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Induced Circuits in Graphs on Surfaces

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ABSTRACT. We show that for any fixed surface S there exists a polynomial-time algorithm to test if there exists an induced circuit traversing two given vertices r and s of an undirected graph G embedded on S. (An *induced circuit* is a circuit without chords.) The general problem (not fixing S) is NP-complete. In fact, for each fixed surface S there exists a polynomial-time to find a maximum number of $r-s$ paths in G such that any two form an induced circuit.

1. Introduction

In this paper we show that the following problem is solvable in polynomial time, for any fixed compact surface S:

(1) given: an undirected graph $G=(V,E)$ embedded on S and two vertices r and s of G;
find: an induced circuit in G that traverses r and s.

An *induced circuit* is a circuit having no chords. The problem is NP-complete for general undirected graphs, as was shown by Bienstock [1]. In [2] the problem was shown to be solvable in polynomial time for planar graphs. In fact we show that for any fixed compact surface S the problem:

(2) given: an undirected graph $G=(V,E)$ embedded on S and two vertices r and s of G;
find: a maximum number of $r-s$ paths in G any two of which form an induced circuit;

is solvable in polynomial time.

Our method uses a variant of a method developed in [3] to derive, for any *fixed k*, a polynomial-time algorithm for the k disjoint paths problem in directed

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planar graphs. (This problem is NP-complete for general directed graphs, even for $k = 2$.) The present method is based on cohomology over free boolean groups.

2. Free boolean groups

The *free boolean group* B_k is the group generated by $g_1, g_2, \ldots, g_k$, with relations $g_j^2 = 1$ for $j = 1, \ldots, k$. So B_k consists of all words $b_1 b_2 \ldots b_t$ where $t \geq 0$ and $b_1, \ldots, b_t \in \{g_1, \ldots, g_k\}$ such that $b_i \neq b_{i-1}$ for $i = 2, \ldots t$. The product $x \cdot y$ of two such words is obtained from the concatenation xy by deleting iteratively all occurrences of any pair $g_j g_j$. This defines a group, with unit element 1 equal to the empty word $\emptyset$.

We call $g_1, \ldots, g_k$ *generators* or *symbols.* Note that

$$B_1 \subset B_2 \subset B_3 \subset \cdots. \tag{3}$$

The *size* $|x|$ of a word x is the number of symbols occurring in it, counting multiplicities. A word y is called a *segment* of word w if $w = xyz$ for certain words x, z. If $w = yz$ for some word z, y is called a *beginning segment* of w, denoted by $y \leq w$. This partial order gives trivially a lattice if we extend B_k with an element ∞ at infinity. Denote the meet and join by $\wedge$ and $\vee$.

We prove two useful lemmas.

LEMMA 1. *For all $x, y, z \in B_k$ one has:*

$$x \leq y \cdot z \text{ and } z \leq y^{-1} \cdot x \Longleftrightarrow x^{-1} \cdot y \cdot z = 1 \text{ or } y = xwz^{-1} \text{ for some word } w. \tag{4}$$

Proof. $\Longleftarrow$ being easy, we show $\Longrightarrow$. Let $w := x^{-1} \cdot y \cdot z$. As $x \leq y \cdot z$, $y \cdot z = xw$; and as $z \leq y^{-1} \cdot x$, $y^{-1} \cdot x = zw^{-1}$, that is, $x^{-1} \cdot y = wz^{-1}$. Hence if $w \neq 1$ then $xwz^{-1} = x \cdot w \cdot z^{-1} = y$. ∎

LEMMA 2. *Let $x, y \in B_k$. If $x \not\leq y$ then the first symbol of x^{-1} is equal to the first symbol of $x^{-1} \cdot y$.*

Proof. Let $z := x \wedge y$. So $x^{-1} \cdot y$ is the concatenation of $x^{-1} \cdot z$ and $z^{-1} \cdot y$. Since $x^{-1}z \neq 1$, the first symbol of $x^{-1} \cdot y$ is equal to the first symbol of $x^{-1} \cdot z$. Since $x^{-1}z \neq 1$ and $z \leq x$, the first symbol of $x^{-1} \cdot z$ is equal to the first symbol of x^{-1}. Hence the first symbol of x^{-1} is equal to the first symbol of $x^{-1} \cdot y$. ∎

3. The cohomology feasibility problem for free boolean groups

Let $D = (V, A)$ be a weakly connected directed graph, let $r \in V$, and let $(G, \cdot)$ be a group. Two functions $\phi, \psi : A \longrightarrow G$ are called *r-cohomologous* if there exists a function $f : V \longrightarrow G$ such that

(5) (i) $f(r) = 1$;
(ii) $\psi(a) = f(u)^{-1} \cdot \phi(a) \cdot f(w)$ for each arc $a = (u, w)$.

This clearly gives an equivalence relation.

Consider the following *cohomology feasibility problem (for free boolean groups)*:

(6) given: a weakly connected directed graph $D = (V, A)$, a vertex r, and a function $\phi : A \longrightarrow B_k$;

find: a function $\psi : A \longrightarrow B_k$ such that ψ is r-cohomologous to ϕ and such that $|\psi(a)| \leq 1$ for each arc a (if there is one).

We give a polynomial-time algorithm for this problem. The running time of the algorithm is bounded by a polynomial in $|A| + \sigma + k$, where σ is the maximum size of the words $\phi(a)$ (without loss of generality, $\sigma \geq 1$).

We may assume that with each arc $a = (u, w)$ also $a^{-1} := (w, u)$ is an arc of D, with $\phi(a^{-1}) = \phi(a)^{-1}$.

Note that, by the definition of r-cohomologous, equivalent to finding a ψ as in (6), is finding a function $f : V \longrightarrow B_k$ satisfying:

(7) (i) $f(r) = 1$;

(ii) for each arc $a = (u, w)$: $|f(u)^{-1} \cdot \phi(a) \cdot f(w)| \leq 1$.

We call such a function f *feasible.*

It turns out to be useful to introduce the concept of 'pre-feasible' function. A function $f : V \longrightarrow B_k$ is *pre-feasible* if

(8) (i) $f(r) = 1$;

(ii) for each arc $a = (u, w)$: if $|f(u)^{-1} \cdot \phi(a) \cdot f(w)| > 1$ then $\phi(a) = f(u)yf(w)^{-1}$ for some word y.

Pre-feasibility behaves nicely with respect to the partial order $\leq$ on the set B_k^V of all functions $f : V \longrightarrow B_k$ induced by the partial order $\leq$ on B_k as: $f \leq g \Leftrightarrow f(v) \leq g(v)$ for each $v \in V$. It is easy to see that B_k^V forms a lattice if we add an element ∞ at infinity. Let $\wedge$ and $\vee$ denote the meet and join. Then:

PROPOSITION 1. *If f_1 and f_2 are pre-feasible, then so is $f := f_1 \wedge f_2$.*

Proof. Clearly $f(r) = 1$. Suppose $|f(u)^{-1} \cdot \phi(a) \cdot f(w)| > 1$ for some arc $a = (u, w)$. We show $\phi(a) = f(u)yf(w)^{-1}$ for some y. By (4) we may assume by symmetry that $f(u) \not\leq \phi(a) \cdot f(w)$. Since $f(w) = f_1(w) \wedge f_2(w)$, there is an $i \in \{1, 2\}$ such that $f(u)^{-1} \cdot \phi(a) \cdot f_i(w)$ contains $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ as a beginning segment. Without loss of generality, $i = 1$. So $|f(u)^{-1} \cdot \phi(a) \cdot f_1(w)| > 1$. As $f(u) \not\leq \phi(a) \cdot f(w)$, by Lemma 2, the first symbols of $f(u)^{-1}$ and $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ are equal. Since $f(u)^{-1} \cdot \phi(a) \cdot f(w) \leq f(u)^{-1} \cdot \phi(a) \cdot f_1(w)$, it follows that the first symbols of $f(u)^{-1}$ and $f(u)^{-1} \cdot \phi(a) \cdot f_1(w)$ are equal. So $f_1(u)^{-1} \cdot \phi(a) \cdot f_1(w)$ contains $f(u)^{-1} \cdot \phi(a) \cdot f_1(w)$ as segment. Hence $|f_1(u)^{-1} \cdot \phi(a) \cdot f_1(w)| > 1$. As f_1 is pre-feasible, $\phi(a) = f_1(u)y'f_1(w)^{-1}$ for some y'. Since $f(u) \leq f_1(u)$ and $f(w) \leq f_1(w)$ this implies $\phi(a) = f(u)yf(w)^{-1}$ for some y. ∎

So for any function $f : V \longrightarrow B_k$ there exists a unique smallest pre-feasible function $\bar{f} \geq f$, provided there exists at least one pre-feasible function $g \geq f$. If no such g exists we set $\bar{f} := \infty$. In the next section we show that $\bar{f}$ can be found in polynomial time for any given f.

We first note:

PROPOSITION 2. *If $\bar{f}$ is finite then*

$$(9) \quad \begin{array}{ll} \text{(i)} & f(r) = 1; \\ \text{(ii)} & |f(v)| < (\sigma+1)|V| \textit{ for each vertex } v; \\ \text{(iii)} & f(u) \leq \phi(a) \cdot f(w) \textit{ or } f(w) \leq \phi(a)^{-1} \cdot f(u) \textit{ for each} \\ & \textit{arc } a = (u,w) \textit{ with } |f(u)^{-1} \cdot \phi(a) \cdot f(w)| > 1. \end{array}$$

Proof. Let $\bar{f}$ be finite. Trivially $f(r) \leq \bar{f}(r) = 1$. Moreover, let $a_1, \ldots, a_t$ form a simple path from r to v. By induction on t one shows $|\bar{f}(v)| \leq (\sigma+1)t$. (Indeed, let $a_t = (u,v)$. If $|\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(v)| \leq 1$ then by induction $|\bar{f}(u)| \leq (\sigma+1)(t-1)$, and hence $|\bar{f}(v)| \leq \bar{f}(u)| + |\phi(a)| + 1 \leq (\sigma+1)t$. If $|\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(v)| > 1$ then by (8) $\bar{f}(v)$ is a segment of $\phi(a)$ and hence $|\bar{f}(v)| \leq \sigma \leq (\sigma+1)t$.) So $|f(v)| \leq |\bar{f}(v)| < (\sigma+1)|V|$.

To see (iii), assume that $f(u) \not\leq \phi(a) \cdot f(w)$ and $f(w) \not\leq \phi(a^{-1}) \cdot f(u)$. So by Lemma 2 the first symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the first symbol of $f(u)^{-1}$. Similarly, the last symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the last symbol of $f(w)$. Since $f(u) \leq \bar{f}(u)$ and $f(w) \leq \bar{f}(w)$, it follows that $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is a segment of $\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)$. So $|\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)| > 1$. As $\bar{f}$ is pre-feasible this implies that $\phi(a) = \bar{f}(u)y\bar{f}(w)^{-1}$ for some y. Hence, since $f \leq \bar{f}$, $\phi(a) = f(u)y'f(w)^{-1}$ for some y'. So $f(u) \leq f(u)y' = \phi(a) \cdot f(w)$, contradicting our assumption. ∎

4. A subroutine finding $\bar{f}$

Let input $D = (V, A), r, \phi$ for the cohomology feasibility problem (6) be given. We may assume that for any arc $a = (u, w)$, $a^{-1} = (w, u)$ is also an arc of D, with $\phi(a^{-1}) = \phi(a)^{-1}$. Let moreover $f : V \longrightarrow B_k$ be given.

If f is pre-feasible output $\bar{f} := f$. If f violates (9) output $\bar{f} := \infty$. If none of these applies, perform the following iteration:

Iteration: Choose an arc $a = (u, w)$ satisfying $|f(u)^{-1} \cdot \phi(a) \cdot f(w)| > 1$ and $f(w) \not\leq \phi(a)^{-1} \cdot f(u)$. (Such an arc exists by (4). As (9)(iii) is not violated, we know $f(u) \leq \phi(a) \cdot f(w)$.)

Let x be obtained from $\phi(a) \cdot f(w)$ by deleting the last symbol; reset $f(u) := x$,

and iterate.

PROPOSITION 3. *At each iteration, $\sum_v |f(v)|$ strictly increases.*

Proof. Since $f(u) \leq \phi(a) \cdot f(w)$ and $|f(u)^{-1} \cdot \phi(a) \cdot f(w)| > 1$, x is strictly larger than the original $f(u)$. ∎

This directly implies:

PROPOSITION 4. *After at most $(\sigma + 1)|V|^2$ iterations the subroutine stops.*

Proof. After $(\sigma + 1)|V|^2$ iterations, by Proposition 3 there exists a vertex u such that $|f(u)| \geq (\sigma + 1)|V|$. Then (9)(ii) is violated. ∎

Moreover we have:

PROPOSITION 5. *In the iteration, resetting f does not change $\bar{f}$.*

Proof. We must show that $x \leq \bar{f}(u)$ if $\bar{f}$ is finite. If there exists y such that $\phi(a) = \bar{f}(u) y \bar{f}(w)^{-1}$ then

$$f(w) \leq \bar{f}(w) \leq \bar{f}(w) y^{-1} = \phi(a)^{-1} \cdot \bar{f}(u) \leq \phi(a)^{-1} \cdot f(u) \tag{10}$$

(since $f(u) \leq \bar{f}(u) \leq \phi(a)$). This contradicts the choice of a in the iterations. Therefore, since $\bar{f}$ is pre-feasible, we know $|\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)| \leq 1$.

Since $f(w) \not\leq \phi(a^{-1}) \cdot f(u)$, by Lemma 2 the last symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the last symbol of $f(w)$. Hence (since $f(w) \leq \bar{f}(w)$) $f(u)^{-1} \cdot \phi(a) \cdot f(w) \leq f(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)$. Since $f(u) \leq \phi(a) \cdot f(w)$ it follows that $\phi(a) \cdot f(w) \leq \phi(a) \cdot \bar{f}(w)$. Let y be obtained from $\phi(a) \cdot \bar{f}(w)$ by deleting the last symbol. Then $x \leq y \leq \bar{f}(u)$, since $|\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)| \leq 1$. ∎

5. Algorithm for the cohomology feasibility problem

Let input $D = (V, A), r, \phi$ for the cohomology feasibility problem (6) be given. Again we may assume that for each arc $a = (u, w)$, $a^{-1} = (w, u)$ is also an arc, with $\phi(a^{-1}) = \phi(a)^{-1}$. We find a feasible function f (if there is one) as follows.

Let W be the set of pairs (v, x) with $v \in V$ and $x \in B_k$ such that there exists an arc $a = (v, w)$ with $1 \neq x \leq \phi(a)$. For every $(v, x) \in W$ let $f_{v,x}$ be the function defined by: $f_{v,x}(v) := x$ and $f_{v,x}(v') := 1$ for each $v' \neq v$. Let E be the set of pairs $\{(v, x), (v', x')\}$ from W for which $\bar{f}_{v,x} \vee \bar{f}_{v',x'}$ is finite and pre-feasible. Let E' be the set of pairs $\{(u, x), (w, z)\}$ from W for which there is an arc $a = (u, w)$ with $\phi(a) = xz^{-1}$. We search for a subset X of W such that each pair in X belongs to E and such that X intersects each pair in E'. This is a special case of the 2-satisfiability problem, and hence can be solved in polynomial time.

PROPOSITION 6. *If X exists then the function $f := \bigvee_{(v,x) \in X} \bar{f}_{v,x}$ is feasible. If X does not exist then there is no feasible function.*

Proof. First assume X exists. Since $\bar{f}_{v,x} \vee \bar{f}_{v',x'}$ is finite and pre-feasible for each two $(v, x), (v', x')$ in X, f is finite and $f(r) = 1$. Moreover, suppose $|f(u)^{-1} \cdot \phi(a) \cdot$

$f(w)| > 1$ for some arc $a = (u, w)$. By definition of f there are $(v, x), (v', x') \in X$ such that $f(u) = \bar{f}_{v,x}(u)$ and $f(w) = \bar{f}_{v',x'}(w)$ for $(v, x), (v', x') \in X$. As $\bar{f}_{v,x} \vee \bar{f}_{v',x'}$ is pre-feasible, $\phi(a) = \bar{f}_{v,x}(u) y \bar{f}_{v',x'}(w)^{-1}$ for some y. Then $|y| > 1$. Split $y = bc^{-1}$ with b and c nonempty. Then $(u, f(u)b) \in X$ or $(w, f(w)c) \in X$ since X intersects each pair in E'. If $(u, f(u)b) \in X$ then $f(u)b = f_{u,f(u)b}(u) \leq \bar{f}_{u,f(u)b}(u) \leq f(u)$, a contradiction. If $(w, f(w)c) \in X$ one obtains similarly a contradiction.

Assume conversely that there exists a feasible function f. Let X be the set of pairs $(v, x) \in X$ with the property that $x \leq f(v)$. Then X intersects each pair in E'. For suppose that for some arc $a = (u, w)$ with $\phi(a) = xz^{-1}$ and $x \neq 1 \neq z$, one has $(u, x) \notin X$ and $(w, z) \notin X$, that is, $x \not\leq f(u)$ and $z \not\leq f(w)$. This however implies $|f(u)^{-1} \cdot \phi(a) \cdot f(w)| \geq 2$, a contradiction.

Moreover, each pair in X belongs to E. For let $(v, x), (v', x') \in X$. We show that $\{(v, x), (v', x')\} \in E$, that is, $f' := \bar{f}_{v,x} \vee \bar{f}_{v',x'}$ is pre-feasible. As $\bar{f}_{v,x} \leq f$ and $\bar{f}_{v',x'} \leq f$, f' is finite and $f'(r) = 1$. Consider an arc $a = (u, w)$ with $|f'(u)^{-1} \cdot \phi(a) \cdot f'(w)| > 1$. We may assume $f'(u) = \bar{f}_{v,x}(u)$ and $f'(w) = \bar{f}_{v',x'}(w)$ (since $\bar{f}_{v,x}$ and $\bar{f}_{v',x'}$ themselves are pre-feasible). To show $\phi(a) = f'(u) y f'(w)^{-1}$ for some y, by (4) we may assume $f'(w) \not\leq \phi(a^{-1}) \cdot f'(u)$. So by Lemma 2, the last symbol of $f'(u)^{-1} \cdot \phi(a) \cdot f'(w)$ is equal to the last symbol of $f'(w)$.

Suppose now that $f'(u) \not\leq \phi(a) \cdot f'(w)$. Then by Lemma 2, the first symbol of $f'(u)^{-1} \cdot \phi(a) \cdot f'(w)$ is equal to the first symbol of $f'(u)^{-1}$. Since $f' \leq f$ this implies that $f'(u)^{-1} \cdot \phi(a) \cdot f'(w)$ is a segment of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$. This contradicts the fact that $|f(u)^{-1} \cdot \phi(a) \cdot f(w)| \leq 1$.

So $f'(u) \leq \phi(a) \cdot f'(w)$. As $f_{v',x'}(u) \leq f'(u)$ and $|f'(u)^{-1} \cdot \phi(a) \cdot f'(w)| > 1$ it follows that $|\bar{f}_{v',x'}(u)^{-1} \cdot \phi(a) \cdot f'(w)| > 1$. As $f'(w) = \bar{f}_{v',x'}(w)$ we have $|\bar{f}_{v',x'}(u)^{-1} \cdot \phi(a) \cdot \bar{f}_{v',x'}(w)| > 1$. As $\bar{f}_{v',x'}$ is pre-feasible, $\phi(a) = \bar{f}_{v',x'}(u) y \bar{f}_{v',x'}(w)^{-1}$ for some y. So $f'(u) \leq \phi(a) \cdot f'(w) = \bar{f}_{v',x'}(u) y$. Hence $\bar{f}_{v',x'}(u) y = f'(u) y'$ for some y'. It follows that $\phi(a) = f'(u) y' f'(w)^{-1}$. ∎

Thus we have:

THEOREM 1. *The cohomology feasibility problem for free boolean groups is solvable in time bounded by a polynomial in* $|A| + \sigma + k$.

6. Graphs on surfaces and homologous functions

Let $G = (V, E)$ be an undirected graph embedded in a compact surface. For each edge e of G choose arbitrarily one of the faces incident with e as the *left-hand face* of e, and the other as the *right-hand face* of e. (They might be one and the same face.) Let $\mathcal{F}$ denote the set of faces of G, and let R be one of the faces of G. We call two functions $\phi, \psi : E \longrightarrow B_k$ *R-homologous* if there exists a function $f : \mathcal{F} \longrightarrow B_k$ such that

(11) (i) $f(R) = 1$;

(ii) $f(F)^{-1} \cdot \phi(e) \cdot f(F') = \psi(e)$ for each edge e, where F and F' are the left-hand and right-hand face of e respectively.

The relation to cohomologous is direct by duality. The *dual* graph $G^* = (\mathcal{F}, E^*)$ of G has as vertex set the collection $\mathcal{F}$ of faces of G, while for any edge e of G there is an edge e^* of G^* connecting the two faces incident with e. Let D^* be the directed graph obtained from G^* by orienting each edge e^* from the left-hand face of e to the right-hand face of e. Define for any function ϕ on E the function ϕ^* on E^* by $\phi^*(e^*) := \phi(e)$ for each $e \in E$. Then ϕ and ψ are R-homologous (in G), if and only if ϕ^* and ψ^* are R-cohomologous (in D^*).

7. Enumerating homology classes

Let $G = (V, E)$ be an undirected graph embedded in a surface and let $r, s \in V$, such that no loop is attached at r or s. We call a collection $\Pi = (P_1, \dots, P_k)$ of $r - s$ walks an $r - s$ *join* (of *size* k) if:

(12) (i) each P_i traverses r and s only as first and last vertex respectively;

(ii) each edge is traversed at most once by the $P_1, \dots, P_k$;

(iii) P_i does not cross itself or any of the other P_j;

(iv) $P_1, \dots, P_k$ occur in this order cyclically at r.

Note that any solution of (2) can be assumed to be an $r - s$ join.

For any $r - s$ join $\Pi = (P_1, \dots, P_k)$ let $\phi_\Pi : E \longrightarrow B_k$ be defined by:

(13)
$$\begin{aligned} \phi_\Pi(e) &:= g_i \quad \text{if walk } P_i \text{ traverses } e \ (i = 1, \dots, k); \\ &:= 1 \quad \text{if } e \text{ is not traversed by any of the } P_i. \end{aligned}$$

Let R be one of the faces of G. Note that if ϕ is R-homologous to ϕ_Π then for each vertex $v \neq r, s$ we have

(14)
$$\phi(e_1)^{\varepsilon_1} \cdot \ldots \cdot \phi(e_t)^{\varepsilon_t} = 1,$$

where $F_0, e_1, F_1, \dots, F_{t-1}, e_t, F_t$ are the faces and edges incident with v in cyclic order (with $F_t = F_0$), and where $\varepsilon_j := +1$ if F_{j-1} is the left-hand face of e_j and F_j is the right-hand face of e_j, and $\varepsilon_j := -1$ if F_{j-1} is the right-hand face of e_j and F_j is the left-hand face of e_j. (If $F_{j-1} = F_j$ we should be more careful.) This follows from the fact that (14) holds for $\phi = \phi_\Pi$ and that (14) is invariant for R-homologous functions.

We now consider the following problem:

(15) given: a connected undirected graph cellularly embedded on a surface S, vertices r, s of G, such that $G-\{r,s\}$ is connected and r and s are not connected by an edge, a face R of G, and a natural number k;

find: functions $\phi_1, \ldots, \phi_N : E \longrightarrow B_k$ such that for each $r-s$ join Π of size k, ϕ_Π is R-homologous to at least one of $\phi_1, \ldots, \phi_N$.

(A graph is *cellularly embedded* if each face is homeomorphic with an open disk.)

THEOREM 2. *For any fixed surface S, problem* (15) *is solvable in time bounded by a polynomial in $|V|+|E|$.*

Proof. If e is any edge connecting two different vertices $\neq r, s$, we can contract e. Any solution of (15) for the modified graph directly yields a solution for the original graph (by (14)). So we may assume $V = \{r, s, v\}$ for some vertex v. Similarly, we may assume that G has no loops that bound an open disk.

Call two edges *parallel* if and only if they form the boundary of an open disk in S not containing R. Let p be the number of parallel classes and let f' denote the number of faces that are bounded by at least three edges. So $2p \geq 3f'$. By Euler's formula, $4 + f' \geq p + \chi(S)$, where $\chi(S)$ denotes the Euler characteristic of S. This implies $12 + 2p \geq 12 + 3f' \geq 3p + 3\chi(S)$ and hence $p \leq 12 - 3\chi(S)$. That is, for fixed S, p is bounded.

Let E' be a subset of E containing one edge from every parallel class. Note that any B_k-valued function on E is R-homologous to a B_k-valued function that has value 1 on all edges not in E'.

Let $\Pi = (P_1, \ldots, P_k)$ be an $r-s$ join such that no P_i traverses two edges e, e' consecutively that are parallel. For any 'path' e, v, e' in E' of length two, with e and e' incident with vertex v and e and e' not parallel, let $f(\Pi, e, v, e')$ be the number of times the P_i contain $\tilde{e}, v, \tilde{e'}$, for some $\tilde{e}$ parallel to e and some $\tilde{e'}$ parallel to e'. (Here e or e' is assumed to have an orientation if it is a loop.)

Now up to R-homology and up to a cyclic permutation of the indices of $P_1, \ldots, P_k$, Π is fully determined by the numbers $f(\Pi, e, v, e')$. This follows directly from the fact that the P_i do not have (self-)crossings.

So to enumerate $\phi_1, \ldots, \phi_N$ it suffices to choose for each path e, v, e' a number $g(e, v, e') \leq |E|$. Since $|E'| = p \leq 9 - 3\chi(S)$ there are at most $(|E|+1)^{(12-3\chi(S))^2}$ such choices. For each choice we can find in polynomial time an $r-s$ join Π with $f(\Pi, e, v, e') = g(e, v, e')$ for all e, v, e' if it exists. Enumerating the ϕ_Π gives the required enumeration. ∎

8. Induced circuits

THEOREM 3. *For each fixed surface S, there is a polynomial-time algorithm that gives for any graph $G = (V, E)$ embedded on S and any two vertices r, s of*

G a maximum number of $r-s$ paths each two of which form an induced circuit.

Proof. It suffices to show that for each fixed natural number k we can find in polynomial time k $r-s$ paths each two of which form an induced circuit, if they exist.

We may assume that $G - \{r, s\}$ is connected, that r and s are not connected by an edge, and that G is cellularly embedded. Choose a face R of G arbitrarily. By Theorem 2 we can find in polynomial time a list of functions $\phi_1, \ldots, \phi_N : A \longrightarrow B_k$ such that for each $r-s$ join Π, ϕ_Π is R-homologous to at least one of the ϕ_j.

Consider the (directed) dual graph $D^* = (\mathcal{F}, A^*)$ of G (see Section 6). We extend D^* to a graph $D^+ = (\mathcal{F}, A^+)$ as follows.

For every pair of vertices F, F' of D^* and every $F - F'$ path π (not necessarily directed) on the boundary of one face or of two adjacent faces of D^*, extend the graph with an arc a_π from F to F'. (Note that there are only a polynomially bounded number of such paths.) For each $\phi : A \longrightarrow B_k$ define $\phi^+ : A^+ \longrightarrow B_k$ by $\phi^+(e^*) := \phi(e)$ and

(16) $$\phi^+(a_\pi) := \phi(e_1)^{\varepsilon_1} \cdot \ldots \cdot \phi(e_t)^{\varepsilon_t}$$

for any path $\pi = (e_1^*)^{\varepsilon_1} \ldots (e_t^*)^{\varepsilon_t}$. (Here $\varepsilon_1, \ldots, \varepsilon_t \in \{+1, -1\}$.)

By Theorem 1 we can find, for each $j = 1, \ldots, N$ in polynomial time a function ϑ satisfying

(17) (i) ϑ is R-cohomologous to ϕ_j^+ in D^+, and
(ii) $|\vartheta(b)| \leq 1$ for each arc b of D^+,

provided that such a ϑ exists.

If we find a function ϑ, for $i = 1, \ldots, k$ let Q_i be a shortest $r-s$ path traversing only the set of edges e of G with $\vartheta(e^*) = g_i$. If such paths $Q_1, \ldots, Q_k$ exist, and any two of them form an induced circuit, we are done (for the current value of k).

We claim that, doing this for all $\phi_1, \ldots, \phi_N$, we find paths as required, if they exist. For let $\Pi := (P_1, \ldots, P_k)$ form a collection of k $r-s$ paths any two of which form an induced circuit. Since Π is an $r-s$ join, there exists a $j \in \{1, \ldots, N\}$ such that ϕ_Π and ϕ_j are R-homologous.

We first show that there exists a function ϑ satisfying (17), viz. $\vartheta := \phi_\Pi^+$. To see this, we first show that ϕ_Π^+ is R-cohomologous to ϕ_j^+ in D^+. Indeed, ϕ_Π and ϕ_j are R-homologous in G. Hence there exists a function $f : \mathcal{F} \longrightarrow B_k$ such that $f(R) = 1$ and such that

(18) $$f(F)^{-1} \cdot \phi_\Pi(e) \cdot f(F') = \phi_j(e)$$

for each edge e, where F and F' are the left-hand and right-hand face of e respectively. This implies:

(19) $$f(F)^{-1} \cdot \phi_\Pi^+(e^*) \cdot f(F') = \phi_j^+(e^*).$$

Moreover, for every pair of vertices F_0, F_t of D^* and every $F_0 - F_t$ path $\pi = (e_1^*)^{\varepsilon_1} \dots (e_t^*)^{\varepsilon_t}$ in D^* on the boundary of at most two faces of D^* we have (assuming $(e_i^*)^{\varepsilon_i}$ runs from F_{i-1} to F_i for $i = 1, \dots, t$):

$$\begin{aligned}(20)\qquad & f(F_0)^{-1} \cdot \phi_\Pi^+(a_\pi) \cdot f(F_t) \\ &= (f(F_0)^{-1} \cdot \phi_\Pi(e_1)^{\varepsilon_1} f(F_1)) \cdot (f(F_1)^{-1} \cdot \phi_\Pi(e_2)^{\varepsilon_2} f(F_2)) \cdot \\ &\dots \cdot (f(F_{t-1})^{-1} \cdot \phi_\Pi(e_t)^{\varepsilon_1} f(F_t)) \\ &= \phi_j(e_1)^{\varepsilon_1} \cdot \phi_j(e_2)^{\varepsilon_2} \cdot \dots \cdot \phi_j(e_t)^{\varepsilon_t} = \phi_j^+(a_\pi).\end{aligned}$$

So ϕ_Π^+ and ϕ_j^+ are R-cohomologous.

Next we show that $|\phi_\Pi^+(b)| \leq 1$ for each arc b of D^+. Indeed, for any edge e of G we have $\phi_\Pi^+(e^*) = \phi_\Pi(e) \in \{1, g_1, \dots, g_k\}$. So $|\phi_\Pi^+(e^*)| \leq 1$. Moreover, for any path $\pi = (e_1)^{\varepsilon_1}(e_2)^{\varepsilon_2} \dots (e_t)^{\varepsilon_t}$ as above, $\phi_\Pi^+(a_\pi) = \phi_\Pi(e_1)^{\varepsilon_1} \cdot \dots \cdot \phi_\Pi(e_t)^{\varepsilon_t}$. Since there exist two vertices v', v'' of G such that each of $e_1, \dots, e_t$ is incident with at least one of v', v'', we know that there exists at most one $i \in \{1, \dots, k\}$ such that P_i traverses at least one of the edges $e_1, \dots, e_t$. Hence there is at most one generator occurring in $\phi_\Pi(e_1)^{\varepsilon_1} \cdot \dots \cdot \phi_\Pi(e_t)^{\varepsilon_t}$. That is, $|\phi_\Pi^+(a_\pi)| \leq 1$. This shows that $\vartheta := \phi_\Pi^+$ satisfies (17).

Conversely, we must show that if ϑ satisfies (17), then ϑ gives paths $Q_1, \dots, Q_k$ as above. Indeed, since ϑ is R-cohomologous to ϕ_Π^+, for each $i = 1, \dots, k$, the set of edges e of G with $\vartheta(e^*) = g_i$ contains an $r - s$ path (since $\zeta := \phi_\Pi^+$ has the property that the subgraph $(V, \{e \in E | \zeta(e^*)$ contains the symbol g_i an odd number of times$\})$ of G has even degree at each vertex except at r and s, and since this property is maintained under R-cohomology). Choose for each i such a path Q_i. Suppose that, for some $i \neq j$, there exists an edge $e = \{v, v'\}$ with Q_i traversing v and Q_j traversing v' $(v, v' \notin \{r, s\})$. Then there exist faces F_0 and F_t of G and an $F_0 - F_t$ path $\pi = (e_1)^{\varepsilon_1} \dots (e_t)^{\varepsilon_t}$ in D^* on the boundary of the faces v and v' of D^* such that $\vartheta(e_1^*)^{\varepsilon_1} \cdot \dots \vartheta(e_t^*)^{\varepsilon_t}$ contains both symbol g_i and symbol g_j. Now

$$(21)\qquad \vartheta(a_\pi) = \vartheta(e_1^*)^{\varepsilon_1} \cdot \dots \cdot \vartheta(e_t^*)^{\varepsilon_t},$$

since this equation is invariant under R-cohomology and since it holds when ϑ is replaced by ϕ_Π^+. So $\vartheta(a_\Pi)$ contains both symbol g_i and g_j. This contradicts the fact that $|\vartheta(a_\pi)| \leq 1$.

So there is no edge connecting internal vertices of Q_i and Q_j. Replacing each Q_i by a chordless path Q_i' in G that uses only vertices traversed by Q_i, we obtain paths as required. ∎

We refer to [4] for an extension of the methods described above.

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