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FULL LENGTH PAPER

The traveling salesman problem on cubic and subcubic graphs

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Abstract We study the traveling salesman problem (TSP) on the metric completion of cubic and subcubic graphs, which is known to be NP-hard. The problem is of interest because of its relation to the famous 4/3-conjecture for metric TSP, which says that the integrality gap, i.e., the worst case ratio between the optimal value of a TSP instance and that of its linear programming relaxation (the subtour elimination relaxation), is 4/3. We present the first algorithm for cubic graphs with approximation ratio 4/3. The proof uses polyhedral techniques in a surprising way, which is of independent interest. In fact we prove constructively that for any cubic graph on n vertices a tour of length $4n/3 - 2$ exists, which also implies the 4/3-conjecture, as an upper bound, for this class of graph-TSP. Recently, Mömke and Svensson presented an algorithm

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that gives a 1.461-approximation for graph-TSP on general graphs and as a side result a 4/3-approximation algorithm for this problem on subcubic graphs, also settling the 4/3-conjecture for this class of graph-TSP. The algorithm by Mömke and Svensson is initially randomized but the authors remark that derandomization is trivial. We will present a different way to derandomize their algorithm which leads to a faster running time. All of the latter also works for multigraphs.

Keywords Traveling salesman problem · Approximation algorithms · Subtour elimination · Matching polytope · Polyhedral combinatorics

Mathematics Subject Classification 90B10 · 90C10 · 90C27 · 90C59

1 Introduction

Given a complete undirected graph $G = (V, E)$ with vertex set V, $|V| = n$, and edge set E, with non-negative edge costs $c \in \mathbf{R}^E$, $c \neq 0$, the well-known *Traveling Salesman Problem* (TSP) is to find a Hamilton cycle in G of minimum cost. When the costs satisfy the triangle inequality, i.e., when $c_{ij} + c_{jk} \geq c_{ik}$ for all $i, j, k \in V$, we call the problem *metric*. A special case of the metric TSP is the so-called *graph-TSP*, where, given an undirected, unweighted underlying graph $G = (V, E)$, a complete weighted graph on V is formed by defining the cost between two vertices as the number of edges on the shortest path between them. This new graph is known as the *metric completion* of G. Equivalently, this can be formulated as the problem of finding a spanning Eulerian multi-subgraph $H = (V, E')$ of G with a minimum number of edges. Note that any spanning Eulerian multi-subgraph yields an Euler tour which can be transformed into a TSP tour of at most the same length. Conversely, any TSP tour in the metric completion gives a spanning Eulerian multi-subgraph if we replace any edge (u, v) in the tour by the edges of a shortest path from u to v in G.

The TSP is well-known to be NP-hard [25], even for the special cases of graph-TSP. As noticed in [21], APX-hardness follows rather straightforwardly from the APX-hardness of (weighted) graphs with edges of length 1 or 2 ((1,2)-TSP) (Papadimitriou and Yannakakis [31]), even if the maximum degree is 6.

In general, the TSP cannot be approximated in polynomial time to within any constant unless $P = NP$, however for the metric TSP there is the elegant algorithm due to Christofides [12] from 1976 which gives a 3/2-approximation. Surprisingly, in over three decades no one has found an approximation algorithm which improves upon this bound of 3/2, and the quest for finding such improvements is one of the most challenging research questions in combinatorial optimization.

A related approach for approximating TSP is to study the *integrality gap* $\alpha(TSP)$, which is the worst-case ratio between the optimal solution for a TSP instance and the optimal solution to its linear programming relaxation, the so-called *Subtour Elimination Relaxation* (henceforth SER) (see [5] for more details), which is described as

$$\min \sum_{e \in E} c_e x_e \text{ such that}$$
$$x(\delta(v)) = 2 \text{ for } v \in V, x(\delta(S)) \geq 2 \text{ for } \emptyset \neq S \subsetneq V, \text{ and } x \geq 0.$$

Here, $x(\delta(S))$ denotes the sum of the x_e over all edges e in the cut $(S, \bar{S})$ defined as the set of edges $e = uv$ with $u \in S$ and $v \in \bar{S}$. The value $\alpha(TSP)$ gives one measure of the quality of the lower bound provided by SER for the TSP. Moreover, a polynomial-time constructive proof for value $\alpha(TSP)$ would provide an $\alpha(TSP)$-approximation algorithm for the TSP.

For metric TSP, it is known that $\alpha(TSP)$ is at most 3/2 (see Shmoys and Williamson [34], Wolsey [35]), and is at least 4/3. A ratio of 4/3 is reached asymptotically by the family of graph-TSP problems consisting of two vertices joined by three paths of length k; see also [5] for a similar family of graphs giving this ratio. However, the exact value of $\alpha(TSP)$ is not known, and there is the following well-known conjecture, which dates back to the early 1980's (see for example [20]):

Conjecture 1 *For the metric TSP, the integrality gap $\alpha(TSP)$ for SER is 4/3.*

As with the quest to improve upon Christofides' algorithm, the quest to prove or disprove this conjecture has been open for almost 30 years, with very little progress made.

A graph $G = (V, E)$ is *cubic* if all of its vertices have degree 3, and *subcubic* if they have degree at most 3. A *multigraph* is one in which multiple copies of edges (i.e., parallel edges) are allowed between vertices (but loops are not allowed) and a graph is called *simple* if there are no multiple copies of edges. A *cycle* in a graph is a closed path having no repetition of vertices. A *cycle cover* (also sometimes referred to as a 2-factor or a perfect 2-matching) of G is a set of vertex disjoint cycles that together span all vertices of G. A *perfect matching* M of a graph G is a set of vertex-disjoint edges of G that together span all vertices of G.

In this paper we study the graph-TSP problem on cubic and subcubic graphs. Note that the integrality gap instances described above are graph-TSP instances on bridgeless subcubic graphs. Also, solving the graph-TSP on such graphs would solve the problem of deciding whether a given bridgeless cubic graph G has a Hamilton cycle, which is known to be NP-complete, even if G is also planar (Garey et al. [19]) or bipartite (Akiyama et al. [2]). In [14] there is an unproven claim that (1,2)-TSP is APX-hard when the graph of edges of length 1 is cubic, which would imply APX-hardness of graph-TSP on cubic graphs. Also note that the 3/2-ratio of Christofides' algorithm is tight for cubic graph-TSP (see [11]).

In 2005, Gamarnik et al. [18] provided the first approximation improvement over Christofides' algorithm for graph-TSP on 3-edge-connected cubic graphs. They provide a polynomial-time algorithm that finds a Hamilton cycle of cost at most τn for $\tau = (3/2 - 5/389) \approx 1.487$. Since n is a lower bound for the optimal value for graph-TSP on such graphs, this results in a τ-approximation for the graph-TSP. Moreover, since n is also a lower bound for the associated SER,[1] it proves that the integrality gap $\alpha(TSP)$ is at most τ for such problems.

Only recently, the work by Gamarnik et al. has been succeeded by a sudden outburst of results on the approximation of graph-TSP and its SER. In 2009 and 2010, polynomial-time algorithms that find triangle- and square-free cycle covers for cubic

[1] To see that n is a lower bound for SER, sum all of the so-called "degree constraints" for SER. Dividing the result by 2 shows that the sum of the edge variables in any feasible SER solution equals n.

3-edge-connected graphs have been developed (see [6,8] and [23]). These papers do not explicitly study the graph-TSP problem, but as a by-product, these algorithms provide a cycle cover with at most $n/5$ cycles, and, as we will argue in Sect. 2, thus give a $(1.4n - 2)$-approximation.

Shortly after, improvements for graph-TSP were obtained independently by Aggarwal et al. [1], Oveis Gharan et al. [30] and by us [10,11]. Aggarwal et al. gave a 4/3-approximation for 3-edge-connected cubic graphs by constructing a TSP tour of length at most $4n/3 - 2$ when $n \geq 6$. The proof is based on the idea of finding a triangle- and square-free cycle cover, and then shrinking and "splitting off" certain 5-cycles in the cover.

Around the same time, Oveis Gharan et al. [30] presented a randomized $(3/2 - \epsilon)$-approximation for graph-TSP for some $\epsilon > 0$, which was the first polynomial-time algorithm with an approximation ratio strictly less than 3/2 for graph-TSP on general graphs. Their approach is very different from any of the others.

Independently of all this, we made the next improvement (see Boyd et al. [11]) by showing that every bridgeless cubic graph has a TSP tour of length at most $4n/3 - 2$ when $n \geq 6$. This was the first result which showed that Conjecture 1 is true for graph-TSP, as an upper bound, on cubic bridgeless graphs and it automatically implies a 4/3-approximation algorithm for this class of graph-TSP. The results extend to all cubic graphs. They have appeared in a preliminary form in [11]. The proof of the $(4n/3-2)$-bound uses polyhedral techniques in a surprising way, which may be more widely applicable. We present a complete proof of the result in Sect. 2.

In [11] we also show a bound of $7n/5 - 4/5$ on the length of a graph-TSP tour for subcubic bridgeless graphs. We conjectured that the true bound should be $(4n/3-2/3)$, which is equal to the well-known lower bound for this class of graphs (see Sect. 4). For reasons that become clear below we do not give the details of this result here but instead refer to the extended version [9] of [11] for its proof. All the above results are extended to graphs with bridges. For cubic graphs we show this extension at the end of Sect. 2.

Recently, Mömke and Svensson [26,27] came up with a powerful new approach, which enabled them to prove a 1.461-approximation for graph-TSP for general graphs. In the context of the present paper it is interesting that their approach led to a bound of $(4n/3 - 2/3)$ on the graph-TSP tour for all subcubic bridgeless graphs, thus improving upon our above-mentioned $(7n/5 - 4/5)$-bound and settling our conjecture affirmatively. Mucha [28] refined the analysis and obtained an approximation ratio of 13/9 for general graph-TSP. Very recently, Sebö and Vygen [33] announced a 7/5-approximation algorithm for general graph-TSP together with a 3/2-approximation for the path version in arbitrary metrics.

Another result of this paper is an alternative derandomization for Mömke and Svensson's randomized algorithm [26,27]. It is not only simpler but has also a substantially faster running time than the obvious derandomization; for example it has complexity $O(n^2 \log n)$ for (sub)cubic graphs rather than $O(n^6)$. In Sect. 3.1 we explain the basic ingredients of Mömke and Svensson's algorithm in order to be able to describe our result in Sect. 3.2.

We conclude this section with a survey of some of the other relevant literature. Grigni et al. [21] give a polynomial-time approximation scheme (PTAS) for graph-TSP on planar graphs (this was later extended to a PTAS for the weighted planar graph-TSP by Arora et al. [3]). For graph G containing a cycle cover with no triangles, Fotakis and Spirakis [16] show that graph-TSP is approximable in polynomial time within a factor of $17/12 \approx 1.417$ if G has diameter 4 (i.e., the longest path has length 4), and within 7/5 = 1.4 if G has diameter 3. For graphs that do not contain a triangle-free cycle cover they show that if G has diameter 3, then it is approximable in polynomial time within a factor of $22/15 \approx 1.467$. For graphs with diameter 2 (i.e., TSP(1,2)), a $7/6 \approx 1.167$-approximation for graph-TSP was achieved by Papadimitriou and Yannakakis [31], and improved to $8/7 \approx 1.143$ by Berman and Karpinski [7].

2 A 4/3-approximation for bridgeless cubic graphs

In this section, we will prove the following:

Theorem 1 *Every bridgeless simple cubic graph $G = (V, E)$ with $n \geq 6$ has a graph-TSP tour of length at most $\frac{4}{3}n - 2$.*

We begin by giving some definitions, and preliminary results. For any vertex subset $S \subseteq V$, $\delta(S) \subseteq E$, defined as the set of edges connecting S and $V \setminus S$, is called the *cut* induced by S. A cut of cardinality k is called a *k-cut* if it is minimal in the sense that it does not contain any cut as a proper subset. A *k-cycle* is a cycle containing k edges, and a *chord* of a cycle of G is an edge not in the cycle, but with both ends u and v in the cycle. An *Eulerian subgraph* of G is a connected subgraph where multiple copies of the edges are allowed, and all vertices have even degree. Note that such a subgraph has an Eulerian tour of length equal to its number of edges, which can be "short-cut" into a TSP tour of the same length for the associated graph-TSP problem.

A crucial result needed for our algorithm and main theorem, also applied by Gamarnik et al. [18], is Petersen's Theorem [32] stating that any bridgeless cubic graph can be partitioned into a cycle cover and a perfect matching. We give a useful strengthened form of it below in Lemma 1.

For any edge set $F \subseteq E$, the *incidence vector of F* is the vector $\chi^F \in \{0, 1\}^E$ defined by $\chi^F_e = 1$ if $e \in F$, and 0 otherwise. For any edge set $F \subseteq E$ and $x \in \mathbf{R}^E$, let $x(F) = \sum_{e \in F} x_e$. Given graph G, the associated *perfect matching polytope*, $P^M(G)$, is the convex hull of all incidence vectors of the perfect matchings of G, which Edmonds [15] shows to be given by:

$$\begin{aligned} x(\delta(v)) &= 1, \quad \forall v \in V, \\ x(\delta(S)) &\geq 1, \quad \forall S \subset V, \ |S| \text{ odd}, \\ 0 \leq x_e &\leq 1, \quad \forall e \in E. \end{aligned}$$

Using this linear description and similar methods to those found in [24] and [29], we have the following strengthened form of Petersen's Theorem, in which we use the notion of a *3-cut perfect matching*, which is a perfect matching that intersects every 3-cut of the graph in exactly one edge:

Lemma 1 *Let $G = (V, E)$ be a bridgeless cubic graph and let $x^* = \frac{1}{3}\chi^E$. Then x^* can be expressed as a convex combination of incidence vectors of 3-cut perfect* *matchings, i.e., there exist* 3*-cut perfect matchings* M_i *,* $i = 1, 2, ..., k$ *of* G *and positive real numbers* λ_i *,* $i = 1, 2, ..., k$ *such that*

$$x^* = \sum_{i=1}^{k} \lambda_i (\chi^{M_i}) \text{ and } \sum_{i=1}^{k} \lambda_i = 1. \tag{1}$$

Proof Since both sides of any 2-cut in a cubic graph have an even number of vertices, it is easily verified that x^* satisfies the linear description above, and thus lies in $P^M(G)$. It follows that x^* can be expressed as a convex combination of perfect matchings of G, i.e., there exist perfect matchings M_i, $i = 1, 2, \ldots, k$ of G and positive real numbers λ_i, $i = 1, 2, \ldots, k$ such that (1) holds. Further, for any 3-cut, the cut constraint in $P^M(G)$ is tight for x^*, which implies that it is tight for any of the perfect matchings M_i in the convex combination. □

The perfect matchings $M_i, i = 1, 2, \ldots k$, of Lemma 1 will be used in the proof of our main theorem. Algorithmic versions of Caratheodory's theorem (see for example Theorem 6.5.11 in [22]) say that we can find such a set $\mathcal{M}$ of matchings in polynomial time. Barahona [4] provides an algorithm to find for any point in $P^M(G)$ a set of perfect matchings for expressing the point as a convex combination of their incidence vectors in $O(n^6)$ time, and with $k \leq 7n/2 - 1$, for any graph G.

The idea we will use in the proof of our main theorem is as follows: By Petersen's Theorem we know we can always find a cycle cover of G. Suppose that we can find such a cycle cover that has no more than $n/6$ cycles. Then, contracting the cycles, adding a doubled spanning tree in the resulting graph and uncontracting the cycles would yield a spanning Eulerian multi-subgraph with no more than $n + 2(n/6 - 1) = 4n/3 - 2$ edges. Together with the obvious lower bound of n on the length of any optimal graph-TSP tour, this yields an approximation ratio of 4/3. However, such a cycle cover does not always exist (for example, consider the Petersen graph).[2]

Instead, we take the k cycle covers associated with the 3-cut matchings of Lemma 1 and combine their smaller cycles into larger cycles or Eulerian subgraphs, such as to obtain k covers of G with Eulerian subgraphs which, together with the double spanning tree, result in k spanning Eulerian subgraphs of G having an average number of edges of at most $4n/3$. Unless stated otherwise, an Eulerian subgraph is connected. For the construction of larger Eulerian subgraphs the following lemma will be useful.

Lemma 2 *Let* H_1 *and* H_2 *be connected Eulerian subgraphs of a (sub)cubic graph such that* H_1 *and* H_2 *have at least two vertices in common and let* H_3 *be the* sum *of* H_1 *and* H_2*, i.e., the union of their vertices and the sum of their edges, possibly giving rise to parallel edges. Then we can remove two edges from* H_3 *such that it stays connected and Eulerian.*

Proof Let u and v be in both subgraphs. The edge set of H_3 can be partitioned into edge-disjoint (u, v)-walks P_1, P_2, P_3, P_4. Vertex u must have two parallel edges which are on different paths, say $e_1 \in P_1$ and $e_2 \in P_2$. When we remove e_1 and e_2, the graph

[2] Note that a bound of $n/6 + 1$ cycles is sufficient for the the 4/3-approximation and it follows from results in [27] (see Sect. 3) that such a cycle cover indeed always exists.

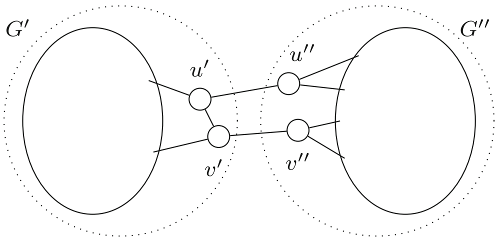

Fig. 1 An illustration for Lemma 3

stays Eulerian. Moreover, it stays connected since u and v are still connected by P_3 and P_4 and, clearly, each vertex on P_1 and P_2 remains connected to v. □

The following lemma, which applies to any graph, allows us to preprocess our graph by removing certain subgraphs. Removing these beforehand simplifies the proof as it allows us to avoid a recursive argument. (See Fig. 1 for an illustration of the structure described in the lemma.)

Lemma 3 *Assume that removing edges $u'u''$ and $v'v''$ from graph $G = (V, E)$ decomposes it into two graphs $G' = (V', E')$ and $G'' = (V'', E'')$ with $u', v' \in V'$, and $u'', v'' \in V''$ and such that:*

1. *$u'v' \in E$ and $u''v'' \notin E$.*
2. *there is a spanning Eulerian subgraph T' of G' with at most $4|V'|/3 - 2$ edges.*
3. *there is a spanning Eulerian subgraph T'' of $G'' \cup u''v''$ with at most $4|V''|/3 - 2$ edges.*

Then there is a spanning Eulerian subgraph T of G with at most $4|V|/3 - 2$ edges.

Proof If T'' does not use edge $u''v''$ then we take edge $u'u''$ doubled and add subgraph T'. If T'' uses edge $u''v''$ once then we remove it and add edges $u'u''$, $v'v''$ and $u'v'$ and subgraph T'. If T'' uses edge $u''v''$ twice then we remove both copies and add edge $u'u''$ doubled, $v'v''$ doubled, and subgraph T'. □

We use Lemma 3 to remove all subgraphs of the form shown in Fig. 2, which we call a *p-rainbow* subgraph. In such subgraphs there is a path $u_0, u_1, \ldots, u_{p+1}$ and path $v_0, v_1, \ldots, v_{p+1}$ for some $p \geq 1$, and a 4-cycle u_0, a, v_0, b with chord ab. Furthermore, there are edges $u_i v_i$ for each $i \in \{1, 2, \ldots, p\}$ but there is no edge between u_{p+1} and v_{p+1}. The figure shows a p-rainbow for $p = 2$. For general p, the 2-cut of Lemma 3 is given by $u' = u_p$, $u'' = u_{p+1}$, $v' = v_p$, and $v'' = v_{p+1}$.

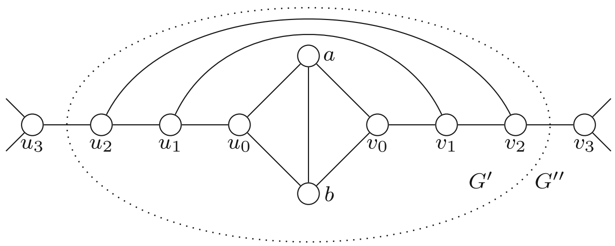

Fig. 2 In this p-rainbow example, $p = 2$ and $u' = u_2$, $u'' = u_3$, $v' = v_2$, and $v'' = v_3$

If G contains a p-rainbow G', $p \geq 1$, then we remove G' and add edge $u''v''$ to the remaining graph G''. Note that G'' is also a simple bridgeless cubic graph. We repeat this until there are no more p-rainbows in G'' for any $p \geq 1$. If the final remaining graph G'' has at least 6 vertices, then assuming G'' has a spanning Eulerian subgraph with at most $4/3|V''| - 2$ edges, we can apply Lemma 3 repeatedly to obtain such a subgraph of length at most $4n/3 - 2$ for the original graph G. If the final remaining graph G'' has less than 6 vertices, then it must have 4 vertices, since it is cubic, hence it forms a complete graph on 4 vertices. In this case we take the Hamilton path from u'' to v'' in G'' and match it with the Hamilton path of the p-rainbow that goes from u_p to v_p to obtain a Hamilton cycle of the graph G'' with the edge $u''v''$ replaced by the p-rainbow. We can then apply Lemma 3 repeatedly to obtain a spanning Eulerian subgraph of G with at most $4n/3-2$ edges. Note that removing all the rainbows can be done in $O(n \log n)$ time, since given any edge of the graph one can check in time linear in the size of the rainbow if the edge is contained in a rainbow. Also, constructing the tour in G given the tour in G'' takes no more than $O(n \log n)$ time.

Proof of Theorem 1 By the above discussion, we assume that there are no p-rainbow subgraphs in G. By Lemma 1, there exist 3-cut perfect matchings $M_1, \ldots, M_k$ and positive real numbers $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = 1$ and $\frac{1}{3}\chi^E = \sum_{i=1}^{k} \lambda_i(\chi^{M_i})$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be the cycle covers of G corresponding to $M_1, M_2, \ldots M_k$. Since each M_i is a 3-cut perfect matching, each $\mathcal{C}_i$ intersects each 3-cut of G in exactly 2 edges, and hence contains neither a 3-cycle nor a 5-cycle with a chord.

If some $\mathcal{C}_i$ has no more than $n/6$ cycles, then we are done, by the argument given earlier. Otherwise we manipulate each of the cycle covers by operations (i) and (ii) below, which we will show to be well-defined. First operation (i) will be performed as long as possible. Then operation (ii) will be performed as long as possible.

(i) If two cycles C_i and C_j of the cycle cover intersect a (chordless) cycle C of length 4 in G (the original graph) then combine them into a single cycle on $V(C_i) \cup V(C_j)$. (See Fig. 3 for an illustration.)

The details of operation (i) are as follows: Assume that u_1u_2 and v_1v_2 are matching edges on C and u_1v_1 is an edge of C_i and u_2v_2 is an edge of C_j. Deleting the latter two edges and inserting the former two yields a single cycle of length equal to the sum of the lengths of C_i and C_j. Notice that operation (i) always leads to cycles of length at least 8. Hence after operation (i) is finished we still have a cycle cover. Operation (ii) below combines cycles into Eulerian subgraphs and subsequently Eulerian subgraphs

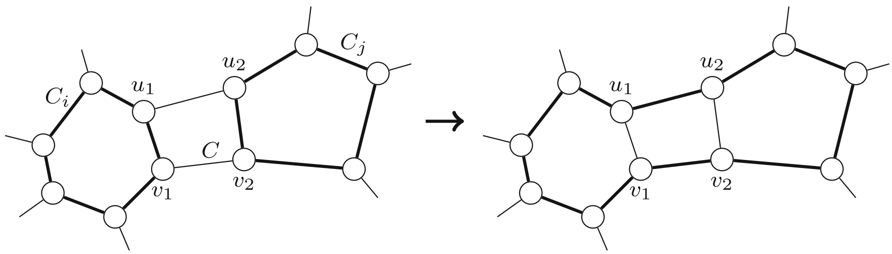

Fig. 3 Operation (i)

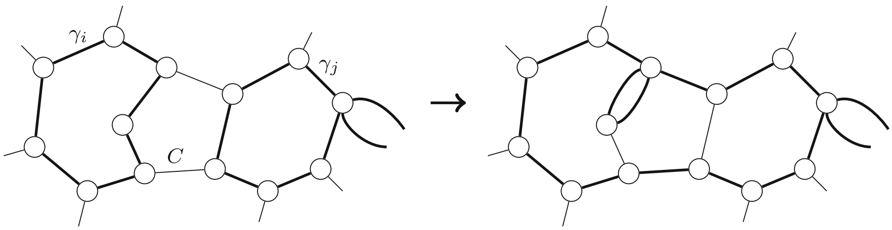

Fig. 4 Operation (ii)

into larger Eulerian subgraphs, turning the cycle covers into Eulerian subgraph covers. Both types of cover we call simply a *cover* and their elements (cycles and Eulerian subgraphs) we call *components*.

(ii) If two components γ_i and γ_j of the cycle cover or the Eulerian subgraph cover, each having at least 5 vertices, intersect a (chordless) cycle C of length 5 in G (the original graph) then combine them into a single Eulerian subgraph where the number of edges is 1 plus the number of edges of γ_i and γ_j. (See Fig. 4 for an illustration.)

The details of operation (ii) are as follows. First note that for any cycle C, its vertex set $V(C)$ has the following (trivial) property:

$\mathcal{P}$: Each $v \in V(C)$ has at least two other vertices $u, w \in V(C)$ such that $vu \in E$ and $vw \in E$.

If two vertex sets both satisfy $\mathcal{P}$ then their union also satisfies $\mathcal{P}$. Since the vertex set of each component γ constructed by operations (i) or (ii) is a result of taking unions of vertex sets of cycles, each such γ has property $\mathcal{P}$. In particular, since G is cubic, this implies that the two components γ_i and γ_j share 2 and 3 vertices with C, respectively (note that they cannot each share exactly 2 vertices, as this would imply that a vertex of C is not included in the cover). We first merge γ_1 and C as in Lemma 2 and remove 2 edges, and then merge the result with γ_2, again removing 2 edges. Altogether we added the 5 edges of C and removed 4 edges.

Operation (ii) leads to Eulerian subgraphs with at least 10 vertices. Thus, any Eulerian subgraph with at most 9 vertices is a cycle. At the completion of operations (i) and (ii), let the resulting Eulerian subgraph covers be $\Gamma_1, \ldots, \Gamma_k$.

Given $\Gamma_1, \ldots, \Gamma_k$, we bound for each vertex its average contribution to the number of edges in the Eulerian subgraphs weighted by the λ_i's. We define the *contribution* of a vertex v which in cover Γ_i lies on an Eulerian subgraph with ℓ edges and h vertices as $z_i(v) = \frac{\ell+2}{h}$; the 2 in the numerator is added for the cost of the double edge to connect the component to the others in final spanning Eulerian subgraph. Note that $\sum_{v \in V} z_i(v)$ is equal to the number of edges in a spanning Eulerian subgraph which results from Γ_i, plus 2. Let $z(v) = \sum_i \lambda_i z_i(v)$, then the average number of edges in the resulting Eulerian subgraphs is,

$$-2 + \sum_i \lambda_i \sum_{v \in V} z_i(v) = -2 + \sum_{v \in V} z(v).$$

We will show that $z(v) \leq 4/3$ for all $v \in V$.

Observation 1 *For any vertex v and $i \in \{1, 2, \ldots, k\}$, the contribution $z_i(v)$ is*

(a) at most $\frac{h+2}{h}$, where $h = \min\{t, 10\}$ and v is on a cycle of length at least t after operation (i).
(b) at most 13/10 if operation (ii) was applied to some component containing v.

Proof (Observation 1) Assume that v is on a Eulerian subgraph γ in Γ_i of g vertices. First we prove (b). If operation (ii) was applied to some component containing v, then vertex v was on a cycle of length at least 5 after operation (i). Each application of (ii) adds at least 5 vertices to the component of v. Hence, the number of times that (ii) was applied to the component of v is at most $g/5 - 1$. Since each application adds exactly one edge, the number of edges in γ is at most $g + g/5 - 1$. Hence,

$$z_i(v) \leq \frac{g + g/5 + 1}{g} = \frac{12}{10} + \frac{1}{g} \leq \frac{13}{10}.$$

We use a similar argument to prove (a). Clearly, $g \geq h$. If γ is a cycle then the contribution of v in Γ_i is $(g + 2)/g \leq (h + 2)/h$ and (a) is true. If γ is not a cycle then this Eulerian subgraph was composed by operation (ii) applied to cycles, each of length at least 5 and one of these had length at least h. Hence, the number of these cycles is at most $1 + (g - h)/5$. Since every application of operation (ii) adds one edge extra, the number of edges in γ is at most $g + (g - h)/5$. Hence, since $h \leq 10$,

$$z_i(v) \leq \frac{g + (g - h)/5 + 2}{g} \leq \frac{g + (g - h)/(h/2) + 2}{g} = \frac{h + 2}{h}.$$

□

Note the subtleties in Observation 1: If v is on a cycle of length at least t after operation (i), and $t \leq 10$, then (a) says that $z_i(v)$ is at most $(t + 2)/t$. If $t > 10$, then (a) says that its contribution is at most 12/10. And finally, if t is 5 or 6 and we know that operation (ii) was applied to some component containing v, then (b) allows us to improve the upper bound on $z_i(v)$ to $13/10$ (for other values of t, (b) does not give an improvement).

From now on we fix any vertex v. Suppose that there is no ℓ such that v is on a 4-cycle or a 5-cycle of Γ_ℓ. Then using Observation 1, we have $z_i(v) \leq \max\{8/6, 13/10\} = 4/3$ for every cover Γ_i, and thus $z(v) \leq 4/3$ and we are done.

Now suppose there exists an ℓ such that v is on a 4-cycle C of Γ_ℓ. Then C must be present in $\mathcal{C}_\ell$ as well. First assume that C is chordless in G. Then all four edges adjacent to C are in the perfect matching M_ℓ that corresponds to $\mathcal{C}_\ell$.

Observation 2 *For any pair of vertices on a chordless cycle of G that appears in any $\mathcal{C}_i$, any path between the two that does not intersect the cycle has length at least* 3.

We partition the set $\mathcal{C}_1, \ldots, \mathcal{C}_k$ according to the way the corresponding M_i's intersect the cycle C. Define sets X_0, X_1, X_2 where $X_j = \{i \mid |C \cap M_i| = j\}$ for $j = 0, 1, 2$. Let $x_t = \sum_{i \in X_t} \lambda_i$, $t = 0, 1, 2$. Clearly $x_0 + x_1 + x_2 = 1$. Since each of the four edges adjacent to C receives total weight 1/3 in the matchings, we have that

$4x_0 + 2x_1 = 4/3 \Rightarrow x_0 = 1/3 - x_1/2$. Since each of the edges of C receives total weight 1/3 in the matchings, $x_1 + 2x_2 = 4/3 \Rightarrow x_2 = 2/3 - x_1/2$.

Clearly, for any $i \in X_0$, v lies on cycle C in $\mathcal{C}_i$, and thus by Observation 1(a), $z_i(v) \leq 6/4$, since v lies on a cycle of length at least 4 after operation (i). By Observation 2, for any $i \in X_1$, v lies on a cycle of length at least 6 in $\mathcal{C}_i$, and thus by Observation 1(a), $z_i(v) \leq 8/6$. For any $i \in X_2$, if C is intersected by one cycle in $\mathcal{C}_i$, then this cycle has length at least 8 by Observation 2. If for $i \in X_2$, C is intersected by two cycles of length at least 4 each, then, after performing operation (i), v will be on a cycle of length at least 8. Thus using Observation 1(a) one more time, we obtain

$$\begin{aligned} z(v) &\leq x_0 6/4 + x_1 8/6 + x_2 10/8 \\ &= (1/3 - x_1/2)6/4 + x_1 8/6 + (2/3 - x_1/2)10/8 \\ &= 4/3 + x_1(8/6 - 6/8 - 10/16) = 4/3 - x_1/24 \leq 4/3. \end{aligned}$$

We prove now that $z(v) \leq 4/3$ also if C is a 4-cycle with a chord. Let us call the vertices on the cycle u_0, a, v_0, b, let ab be the chord, and v is any of the four vertices. If $u_0v_0 \in E$, then $G = K_4$ (the complete graph on 4 vertices), contradicting the assumption that $n \geq 6$. Thus edges u_0u_1 and v_0v_1 exist, with $u_1, v_1 \notin C$. Notice that $u_1 \neq v_1$ since otherwise G would contain a bridge, contradicting 2-connectedness. Let C' be the cycle containing v in some cycle cover $\mathcal{C}_i$. If C' does not contain edge u_0u_1 then $C' = C$. If, on the other hand, $u_0u_1 \in C'$ then also $v_0v_1 \in C'$ and $ab \in C'$. Note that $u_1v_1 \notin E$ since otherwise we have a p-rainbow subgraph as in Fig. 2, and we are assuming that we do not have any such subgraphs. Consequently, C' cannot have length exactly 6. It also cannot have length 7 since then a 3-cut with 3 matching edges would occur. Therefore, any cycle containing u_0u_1 has length at least 8. Applying Observation 1(a) twice we conclude that $z(v) \leq 1/3 \cdot 6/4 + 2/3 \cdot 10/8 = 4/3$.

Now assume there exists a (chordless) 5-cycle C containing v in some Γ_ℓ. Note that we can assume that no $w \in C$ is on a 4-cycle of G, otherwise operation (i) would have been applied and the component of v in Γ_ℓ would have size larger than 5. Note further that C is present in $\mathcal{C}_\ell$ as well. The proof for this case is rather similar to the case for the chordless 4-cycle. Let X_j be the set $\{i \mid |C \cap M_i| = j\}$, for $j = 0, 1, 2$. Let $x_t = \sum_{i \in X_t} \lambda_i$, $t = 0, 1, 2$. Again, we have $x_0 + x_1 + x_2 = 1$. Clearly, for any $i \in X_0$, v lies on C in $\mathcal{C}_i$ and for $i \in X_1$, v lies on a cycle of length at least 7 by Observation 2. Hence, by Observation 1(a) we have $z_i(v) \leq 7/5$ for $i \in X_0$ and $z_i(v) \leq 9/7$ for $i \in X_1$. For any $i \in X_2$ there are two possibilities: either C is intersected by one cycle in $\mathcal{C}_i$, which, by Observation 2, has length at least 9, or C is intersected in $\mathcal{C}_i$ by two cycles, say C_1 and C_2. In the first case we have $z_i(v) \leq 11/9$ by Observation 1(a). In the second case, as argued before, we can assume that no $w \in C$ is on a 4-cycle of G. Hence, C_1 and C_2 each have at least 5 vertices and operation (ii) will be applied, unless C_1 and C_2 end up in one large cycle by operation (i). In the first case we apply Observation 1(b) and get $z_i(v) \leq 13/10$, and in the second case we apply Observation 1(a): $z_i(v) \leq 12/10$. Hence, for any $i \in X_2$ we have $z_i(v) \leq \max\{11/9, 12/10, 13/10\} = 13/10$.

$$\begin{aligned} z(v) &\leq x_0 7/5 + x_1 9/7 + x_2 13/10 \\ &\leq x_0 7/5 + x_1 13/10 + x_2 13/10 \\ &= x_0 7/5 + (1 - x_0) 13/10 = 13/10 + x_0 1/10 \\ &\leq 13/10 + 1/30 = 4/3. \end{aligned}$$

□

As previously mentioned, Barahona [4] provides a polynomial-time algorithm which finds a set of at most $7n/2 - 1$ perfect matchings such that $\frac{1}{3}\chi^E$ can be expressed as a convex combination of the incidence vectors of these matchings. This algorithm runs in $O(n^6)$ time. Operations (i) and (ii) can be implemented to run in $O(n)$ time for each of these perfect matching. (Note that operations (i) and (ii) are applied only $O(n)$ time since the number of components is reduced by one at each step.) Hence, given the $7n/2 - 1$-perfect matchings, the running time of our algorithm is $O(n^2)$. As shown in the proof of Lemma 1, these matchings will automatically be 3-cut perfect matchings. Once we have this set of perfect matchings then applying operations (i) and (ii) on the corresponding cycle covers gives at least one tour of length at most $4n/3 - 2$ according to the above theorem. As any tour has length at least n for graph-TSP, we have the following approximation result:

Corollary 1 *For graph-TSP on simple bridgeless cubic graphs there exists a polynomial-time* 4/3*-approximation algorithm.*

As n is a lower bound on the value of SER for graph-TSP it also follows that, as an upper bound, Conjecture 1 is true for this class of problems, i.e.,

Corollary 2 *For graph-TSP on simple bridgeless cubic graphs the integrality gap for SER is at most* 4/3.

We remark that the largest ratio we found so far for $\alpha(TSP)$ on simple bridgeless cubic examples is 7/6 (see Sect. 4).

We extend the analysis to any connected cubic graph by studying bridges. Deleting the bridges of a graph splits it into separate components each of which is either a single vertex or a bridgeless graph where all vertices have degree 3, except for the ones that were incident to a bridge, which have degree 2. We start with a crucial lemma.

Lemma 4 *For a connected bridgeless simple graph consisting of* $k - m$ *vertices of degree* 3 *and* $m \geq 1$ *vertices of degree* 2*, a TSP tour of length at most* $(4/3)k + (2/3)m - 2$ *can be constructed.*

Proof We begin by observing that the theorem is true for $k \leq 6$ and $m = 1$. It is easy to verify that there is only one such a graph and it has 5 vertices and a tour of length $5 < (4/3)5 + 2/3 - 2 = 5 + 1/3$.

Now, suppose the lemma is not true. Then there is a smallest counterexample, i.e., a graph with a minimum number m of vertices of degree 2. Take such a graph $H = (V, E)$ and take any vertex v of degree 2. Let u and w be its neighbours. We distinguish between the cases $uw \notin E$ and $uw \in E$.

First, assume that $uw \notin E$. We delete v (and its incident edges uv and vw) from the graph and add edge uw. The resulting graph has $k - 1$ vertices of which $m - 1$ have degree 2.

If $m \geq 2$ then this graph must have a tour of length at most $(4/3)(k-1) + (2/3)(m-1) - 2 = (4/3)k + (2/3)m - 4$, otherwise we would have a smaller counterexample. The same bound applies if $m = 1$ since then we must have $k \geq 7$ and by Theorem 1 the length of a tour on the resulting cubic graph is bounded by $(4/3)(k-1) - 2 = (4/3)k + (2/3)m - 4$. In either case, suppose that this tour contains uw. Deleting this edge from the tour and inserting the edges uv and vw yields a tour containing v. Now suppose that this tour does not contain uw, then adding the edge uv twice to the tour again incorporates v into the tour. In both cases at most two edges are added to incorporate vertex v, yielding a TSP tour of length at most $(4/3)k + (2/3)m - 4 + 2 = (4/3)k + (2/3)m - 2$, contradicting the fact that we had a counterexample.

Now assume that $uw \in E$. We remove edge uw and add a vertex z to the graph, together with edges uz, vz and wz. The resulting graph has $k + 1$ vertices but only $m - 1$ of them have degree 2. As argued in the first case, it must have a tour of length at most $(4/3)(k+1) + (2/3)(m-1) - 2$. It is easy to verify that at least one edge less is needed for a TSP tour on the graph H without the vertex z. Thus, the resulting TSP tour has length at most $(4/3)(k+1) + (2/3)(m-1) - 2 - 1 = (4/3)k + (2/3)m - 7/3$. Again this contradicts the existence of the counterexample with m vertices of degree 2. Turning this proof in an algorithm is straightforward. First, we make the graph cubic as described in the proof. Then, given a TSP tour in the cubic graph, the proof shows how to find the tour in the original graph. The shrinking and extending operations take only $O(n)$ time in total. □

Theorem 2 *For a connected cubic simple graph on n vertices with b bridges and s vertices incident to more than one bridge, a TSP tour of length at most $(4/3)(n + b - s) - 2$ can be constructed.*

Proof Removing the bridges yields $b + 1$ bridgeless components, s of them being single vertex components. Thus, there are $2b - 3s$ vertices with degree 2. Using Lemma 4, a TSP tour can be constructed of length at most $(4/3)(n - s) + (2/3)(2b - 3s) - 2(b + 1 - s) + 2b = (4/3)(n + b - s) - 2$. □

The proof of Lemma 4 shows constructively how to handle bridges. Since an optimal tour on a graph with b bridges has at least $n+2b-s$ edges, we can extend the corollaries above to graphs with bridges:

Corollary 3 *For graph-TSP on simple cubic graphs, there exists a polynomial-time* 4/3*-approximation algorithm, and the integrality gap for SER is at most* 4/3. □

3 A faster derandomization of the Mömke–Svensson algorithm

As discussed in Sect. 1, Mömke and Svensson [27] present a 1.461-approximation algorithm for graph-TSP on general graphs, and give a bound of $4n/3 - 2/3$ on the graph-TSP tour for all subcubic bridgeless graphs. Their method is different from all previous methods in that it is based on detecting a set of removable edges R. This algorithm is presented as a randomized algorithm, but Mömke and Svensson notice that there is an obvious derandomization. Our contribution is to show that there is an easier way to derandomize the algorithm than stated in [27], giving a considerable reduction in running time of the overall algorithm. To explain our result we need to briefly describe some essential ingredients of the method proposed in [27] for detecting the set R. We will see that this works out particularly nicely for cubic graphs.

3.1 A short description of the Mömke–Svensson algorithm

If $G = (V, E)$ is cubic, then the algorithm in [27] starts with finding an arbitrary depth first search (henceforth DFS) tree T of G using any vertex r as the root. For general graphs, the graph G is first reduced by solving its LP-relaxation and taking the edges of the support only. It is then reduced further by solving a minimum cost circulation problem. This reduction of G is the most complicated part in the analysis in [27] and we do not discuss it here. We denote the reduced graph simply by G and let T be a DFS tree in G. (For the non-cubic case, T is not arbitrary.)

For the moment, consider the edges of T to be directed away from r. The set of remaining edges is denoted by B. They are back edges which are directed towards the root r. By the properties of DFS trees, each back edge $b = xy$ forms a unique directed cycle together with the path from y to x on T. Let t_b be the unique edge in T on that cycle whose tail is incident with the head y of b, and let $T_B = \{t_b : b \in B\}$. The set of removable edges is $R = B \cup T_B$. Each arc $e \in T_B$ is made part of a *pair*: its partner is chosen arbitrarily from amongst the back edges $b \in B$ such that $e = t_b$. Thinking of everything undirected again then notice that, given a pair $\{b \in B, t_b \in T_B\}$, $(T \setminus t_b) \cup b$ forms a different spanning tree of G. In fact, essentially Mömke and Svensson show indirectly that any number of partnered edges t_b and b can be swapped, and the result will still be another spanning tree.

Lemma 5 [27] *Let T_J be a subset of the edges in T_B, with corresponding partner back edges $J \subseteq B$. Let T^* be the result of taking spanning tree T, removing the edges of T_J, and adding the edges of J. Then T^* is also a spanning tree of G.* □

Let $M \subset E$ have the property that for any $b \in B \cap M$ it holds that $t_b \notin M$. Then, deleting all edges in $M \cap R$ from G, Lemma 5 implies that the graph remains connected. If in addition, M has the property that the multigraph with edge set $E \cup M$ is Eulerian, then a spanning Eulerian multigraph is obtained by deleting all edges in $M \cap R$ and doubling all edges in $M \setminus R$. The number of edges is

$$|E| - |M \cap R| + |M \setminus R|. \tag{2}$$

Determining a set $M \subset E$ which satisfies both of these properties and minimizes (2) is easy for (sub)cubic graphs but much harder for general graphs. If G is cubic then any perfect matching M clearly satisfies both properties. For general graphs, any vertex which has a degree unequal to three is replaced by a gadget such that the extended graph G' becomes cubic and 2-connected. Then, any perfect matching M' in G' defines a T-join $M = M' \cap E$ of the odd-degree vertices in G. Moreover, the gadgets used in [27] ensure that for any $b \in B \cap M$ it holds that $t_b \notin M$.

Mömke and Svensson now sample a perfect matching such that each edge is picked in the perfect matching with probability 1/3. Doing so, the expected value of (2) is

$$|E| - |R|/3 + (|E| - |R|)/3 = \tfrac{4}{3}|E| - \tfrac{2}{3}|R|. \tag{3}$$

3.2 Finding the best matching

The running time of Mömke and Svensson's algorithm is dominated by the problem of finding the set of perfect matchings from which to sample. Sampling is done as follows. Let $x^* = \frac{1}{3}\chi^{E'}$, where $\chi^{E'}$ is the vector of $|E'|$ ones. Then x^* can be expressed as a convex combination of incidence vectors of perfect matchings in G'. (We used a stronger statement in Lemma 1.) Barahona's algorithm [4] finds such a set in $O(n^6)$ time. It is not clear if sampling a perfect matching from the set is any easier than finding the set. One can find the matching from this set with minimum value for (2) in $O(n^2)$ time by computing this value for each of the $O(n)$ perfect matchings given by Barahona's algorithm.

Instead, we propose a very simple direct way to find a perfect matching with minimum value for (2) by solving a minimum weight perfect matching problem. We assign weights to the edges in G' based on R:

$$c_e = \begin{cases} -1 & \text{if } e \in R, \\ 1 & \text{if } e \in E\backslash R, \\ 0 & \text{if } e \in E'\backslash E. \end{cases} \tag{4}$$

Since $x^* = \frac{1}{3}\chi^{E'}$ can be expressed as a convex combination of incidence vectors of perfect matchings, there must be a perfect matching M^* of weight at most $c(E')/3$, where $c(E')$ is the total weight of G'. (For a generalization of this idea, see Naddef and Pulleyblank [29], Theorem 4.) Now consider the multigraph H we obtain by taking G, removing the edges of $M^* \cap R$, and doubling the edges of $M^* \cap E\backslash R$. Then H is a spanning Eulerian subgraph of G and the number of edges in H is

$$|E| + c(M^*) \le |E| + c(E')/3 = |E| + (|E| - 2|R|)/3 = \tfrac{4}{3}|E| - \tfrac{2}{3}|R|,$$

which is the same as (3).

If $G = (V, E)$ is a bridgeless (sub)cubic multigraph (i.e., all vertices in G have degree 2 or 3) with n_2 vertices of degree 2 and n_3 vertices of degree 3, then $|E| = (2n_2 + 3n_3)/2$ and $|R| = 2(|E| - |T|) - 1 = n_3 + 1$. The number of edges in the Eulerian subgraph is bounded by

$$\frac{4}{3}|E| - \frac{2}{3}|R| = \frac{4}{3}n - \frac{2}{3}.$$

Instead of using a gadget for vertices of degree 2 we can replace each maximal path of degree-two vertices by a single edge, obtaining a reduced cubic graph $G' = (V', E')$, and simply use the algorithm for the cubic case, i.e., with the DFS tree and set of removable edges R defined as above, but with weighted edges. We assign a weight of $q_e - 1$ for each edge $e \in R$ and weight $q_e + 1$ for each edge $e \in E'\backslash R$, where q_e is the number of degree-two vertices that the edge e represents.

The running time of the algorithm described above is dominated by the time required to find a minimum cost perfect matching. This step can be performed in $O(n(|E| + n\log n))$ time (see [17]), which is $O(n^2 \log n)$ for cubic graphs.

Theorem 3 *Let $G = (V, E)$ be a bridgeless subcubic multigraph with n vertices. There is a $O(n^2 \log n)$ algorithm that finds a spanning Eulerian multi-subgraph H of G with at most $4n/3 - 2/3$ edges.*

We conclude this section with two interesting theorems that follow quite easily from the work of Mömke and Svensson [27]. Note that a restriction of their algorithm is that all edges that are removed from the graph are taken from a perfect matching. This gives the following nice corollary.

Theorem 4 *Any connected bridgeless cubic multigraph has a cycle cover with at most $\lfloor n/6 + 2/3 \rfloor$ cycles.*

Proof Let G be a connected bridgeless cubic multigraph with removable set R. In the analysis above we have seen that there is a perfect matching M with $|M \cap R| \geq |R|/3$. Removing $M \cap R$ from the graph leaves a connected graph. Hence, removing the whole set M leaves a cycle cover for which the number of components (cycles) is at most

$$|M| - |M \cap R| + 1 \leq n/2 - |R|/3 + 1 = n/2 - (n+1)/3 + 1 = n/6 + 2/3.$$

□

Given a cycle cover $\mathcal{C}$ of a connected graph, we can find a spanning tree with at most $|\mathcal{C}|$ leaves as follows. First, add $|\mathcal{C}| - 1$ edges to connect the cycles. Then, in each cycle remove one edge that is adjacent to a connecting matching edge. The result is a spanning tree with at most $|\mathcal{C}|$ leaves.

Theorem 5 *Any connected bridgeless cubic multigraph has a spanning tree with at most $\lfloor n/6 + 2/3 \rfloor$ leaves.*

4 Epilogue

Very recently, remarkable progress has been made on the approximability of graph-TSP. In the table below we show the present state of knowledge. It contains: (1st column) lower bounds on the length of graph-TSP tours on n vertices, for n large enough, (2nd column) upper bounds which we know how to construct, (3rd column) lower bounds on the integrality gap of SER, (4th column) upper bounds on the integrality gap of SER, and (last column) upper bounds on the best possible approximation ratio. The bounds apply to bridgeless graphs, because they are the crucial ones within the classes. All lower bounds hold for simple graphs.

	TSP l.b.	TSP u.b.	SER l.b.	SER u.b.	Approx.
General graphs	$2n - 4$	$2n - 2$	4/3	7/5	7/5
Subcubic graphs	$4n/3 - 2/3$	$4n/3 - 2/3$	4/3	4/3	4/3
Cubic graphs	$11n/9 - 8/9$	$4n/3 - 2$	7/6	4/3	4/3

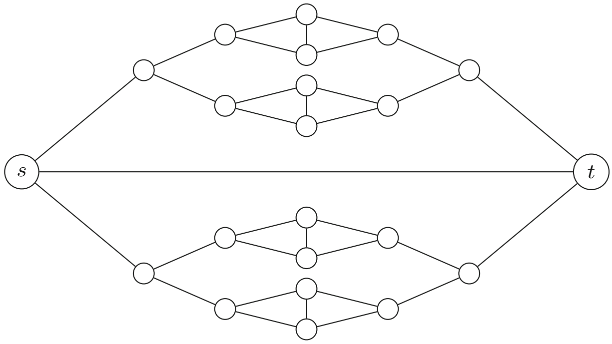

Fig. 5 Family of cubic graphs for which the optimal graph-TSP tour has length $11n/9 - 8/9$

The lower bound for general graphs in the first column is given by the complete bipartite graph $K_{2,n-2}$ (on 2 and $n-2$ vertices). The lower bound for subcubic graphs is well-known and given by a graph on $n = 3k+2$ vertices consisting of 3 paths of $k+1$ edges each with common endpoints. The optimal tour has length $4k+2 = 4n/3-2/3$. The cubic lower bound we prove in Lemma 6. Notice that if we do not restrict to simple graphs then the subcubic lower bound gives a lower bound of $4n/3-2/3$ for the cubic case by duplicating edges on the paths alternately.

Lemma 6 *For any n_1 there is a simple cubic bridgeless graph on $n > n_1$ vertices such that the optimal tour has length at least* $11n/9 - 8/9$.

Proof Take two complete binary trees and connect their leaves as in Fig. 5 and add an edge between the two roots s and t. Let $2k+2$ be the distance from s to t not using edge st. Denote the corresponding graph by F_k. The example shows F_2. In general $k \geq 1$ and $n = 6 \cdot 2^k - 2$. Now let us compute an optimal TSP tour. Let $T(k)$, $P(k)$ be the length of the shortest connected Eulerian subgraph in F_k using edge st, respectively 0 and 1 times. Then, $T(1) = 10$ and $P(1) = 12$. Consider a minimum spanning connected Eulerian subgraph in F_k. If it does not contain edge st, then the Eulerian subgraph either contains exactly one copy of each of the four edges incident to st or three of these four edges doubled. In the first case $T(k) = 4+2(P(k-1)-1)$ and in the latter we have $T(k) = 6+2T(k-1)$. Hence, $T(k) = \min\{6+2T(k-1), 2+2P(k-1)\}$.

If the Eulerian subgraph does contain edge st then it is easy to see that $P(k) = 5 + T(k-1) + (P(k-1) - 1) = 4 + T(k-1) + P(k-1)$. Given the initial values $T(1) = 10$ and $P(1) = 12$ the values that follow from these equations are uniquely defined. One may verify that the following functions satisfy the equations.

$$T(k) = 22/3 \cdot 2^k - 14/3,\ P(k) = 22/3 \cdot 2^k - 8/3 \text{ for odd } k,$$
$$T(k) = 22/3 \cdot 2^k - 10/3,\ P(k) = 22/3 \cdot 2^k - 10/3 \text{ for even } k.$$

For even k the length of the optimal tour is $22/3 \cdot 2^k - 10/3 = 11n/9 - 8/9$. □

Recently, Correa et al. [13] showed that any 2-connected cubic graphs has a tour of length at most $(4/3 - \epsilon)n$ for some $\epsilon > 0$. Hence, the integrality gap of SER is strictly less than 4/3. We believe that graph-TSP is APX-hard also for cubic graphs.

The lower bound of 7/6 on the integrality gap for cubic graphs is attained by the following graph. Connect two points by three equally long paths. Then replace every vertex of degree 2 by a 4-cycle with a chord so as to make the graph cubic.

Of course, the main research challenges remain to prove Conjecture 1 or to show a 4/3-approximation algorithm. For general metric TSP even an approximation ratio strictly less than 3/2 is still wide open. For graph-TSP, Mömke and Svensson [26,27] and Sebö and Vygen [33] have made promising and important steps.

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