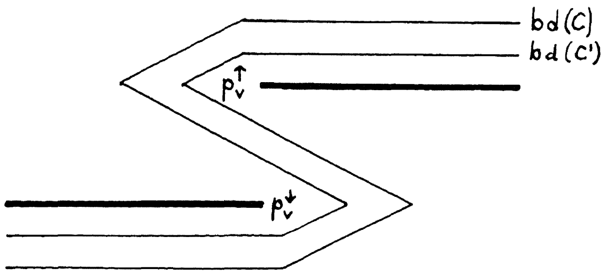

FIGURE 81

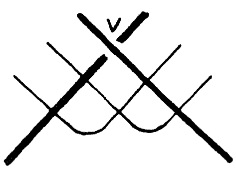

FIGURE 82

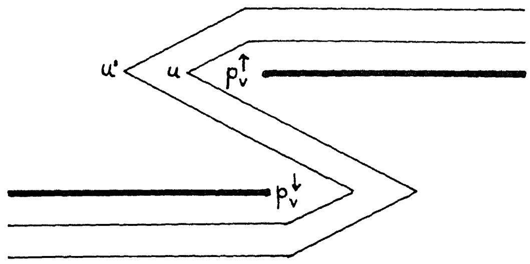

FIGURE 83

We also note:

CLAIM 9. *Let e and e' be two edges of G incident with a vertex v of G, such that e and e' are neighbouring in the cyclic order of edges incident with v. Let $w \in W$ and $u \in U$ be near to v such that w projects to e and u projects to e'. Then u is on a Z-type curve, and all points in U near to v project to e or e'.*

Proof. We may assume that $\pi(w)$ is the point in $\pi[W] \cap e$ that is closest to v, and that $\pi(u)$ is the point in $\pi[U]$ on e' that is closest to v. (Note that if $u \in U$ is closer to v than $u' \in U$, and if u is on a Z-type curve, then also u' is a Z-type curve. See, e.g., Fig. 83. So if the closest point in U near to v along a given edge is on a Z-type curve, then all points in U near to v along this edge are on a Z-type curve.)

If the arrow crossing $\varepsilon_{w'}$ points towards v then by Claim 5 we have Fig. 84 (up to exchanging up and down and left and right in this figure). Then u should be in a Z-type curve, since ε_1 and ε_2 cannot lead to each other by Claim 5. The second point in U on this Z-type curve should be in $\pi^{-1}[e]$.

If the arrow crossing $\varepsilon_{w'}$ points away from v we have Fig. 85 (up to exchanging up and down and left and right in this figure), as we start like in Fig. 67.

In case (a), ε_1 and ε_2 should lead to each other (as there are no points

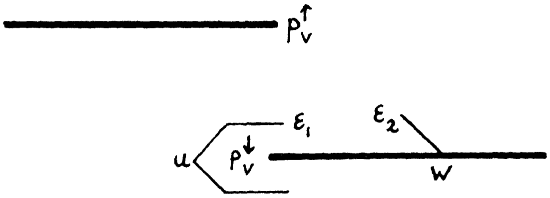

FIGURE 84

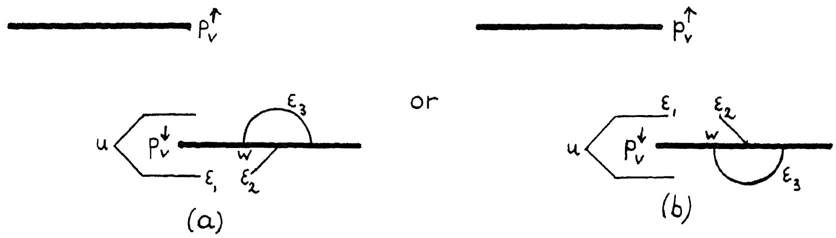

FIGURE 85

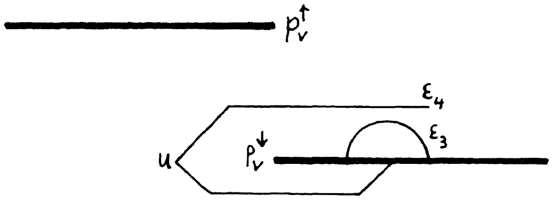

FIGURE 86

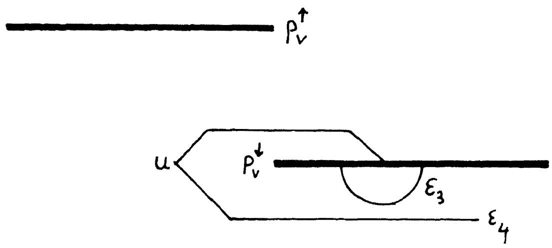

FIGURE 87

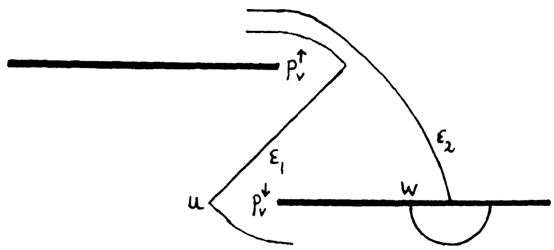

FIGURE 88

38

in $\pi[U]$ on e closer to v than $\pi(u)$). Then u should be on a Z-type curve, since otherwise we would have Fig. 86, contradicting Claim 6 (as ε_3 and ε_4 are on the boundary of the same component in $\mathscr{C}$). So u is on a Z-type curve, and the second point in U on this curve is in $\pi^{-1}[e]$.

In case (b), if ε_1 and ε_2 lead to each other we would have Fig. 87, again leading to a contradiction to Claim 6. So ε_1 and ε_2 do not lead to each other, and hence we have Fig. 88. So u is on a Z-type curve, and the second point in U on this curve is in $\pi^{-1}[e]$. ∎

As consequence one has:

> Let e and e' be two edges of G incident with a vertex v of G, such that e and e' are neighbouring in the cyclic order of edges incident with v. Let $w \in W$ and $u \in U$ be near to v such that w projects to e and u projects to e'. Let F be the face of G incident with v, e and e'. Then there is no directed path in H from $\pi(w)$ to $\pi(u)$ or from $\pi(u)$ to $\pi(w)$ that is contained in F. (34)

This follows from the fact that by Claim 9, u should be on a Z-type curve and hence in F we have Fig. 89 (up to symmetry). So H cannot have a directed path as described.

Another consequence of Claim 9 is:

> Let e and e' be two edges of G incident with a vertex v of G, such that e and e' are neighbouring in the cyclic order of edges incident with v and such that both e and e' contain points in $\pi[W]$ near to v. Then each point in U near to v projects to $e \cup e'$. (35)

For suppose to the contrary that there exists a point u near to v that projects to edge e'' incident with v, with $e'' \notin \{e, e'\}$. Let e'' be neighbouring

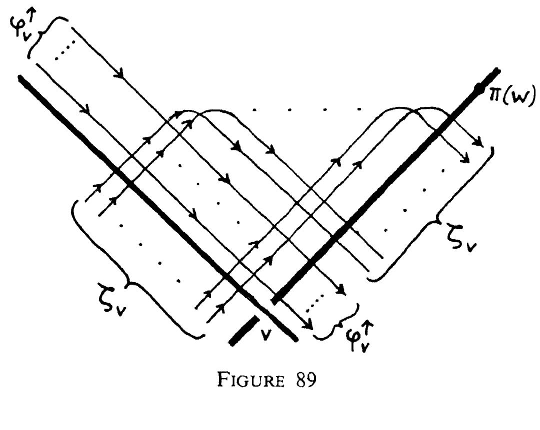

FIGURE 89

e', say, in the cyclic order of edges incident with v. By Claim 9, u is on a Z-type curve, and all points in U near to v project to $e' \cup e''$. Hence e' contains a point, u' say, in $\pi[U]$ near to v. Applying Claim 9 again, there exists a point in U near to v projecting to e. This contradicts the fact that all points in U near to v project to at most two edges of G incident with v.

A lower bound for $\sum_{k=1}^{\infty} \chi(R_{2k})$. Define for any k,

$$R_k := \text{closure of } \{x \in \mathbb{R}^2 \mid \omega(x) \geqslant k\}. \tag{36}$$

So $R_k = \emptyset$ if k is large enough.

Let ρ be the number of Z-type segments (Fig. 43). We show that the Euler characteristic $\chi(R_{2k})$ of the sets R_{2k} satisfy:

CLAIM 10. $4\sum_{k=1}^{\infty} \chi(R_{2k}) \geqslant 2|V_{\text{even}}| + |W_{\text{odd}}| + |U| + 2|W_{\text{odd}}^{-}| + 2\eta + 2\rho$.

Proof. We first prove:

SUBCLAIM 10a. $\sum_{k=1}^{\infty} \chi(R_{2k}) = \frac{1}{2}|\mathscr{C}_1| - \frac{1}{2}b(K) - \sum_{v \in VG} \lfloor(\omega(v) - 1)/2\rfloor + \frac{1}{2}|W_{\text{odd}}| + \frac{1}{2}|U|$. *Here* $\lfloor\ \rfloor$ *denotes lower integer part.*

Proof. We first show that for each face F of G,

$$\sum_{k=1}^{\infty} \chi(R_{2k} \cap F) = \lfloor \tfrac{1}{2}\kappa_F \rfloor, \tag{37}$$

where κ_F denotes the number of components in $\mathscr{C}_1$ contained in $\pi^{-1}[F]$. Note that κ_F is odd, if and only if F is odd.

For any $\mathscr{D} \subseteq \{C \in \mathscr{C} \mid C \subseteq \pi^{-1}[F]\}$ and $x \in \mathbb{R}^2$ let $\omega^{\mathscr{D}}(x) := |\pi^{-1}(x) \cap \bigcup_{D \in \mathscr{D}} D|$ and $R_k^{\mathscr{D}} := F \cap \text{closure of } \{x \in \mathbb{R}^2 \mid \omega^{\mathscr{D}}(x) \geqslant k\}$. We show by induction on $|\mathscr{D}|$ that

$$\sum_{k=1}^{\infty} \chi(R_{2k}^{\mathscr{D}}) = \lfloor \tfrac{1}{2} |\mathscr{D} \cap \mathscr{C}_1| \rfloor. \tag{38}$$

The case $\mathscr{D} = \{C \in \mathscr{C} \mid C \subseteq \pi^{-1}[F]\}$ is (37).

For $\mathscr{D} = \emptyset$, (38) is trivial since $R_{2k}^{\emptyset} = \emptyset$ for all $k \geqslant 1$. Next let $C \in \mathscr{C} \setminus \mathscr{D}$ with $C \subseteq \pi^{-1}[F]$. Let $\mathscr{D}' := \mathscr{D} \cup \{C\}$. If C belongs to $\mathscr{C}_0$ then $\chi(\pi[C]) = 0$ (since C is the union of two disks above each other and since $\pi[B(C)]$ contributes -1 to $\chi(\pi[C])$). Moreover, for each $k \geqslant 1$ one has $R_{2k}^{\mathscr{D}'} = R_{2k}^{\mathscr{D}} \cup (R_{2k-2}^{\mathscr{D}} \cap \pi[C])$. Hence

$$\sum_{k=1}^{\infty} \chi(R_{2k}^{\mathscr{S}'}) = \sum_{k=1}^{\infty} \chi(R_{2k}^{\mathscr{S}} \cup (R_{2k-2}^{\mathscr{S}} \cap \pi[C]))$$

$$= \sum_{k=1}^{\infty} (\chi(R_{2k}^{\mathscr{S}}) + \chi(R_{2k-2}^{\mathscr{S}} \cap \pi[C]) - \chi(R_{2k}^{\mathscr{S}} \cap \pi[C]))$$

$$= \chi(R_0^{\mathscr{S}} \cap \pi[C]) + \sum_{k=1}^{\infty} \chi(R_{2k}^{\mathscr{S}}) = \chi(\pi[C]) + \sum_{k=1}^{\infty} \chi(R_{2k}^{\mathscr{S}})$$

$$= \sum_{k=1}^{\infty} \chi(R_{2k}^{\mathscr{S}}) = \lfloor \tfrac{1}{2} |\mathscr{D} \cap \mathscr{C}_1| \rfloor = \lfloor \tfrac{1}{2} |\mathscr{D}' \cap \mathscr{C}_1| \rfloor. \tag{39}$$

Here we use the fact that $R_{2k}^{\mathscr{S}} \subseteq R_{2k-2}^{\mathscr{S}}$ and that $\chi(A) + \chi(B) = \chi(A \cap B) + \chi(A \cup B)$ for all A, B.

If C belongs to $\mathscr{C}_1$ let

$$R_C := \{x \in F \mid |\pi^{-1}(x) \cap C| \geqslant 2\}. \tag{40}$$

Using the observations following Claim 6 one sees that $\chi(R_C) = 0$. If $|\mathscr{D} \cap \mathscr{C}_1|$ is even then for each $k \geqslant 1$ one has $R_{2k}^{\mathscr{S}'} = R_{2k}^{\mathscr{S}} \cup (R_{2k-2}^{\mathscr{S}} \cap R_C)$ and then $\sum_{k=1}^{\infty} \chi(R_{2k}^{\mathscr{S}'}) = \lfloor \frac{1}{2} |\mathscr{D}' \cap \mathscr{C}_1| \rfloor$ follows similarly as in (39).

If $|\mathscr{D} \cap \mathscr{C}_1|$ is odd then for each $k \geqslant 2$ one has $R_{2k}^{\mathscr{S}'} = R_{2k-2}^{\mathscr{S}} \cup (R_{2k-4}^{\mathscr{S}} \cap R_C)$ while $R_2^{\mathscr{S}'} = F$. Hence

$$\sum_{k=1}^{\infty} \chi(R_{2k}^{\mathscr{S}'}) = \chi(F) + \sum_{k=2}^{\infty} \chi(R_{2k-2}^{\mathscr{S}} \cup (R_{2k-4}^{\mathscr{S}} \cap R_C))$$

$$= 1 + \sum_{k=1}^{\infty} \chi(R_{2k}^{\mathscr{S}} \cup (R_{2k-2}^{\mathscr{S}} \cap R_C)) = 1 + \lfloor \tfrac{1}{2} \mathscr{D} \cap \mathscr{C}_1| \rfloor$$

$$= \lfloor \tfrac{1}{2} |\mathscr{D}' \cap \mathscr{C}_1| \rfloor. \tag{41}$$

(Here the third equality follows similarly as in (39).) This shows (38) inductively.

Adding up (37) over all faces F of G gives

$$\sum_{k=1}^{\infty} \chi(R_{2k} \backslash G) = \sum_{F \in FG} \lfloor \tfrac{1}{2} \kappa_F \rfloor = \tfrac{1}{2} |\mathscr{C}_1| - \tfrac{1}{2} b(K). \tag{42}$$

We next consider $\sum_{k=1}^{\infty} \chi(R_{2k} \cap G)$. For any vertex v of H, let $\tilde{\mu}(v)$ be the largest integer k such that v belongs to R_k. So $\tilde{\mu}(v)$ is equal to the maximum value of $\mu(F)$, where F ranges over all faces of H incident with v.

Note that, for each edge e of H with $e \subset G$, $\mu(e)$ (defined as the minimum value of $\omega(x)$ over $x \in e$) is equal to the largest integer k such that e is contained in R_k. Hence

$$\sum_{k=1}^{\infty} \chi(R_{2k} \cap G) = \sum_{v \in VH, v \in G} \left\lfloor \frac{\tilde{\mu}(v)}{2} \right\rfloor - \sum_{e \in EH, e \subset G} \left\lfloor \frac{\mu(e)}{2} \right\rfloor. \tag{43}$$

Consider a vertex v of H. If v is also a vertex of G, then $\tilde{\mu}(v) = \omega(v)$. Let e_1, e_2, e_3, e_4 be the edges of H incident with v. We can choose indices so that $\mu(e_1) = \mu(e_2) = \omega(v)$ and $\mu(e_3) = \mu(e_4) = \omega(v) - 1$ (cf. Fig. 18). Hence

$$\left\lfloor \frac{\tilde{\mu}(v)}{2} \right\rfloor - \frac{1}{2} \sum_{i=1}^{4} \left\lfloor \frac{\mu(e_i)}{2} \right\rfloor = -\left\lfloor \frac{\omega(v) - 1}{2} \right\rfloor. \tag{44}$$

If $v = \pi(u)$ for some $u \in U$, let e_1 and e_2 be the two edges of H incident with v that are contained in G. We can choose indice so that $\mu(e_1) = \tilde{\mu}(v)$ and $\mu(e_2) = \tilde{\mu}(v) - 2$ (cf. Fig. 20). Hence

$$\left\lfloor \frac{\tilde{\mu}(v)}{2} \right\rfloor - \frac{1}{2} \left\lfloor \frac{\mu(e_1)}{2} \right\rfloor - \frac{1}{2} \left\lfloor \frac{\mu(e_2)}{2} \right\rfloor = \frac{1}{2}. \tag{45}$$

If $v = \pi(w)$ for some $w \in W$, then $\tilde{\mu}(v) = \omega(v) + 1$. Let e_1 and e_2 be the two edges of H incident with v that are contained in G. We can choose indices so that $\mu(e_1) = \tilde{\mu}(v)$ and $\mu(e_2) = \tilde{\mu}(v) - 1$ (cf. Fig. 19). Hence

$$\left\lfloor \frac{\tilde{\mu}(v)}{2} \right\rfloor - \frac{1}{2} \left\lfloor \frac{\mu(e_1)}{2} \right\rfloor - \frac{1}{2} \left\lfloor \frac{\mu(e_2)}{2} \right\rfloor = \frac{1}{2} \tag{46}$$

if $\tilde{\mu}(v)$ is even, i.e., if $w \in W_{\text{odd}}$. Similarly,

$$\left\lfloor \frac{\tilde{\mu}(v)}{2} \right\rfloor - \frac{1}{2} \left\lfloor \frac{\mu(e_1)}{2} \right\rfloor - \frac{1}{2} \left\lfloor \frac{\mu(e_2)}{2} \right\rfloor = 0 \tag{47}$$

if $\tilde{\mu}(v)$ is odd, i.e., if $w \in W_{\text{even}}$.

Adding up (44) over all $v \in VG$, (45) over all $u \in U$, (46) over all $w \in W_{\text{odd}}$, and (47) over all $w \in W_{\text{even}}$, gives by (43),

$$\sum_{k=1}^{\infty} \chi(R_{2k} \cap G) = -\sum_{v \in VG} \left\lfloor \frac{\omega(v) - 1}{2} \right\rfloor + \frac{1}{2} |W_{\text{odd}}| + \frac{1}{2} |U|. \tag{48}$$

Combined with (42), this gives the claimed equality. ∎

Multiplying by 4 gives

$$4 \sum_{k=1}^{\infty} \chi(R_{2k}) = 2 |\mathscr{C}_1| - 2b(K) - 4 \sum_{v \in VG} \left\lfloor \frac{\omega(v) - 1}{2} \right\rfloor + 2 |W_{\text{odd}}| + 2 |U|. \tag{49}$$

Rewriting the right hand side gives (since $\Sigma_{v\in VG}\lfloor(\omega(v)-1)/2\rfloor = \sum_{v\in VG}\frac{1}{2}(\omega(v)-1)-\frac{1}{2}|V_{\text{even}}|$)

$$2\left(|\mathscr{C}|-\tfrac{1}{2}|W|-\sum_{v\in VG}(\omega(v)-1)\right)$$
$$-2b(K)-2|\mathscr{C}_0|+2|V_{\text{even}}|+|W|+2|W_{\text{odd}}|+2|U|. \qquad (50)$$

The first term here contains the Euler characteristic of Σ as it can be expressed as follows.

SUBCLAIM 10b. $\chi(\Sigma)=|\mathscr{C}|-\frac{1}{2}|W|-\sum_{v\in VG}(\omega(v)-1)$.

Proof. Since each component in $\mathscr{C}$ is an open disk, one has

$$\chi(\Sigma\backslash\pi^{-1}[G])=|\mathscr{C}|. \qquad (51)$$

Moreover,

$$\chi(\Sigma\cap\pi^{-1}[G])=\chi(\Gamma)=-\tfrac{1}{2}|W|-\sum_{v\in VG}(\omega(v)-1). \qquad (52)$$

This follows from the fact that all vertices of Γ in $W\cup P$ have degree 3, and all vertices of Γ in $\pi^{-1}[V]\backslash P$ have degree 4. All other vertices of Γ have degree 2. Hence

$$\chi(\Gamma)=|V\Gamma|-|E\Gamma|$$
$$=|W\cup P|-\tfrac{3}{2}|W\cup P|+\sum_{v\in VG}(\omega(v)-2)-\tfrac{4}{2}\sum_{v\in VG}(\omega(v)-2)$$
$$=-\tfrac{1}{2}|W|-\sum_{v\in VG}(\omega(v)-1), \qquad (52)$$

since $|\pi^{-1}(v)\backslash P|=\omega(v)-2$ for each vertex v of G and since $|P|=2|VG|$.

Combining (51) and (52) gives the claimed equality. ▌

So (50) is equal to

$$2\chi(\Sigma)-2b(K)-2|\mathscr{C}_0|+2|V_{\text{even}}|+|W|+2|W_{\text{odd}}|+2|U|. \qquad (54)$$

As $\chi(\Sigma)\geqslant b(K)-v(K)=b(K)-|V|$ by assumption (ii) in the lemma, this is at least

$$-2|V|-2|\mathscr{C}_0|+2|V_{\text{even}}|+|W|+2|W_{\text{odd}}|+2|U|. \qquad (55)$$

Now $|\mathscr{C}_0|$ satisfies the following equation (recall that ρ is the number of Z-type segments (Fig. 43) and that ζ is the number of Z-type curves in Fig. 33):

SUBCLAIM 10c. $|\mathscr{C}_0| = \frac{1}{2}|W| + |U| - \rho - \zeta$.

Proof. For any C in $\mathscr{C}_0$, the boundary bd(C) of C should have exactly two turning points. Such a turning point should occur at a point in W or U.

In fact, each point w serves as turning point for exactly one component in $\mathscr{C}$. For let $w \in W^{\uparrow}$, say as in Fig. 90. Then there is one component C, say, in $\mathscr{C}$ that is incident with τ and σ, and one component C', say, in $\mathscr{C}$ that is incident with τ and σ'. C and C' are at different sides of τ.

Now w can serve as turning point only for C. In fact, w is a turning point for some C in $\mathscr{C}_0$ if and only if w is not contained in some Z-type segment. So exactly $|W| - 2\rho$ points in W serve as turning points for components in $\mathscr{C}_0$.

Any point u in U is turning point for at least one component in $\mathscr{C}_0$ (viz. in the face F_2 or F_4 as in Fig. 35). In fact, u is turning point of two components in $\mathscr{C}_0$, if and only if u is not on a Z-type curve.

Since there are ζ Z-type curves, and each of them contains two points in U, it follows that the points in U make $2|U| - 2\zeta$ turning points for components in $\mathscr{C}_0$.

So

$$2|\mathscr{C}_0| = |W| - 2\rho + 2|U| - 2\zeta, \tag{56}$$

and the claimed equality follows. ∎

Therefore, (55) is equal to

$$\begin{aligned} -2|V| - |W| - 2|U| + 2\rho + 2\zeta + 2|V_{\text{even}}| + |W| + 2|W_{\text{odd}}| + 2|U| \\ = -2|V| + 2\rho + 2\zeta + 2|V_{\text{even}}| + 2|W_{\text{odd}}|. \end{aligned} \tag{57}$$

Rewriting gives

$$\begin{aligned} &-2|V| + |W_{\text{odd}}| + 2|W^-_{\text{odd}}| + \tfrac{1}{2}(|W^+| - |W^-|) \\ &\quad + \tfrac{1}{2}(|W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}|) \\ &\quad + 2\rho + 2\zeta + 2|V_{\text{even}}| \end{aligned} \tag{58}$$

(since

$$\begin{aligned} |W_{\text{odd}}| &= |W^-_{\text{odd}}| + |W^+_{\text{odd}}| = 2|W^-_{\text{odd}}| + \tfrac{1}{2}(2|W^+_{\text{odd}}| - 2|W^-_{\text{odd}}|) \\ &= 2|W^-_{\text{odd}}| + \tfrac{1}{2}((|W^+_{\text{odd}}| + |W^+_{\text{even}}| - |W^-_{\text{odd}}| - |W^-_{\text{even}}|) \\ &\quad + (|W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}|)) \\ &= 2|W^-_{\text{odd}}| + \tfrac{1}{2}(|W^+| - |W^-|) \\ &\quad + \tfrac{1}{2}(|W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}|)). \end{aligned} \tag{59}$$

That this rewriting is helpful is seen by the following two subclaims.

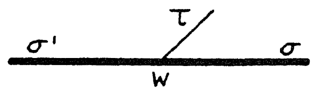

FIGURE 90

SUBCLAIM 10d. $|W^+| - |W^-| = 2v(K)$.

Proof. One directly derives from (19)

$$\tau(K, \Sigma) = |W^+| - |W^-| + 2w(K). \tag{60}$$

Since $\tau(K, \Sigma) = 2(v(K) + w(K))$ by assumption (iii) in the lemma, we have the required equality. ∎

SUBCLAIM 10e. $|W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}| = 2v(K) + 2\varphi + 4\eta$.

Proof. Consider a component e of $K \backslash P$. Let it connect $p^{\downarrow}_{v}$ and $p^{\uparrow}_{v'}$ as in Fig. 91. In this figure, $\alpha, \beta, \alpha', \beta'$ denote the μ-values in the corresponding faces of H incident with v and v'. Note that α and α' are even.

Define $\xi(e, v) := 1$ if $\beta = \alpha + 1$ and $\xi(e, v) := 0$ if $\beta = \alpha - 1$. Similarly, define $\xi(e, v') := 1$ if $\beta' = \alpha' + 1$ and $\xi(e, v') := 0$ if $\beta' = \alpha' - 1$. (So $\xi(e, v)$ indicates at which side of $p^{\uparrow}_{v}$ the surface Σ is attached. Similarly for $\xi(e, v')$.)

Let $v^{\uparrow}(e)$ denote the number of points in U that are above e (they necessarily are near to v) and let $v^{\downarrow}(e)$ denote the number of points in U that are under e (necessarily near to v'). Let $v(e) := v^{\uparrow}(e) + v^{\downarrow}(e)$.

For any $x \in \mathbb{R}^3$, let $\kappa(x)$ denote the number of points in Σ strictly under x, minus the number of points in Σ strictly above x.

We show

$$\kappa(p^{\uparrow}_{v'}) - \kappa(p^{\downarrow}_{v}) = \xi(e, v) + \xi(e, v') + 2v(e) + |W^+_{\text{odd}} \cap e| + |W^-_{\text{even}} \cap e| - |W^-_{\text{odd}} \cap e| - |W^+_{\text{even}} \cap e|. \tag{61}$$

Indeed, when traversing e from $p^{\downarrow}_{v}$ to $p^{\uparrow}_{v'}$, near $p^{\downarrow}_{v}$ the number of levels above deleted is $\xi(e, v) + 2v^{\uparrow}(e)$, while near $p^{\uparrow}_{v'}$ the number of levels under added is $\xi(e, v') + 2v^{\downarrow}(e)$.

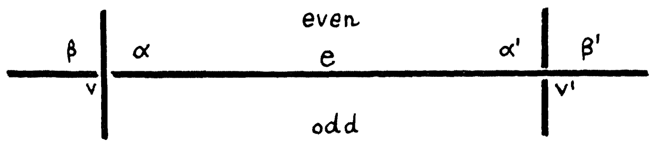

FIGURE 91

Moreover, traversing any point w in W^+_{odd}, if $w \in W^\uparrow$, then one level above is deleted (cf. Figs. 12(a) and 19(a), where α is odd), and if $w \in W^\downarrow$, then one level under is added (cf. Figs. 12(c) and 19(b) where α is odd).

Similarly, traversing any point w in W^-_{odd}, if $w \in W^\uparrow$, then one level above is added (cf. Figs. 12(b) and 19(b), where α is odd), and if $w \in W^\downarrow$, then one level under is deleted (cf. Figs. 12(d) and 19(a), where α is odd).

Symmetric statements hold for $w \in W^-_{\text{even}}$ and $w \in W^+_{\text{even}}$. This shows (61).

Now, for any $v \in VG$, if e and e' are the two components of $K \backslash P$ incident with $p^\downarrow_v$, then $\xi(e, v) + \xi(e', v) = 1$. Similarly for $p^\uparrow_v$.

Hence, adding up (61) over all components e of $K \backslash P$ we obtain

$$2\left(\sum_{v \in VG} \kappa(p^\uparrow_v) - \sum_{v \in VG} \kappa(p^\downarrow_v)\right)$$
$$= 2v(K) + 2\,|U| + |W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}|. \tag{62}$$

Now from Figs. 33 and 34 we see that for any $v \in VG$,

$$\kappa(p^\uparrow_v) = (\beta^\downarrow_v + \zeta_v + \varphi^\downarrow_v + 1 + \varphi^\downarrow_v + \zeta_v + \eta_v + \varphi^\uparrow_v) - (\varphi^\uparrow_v + \zeta_v + \beta^\uparrow_v)$$
and
$$\kappa(p^\downarrow_v) = (\beta^\downarrow_v + \zeta_v + \varphi^\downarrow_v) - (\varphi^\downarrow_v + \zeta_v + \eta_v + \varphi^\uparrow_v + 1 + \varphi^\uparrow_v + \zeta_v + \beta^\uparrow_v). \tag{63}$$

Hence

$$\kappa(p^\uparrow_v) - \kappa(p^\downarrow_v) = 2\varphi_v + 2\zeta_v + 2\eta_v + 2. \tag{64}$$

Therefore

$$|W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}|$$
$$= 4\varphi + 4\zeta + 4\eta + 4v(K) - 2v(K) - 2|U| = 2v(K) + 2\varphi + 4\eta \tag{65}$$

since $|U| = \varphi + 2\zeta$ by (27). ∎

Subclaims 10d and 10e imply that (58) is equal to

$$-2\,|V| + |W_{\text{odd}}| + 2\,|W^-_{\text{odd}}| + |V| + |V|$$
$$+ \varphi + 2\eta + 2\rho + 2\zeta + 2\,|V_{\text{even}}|, \tag{66}$$

which equals

$$|W_{\text{odd}}| + 2\,|W^-_{\text{odd}}| + \varphi + 2\eta + 2\rho + 2\zeta + 2\,|V_{\text{even}}|. \tag{67}$$

By (27) this is equal to the right hand side in Claim 10. ∎

An equality for $\sum_{k=1}^{\infty} \delta(R_{2k})$. For any subset R of $\mathbb{R}^2$ that is the closure of the union of some faces of H, let $\delta(R)$ denote the number of times edges of G "leave" R. To be precise, for any edge e of G, we say that *e leaves R*

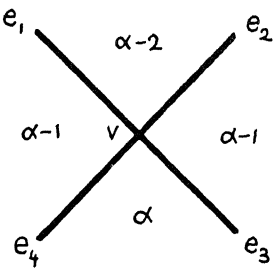

FIGURE 92

at v if $v \in \mathrm{bd}(R)$ and $v \in \overline{e \backslash R}$. Let $\delta(R, e)$ denote the number of times that e leaves R (that is, the number of v such that e leaves R at v), and define

$$\delta(R) := \sum_{e \in EG} \delta(R, e). \tag{68}$$

So if one makes a set of closed curves in $\mathbb{R}^2 \backslash R$ close to the boundary components of R, then these curves will have $\delta(R)$ crossings with G.

CLAIM 11. $\sum_{k=1}^{\infty} \delta(R_{2k}) = 2\,|V_{\text{even}}| + |W_{\text{odd}}| + |U|$.

Proof. Consider a vertex v of H. Let $\alpha := \omega(v)$. First let $v \in VG$. Consider the neighbourhood of v as in Fig. 92. If α is even, then $v \in \mathrm{bd}(R_{2k}) \Leftrightarrow 2k = \alpha$, and only e_1 and e_2 leave R_α at v. If α is odd, then $v \in \mathrm{bd}(R_{2k}) \Leftrightarrow 2k = \alpha - 1$, and no edge of G leaves $R_{\alpha-1}$ at v.

Next let $v \in \pi[U]$. Consider the neighbourhood of v as in Fig. 93. If α is even, then $v \in \mathrm{bd}(R_{2k}) \Leftrightarrow 2k = \alpha$, and edge e leaves R_α at v. If α is odd, then $v \in \mathrm{bd}(R_{2k}) \Leftrightarrow 2k = \alpha \pm 1$, and edge e leaves $R_{\alpha+1}$ at v, but e does not leave $R_{\alpha-1}$ at v.

Finally, let $v \in \pi[W]$. Consider the neighbourhood of v as in Fig. 94. If α is even, then $v \in \mathrm{bd}(R_{2k}) \Leftrightarrow 2k = \alpha$, and no edge of G leaves R_α at v. If α is odd, then $v \in \mathrm{bd}(R_{2k}) \Leftrightarrow 2k = \alpha + 1$, and edge e leaves $R_{\alpha+1}$ at v.

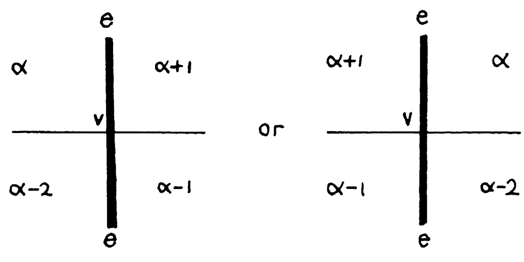

FIGURE 93

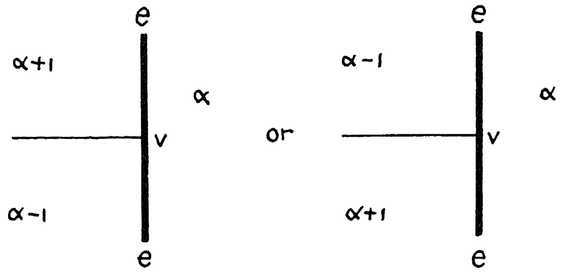

FIGURE 94

Adding up over all vertices v of H on G we obtain the claim. ▮

The remainder of the proof makes the following intuitive argument precise. By Claims 10 and 11,

$$\sum_{k=1}^{\infty} \delta(R_{2k}) = 2\,|V_{\text{even}}| + |W_{\text{odd}}| + |U| \leqslant \sum_{k=1}^{\infty} 4\chi(R_{2k}). \tag{69}$$

On the other hand, roughly speaking, since G is well-connected, for each k, $\delta(R_{2k}) \geqslant 4\chi(R_{2k})$, since there are at least four edges of G leaving any component of any R_{2k}. Equality throughout then should imply the existence of the isotopy bringing Σ to Σ_K as required.

The graph H′. Let H' be defined by

$$H' := \bigcup_{k=1}^{\infty} \text{bd}(R_{2k}). \tag{70}$$

That is, H' is the subgraph of H consisting of those edges e of H for which $\lfloor \frac{1}{2}\mu(F) \rfloor$ and $\lfloor \frac{1}{2}\mu(F') \rfloor$ differ (by 1), where F and F' are the faces of H incident with e. So H' contains all of $H \backslash G$, while an edge e of H on G is in H', if and only if $\mu(e)$ is even. H' inherits the orientation from H (cf. Fig. 23).

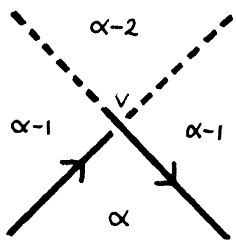

FIGURE 95

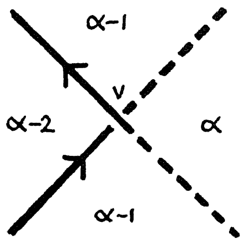

FIGURE 96

Consider a vertex v of H'. Let $\alpha := \omega(v)$. If v is also a vertex of G, then v is incident with two edges of H'. If $v \in V_{\text{even}}$ then the neighbourhood of v in H' is as in Fig. 95. (The interrupted lines in the figure are part of G not in H'.)

If $v \in V_{\text{odd}}$ then it is as in Fig. 96. If $v \in \pi[U]$ and α is even, the neighbourhood of v is as in Fig. 97. If $v \in \pi[U]$ and α is odd, it is as in Fig. 98. If $v \in \pi[W]$ and α is even, the neighbourhood of v is as in Fig. 99. If $v \in \pi[W]$ and α is odd, it is as in Fig. 100. Finally, if $v \notin G$ (that is, v is in a face of G), then the neighbourhood is as in Fig. 101.

Note that any edge e of H' that is on the boundary of an odd face of G is oriented counter-clockwise with resect to that face. (So it is oriented clockwise with respect to even faces, except for the unbounded face.)

Note moreover that H' is "Eulerian"; that is, each vertex of H' has the same number of arcs oriented inwards as outwards. So the edge set of H' can be decomposed into simple directed circuits. (*Simple* means: not traversing any point more than once.) Also, at each vertex v of H', the incoming arcs occur consecutively in the cyclic ordering of arcs incident with v.

CLAIM 12. *Let $D_1, ..., D_t$ be a decomposition of the edge set of H' into simple directed circuits such that $D_1, ..., D_s$ are oriented clockwise and $D_{s+1}, ..., D_t$ are oriented counter-clockwise. Then*:

$$s-(t-s)=\sum_{k=1}^{\infty} \chi(R_{2k}). \tag{71}$$

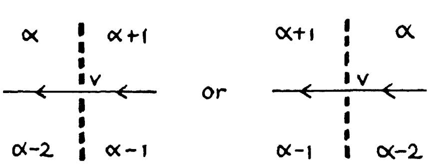

FIGURE 97

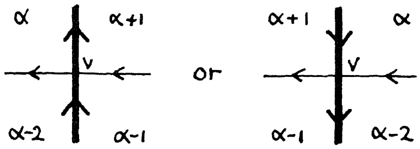

Figure 98

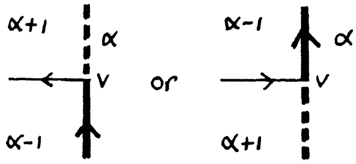

Figure 99

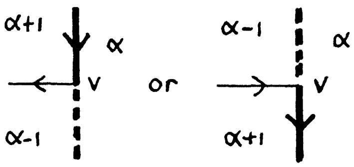

Figure 100

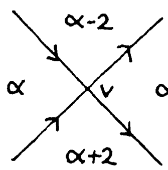

Figure 101

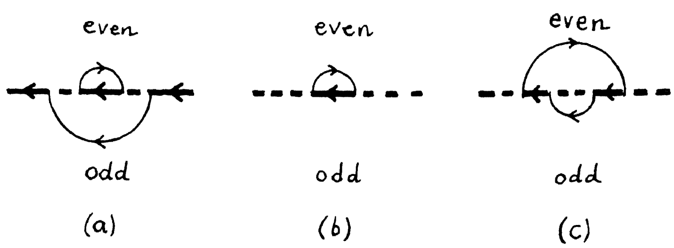

Figure 102

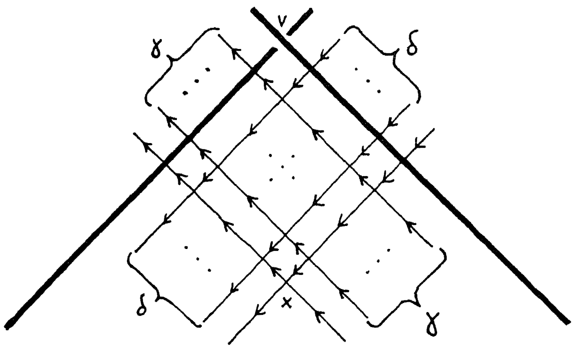

FIGURE 103

Proof. By [2, Lemma 6.3], the number $s-(t-s)$ is independent of the choice of the D_i (it is equal to the Whitney degree of H' considered as a set of disjoint oriented plane curves). Taking for the D_i the boundaries of the components of the R_{2k} we obtain (71). (I am grateful to François Jaeger for pointing out this argument to me; it replaces my original invalid argumentation.) ∎

For any simple closed curve D in $\mathbb{R}^2$ we denote

$$R(D) := \text{closed region enclosed by } D. \qquad (72)$$

We show:

CLAIM 13. *Let D be a simple directed circuit in H', oriented clockwise, with $VG \cap R(D) = \emptyset$. Then D is the circuit in one of the configurations in Fig. 102. (In Fig. 102 the interrupted line is part of G not in H'.)*

Proof. First observe that D should intersect G. Indeed, if D contains $\pi[Q]$ for some component Q of $B(C)$ for some $C \in \mathscr{C}$, then D intersects G as $\overline{\pi[Q]}$ intersects G. If D traverses consecutively parts of $\pi[Q]$ and $\pi[Q']$ say, for some components Q of $B(C)$ and Q' of $B(C')$, for some $C, C' \in \mathscr{C}$, then it contains a crossing x of Q and Q' near to some vertex v of G, as in Fig. 78, 80, or 82. But then v belongs to $R(D)$, as one has Fig. 103

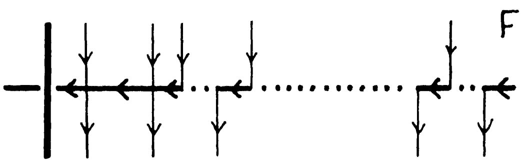

FIGURE 104

(with $\gamma, \delta \geqslant 0$). So by the orientation of H', there is no way for D to avoid enclosing v.

So D intersects G. If D would traverse a vertex of H' in $\pi[U]$ or $\pi[W]$ near to any vertex v of G, then D is either as in Fig. 102(b) or $R(D)$ would contain v, again because by the orientation of H' there is no way for D to avoid enclosing v (cf. Figs. 70, 71, 72, 73, and 74). For instance, at the left hand part of Fig. 70 the graph H' is as in Fig. 104. (The edges in G are oriented in the way indicated since face F is even.) So D should be of one of the types given in Fig. 102. ∎

CLAIM 14. *Let D be a simple directed circuit in H', oriented clockwise, such that $VG \cap R(D) = \{v\}$ for some vertex v of G. Then D is a component of H', and it is the directed circuit in one of the configurations in Fig.* 105.

Proof. Again if D traverses some point in $\pi[U]$ or $\pi[W]$ near to a vertex v' of G, then either it is of type (b) in Fig. 102 (see Figs. 72, 73, and 74) which is ruled out since $VG \cap R(D) \neq \emptyset$, or v' belongs to $R(D)$ (cf.

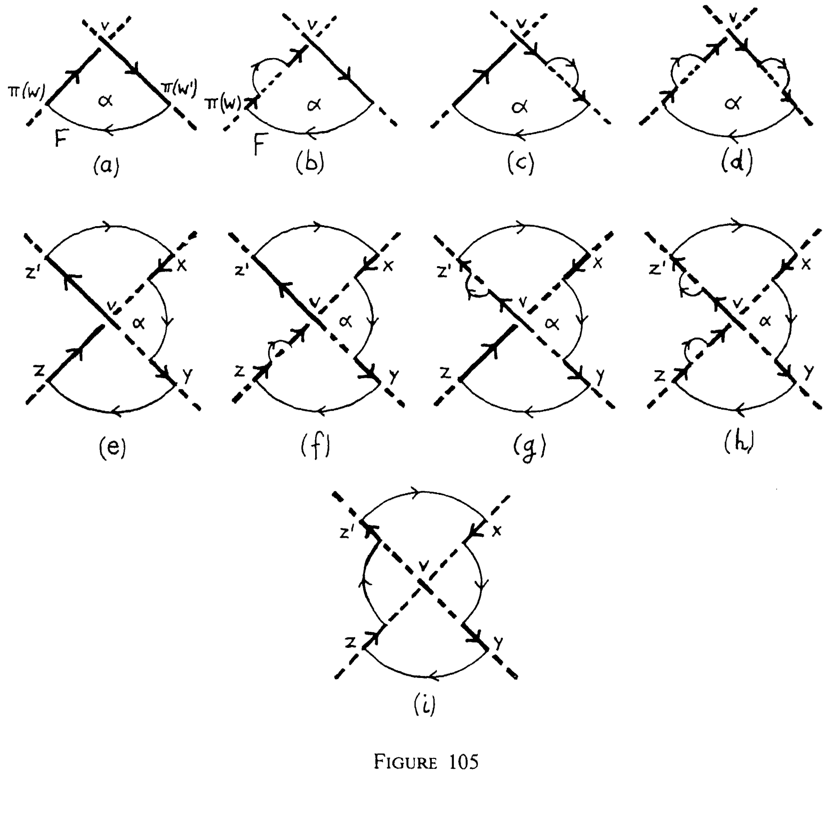

FIGURE 105

Figs. 70, 71, 72, 73, and 74), implying $v' = v$. Hence D cannot traverse any other points in $\pi[U]$ and $\pi[W]$ than those near to v.

So if D intersects one of the edges e incident with v, it intersects e in one of the ways given in Fig. 106.

Let e_1, e_2, e_3, e_4 be the edges incident with v in clockwise order. First suppose that D does not traverse v. Let D_i be the part of D connecting e_i and e_{i+1}, for $i = 1, 2, 3, 4$ (taking indices mod 4). By (34), since each D_i cannot traverse any edge distinct from e_1, e_2, e_3, e_4, it either connects two points in $\pi[U]$ or connects two points in $\pi[W]$. Since at most two neighbouring edges among e_1, e_2, e_3, e_4, say e_1 and e_2, can contain points in $\pi[U]$ near to v, we know that each of D_2, D_3, D_4 connects two points in $\pi[W]$. Then there does not exist a point in $\pi[U]$ near to v. For suppose e_2 (say) contains a point u in $\pi[U]$ near to v. Since e_3 contains a point in $\pi[W]$ near to v, Claim 9 implies that e_3 contains a point in $\pi[U]$ near to v. This contradicts our assumption that each point in U near to v projects to e_1 or e_2.

So all vertices of H traversed by D belong to $\pi[W]$, and hence each crossing is of type (b) or (c) of Fig. 106. Therefore, we have Fig. 105(i).

Second suppose that D traverses v and that v belongs to V_{even}. Then D contains Fig. 107 (see Fig. 95). As $\omega(v) = \alpha$, the vertices in $\pi[U]$ near to v are on e_1 and e_2 only. As D is oriented clockwise, it does not intersect e_1 or e_2. Hence D contains both (d) or (e) of Fig. 106, and (f) or (g) of Fig. 106. Therefore, D does not traverse any point in $\pi[U]$, and hence D is a component of H'. Moreover, D is of type (a), (b), (c), or (d) in Fig. 105.

Finally, suppose D traverses v and v belongs to V_{odd}. Then D contains Fig. 108 (see Fig. 96). As $\omega(v) = \alpha$, the vertices in $\pi[U]$ near to v are on e_1 and e_2 only. So D intersects e_1 as in configuration (f) or (g) of Fig. 106, intersects e_2 as in (d) or (e) of Fig. 106, intersects e_3 as in (c) of Fig. 106, and intersects e_4 as in (b) (since in (h) and (i) of Fig. 106 D traverses a

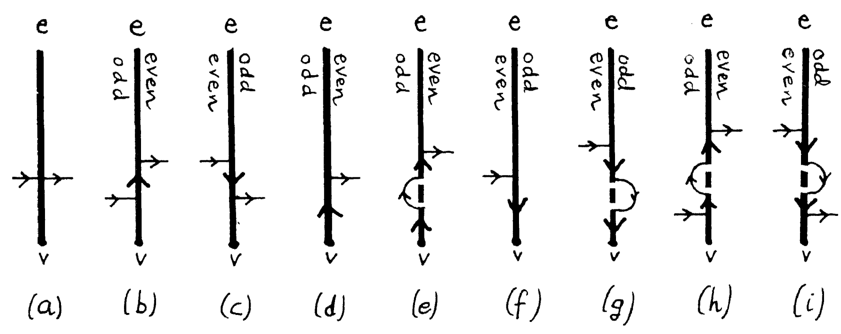

FIGURE 106

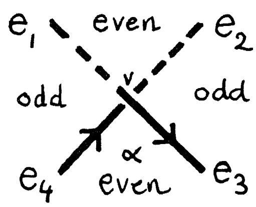

FIGURE 107

point in $\pi[U]$ (see Figs. 72, 73, and 74)). So both e_3 and e_4 contain points in $\pi[W]$, implying that there is no point in U near to v, by (35) (since all points in U near to v project to $e_1 \cup e_2$). So D does not traverse any point in $\pi[U]$, and hence D is a component of H'. Moreover, D is of type (e), (f), (g), or (h) in Fig. 105. ▌

We call any of the components in Figs. 102 and 105 *small components.* Note that Claim 14 implies:

$$\text{If } D \text{ is a small component with } VG \cap R(D) = \{v\}, \text{ then there is no point in } U \text{ near to } v. \tag{73}$$

To see this for Fig. 105(a), consider Fig. 109. Observe that all points in U near to v project to $e \cup e'$. So by (35) there is no point in U near to v. Similarly for the other configurations in Fig. 105.

Also note that each of the configurations (a), (b), (c), (d) in Fig. 105 implies that v belongs to V_{even}, and that each of (e), (f), (g), (h) in Fig. 105 implies that v belongs to V_{odd}. Moreover, Fig. 105(a) as seen from F is as in Fig. 110. This can be seen as follows. Seen from F we have Fig. 111. (Note that $w \in W^{\uparrow}$ and $w' \in W^{\downarrow}$ by Claim 5.) Now there is no point in $\pi[W]$ between $\pi(w)$ and v in Fig. 105(a), since otherwise the $\pi(w) - v$ part would not be fully contained in H'. So ε_1 should lead to $p_v^{\uparrow}$ or a point above $p_v^{\uparrow}$ (by Claim 5), and ε_2 cannot lead to part l. Moreover, there is no point in $\pi[U]$ near to v on one of these edges. Hence ε_1 and ε_2 should lead to each other. Similarly, ε_3 and ε_4 should lead to each other, and hence we have Fig. 110.

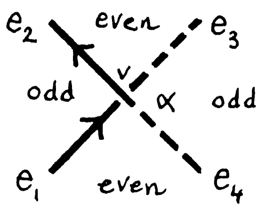

FIGURE 108

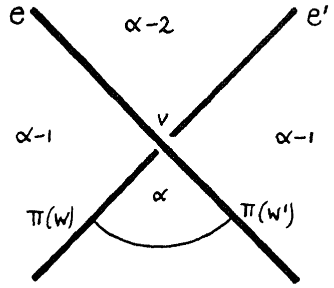

FIGURE 109

Similarly, Fig. 105(b) as seen from F is as in Fig. 112. (Note that w belongs to $w^{\downarrow}$, as if $w \in W^{\uparrow}$ then by (25) $w \in W^{+}$, contradicting the fact that Fig. 68 does not occur.)

Similarly for Figs. 105(c) and (d).

The graph Δ'. Consider the set

$$\Delta' := \left(\Delta \cup \bigcup_{v \in VG} e_v\right) \Big\backslash \bigcup \{\sigma \mid \sigma \text{ segment on } K \text{ with } \mu(\sigma) \text{ odd}\}. \quad (74)$$

(As before, e_v denotes the open line segment connecting $p_v^{\downarrow}$ and $p_{v'}^{\uparrow}$.)

Each point in $P \cup W$ is incident with two segments on K, one with even μ-value and one with odd μ-value. Hence Δ' is a 2-regular graph embedded in $\mathbb{R}^3$. So each component of Δ' is a circuit.

Note that

$$H' = \pi[\Delta']. \quad (75)$$

The orientation of H' induces an orientation of Δ', in which each component is a directed circuit. It is easy to see that this is obtained when each line segment e_v is oriented from $p_v^{\downarrow}$ to $p_v^{\uparrow}$.

The length function l. For each edge e of H' define the "length" $l(e)$ of e by

$$\begin{aligned} l(e) &:= |\bar{e} \cap V_{\text{even}}| && \text{if } e \subseteq G, \\ &:= |\bar{e} \cap G| && \text{if } e \text{ is contained in an even face of } G, \quad (76) \\ &:= 0 && \text{if } e \text{ is contained in an odd face of } G. \end{aligned}$$

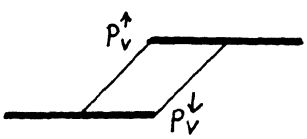

FIGURE 110

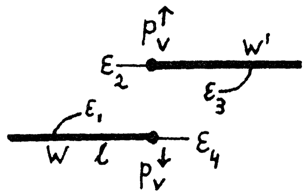

FIGURE 111

For any $H'' \subseteq H'$ define

$$l(H'') := \sum_{e \in EH', e \subseteq H''} l(e). \tag{77}$$

Then:

CLAIM 15. *Let R be a closed region in $\mathbb{R}^2$ such that the boundary* $\text{bd}(R)$ *of R is part of H' in such a way that R is at the right hand side of any edge e of H' on* $\text{bd}(R)$. *Then*

$$l(\text{bd}(R)) = \delta(R). \tag{78}$$

Proof. Since for any vertex v of H' of degree 4 the edges incident with v are oriented as in Fig. 113, $\text{bd}(R)$ consists of pairwise disjoint simple directed circuits.

For any vertex v of H on $G \cap \text{bd}(R)$, define $\vartheta(v)$ as follows. If $v \in VG$, let $\vartheta(v) := 2$ if $v \in V_{\text{even}}$ and $\vartheta(v) := 0$ if $v \in V_{\text{odd}}$. If $v \notin VG$, let $\vartheta(v)$ be the number of edges $e \subseteq \text{bd}(R)$ with $v \in \bar{e}$, and e being contained in an even face of G. By definition of l,

$$l(\text{bd}(R)) = \sum_{v \in VH \cap G \cap \text{bd}(R)} \vartheta(v). \tag{79}$$

Now, on the other hand, define for any $v \in VH \cap G \cap \text{bd}(R)$, $\vartheta'(v)$ as the number of edges e of H contained in G such that $v \in \bar{e}$ and $e \cap R = \emptyset$. So

$$\delta(R) = \sum_{v \in VH \cap G \cap \text{bd}(R)} \vartheta'(v). \tag{80}$$

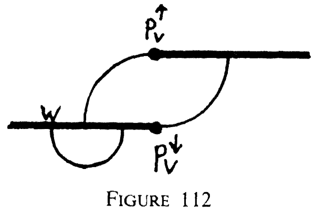

FIGURE 112

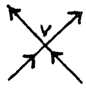

FIGURE 113

We show that $\vartheta(v) = \vartheta'(v)$ for each $v \in VH \cap G \cap \text{bd}(R)$, implying (78) by (79) and (80).

Let $v \in VH \cap G \cap \text{bd}(R)$ and let $\alpha := \omega(v)$. If $v \in VG$ and α is even then we have Fig. 114 (cf. Fig. 95). We see that $\vartheta'(v) = 2 = \vartheta(v)$.

If $v \in VG$ and α is odd then we have Fig. 115 (cf. Fig. 96) and we see that $\vartheta'(v) = 0 = \vartheta(v)$.

If $v \in \pi[U]$ and α is even, then (up to symmetry) we have Fig. 116 (cf. Fig. 97). We see that $\vartheta'(v) = 1 = \vartheta(v)$.

If $v \in \pi[U]$ and α is odd then (up to symmetry) one of the configurations in Fig. 117 applies (cf. Fig. 98). Then $\vartheta(v) = 1, 0, 1$ and 0 respectively; similarly $\vartheta'(v) = 1, 0, 1$ and 0 respectively.

If $v \in \pi[W]$ and α is even then one of the configurations in Fig. 118 applies (cf. Fig. 99). We see that $\vartheta'(v) = 0 = \vartheta(v)$.

Finally, if $v \in \pi[W]$ and α is odd then one of the configurations in Fig. 119 applies (cf. Fig. 100). Now $\vartheta'(v) = 1 = \vartheta(v)$. ▌

We next show:

CLAIM 16. *Each simple directed circuit D in H' with $VG \cap R(D) \neq \emptyset$ is oriented clockwise, has length $l(D) = 4$, and satisfies one of the following:*

(i) $|VG \cap R(D)| = 1$;

(ii) $|VG \backslash R(D)| = 1$; (81)

(iii) *$VG \subseteq R(D)$, and there are two edges e, e' of G on the boundary of the unbounded face F_0 such that each of e and e' leaves $R(D)$ twice.*

Moreover, $W^-_{\text{odd}} = \emptyset$, $\eta = 0$, and the configuration in Fig. 120 *does not occur.*

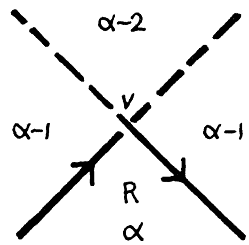

FIGURE 114

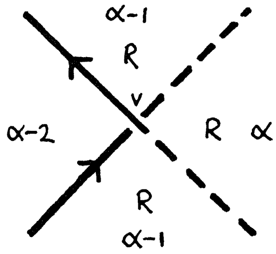

FIGURE 115

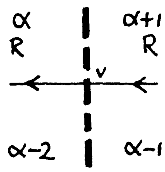

FIGURE 116

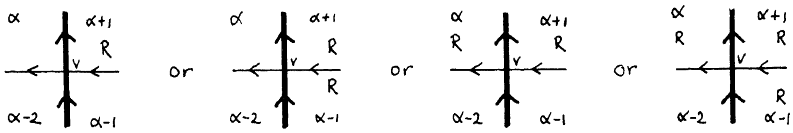

FIGURE 117

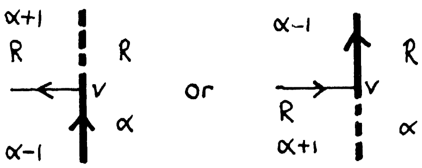

FIGURE 118

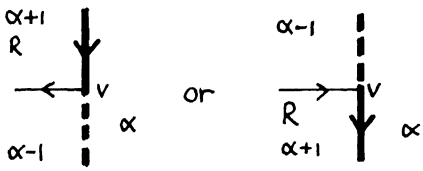

FIGURE 119

FIGURE 120

Proof. For any oriented curve Q, let x_Q be its beginning point and y_Q be its end point (these points are not part of Q if Q is an open curve). Note that for each $C \in \mathscr{C}$ contained in $\pi^{-1}[F_0]$ one has $C \in \mathscr{C}_0$. Hence there is no Z-type curve or segment "seen" from F_0.

We first show the following (where we use the fact that the unbounded face F_0 of G is bounded by at least four edges of G):

SUBCLAIM 16a. *There exist vertices v_1 and v_2 of G on the boundary of the unbounded face F_0 such that v_1 and v_2 are not adjacent in G, and such that for each $i \in \{1, 2\}$ and for each component Q of $\Delta' \cap \pi^{-1}[F_0]$, if the $\pi(x_Q) - \pi(y_Q)$ part of* bd(F_0) *(in clockwise orientation) contains v_i, then one of x_Q, y_Q is near to v_i.*

Proof. Note that, by the observations on Figs. 70–74, for each component Q of $\Delta' \cap \pi^{-1}[F_0]$, x_Q or y_Q is near to the nearest vertex on the $\pi(x_Q) - \pi(y_Q)$ part of bd(F_0). Also note that Q is a component of $B(C)$ for some $C \in \mathscr{C}_0$, as $\pi^{-1}[F_0]$ does not contain any component in $\mathscr{C}_1$.

If for each Q the $\pi(x_Q) - \pi(y_Q)$ part of bd(F_0) contains at most two vertices of G, we can take any two nonadjacent vertices v_1, v_2 of G on bd(F_0).

If for at least one such component Q the $\pi(x_Q) - \pi(y_Q)$ part of bd(F_0) contains more than two vertices of G, choose Q maximal in the sense that the $\pi(x_Q) - \pi(y_Q)$ part of bd(F_0) is as large as possible. Then we choose v_1 and v_2 so that x_Q is near to v_1 and y_Q is near to v_2.

Now v_1 and v_2 have the required properties. For suppose that for some component Q' of $\Delta' \cap \pi^{-1}[F_0]$ the $\pi(x_{Q'}) - \pi(y_{Q'})$ part of bd(F_0) contains v_1 in such a way that neither $x_{Q'}$ nor $y_{Q'}$ is near to v_1. If $\pi[Q']$ does not cross $\pi[Q]$ then the $\pi(x_{Q'}) - \pi(y_{Q'})$ part of bd(F_0) would be larger than the $\pi(x_Q) - \pi(y_Q)$ part of bd(F_0), contradicting the choice of Q.

So $\pi[Q']$ crosses $\pi[Q]$. Hence $\pi[Q']$ should cross $\pi[Q]$ near to v_1 or near to v_2 (by (32)). If $\pi[Q']$ crosses $\pi[Q]$ near to v_1, then $y_{Q'}$ is near to v_1. If $\pi[Q']$ crosses $\pi[Q]$ near to v_2, then $x_{Q'}$ should be near to v_2. Hence v_2 is contained in the $\pi(x_{Q'}) - \pi(y_{Q'})$ part of bd(F_0). Since also v_1 is contained in the $\pi(x_{Q'}) - \pi(y_{Q'})$ part of bd(F_0), $\pi[Q']$ should have a second crossing with $\pi[Q]$. This crossing should be near to v_1, and hence $y_{Q'}$ is near to v_1.

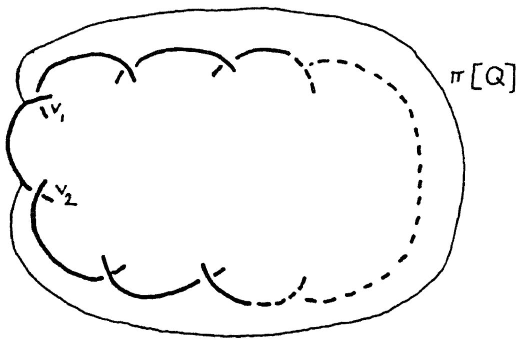

FIGURE 121

The proof is similar for the case where the $\pi(x_Q)-\pi(y_Q)$ part of $\mathrm{bd}(F_0)$ contains v_2.

Moreover, v_1 and v_2 are nonadjacent since otherwise we would have Fig. 121. Let $C\in\mathscr{C}$ with $B(C)=Q$. As $C\in\mathscr{C}_0$, we see Fig. 122 from F_0 on $\pi^{-1}[\mathrm{bd}(F_0)]$. (The turning points might also be on the $p_{v_1}^{\uparrow}-p_{v_2}^{\downarrow}$ part—see Figs. 75, 76.)

By our choice of Q there is no component $C'\in\mathscr{C}_0$ contained in $\pi^{-1}[F_0]$ such that $\mathrm{bd}(C')$ encloses $\mathrm{bd}(C)$ on the cylinder $\pi^{-1}[\mathrm{bd}(F_0)]$. So we can apply an isotopy in S^3 to C so that the boundary of C encloses (part of) l only, as in Fig. 123. (Again, the turning points might be on the $p_{v_1}^{\uparrow}-p_{v_2}^{\downarrow}$ part.)

This makes

$$\sum_{v\in VG\cap \mathrm{bd}(F_0)} \omega(v) \tag{82}$$

smaller, contradicting the minimality assumption (23)(ii). ∎

Let $\mathscr{D}$ denote the collection of all boundary components D of all R_{2k} that are oriented clockwise such that $VG\cap R(D)\neq\emptyset$. Let ρ' denote the number of small components of the types given in Fig. 102. So by Claims 12 and 13,

$$\sum_{k=1}^{\infty} \chi(R_{2k})\leqslant|\mathscr{D}|+\rho'. \tag{83}$$

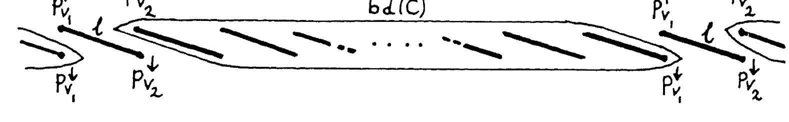

FIGURE 122

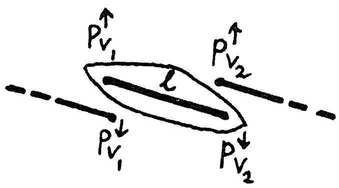

FIGURE 123

Let e_1', e_1'' be the two edges of G incident with v_1 on $\text{bd}(F_0)$, and let e_2', e_2'' be the two edges of G incident with v_2 on $\text{bd}(F_0)$.

For any simple directed circuit D let again $R(D)$ denote the closed region enclosed by D. Moreover, let $r_1(D)$ be equal to the number of sets among e_1', $\{v_1\}$, e_1'' that are contained in $R(D)$. So $r_1(D) \in \{0, 1, 2, 3\}$. Similarly, let $r_2(D)$ be equal to the number of sets among e_2', $\{v_2\}$, e_2'' that are contained in $R(D)$.

This is used in showing:

SUBCLAIM 16b. *There is no crossing* (*i.e.*, *vertex of H of degree* 4) *in F_0 near to v_1 or near to v_2.*

Proof. Suppose the subclaim is not true, and suppose without loss of generality that there exists a crossing in F_0 near to v_1. This implies

$$(\#D \in \mathscr{D} \mid r_1(D) = 3) < (\#D \in \mathscr{D} \mid r_1(D) = 1). \tag{84}$$

The reason is that the crossings in F_0 near to v_1 are locally as in Fig. 124. (There are no Z-type curves near v_1, since $\pi^{-1}[F_0]$ contains no component in $\mathscr{C}_1$, as $\mu(F_0) = 0$.)

It implies that v_1 belongs to V_{even} (by (32) and (33)) and that the curves

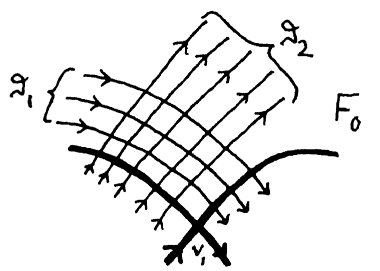

FIGURE 124

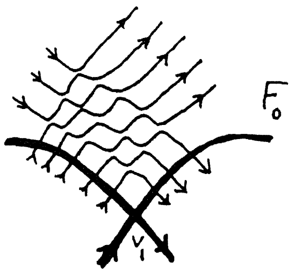

FIGURE 125

in $\mathscr{D}$ are locally as in Fig. 125. Let there be ϑ_1 curves Q in $\Delta' \cap \pi^{-1}[F_0]$ with y_Q near to v_1 and let there be ϑ_2 curves Q in $\Delta' \cap \pi^{-1}[F_0]$ with x_Q near to v_1 (cf. Fig. 124). Then there are $\min\{\vartheta_1, \vartheta_2\}$ curves $D \in \mathscr{D}$ with $r_1(D) = 3$, $|\vartheta_1 - \vartheta_2|$ curves $D \in \mathscr{D}$ with $r_2(D) = 2$, and $\min\{\vartheta_1, \vartheta_2\} + 1$ curves $D \in \mathscr{D}$ with $r_1(D) = 1$ (cf. Figs. 125 and 95). Hence we have (84). (Note that by the conditions in Subclaim 16a, all D in $\mathscr{D}$ with $r_1(D) = 3$ occur (partly) in Fig. 125.)

Moreover,

$$\text{for no curve } D \in \mathscr{D} \text{ one has } VG \cap R(D) = \{v_1\}, \tag{85}$$

since otherwise by Claim 14 and (73) there are no points in U near to v_1, and hence there are no crossings near to v_1.

Now we distinguish two cases.

CASE 1. *There exists a crossing in F_0 near to v_2.* This similarly implies:

$$(\# D \in \mathscr{D} \mid r_2(D) = 3) < (\# D \in \mathscr{D} \mid r_2(D) = 1) \tag{86}$$

and v_2 belongs to V_{even}.

Now for each $D \in \mathscr{D}$ we have

$$l(D) \geqslant 8 - 2\,|\,r_1(D) - 1| - 2\,|r_2(D) - 1|. \tag{87}$$

To see this, let $\tilde{D}$ be a closed curve encircling D and very close to D, in such a way that $\tilde{D}$ has exactly $l(D) = \delta(R(D))$ crossings with G. Then showing (87) is simple case-checking, using the facts that $R(D)$ should contain at least one vertex of G, and that hence, by the well-connectedness of K, $\tilde{D}$ should cross G often enough; that is

$$\text{if } \varnothing \neq VG \cap R(D) \neq VG \text{ then } l(D) \geqslant 4; \text{ if } |VG \cap R(D)| \geqslant 2 \text{ and } |VG \setminus R(D)| \geqslant 2 \text{ then } l(D) \geqslant 6. \tag{88}$$

[Indeed, to check (87), we use the following observation:

> Let $\mathscr{G}$ be a 4-edge-connected planar graph embedded in $\mathbb{R}^2$, and let $\mathscr{B}$ be a simple closed curve in $\mathbb{R}^2$ not traversing any vertex of $\mathscr{G}$. Let t be the number of edges of $\mathscr{G}$ incident with the unbounded face $\mathscr{F}_0$ that are crossed at least once by $\mathscr{B}$. Then $\mathscr{B}$ has at least $2t$ crossings with $\mathscr{G}$. (89)

(*Proof.* Decompose $\mathscr{B}$ into curves $\mathscr{B}_1, ..., \mathscr{B}_s$, where each $\mathscr{B}_i$ has both ends in $\mathscr{F}_0$ and has exactly two crossings with the boundary of $\mathscr{F}_0$. Let λ_i be the number of edges incident with $\mathscr{F}_0$ crossed by $\mathscr{B}_i$ (so $1 \leqslant \lambda_i \leqslant 2$), and let μ_i be the number of crossings of $\mathscr{B}_i$ with $\mathscr{G}$. Then, since $\mathscr{G}$ is 4-edge-connected, $\mu_i \geqslant 2$ if $\lambda_i = 1$ and $\mu_i \geqslant 4$ if $\lambda_i = 2$. That is, $\mu_i \geqslant 2\lambda_i$. Hence $\mu_1 + \cdots + \mu_s \geqslant 2(\lambda_1 + \cdots + \lambda_s) \geqslant 2t$.)

We may assume without loss of generality that $r_2(D) \leqslant r_1(D)$. First assume $r_2(D) = 0$. If $r_1(D) = 0$, then $\varnothing \neq VG \cap R(D) \neq VG$, implying $l(D) \geqslant 4$ by (88). If $r_1(D) = 1$, then $\tilde{D}$ crosses e_1' and e_1'' and $v_2 \notin R(D)$; also $VG \cap R(D) \neq \{v_1\}$ (by (85)); so $|VG \cap R(D)| \geqslant 2$. If $|V(G) \backslash R(D)| \geqslant 2$ then $l(D) \geqslant 6$ by (88). If $|V(G) \backslash R(D)| \leqslant 1$ then $VG \backslash R(D) \subseteq \{v_2\}$ and hence the end points of e_1' and e_1'' not equal to v_1 belong to $R(D)$. Hence $\tilde{D}$ crosses each of e_1' and e_1'' at least twice. If $\tilde{D}$ would cross G exactly four times then $v_2 \in R(D)$, contradicting the assumption that $r_2(D) = 0$. So $l(D) \geqslant 6$ follows. If $r_1(D) \geqslant 2$, then $v_2 \notin R(D)$, $v_1 \in R(D)$ and hence $l(D) \geqslant 4$ by (88).

Second assume $r_2(D) = 1$. So $\tilde{D}$ crosses both e_2' and e_2'', and hence $l(D) \geqslant 4$ by (89). This gives (87), except if $|r_1(D) - 1| \leqslant 1$.

If $r_1(D) = 1$, then $\tilde{D}$ crosses also both e_1' and e_1''. So $\tilde{D}$ crosses G at least eight times by (89), and hence $l(D) \geqslant 8$.

If $r_1(D) = 2$, then D crosses one of e_1', e_1'' and both of e_2', e_2''. So $\tilde{D}$ crosses G at least six times by (89), and hence $l(D) \geqslant 6$.

Third assume $r_2(D) = 2$. Then $\tilde{D}$ crosses at least one of e_2', e_2''. So $l(D) \geqslant 2$. Hence we have (87), except if $|r_1(D) - 1| \leqslant 1$, that is, if $r_1(D) = 2$. In that case, $\tilde{D}$ crosses at least one of e_1', e_1''. So $\tilde{D}$ crosses G at least four times by (89), and hence $l(D) \geqslant 4$.

If $r_1(D) = r_2(D) = 3$ then (87) is trivial as $l(D) \geqslant 0$.]

Claim 10, (83), (84), (86), (87), and Claims 15 and 11 imply

$$
\begin{aligned}
2\,|V_{\text{even}}| &+ |W_{\text{odd}}| + |U| + 2\,|W^-_{\text{odd}}| + 2\eta + 2\rho - 4\rho' \\
&\leqslant 4\left(\sum_{k=1}^{\infty} \chi(R_{2k})\right) - 4\rho' \leqslant 4\,|\mathscr{D}| \\
&< (2\,|\mathscr{D}| - 2(\#D \in \mathscr{D} \mid r_1(D) = 3) + 2(\#D \in \mathscr{D} \mid r_1(D) = 1)) \\
&\quad + (2\,|\mathscr{D}| - 2(\#D \in \mathscr{D} \mid r_2(D) = 3) + 2(\#D \in \mathscr{D} \mid r_2(D) = 1)) \\
&= \sum_{D \in \mathscr{D}} (4 - 2\,|r_1(D) - 1|) + \sum_{D \in \mathscr{D}} (4 - 2\,|r_2(D) - 1|)
\end{aligned}
$$

$$= \sum_{D \in \mathscr{D}} (8 - 2\,|r_1(D) - 1| - 2\,|r_2(D) - 1|)$$

$$\leqslant \sum_{D \in \mathscr{D}} l(D) \leqslant \left(\sum_{k=1}^{\infty} l(\mathrm{bd}(R_{2k})) \right) - 2\rho'$$

$$= \left(\sum_{k=1}^{\infty} \delta(R_{2k}) \right) - 2\rho' = 2\,|V_{\mathrm{even}}| + |W_{\mathrm{odd}}| + |U| - 2\rho'. \qquad (90)$$

Since $\rho' \leqslant \rho$, this gives a contradiction.

CASE 2. *There is no crossing in F_0 near to v_2.* So

$$(\# D \in \mathscr{D} \mid r_2(D) = 3) = 0. \qquad (91)$$

Now for each $D \in \mathscr{D}$ one has

$$l(D) \geqslant 6 - 2\,|r_1(D) - 1|. \qquad (92)$$

To see this, again let $\tilde{D}$ be a closed curve encircling D and very close to D, in such a way that $\tilde{D}$ has exactly $l(D) = \delta(R(D))$ crossings with G. Then showing (92) is again simple case-checking, using the fact that $R(D)$ contains at least one vertex of G and using the well-connectedness of K.

[If $r_1(D) = 0$ then $v_1 \notin R(D)$, while $VG \cap R(D) \neq \varnothing$, and by (88), $l(D) \geqslant 4$. If $r_1(D) = 1$, then $v_1 \in R(D)$ and $\tilde{D}$ crosses both e_1' and e_1''. If $l(D) = 4$ then either $VG \cap R(D) = \{v_1\}$ contradicting (85), or $\tilde{D}$ would have two crossings with e_1' and two crossings with e_1''; but then $\tilde{D}$ should have more crossings with G as $r_2(D) \leqslant 2$. So $l(D) \geqslant 6$.

If $r_1(D) = 2$ then $v_1 \in R(D)$ and $\tilde{D}$ crosses at least one of e_1', e_1''; say it crosses e_1'. If $l(D) = 2$ then $\tilde{D}$ would have a second crossing with e_1' and no further crossings with G; but this would imply $r_2(D) = 3$. So $l(D) \geqslant 4$.

If $r_1(D) = 3$ then $l(D) \geqslant 2$, since $r_2(D) \leqslant 2$.]

Now by Claim 10, (83), (84), (92), and Claims 15 and 11,

$$2\,|V_{\mathrm{even}}| + |W_{\mathrm{odd}}| + |U| + 2\,|W_{\mathrm{odd}}^-| + 2\eta + 2\rho - 4\rho'$$

$$\leqslant 4 \left(\sum_{k=1}^{\infty} \chi(R_{2k}) \right) - 4\rho' \leqslant 4\,|\mathscr{D}|$$

$$< 4\,|\mathscr{D}| - 2(\# D \in \mathscr{D} \mid r_1(D) = 3) + 2(\# D \in \mathscr{D} \mid r_1(D) = 1)$$

$$= \sum_{D \in \mathscr{D}} (6 - 2\,|r_1(D) - 1|) \leqslant \sum_{D \in \mathscr{D}} l(D) \leqslant \left(\sum_{k=1}^{\infty} l(\mathrm{bd}(R_{2k})) \right) - 2\rho'$$

$$= \left(\sum_{k=1}^{\infty} \delta(R_{2k}) \right) - 2\rho' = 2\,|V_{\mathrm{even}}| + |W_{\mathrm{odd}}| + |U| - 2\rho'. \qquad (93)$$

Since $\rho' \leqslant \rho$, this is a contradiction. ∎

This gives:

SUBCLAIM 16c. *Let D be a simple directed circuit in H', oriented clockwise, and not being a small component as given in Fig. 102. Then $l(D) \geqslant 4$.*

Proof. By Subclaim 16b, $r_1(D) \leqslant 2$ and $r_2(D) \leqslant 2$. If $VG \nsubseteq R(D)$, then, as $R(D)$ contains at least one vertex by Claim 13, $l(D) = \delta(R(D)) \geqslant 4$, by the well-connectedness of K.

If $VG \subseteq R(D)$, then $1 \leqslant r_1(D) \leqslant 2$ and $1 \leqslant r_2(D) \leqslant 2$. So any curve $\tilde{D}$ encircling D and close to D crosses at least one of e_1', e_1'' and at least one of e_2', e_2''. So by (89), $l(D) \geqslant 4$. ∎

Now by Claim 10, (83), Subclaim 16c, and Claims 15 and 11,

$$\begin{aligned} &2\,|V_{\text{even}}| + |W_{\text{odd}}| + |U| + 2\,|W_{\text{odd}}^-| + 2\eta + 2\rho - 2\rho' \\ &\leqslant 4\left(\sum_{k=1}^{\infty} \chi(R_{2k})\right) - 2\rho' \leqslant 4\,|\mathscr{D}| + 2\rho' \leqslant \left(\sum_{D \in \mathscr{D}} l(D)\right) + 2\rho' \\ &\leqslant \sum_{k=1}^{\infty} l(\text{bd}(R_{2k})) = \sum_{k=1}^{\infty} \delta(R_{2k}) = 2\,|V_{\text{even}}| + |W_{\text{odd}}| + |U|. \end{aligned} \tag{94}$$

Since $\rho' \leqslant \rho$, it follows that we have equality throughout in (94). Hence $W_{\text{odd}}^- = \varnothing$ and $\eta = 0$ and $\rho' = \rho$. So Fig. 120 does not occur.

Moreover, H' has no simple directed circuit D that is oriented counter-clockwise. Otherwise we could decompose H' into simple directed circuits $D_1, ..., D_t$ where $D_t = D$, and where for some $s < t$, $D_1, ..., D_s$ are oriented clockwise, and $D_{s+1}, ..., D_t$ are oriented counter-clockwise. This implies by Subclaim 16c and Claim 12,

$$\begin{aligned} &\sum_{k=1}^{\infty} l(\text{bd}(R_{2k})) = l(H') = \sum_{i=1}^{t} l(D_i) \\ &\qquad \geqslant 4s - 2\rho' > 4(s-(t-s)) - 2\rho' = 4\sum_{k=1}^{\infty} \chi(R_{2k}) - 2\rho', \end{aligned} \tag{95}$$

contradicting equality in (94).

It similarly follows that $l(D) = 4$ for each simple directed circuit D not forming a small component as given in Fig. 102. Moreover, by the well-connectedness of K, $|VG \cap R(D)| \leqslant 1$ or $|VG \backslash R(D)| \leqslant 1$. So if $VG \cap R(D) \neq \varnothing$ and (81)(i) and (81)(ii) do not hold, then $VG \subseteq R(D)$. Let e and e' be the two edges incident with v_1 that are incident with F_0. Suppose both e and e' are contained in $R(D)$. Then there are components Q and Q' of $\Delta' \cap \pi^{-1}[F_0]$ so that the $\pi(x_Q) - \pi(y_Q)$ part of $\text{bd}(F_0)$ contains e and the $\pi(x_{Q'}) - \pi(y_{Q'})$ part of $\text{bd}(F_0)$ contains e'. Subclaim 16a then gives that $\pi[Q]$ and $\pi[Q']$ cross near to v_1, contradicting Subclaim 16b. So there is

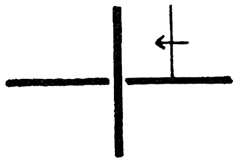

FIGURE 126

an edge, e_1 say, incident with v_1 and F_0 that is not contained in $R(D)$. Similarly, there is an edge, e_2 say, incident with v_2 and F_0 that is not contained in $R(D)$. As $VG \subseteq R(D)$, each of e_1 and e_2 should leave R twice. So we have (81)(iii). ∎

So each component R of each R_{2k} is a closed disk, without holes (the boundary of a hole would be oriented counter-clockwise). By Claim 16 we have $|VG \cap R| \leqslant 1$ or $|VG \backslash R| \leqslant 1$.

CLAIM 17. *The configurations given in Figs.* 105(b)–(i) *do not occur.*

Proof. As Fig. 120 does not occur, we cannot have Figs. 105(b), (c), (d), (f)–(h). Consider a configuration D of type (e) or (i) in Fig. 105, with $R(D)$ minimal (inclusionwise). If it is of type (e) then the point x belongs to W_{odd} and hence $w \in W^+$. So near to v we should have Fig. 126 (cf. Figs. 70, 71, or 73). This would be part of a smaller component, which hence should be of type (a) in Fig. 105. However, D itself traverses v.

If it is of type (i) in Fig. 105, then the points x, y, z, z' belong to W_{odd} and hence to W^+ (since $W^-_{\text{odd}} = \emptyset$ by Claim 16). So again, near to v we should have Fig. 127 (cf. Figs. 70, 71, or 73). Then both w and w' should be part of a smaller component, which hence should be of type (a) in Fig. 105. However, it cannot be the case that both w and w' belong to a component of type (a) in Fig. 105. ∎

Moreover:

CLAIM 18. $W = W^+_{\text{odd}}$ *and* $\varphi = 0$. *The configurations in Figs.* 102(a), (c) *do not occur.*

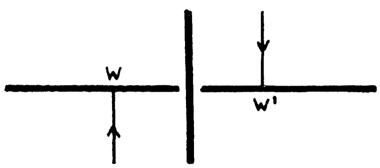

FIGURE 127

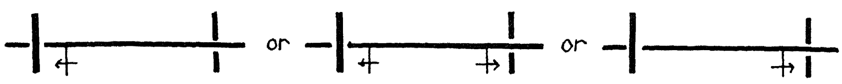

FIGURE 128

Proof. By Claim 16, $W^-_{\text{odd}} = \varnothing$. We next show $W^-_{\text{even}} = \varnothing$. Suppose $W^-_{\text{even}} \neq \varnothing$ and let $w \in W^-_{\text{even}}$. Let e be the edge of G containing $\pi(w)$. So, by Claim 4, all points in W that project to e, belong to W^-_{even}.

Since by Claim 16 Fig. 120 does not occur and since $W^-_{\text{odd}} = \varnothing$, it implies that e is as in Fig. 128 and there are no other points in $\pi[W]$ on e (there might be points in $\pi[U]$ on e) (cf. Figs. 70–74). Without loss of generality, we may assume that Fig. 129 occurs (we may assume this since we can rotate the last configuration in Fig. 128 with respect to a vertical axis and obtain the first). Then on e, left to $\pi(w)$ all edges of H' are entering e from above, and leaving e from below. Moreover, right to $\pi(w)$ all edges of H' are entering e from below, and leaving e from above, as in Fig. 130 (where v' denotes the other end of e). Let R be the component of $R_{\omega(w)}$ with $\pi(w)$ on its boundary. From Fig. 130 we see that e is fully contained in R. So $|VG \cap R| \geqslant 2$ and hence by Claim 16, $|VG \backslash R| \leqslant 1$.

Now first assume that v belongs to V_{even}, as in Fig. 131. For the face values β' and γ' one has $\beta' > \gamma'$ (near to $\pi(w)$), and hence (near to v) one has $\beta > \gamma$. (Possible points in $\pi[U]$ in between do not invalidate the fact that left to d the face value below $\pi[K]$ is larger than that above $\pi[K]$.) Hence $\alpha > \delta$ and $\omega(v) = \alpha$. Let a, b, c, d be the edges of H' as given in Fig. 131. Let R' be the component of $R_{\omega(v)}$ with a, v, b on its boundary. Note that R has c and d on its boundary. So $R' \subseteq R$.

Suppose $VG \cap R' \neq \{v\}$. Then by Claim 16, $|VG \backslash R'| \leqslant 1$. If $|VG \backslash R'| = 1$, let $\{v_0\} = VG \backslash R'$. Since both e and e' leave R', one should have $v_0 = v$, contradicting the fact that v belongs to R'. So $VG \subseteq R'$. Hence e and e' leave R' as in Fig. 132. But in that case R cannot contain R' (cf. Fig. 130).

So we know $VG \cap R' = \{v\}$. Hence by Claim 14, R' is the shaded region in Fig. 133, with $w', w'' \in W^+$ (since $W^-_{\text{odd}} = \varnothing$). We can, by an isotopy in S^3, switch the component C in $\mathscr{C}$ with $B(C)$ having w' and w'' as turning points, to the other side of v. That is, Fig. 133 becomes Fig. 134. However, now $\tilde{w}'$ belongs to W^+ while $w \in W^-$, so they can be cancelled as

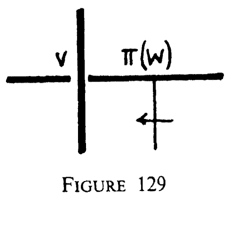

FIGURE 129

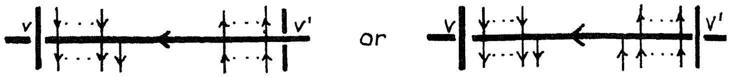

FIGURE 130

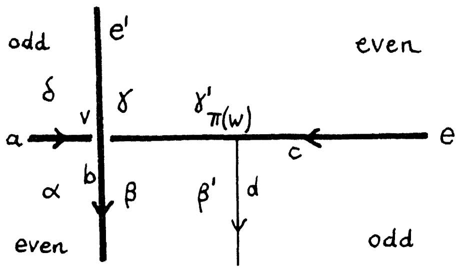

FIGURE 131

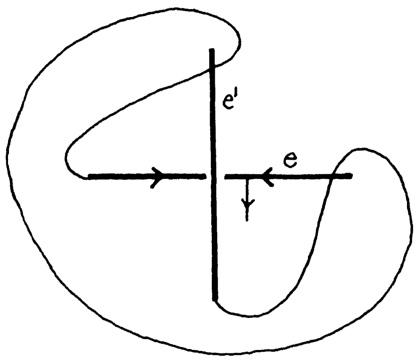

FIGURE 132

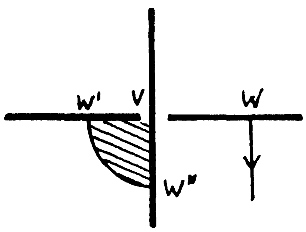

FIGURE 133

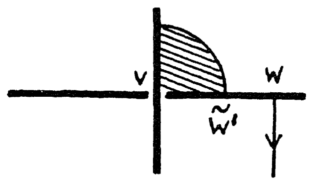

FIGURE 134

in Claim 2. This contradicts the minimality assumption (23)(iv). (The operations described do not change $\omega(v)$ but reduce $|W|$.)

Next assume that v belongs to V_{odd}. Let a, b, c, d be the edges of H' as in Fig. 135. We have $\alpha > \gamma$ (by the same argument as above for the case $v \in V_{even}$). Hence $\beta > \delta$, and therefore $\omega(v) = \alpha$. So there are no points in $\pi[U]$ on part r. It follows that a, b, c, d all are on the boundary of R. Since e is fully contained in R, we know that v and v' belong to R, and hence $|VG \backslash R| \leqslant 1$ by Claim 16.

Let e', e'', e''' be the edges of G as indicated in Fig. 135. We show that e'' is fully contained in R. To see this, we first consider, in Fig. 136, $p_v^{\uparrow}$ and $p_v^{\downarrow}$ as seen from F (see (25)). By Claim 5, ε_1 and ε_2 should lead to each other as in Fig. 137. Hence Σ does not intersect the vertical segment connecting $p_v^{\uparrow}$ and $p_v^{\downarrow}$; that is, $\phi_v + \zeta_v = 0$. So there are no points in U near to v.

Now v forms a "cut point" in R. That is, we can split R into two regions as in Fig. 138. Since $\delta(R) = 4$, we know $\delta(R_1) + \delta(R_2) = 8$. As e and e''' leave R_1 at least once, we know $\delta(R_1) \geqslant 4$. Similarly, as e' and e'' leave R_2 at least once, we know $\delta(R_2) \geqslant 4$. Hence $\delta(R_1) = \delta(R_2) = 4$.

As e and e''' leave R_1 and as $v_1 \in R_1$ (so e leaves R_1 exactly once), we know that $VG \backslash R_1 = \{v\}$. As $R_1 \cap R_2 = \varnothing$ and as $v \notin R_2$, $VG \cap R_2 = \varnothing$. So each of e' and e'' leaves R_2 exactly twice. Since there are no points in U near to v, it follows that there exist points $w', w'', w''' \in W$ as in Fig. 139. As $W_{odd}^{-} = \varnothing$ (Claim 16), w''' belongs to W_{odd}^{+}. Hence $w'' \in W_{even}^{+}$. Now e

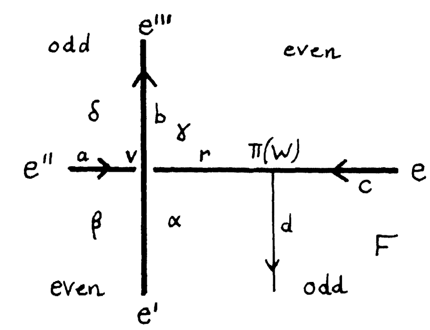

FIGURE 135

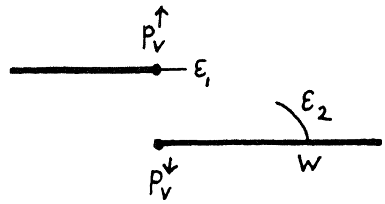

FIGURE 136

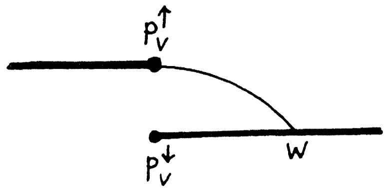

FIGURE 137

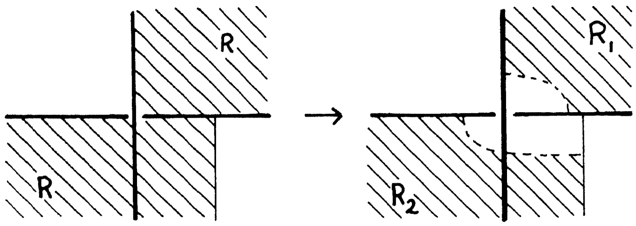

FIGURE 138

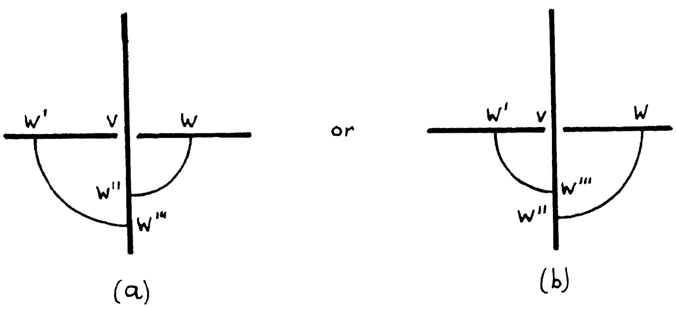

FIGURE 139

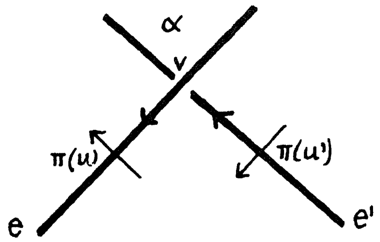

FIGURE 140

should be like in Figs. 70–72. So Fig. 139(a) does not apply. Moreover, since $v \in V_{\text{odd}}$, Fig. 139(b) does not apply. So we have a contradiction.

Concluding, there cannot exist a point $w \in W^-_{\text{even}}$; so $W^- = \varnothing$. Now by Subclaims 10e and 10d (recalling that $\eta = 0$ by Claim 16),

$$|W^+_{\text{odd}}| - |W^+_{\text{even}}| = 2\,|V| + 2\varphi,$$
$$|W^+_{\text{odd}}| + |W^+_{\text{even}}| = 2\,|V|. \tag{96}$$

Hence $W^+_{\text{even}} = \varnothing$ and $\varphi = 0$.

It follows that Figs. 102(a), (c) do not occur, since they involve points in W_{even}. ∎

Since $\phi = 0$, this implies that Figs. 78 and 80 do not occur. In fact, Fig. 82 does not occur either, since:

CLAIM 19. $U = \varnothing$.

Proof. Suppose $U \neq \varnothing$. By Claim 18 this implies that $\zeta_v \neq 0$ for some vertex v. If $v \in V_{\text{odd}}$ consider Fig. 140, where $\alpha = \omega(v)$ and $u, u' \in U$, such that $\pi(u)$ is the point in $\pi[U]$ on e nearest to v and $\pi(u')$ is the point in $\pi[U]$ on e' nearest to v. Let R be the component of $R_{\alpha-1}$ containing v (on its boundary). Since $\pi(u)$ and $\pi(u')$ are on the boundary of R, each of e and e' leaves R at least once. Moreover, e and e' do not both belong to the

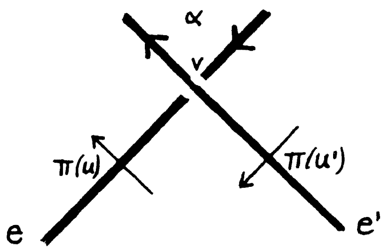

FIGURE 141

boundary of the unbounded face F_0. Now there is no vertex v' such that $VG \backslash R = \{v'\}$, since there are two edges incident with v that leave R at least once, implying $v' = v$, contradicting the fact that v belongs to R. So by Claim 16, $VG \cap R = \{v\}$, that is (by Claim 17), bd(R) is of type (a) in Fig. 105—a contradiction since $v \in V_{\text{odd}}$.

If $v \in V_{\text{even}}$ consider Fig. 141, where $\alpha = \omega(v)$ and $u, u' \in U$, such that $\pi(u)$ is the point in $\pi[U]$ on e nearest to v and $\pi(u')$ is the point in $\pi[U]$ on e' nearest to v. Let R be the component of $R_{\alpha-2}$ containing v. Since $\pi(u)$ and $\pi(u')$ are on the boundary of R, each of e and e' leaves R at least once. Moreover, e and e' do not both belong to the boundary of the unbounded face F_0 (since there are no Z-type curves seen from the unbounded face F_0, as $\pi^{-1}[F_0]$ does not contain any component in $\mathscr{C}_1$). So by Claim 16, $VG \cap R = \{v\}$, that is (by Claim 17), bd(R) is of type (a) in Fig. 105—a contradiction with (73) since there are points in $\pi[U]$ near to v. ∎

As a consequence we have that each vertex of H' has indegree one and outdegree one. That is:

$$\text{Each component of } H' \text{ is a directed circuit.} \tag{97}$$

CLAIM 20. *Each vertex* $v \in V_{\text{even}}$ *is in a component of type* (a) *in Fig.* 105.

Proof. Let $v \in V_{\text{even}}$ and let $\alpha := \omega(v)$ as in Fig. 142. Let R be the component of R_α containing v (on its boundary). Then each of e and e' leaves R at least once. If $VG \cap R = \{v\}$ then bd(R) is of type (a) in Fig. 105 (Claims 14 and 17). If $VG \backslash R = \{v'\}$, then $v' = v$, since e and e' are incident with v' and G is well-connected, contradicting the fact that v belongs to R. So by Claim 16 we may assume that $VG \subseteq R$, and that each of e and e' leaves R exactly twice. Moreover, e and e' both are on the boundary of the unbounded face F_0 of G.

Now R can only be left by edges incident with a point $v' \in V_{\text{even}}$ traversed by bd(R) and by edges containing a point $w \in W$ traversed by bd(R). (If bd(R) contains a point $v' \in V_{\text{odd}}$ then we have Fig. 143, where the shaded region is contained in R—so no edge is leaving R at v'.)

Since e and e' leave R, e and e' should be on the boundary of F_0 and e

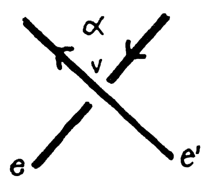

FIGURE 142

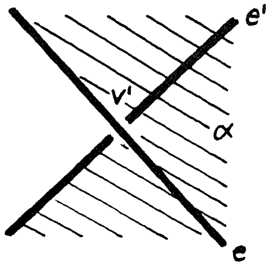

FIGURE 143

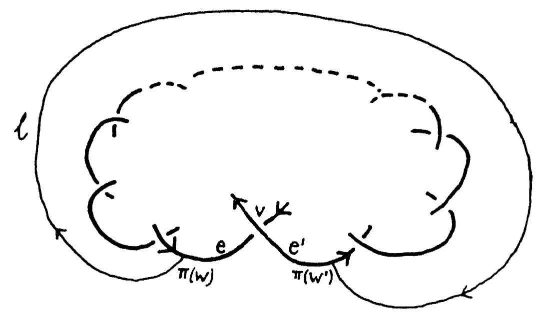

FIGURE 144

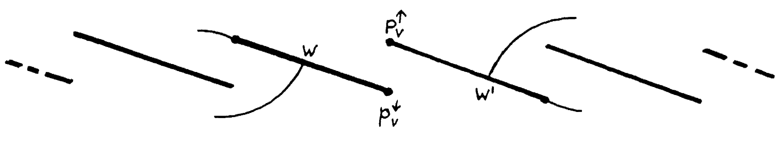

FIGURE 145

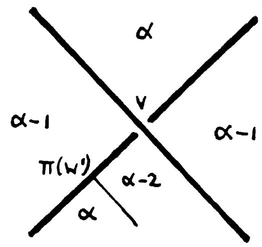

FIGURE 146

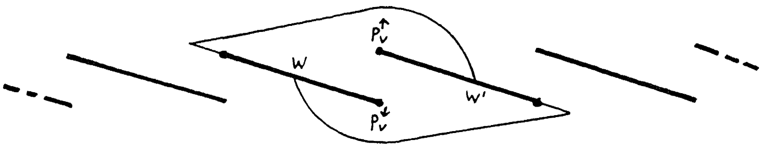

FIGURE 147

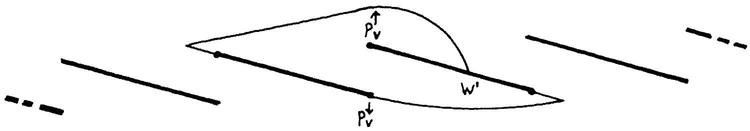

FIGURE 148

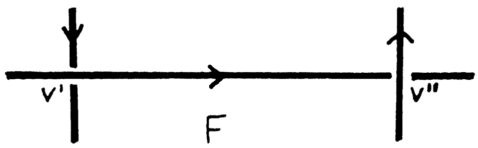

FIGURE 149

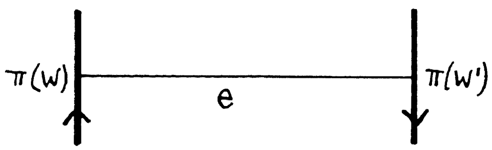

FIGURE 150

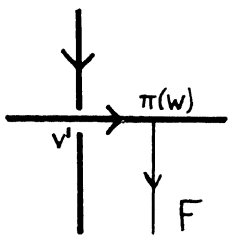

FIGURE 151

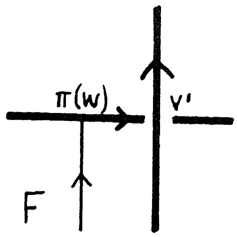

FIGURE 152

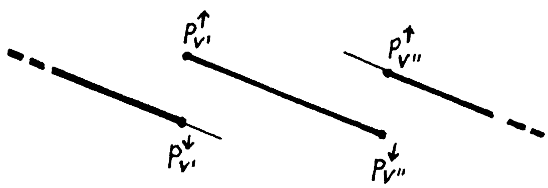

FIGURE 153

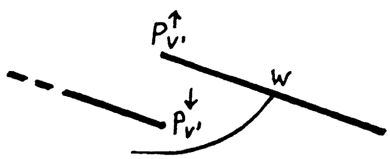

FIGURE 154

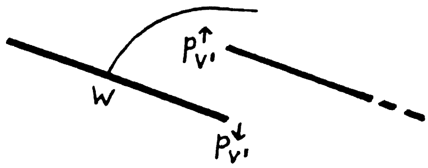

FIGURE 155

and e' should leave R exactly twice. So bd(R) does not contain any other vertex in V_{even} than v, and it should contain a point $\pi(w)$ on e and a point $\pi(w')$ on e' (with $w, w' \in W$), and a curve l in F_0 connecting $\pi(w)$ and $\pi(w')$ as in Fig. 144. Seen from F_0 we have Fig. 145 (by Claim 5). (Note that on e and e' there are no points in W near to v: if $\pi(w')$ would be a point on e near to v, with $w' \in W$, then we would have Fig. 146, assuming without loss of generality that w' is the nearest such point to v (since $W_{\text{even}} = \emptyset$); this implies that we have the given values for μ.) Applying an isotopy we first obtain Fig. 147, and next Fig. 148 (after a shift as in Claim 5). This, however, decreases

$$\sum_{v \in VG \cap \text{bd}(F_0)} \omega(v), \tag{98}$$

contradicting the minimality assumption (23)(ii). ▮

CLAIM 21. $V_{\text{odd}} = \emptyset$.

Proof. Let $v \in V_{\text{odd}}$. Consider the component D of H' containing v. This component consists of a number of edges e of H' each of one of the types (99)–(102):

e runs from v' to v'' for some $v', v'' \in V_{\text{odd}}$ as in Fig. 149; (99)

e runs from $\pi(w)$ to $\pi(w')$ for some $w, w' \in \pi[W]$ as in Fig. 150 (note that $W_{\text{even}} = \emptyset$); (100)

e runs from v' to $\pi(w)$ for some $v' \in V_{\text{odd}}$ and $w \in \pi[W]$ as in Fig. 151; (101)

e runs from $\pi(w)$ to v' for some $w \in \pi[W]$ and $v' \in V_{\text{odd}}$ as in Fig. 152. (102)

For any $x \in \mathbb{R}^3$ let $\lambda(x)$ denote the number of points in Σ strictly above x. Then for any edge of type (99), $\lambda(p^{\uparrow}_{v''}) = \lambda(p^{\uparrow}_{v'})$, since seen from F we have Fig. 153 (by Claim 5). For any edge of type (100), $\lambda(w') = \lambda(w)$, since $\lambda(x)$ is invariant on e. For any edge of type (101), $\lambda(w) = \lambda(p^{\uparrow}_{v'})$, since seen from F we have Fig. 154. For any edge of type (102), $\lambda(p^{\uparrow}_{v'}) = \lambda(w) + 1$, since seen from F we have Fig. 155. Now D traverses at least one point in $\pi[W]$ (since if it would consist only of edges of type (99) then D follows the boundary of an odd face counter-clockwise, contradicting Claim 16). So adding up all changes of $\lambda(x)$ over all edges of H' traversed by D would give a positive number—a contradiction. ▮

It follows that $V = V_{\text{even}}$ and that each $v \in V$ occurs in a component of type (a) in Fig. 105. As $|W| = 2|V|$ (Subclaim 10d), this implies that Fig. 102(b) does not occur. So all components of D are of type (a) in Fig. 105. Hence there exists an isotopy of $\mathbb{R}^3$ bringing Σ to Σ_K. ▮

5. Theorem B

We finally show:

THEOREM B. *Let K and K' be links with well-connected alternating diagrams, such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$, then the diagrams of K and K' are equivalent.*

Proof. Let Φ be an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$. Let $\psi(x) := \Phi(1, x)$ for all $x \in S^3$. So $\psi[\Sigma_K] = \Sigma_{K'}$.

Again, let H_K be the planar graph obtained by putting a vertex in each odd face of $\pi[K]$, joining any two such vertices by an edge if the corresponding odd faces have a crossing in common. So for each vertex v of $\pi[K]$ there is an edge, denoted by ε_v, of H_K. (Recall that e_v denotes the edge on Σ_K connecting $p_v^\uparrow$ and $p_{v'}^\uparrow$.)

The graph $H_{K'}$ is derived similarly from K'. Now ε'_v denotes the edge of $H_{K'}$ corresponding to vertex v of $\pi[K']$. Let e'_v denote the edge in $\Sigma_{K'}$ corresponding to vertex v of $\pi[K']$.

We may assume that any two even faces of $\pi[K]$ have at most one vertex in common. (For suppose that each of $\pi[K]$, $\pi[K']$ has two even faces with at least two vertices in common. Then H_K contains two edges forming a two-edge cut set. By the 3-vertex connectedness of H_K it follows that H_K is a digon or a triangle. It similarly follows that $H_{K'}$ is a digon or a triangle. Then Σ_K and $\Sigma_{K'}$ being isotopic directly implies that K and K' are equivalent.)

For each even face F of $\pi[K]$, we fix a simple closed curve C_F on Σ_K as follows. Let $F_1, ..., F_t$ be the odd faces incident with F, and let $v_1, ..., v_t$ be the vertices of $\pi[K]$ incident with F. Then C_F is a closed curve on Σ_K traversing the faces $D_{F_1}, ..., D_{F_t}$ of Σ_K and crossing each of the edges $e_{v_1}, ..., e_{v_t}$ exactly once, and not traversing any other face of Σ_K or crossing any other edge of Σ_K. (Recall that $D_F = \pi^{-1}[F] \cap \Sigma_K$ for each odd face F of $\pi[K]$.)

Since any two even faces of $\pi[K]$ have at most one vertex in common, we can take the curves C_F in such a way that, for any two even faces F_1, F_2 of $\pi[K]$, C_{F_1} and C_{F_2} have at most one crossing. In fact C_{F_1} and C_{F_2} have exactly one crossing, if and only if $\bar{F}_1$ and $\bar{F}_2$ intersect, viz. in a vertex v of $\pi[K]$. (That is, if and only if F_1 and F_2 are contained in adjacent faces of H_K.) We may assume that this crossing occurs on e_v.

For any even face F of $\pi[K]$, let B_F denote the circuit in H_K bounding the face of H_K containing F.

Now for each even face F of $\pi[K]$, $\psi[C_F]$ is a closed curve on $\Sigma_{K'}$. We may assume that each edge e'_v in $\Sigma_{K'}$ is crossed only a finite number of times by $\psi[C_F]$. For each even face F of $\pi[K]$ and each edge $e = \varepsilon'_v$ of $H_{K'}$ let

$$x(F, e) := \text{number of times } \psi[C_F] \text{ crosses } e'_v. \tag{103}$$

Define for each even face F of $\pi[K]$:

$$B'_F := \{e \in EH_{K'} \mid x(F, e) \text{ is odd}\}. \tag{104}$$

Since $\psi[C_F]$ is a closed curve, it crosses $\text{bd}(D_{F'})$ an even number of times for each odd face F' of $\pi[K']$, and hence B'_F is a cycle (= edge-disjoint union of circuits) in $H_{K'}$.

We show:

CLAIM 22. *For each edge e of $H_{K'}$ there exist even faces $F_1 \neq F_2$ of $\pi[K]$ such that $e \in B'_{F_1} \cap B'_{F_2}$.*

Proof. Consider the homology space over $\mathbb{Z}_2$ of Σ_K. It is generated by the curves C_F, where F ranges over the even faces of $\pi[K]$. To see this, let C be any closed curve on Σ_K. Let A be the set of edges $e = \varepsilon_v$ of H_K with the property that C crosses e_v an odd number of times. Then each vertex of H_K is incident with an even number of edges in A. So A is the symmetric difference (= mod 2 sum) of the boundaries of a collection $\mathscr{F}'$ of faces of H_K. Let $\mathscr{F}$ be the collection of even faces of $\pi[K]$ that are contained in the faces in $\mathscr{F}'$. Then C is homologous over $\mathbb{Z}_2$ to $\sum_{F \in \mathscr{F}} C_F$. To prove this, we may assume that C and the C_F have only a finite number of crossings. Moreover, by slightly shifting we may assume that C and the C_F do not intersect $\text{bd}(\Sigma_K)$. Now $C \cup \bigcup_{F \in \mathscr{F}} C_F$ crosses each e_v an even number of times. Hence we can color, for each odd face F' of $\pi[K]$, the components of $\overline{D_{F'}} \backslash (C \cup \bigcup_{F \in \mathscr{F}} C_F)$ red and blue so that adjacent components have different colors and such that $\text{bd}(D_{F'}) \cap \text{bd}(\Sigma_K)$ is colored red. Doing this for each $D_{F'}$ we obtain a coloring of the components of $\Sigma_K \backslash (C \cup \bigcup_{F \in \mathscr{F}} C_F)$ such that $C \cup \bigcup_{F \in \mathscr{F}} C_F$ separates red and blue. So C and $\bigcup_{F \in \mathscr{F}} C_F$ are homologous over $\mathbb{Z}_2$.

One similarly shows that $\sum_F C_F$ is nullhomologous over $\mathbb{Z}_2$ where F ranges over all even faces of $\pi[K]$.

Now choose an edge e of $H_{K'}$, say $e = \varepsilon'_v$, where v is a vertex of $\pi[K']$. Then $\sum_F x(F, e)$ is even, since $\sum_F C_F$ is nullhomologous on Σ_K, and hence $\sum_F \psi(C_F)$ is nullhomologous on $\psi(\Sigma_K) = \Sigma_{K'}$ (sums ranging over even faces F of $\pi[K]$). So it suffices to show that there exists one even face F of $\pi[K]$ such that $e \in B'_F$.

Let F' be one of the two even faces of $\pi[K']$ incident with vertex v of $\pi[K']$. Then $C_{F'}$ crosses e'_v exactly once. Now $\psi^{-1}[C_{F'}]$ is a closed curve on Σ_K, and hence it is homologous to $\sum_{F \in \mathscr{F}} C_F$ for some collection $\mathscr{F}$ of even faces of $\pi[K]$. Hence $C_{F'}$ is homologous to $\psi[\sum_{F \in \mathscr{F}} C_F]$. Hence there exists an $F \in \mathscr{F}$ such that $\psi[C_F]$ crosses e'_v an odd number of times. So e belongs to B'_F. ∎

Next:

CLAIM 23. *For each even face F of $\pi[K]$ one has $|B_F| = |B'_F|$. Moreover, each edge of $H_{K'}$ is contained in exactly two of the cycles B'_F.*

Proof. For any simple closed curve C' on $\Sigma_{K'}$ and any $e = e'_v$ on $\Sigma_{K'}$ define

$$\gamma(C', e) := [(\text{number of times } C' \text{ crosses } e \text{ in one direction}) - (\text{number of times } C' \text{ crosses } e \text{ in the other direction})]^2. \quad (105)$$

(So here we choose, temporarily, a "left hand side" and a "right hand side" of e. Clearly, the definition is independent of this choice.)

Then

$$\tau(C', \Sigma_{K'}) = \sum_v \gamma(C', e'_v), \quad (106)$$

where v ranges over all vertices of $\pi[K']$. We shall show (106) when C' is orientation-preserving (the extension to the general case is immediate). Consider a crossing of part α (say) of C' with e'_v. Let $\tilde{\alpha}$ be close and parallel to α. Then $\tilde{\alpha}$ makes a positive crossing with α as in Fig. 156. Consider also two crossings of parts α and β (say) of C' with e'_v in the same direction. So part of $\Sigma_{K'}$ looks like Fig. 157. (In Fig. 157 we have displayed only the part of $\Sigma_{K'}$ "in between" α and β.) Let $\tilde{\alpha}$ and $\tilde{\beta}$ be parallel and close to α and β, respectively. Then $\tilde{\alpha}$ and β make a positive crossing, and α and $\tilde{\beta}$ make a positive crossing. Similarly, if α and β cross e'_v in opposite directions we obtain two negative crossings.

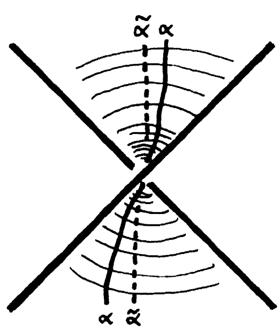

FIGURE 156

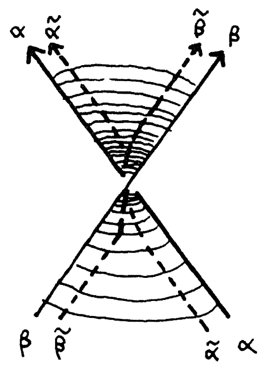

FIGURE 157

Now let λ be the number of times C' crosses e'_v in one direction, and let μ be the number of times C' crosses e'_v in the other direction. Then the number of positive crossings counted in the contribution of v to $\tau(C', \Sigma_{K'})$ is equal to

$$\lambda + \mu + 2\binom{\lambda}{2} + 2\binom{\mu}{2} = \lambda^2 + \mu^2, \tag{107}$$

while the number of negative crossings is $2\lambda\mu$. So the contribution of v to $\tau(C', \Sigma_{K'})$ is equal to $\lambda^2 + \mu^2 - 2\lambda\mu = (\lambda - \mu)^2$. This shows (106).

Moreover, by definitions (103) and (105), for each even face F of $\pi[K]$ and each vertex v of $\pi[K']$, $x(F, \varepsilon'_v)$ is odd if and only if $\gamma(\psi[C_F], e'_v)$ is odd. In particular, if $\varepsilon'_v \in B'_F$ then $\gamma(\psi[C_F], e'_v) \geqslant 1$. Hence for each even face F of $\pi[K]$,

$$|B'_F| \leqslant \sum_v \gamma(\psi[C_F], e'_v) = \tau(\psi[C_F], \Sigma_{K'}) = \tau(C_F, \Sigma_K) = |B_F| \tag{108}$$

(where again v ranges over vertices of $\pi[K']$). Moreover, since by Claim 22 each edge e of $H_{K'}$ is contained in at least two cycles of the form B'_F:

$$\sum_F |B'_F| \geqslant 2v(K') = 2v(K) = \sum_F |B_F|, \tag{109}$$

where F ranges over all even faces of $\pi[K]$.

Combining (108) and (109) gives the claim. ▮

Next we show:

CLAIM 24. *Let F_1 and F_2 be two even faces of $\pi[K]$. Then $|B'_{F_1} \cap B'_{F_2}|$ is odd, if and only if F_1 and F_2 are in adjacent faces of H_K.*

Proof. First assume that F_1 and F_2 are not in adjacent faces of H_K. So by assumption, C_{F_1} and C_{F_2} are disjoint. Then also $\psi[C_{F_1}]$ and $\psi[C_{F_2}]$ are disjoint. We may assume that the projections $\pi[\psi[C_{F_1}]]$ and $\pi[\psi[C_{F_2}]]$ are closed curves in $\mathbb{R}^2$ such that they only cross at vertices of $\pi[K']$, in such a way that near a vertex v of $\pi[K']$ there are

$$x(F_1, \varepsilon'_v) \cdot x(F_2, \varepsilon'_v) \tag{110}$$

crossings of $\pi[\psi[C_{F_1}]]$ with $\pi[\psi[C_{F_2}]]$.

Since the total number of crossings of $\pi[\psi[C_{F_1}]]$ with $\pi[\psi[C_{F_2}]]$ is even, we know that

$$\sum_v x(F_1, \varepsilon'_v) \cdot x(F_2, \varepsilon'_v) \tag{111}$$

is even. Since (111) has the same parity as $|B'_{F_1} \cap B'_{F_2}|$, we know that $|B'_{F_1} \cap B'_{F_2}|$ is even.

If F_1 and F_2 are in adjacent faces, one similarly shows that $|B'_{F_1} \cap B'_{F_2}|$ is odd. ∎

In fact we have:

CLAIM 25. *For any two even faces F_1 and F_2 of $\pi[K]$, $|B'_{F_1} \cap B'_{F_2}| = 1$ if F_1 and F_2 are contained in adjacent faces of H_K, and $|B'_{F_1} \cap B'_{F_2}| = 0$ otherwise.*

Proof. By Claims 23 and 24 and by the well-connectedness of K,

$$\begin{aligned} 2v(K) &= \text{number of pairs } (F_1, F_2) \text{ of two even faces of } \pi[K] \\ &\qquad \text{contained in adjacent faces of } H_K \\ &\leqslant \sum_{(F_1, F_2),\, F_1 \neq F_2} |B'_{F_1} \cap B'_{F_2}| = \sum_{F_1} \Big(\sum_{F_2 \neq F_1} |B'_{F_1} \cap B'_{F_2}| \Big) \\ &= \sum_{F_1} |B'_{F_1}| = \sum_{F_1} |B_{F_1}| = 2v(K). \end{aligned} \tag{112}$$

So the inequality is attained with equality, and the claim follows. ∎

We can now define a function

$$\vartheta : EH_K \to EH_{K'} \tag{113}$$

as follows. For $e \in EH_K$, let F_1 and F_2 be the two even faces of $\pi[K]$ contained in the faces of H_K incident with e. Let

$$B'_{F_1} \cap B'_{F_2} = \{e'\}. \tag{114}$$

Then define $\vartheta(e) := e'$. By Claim 23, this function is one-to-one, and hence onto (since $|EH_K| = |EH_{K'}|$).

Moreover, for each even face F of $\pi[K]$, $\vartheta[B_F] = B'_F$, since

$$\vartheta[B_F] = \bigcup_{F' \neq F} \vartheta[B_F \cap B_{F'}] = \bigcup_{F' \neq F} (B'_F \cap B'_{F'}) = B'_F. \tag{115}$$

So for each cycle B in H_K the set $\vartheta[B]$ is a cycle in $H_{K'}$ (since B is a binary sum of circuits B_F, and hence $\vartheta[B]$ is a binary sum of cycles B'_F).

Now both H_K and $H_{K'}$ are 3-vertex-connected planar graphs (by the well-connectedness of K and K'), with $|VH_K| = b(K) = b(K') = |VH_{K'}|$ and $|EH_K| = v(K) = v(K') = |EH_{K'}|$. Hence, by Whitney's theorem [14], H_K and $H_{K'}$ are the same plane graph, up to rerouting edges through the unbounded face, and up to turning the graph upside down. This implies that the diagrams of K and K' can be obtained from each other by the operations (1). That is, K and K' have equivalent diagrams. ∎

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