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Graphs on the Torus and Geometry of Numbers

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We show that if G is a graph embedded on the torus S and each nonnullhomotopic closed curve on S intersects G at least r times, then G contains at least $\lfloor \frac{3}{4}r \rfloor$ pairwise disjoint nonnullhomotopic circuits. The factor $\frac{3}{4}$ is best possible. We prove this by showing the equivalence of this bound to a bound in the two-dimensional geometry of numbers. To show the equivalence, we study *integer* norms, i.e., norms $\|\cdot\|$ such that $\|x\|$ is an integer for each integer vector x. In particular, we show that each integer norm in two dimensions has associated with it a graph embedded on the torus, and conversely. © 1993 Academic Press, Inc.

1. DESCRIPTION OF RESULTS

Call a closed curve on a surface S *nontrivial* if it is not nullhomotopic. For any graph G embedded on S, the *representativity* (or *face width*) $r(G)$ of G is the minimum of $|C \cap G|$, where C ranges over all nontrivial closed curves on S. We will show the following theorem.

THEOREM 1. (i) *Any graph G embedded on the torus contains at least $\lfloor \frac{3}{4}r(G) \rfloor$ pairwise disjoint nontrivial circuits.*

(ii) *The factor $\frac{3}{4}$ is best possible.*

Here $\lfloor x \rfloor$ denotes the lower integer part of x. A *circuit* is a simple closed curve contained in G. This is related to a result of Brunet, Mohar, and Richter [3] showing that any graph G embedded on any compact orientable surface S contains $\lfloor \frac{1}{2}(r(G)-1) \rfloor$ pairwise disjoint nontrivial circuits.

Remark 1. The representativity of a graph embedded on a surface is recently a focus of attention in the study of minimal genus embeddings of graphs and of graph minors and disjoint paths (see [1, 9, 14–16, 24]). In particular, Robertson and Seymour [14] showed:

> for any compact surface S and any graph H embedded on S there exists a number r so that any graph G embedded on S with representativity at least r contains H as a minor. (1)

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In fact, Robertson and Seymour showed that for any graph H embedded on a compact surface S such that each vertex of H has degree at most three, there exists a number r with the property: for each graph G embedded on S with representativity at least r there exists a homeomorphism $\phi: S \to S$ such that $\phi[H] \subseteq G$. (This implies (1).)

One of the simplest special cases is that for each natural number k there exists a number $r(k)$ such that any graph G embedded on the torus S with representativity at least $r(k)$ contains k pairwise disjoint nontrivial circuits. Theorem 1 asserts that we can take $r(k) = \lceil \frac{4}{3}k \rceil$, where $\frac{4}{3}$ is the best possible factor.

We will show that Theorem 1 is equivalent to the following result in the geometry of numbers: For any symmetric convex body K (i.e., full dimensional compact convex set K with $K = -K$) in $\mathbb{R}^n$, let

$$K^* := \{ y \in \mathbb{R}^n \mid y^{\mathrm{T}}x \leqslant 1 \text{ for each } x \in K \}. \tag{2}$$

As is well known, K^* is again a symmetric convex body, and $(K^*)^* = K$.

Now Theorem 1 is equivalent to:

THEOREM 2. (i) *For any symmetric convex body K in $\mathbb{R}^2$, there exists a nonzero integer vector in K or there exists a nonzero integer vector in $\frac{4}{3} \cdot K^*$.*

(ii) *The factor $\frac{4}{3}$ is best possible.*

Although we assume that this result belongs to the folklore of the geometry of numbers, we were not able to locate a proof in the literature. (The best result in this direction we found in the literature was by Mahler [12] who proved a factor $\sqrt{2}$ replacing $\frac{4}{3}$ in Theorem 2(i).) Therefore, for completeness we describe a proof of Theorem 2 in Section 3 of this paper.

Remark 2. As is well known (cf. Cassels [4], Lekkerkerker [11]), there are several equivalent forms for Theorem 2(i). First, for any symmetric convex body K in $\mathbb{R}^2$ not containing a nonzero integer vector, there exists a nonzero integer vector c such that $c^{\mathrm{T}}x \leqslant \frac{4}{3}$ for each vector x in K.

Second, define for each symmetric convex body K, $\lambda(K)$ to be the smallest value of λ for which $\lambda \cdot K$ contains a nonzero integer vector. Then for any symmetric convex body K in $\mathbb{R}^2$, $\lambda(K) \cdot \lambda(K^*) \leqslant \frac{4}{3}$.

Third, for any norm $\|\cdot\|$ in $\mathbb{R}^n$, the *dual norm* $\|\cdot\|_*$ is defined by

$$\|y\|_* := \sup_x \frac{y^{\mathrm{T}}x}{\|x\|}, \tag{3}$$

where the supremum ranges over all nonzero vectors x in $\mathbb{R}^n$. Then, for any norm $\|\cdot\|$ in $\mathbb{R}^2$, there exist nonzero integer vectors x and y such that $\|x\| \cdot \|y\|_* \leqslant \frac{4}{3}$.

The equivalence of Theorems 1 and 2 is proved with the help of the following theorem given in [19] ([21, 6] gave more direct proofs and extensions to the directed case). Call two closed curves on the torus *freely homotopic* if one can be shifted continuously to the other over the torus (so there is no fixed "base point").

For any two closed curves C and D, let $\mathrm{mincr}(C, D)$ denote the minimum number of crossings of C' and D' (counting multiplicities), where C' and D' range over all closed curves freely homotopic to C and D, respectively.

THEOREM 3. *Let G be a graph embedded on the torus S, and let C be a simple closed curve on S. Then G contains k pairwise disjoint circuits each freely homotopic to C, if and only if each closed curve D on S has at least $k \cdot \mathrm{mincr}(C, D)$ intersections with G (counting multiplicities).*

To describe the equivalence of Theorems 1 and 2, represent the torus as the product $S^1 \times S^1$, where S^1 is the unit circle in the complex plane. For each $(m, n) \in \mathbb{Z}^2$, let $C_{m,n}: S^1 \to S^1 \times S^1$ be the closed curve on the torus defined by

$$C_{m,n}(z) := (z^m, z^n), \tag{4}$$

for $z \in S^1$.

Now, as is well known (cf. [23, Section 6.2.2]), the $C_{m,n}$ form a system of representatives for the free homotopy classes of closed curves on the torus. Moreover,

$$\mathrm{mincr}(C_{m,n}, C_{m',n'}) = |mn' - m'n| \tag{5}$$

for all m, n, m', $n' \in \mathbb{Z}$.

Let G be a graph on the torus, such that each face of G is an open disk, i.e., such that $r(G) > 0$. (This clearly will be no restriction in Theorem 1.) Define for each $(m, n) \in \mathbb{Z}^2$, $f_G(m, n)$ as the minimum number of intersections of C' and G (counting multiplicities), where C' ranges over all closed curves homotopic to $C_{m,n}$.

It is not difficult to see that

$$\begin{aligned} &\text{(i)} \quad f_G(m+m', n+n') \leqslant f_G(m, n) + f_G(m', n'), \\ &\text{(ii)} \quad f_G(km, kn) = |k| \cdot f_G(m, n) \end{aligned} \tag{6}$$

hold for all (m, n), $(m', n') \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$. (The inequality in (i) follows from the fact that if C is freely homotopic to $C_{m,n}$ and C' is freely homotopic to $C_{m',n'}$ and (m, n) and (m', n') are linearly independent, then C has a crossing with C'. We can concatenate C and C' at this crossing so as to obtain a closed curve C'' freely homotopic to $C_{m+m',n+n'}$ with $\text{cr}(G, C'') = \text{cr}(G, C) + \text{cr}(G, C')$, where cr denotes the number of crossings. The equality in (ii) is easy.) Hence there exists a unique norm $\|\cdot\|$ in $\mathbb{R}^2$ with the property that $\|(m, n)\| = f_G(m, n)$ for each $(m, n) \in \mathbb{Z}^2$.

Having this, we give one half of the proof of the equivalence of Theorems 1 and 2:

Proof of the implications Theorem 2(i) $\Rightarrow$ *Theorem* 1(i) *and Theorem* 1(ii) $\Rightarrow$ *Theorem* 2(ii). The representativity $r(G)$ of G is equal to the minimum of $\|(m, n)\|$ over all nonzero integer vectors (m, n). Hence, by Theorem 2(i) (third variant in Remark 2), there exists a nonzero integer vector (m', n') such that $\|(m', n')\|_* \leq \frac{4}{3} r(G)^{-1}$.

By definition (3) of $\|\cdot\|_*$,

$$\frac{(m', n')^{\mathrm{T}}(m, n)}{\|(m, n)\|} \leq \frac{4}{3r(G)} \tag{7}$$

for each nonzero vector (m, n) in $\mathbb{R}^2$. This implies

$$\tfrac{3}{4} r(G)|m'm + n'n| \leq f_G(m, n) \tag{8}$$

for each integer vector (m, n). Therefore, as $|m'm + n'n| = \text{mincr}(C_{m,n}, C_{n', -m'})$, by Theorem 3, G contains $\lfloor \frac{3}{4} r(G) \rfloor$ pairwise disjoint circuits, each freely homotopic to $C_{n', -m'}$. This shows Theorem 1(i).

This construction also shows that Theorem 1(ii) implies Theorem 2(ii), since any better factor in 2(i) would imply a better factor in 1(i). ∎

To see the other implications, we consider integer norms. We call a norm $\|\cdot\|$ in $\mathbb{R}^n$ an *integer norm* if $\|x\|$ is an integer for each x in $\mathbb{Z}^n$.

Above we saw that each graph G embedded on the torus gives an integer norm $\|\cdot\|$ in $\mathbb{R}^2$ such that $f_G(m, n) = \|(m, n)\|$ for each integer vector (m, n). In fact *each* integer norm in $\mathbb{R}^2$ can be constructed in this way:

THEOREM 4. *For each integer norm $\|\cdot\|$ in $\mathbb{R}^2$ there exists a graph G embedded on the torus such that $f_G(m, n) = \|(m, n)\|$ for each integer vector (m, n).*

We will give a proof of this theorem in Section 2 below.

Proof of the implications Theorem 1(i) $\Rightarrow$ *Theorem* 2(i) *and Theorem* 2(ii) $\Rightarrow$ *Theorem* 1(ii). We first show the first implication. Let K be a symmetric convex body in $\mathbb{R}^2$ not containing any nonzero integer vector. We show that $\frac{4}{3} \cdot K^*$ contains a nonzero integer vector.

We may assume that K is a polygon in $\mathbb{R}^2$ with rational vertices (since we can make K slightly larger). Then also K^* is a polygon with rational vertices.

Define the norm $\|\cdot\|$ in $\mathbb{R}^2$ by

$$\|x\| := \min\{\lambda \mid x \in \lambda \cdot K\} = \max\{x^T y \mid y \in K^*\} \tag{9}$$

for $x \in \mathbb{R}^2$. Let t be a common multiple of the dènominators of the components of the vertices of K^*, with the further property that t is also a multiple of four.

Then $t \cdot \|\cdot\|$ is an integer norm, as the maximum in (9) is attained at a vertex of K^*. Hence, by Theorem 4, there exists a graph G embedded on the torus such that $f_G(m, n) = t \cdot \|(m, n)\|$ for each integer vector (m, n).

As K contains no nonzero integer vector, we know that $\|(m, n)\| > 1$ for each nonzero integer vector (m, n), and hence $f_G(m, n) > t$ for each nonzero integer vector (m, n). So the representativity $r(G)$ of G is larger than t.

By Theorem 1(i), G contains $\frac{3}{4}t$ pairwise disjoint nontrivial circuits. They all are mutually freely homotopic; say, they are all freely homotopic to $C_{m,n}$. So, by the necessity of the condition in Theorem 3 and by (5), for each integer vector (m', n'),

$$\tfrac{3}{4}t \cdot |mn' - m'n| = \tfrac{3}{4}t \cdot \operatorname{mincr}(C_{m,n}, C_{m',n'}) \leqslant f_G(m', n') = t \cdot \|(m', n')\|. \tag{10}$$

Hence $\|(n, -m)\|_* \leqslant \frac{4}{3}$, and therefore, $(n, -m)$ belongs to $\frac{4}{3} \cdot K^*$. This shows Theorem 2(i).

Again, any better factor in Theorem 1(i) would imply a better factor in Theorem 2(i). This gives the implication Theorem 2(ii) $\Rightarrow$ Theorem 1(ii). ▌

In Section 2 we will prove Theorem 4 and develop some further results on integer norms in relation to graphs on the torus, and in Sections 3 we give a proof of Theorem 2.

2. Integer Norms and Graphs on the Torus

In this section we give a proof of Theorem 4 above. To this end, we derive some further results. The following theorem follows directly from the "cutting plane theorem" of Chvátal [5]. It is a slight extension of a result of Hoffman [10] for polytopes (extended by Edmonds and Giles [8] to polyhedra, forming the basis for the theory of *total dual integrality*—cf. [18, Chap. 23]).

A *polytope* is the convex hull of a finite set of vectors. A polytope P is called *integer* if each vertex of P is an integer vector.

Theorem 5. *Let C be a nonempty compact convex set in $\mathbb{R}^n$. Then C is an integer polytope, if and only if* $\max\{c^T x \mid x \in C\}$ *is an integer for each integer vector $c \in \mathbb{R}^n$.*

This implies:

THEOREM 6. *For any integer norm* $\|\cdot\|$ *in* $\mathbb{R}^n$ *there exist integer vectors* $y_1, \dots, y_t$ *in* $\mathbb{R}^n$ *such that for each* $x \in \mathbb{R}^n$:

$$\|x\| = \max\{y_1^T x, \dots, y_t^T x\}. \tag{11}$$

Proof. Let $K := \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}$. Hence for each $x \in \mathbb{R}^n$:

$$\|x\| = \max\{x^T y \mid y \in K^*\}. \tag{12}$$

As $\|\cdot\|$ is an integer norm, this maximum is an integer for each integer vector x. Hence, by Theorem 5, K^* is an integer polytope. So we can take for $y_1, \dots, y_t$ the vertices of K^*. ∎

Remark 3. One similarly shows the following related result. Any function $\varphi: \mathbb{Z}^n \to \mathbb{Z}_+$ satisfies

$$\begin{aligned} &\text{(i)} \quad \varphi(x+x') \leqslant \varphi(x) + \varphi(x') && \text{for all} \quad x, x' \in \mathbb{Z}^n, \\ &\text{(ii)} \quad \varphi(k \cdot x) = |k| \cdot \varphi(x) && \text{for all} \quad k \in \mathbb{Z} \text{ and } x \in \mathbb{Z}^n, \end{aligned} \tag{13}$$

if and only if there exist integer vectors $y_1, \dots, y_t$ such that

$$\varphi(x) = \max\{|y_1^T x|, \dots, |y_t^T x|\} \tag{14}$$

for each $x \in \mathbb{Z}^n$.

Equivalently, in terms of groups: Let G be an abelian group. Then any function $\varphi: G \to \mathbb{Z}_+$ satisfies

$$\begin{aligned} &\text{(i)} \quad \varphi(x+x') \leqslant \varphi(x) + \varphi(x') && \text{for all} \quad x, x' \in G, \\ &\text{(ii)} \quad \varphi(k \cdot x) = |k| \cdot \varphi(x) && \text{for all} \quad k \in \mathbb{Z} \text{ and } x \in G, \end{aligned} \tag{15}$$

if and only if there exist homomorphisms $\varphi_1, \dots, \varphi_t: G \to \mathbb{Z}$ such that

$$\varphi(x) = \max\{|\varphi_1(x)|, \dots, |\varphi_t(x)|\} \tag{16}$$

for each $x \in G$.

For integer norms in $\mathbb{R}^2$ we derive from Theorem 6 a further characterization.

THEOREM 7. *A norm* $\|\cdot\|$ *in* $\mathbb{R}^2$ *is integer, if and only if there exist integer vectors* $z_1, \dots, z_k$ *in* $\mathbb{R}^2$ *such that*

$$\|x\| = \tfrac{1}{2} \sum_{i=1}^{k} |z_i^T x| \tag{17}$$

for each $x \in \mathbb{R}^2$ and such that both components of the vector $z_1 + \cdots + z_k$ are even.

Proof. Sufficiency of the condition follows from the fact that, for any integer vector x,

$$\frac{1}{2} \sum_{i=1}^{k} z_i^{\mathrm{T}} x = \left(\tfrac{1}{2}(z_1 + \cdots + z_k)\right)^{\mathrm{T}} x \tag{18}$$

is an integer that differs by an integer value, viz.

$$\sum_{i=1}^{k} \tfrac{1}{2}(|z_i^{\mathrm{T}} x| - z_i^{\mathrm{T}} x), \tag{19}$$

from $\|x\|$ (by (17)).

To see necessity, let $K := \{x \in \mathbb{R}^2 \mid \|x\| \leqslant 1\}$. By Theorem 6, K^* is a polygon with integer vertices, $y_1, ..., y_{2k}$, say, in cyclic order. So $y_{i+k} = -y_i$ for $i = 1, ..., k$. Define

$$z_i := y_{i+1} - y_i \tag{20}$$

for $i = 1, ..., k$. So $z_1 + \cdots + z_k = y_{k+1} - y_1 = 2y_{k+1}$ is an even vector. We show that (17) holds for each $x \in \mathbb{R}^2$.

Since $\|x\| = \max\{y^{\mathrm{T}} x \mid y \in K^*\}$, we know that

$$\|x\| = \max\{y_1^{\mathrm{T}} x, ..., y_{2k}^{\mathrm{T}} x\}. \tag{21}$$

Let the maximum be attained by $y_j^{\mathrm{T}} x$. Without loss of generality, $1 \leqslant j \leqslant k$. It follows that $z_1^{\mathrm{T}} x, ..., z_{j-1}^{\mathrm{T}} x \geqslant 0$ and $z_j^{\mathrm{T}} x, ..., z_k^{\mathrm{T}} \leqslant 0$. Hence

$$\sum_{i=1}^{k} |z_i^{\mathrm{T}} x| = \sum_{i=1}^{j-1} z_i^{\mathrm{T}} x - \sum_{i=j}^{k} z_i^{\mathrm{T}} x. \tag{22}$$

Now $y_1 = -y_{k+1} = -\frac{1}{2}(z_1 + \cdots + z_k)$, implying that $y_j = y_1 + z_1 + \cdots + z_{j-1} = \frac{1}{2}(z_1 + \cdots + z_{j-1} - z_j - \cdots - z_k)$. Hence the right-hand side of (22) is equal to $2y_j^{\mathrm{T}} x = 2\|x\|$. ∎

We are now able to prove:

THEOREM 4. *For each integer norm $\|\cdot\|$ in $\mathbb{R}^2$ there exists a graph G embedded on the torus such that $f_G(m, n) = \|(m, n)\|$ for each integer vector (m, n).*

Proof. Let $\|\cdot\|$ be an integer norm in $\mathbb{R}^2$. By Theorem 7, there exist integer vectors $z_1, ..., z_k$ in $\mathbb{R}^2$ such that

$$\|x\| = \tfrac{1}{2} \sum_{i=1}^{k} |z_i^{\mathrm{T}} x| \tag{23}$$

holds for each $x \in \mathbb{R}^2$ and such that $z_1 + \cdots + z_k$ is an even vector.

We may assume that, for each $i = 1, ..., k$ the two components of z_i are relatively prime (as $z_1, ..., z_k$ need not all be different). Write $z_i = (z_i', z_i'')^{\mathrm{T}}$ for $i = 1, ..., k$.

Again, let $S = S^1 \times S^1$ be the torus. Let $\Pi: \mathbb{R}^2 \to S$ be the usual projection of $\mathbb{R}^2$ to the torus (i.e., $\Pi(x) := (e^{2\pi i x'}, e^{2\pi i x''})$ for each $x = (x', x'')^{\mathrm{T}}$ in $\mathbb{R}^2$). Call a simple closed curve D on S *geodesic* if each component of $\Pi^{-1}[D]$ is a straight line in $\mathbb{R}^2$.

For each $i = 1, ..., k$, let D_i be a geodesic simple closed curve on S freely homotopic to $C_{z_i'', -z_i'}$, in such a way that each two of the D_i are different and no point of S is in more than two of the D_i. So $\mathrm{mincr}(C_{m,n}, D_i) = |mz_i' + nz_i''|$ for all m, n.

Let H be the four-regular graph on the torus formed by the union of $D_1, ..., D_k$. Then one easily checks that for each $(m, n) \in \mathbb{Z}^2$:

> each closed curve C freely homotopic to $C_{m,n}$, not traversing vertices of H, has at least
>
> $$\sum_{i=1}^{k} \mathrm{mincr}(C_{m,n}, D_i) = \sum_{i=1}^{k} |mz_i' + nz_i''| = 2\|(m, n)\|$$
>
> crossings with H; moreover, at least one such curve has exactly this number of crossings with H. (24)

(Indeed, C has at least $\mathrm{mincr}(C_{m,n}, D_i)$ crossings with part D_i of H. This gives the lower bound. Equality can be attained by $C_{m,n}$ itself or a slight shift of it.)

Since $z_1 + \cdots + z_k$ is even, we know that each closed curve on S, not traversing vertices of H, has an even number of crossings with H. So we can color each face of H black or white in such a way that adjacent faces have different colors.

Hence we can construct a "radial" graph G as follows: In each black face F, put a vertex and connect it by (pairwise disjoint) lines through F to each of the vertices of H incident with F. Doing this for each black face of H, we obtain a graph G, called a *radial* graph G.

Now each closed curve on S, intersecting H r times and not intersecting vertices of H, can be shifted slightly so that it intersects G $\tfrac{1}{2}r$ times (in vertices of G). So from (24) we have that $f_G(m, n) = \|(m, n)\|$ for each integer vector (m, n). ∎

Remark 4. The graph G in Theorem 4 need not be unique, but (as was shown in [20]) the minimal such graphs are unique, in the following sense:

Let G be a graph embedded on the torus S. A *minor* of G is any graph obtained from G by a series of deletions and contractions of edges (contracting loops only if they enclose a face). Any minor of G has a natural embedding on S derived from the embedding of G. It is a *proper* minor if at least one edge is deleted or contracted. Call a graph G embedded on the torus S a *kernel* if for each proper minor G' of G one has $f_{G'} \neq f_G$ (i.e., $f_{G'}(m, n) < f_G(m, n)$ for at least one integer vector (m, n)).

So for each integer norm $\|\cdot\|$ in $\mathbb{R}^2$ there exists at least one kernel G on S with $f_G(m, n) = \|(m, n)\|$ for all integer vectors (m, n). Now by the results in [20], for any fixed integer norm $\|\cdot\|$, each two such kernels can be obtained from each other by a series of the following operations:

(i) shifting the graph over the torus;
(ii) taking the surface dual of the graph; (25)
(iii) ΔY-exchange.

Here *ΔY-exchange* means replacing a triangular face F by a vertex in the face connected to the three vertices incident with F, or conversely. (This operation was introduced by Steinitz [22], who called it the θ-process.)

3. Proof of Theorem 2

Although Theorem 2 is nothing but a simple exercise in plane geometry, for completeness we give a proof here. As a preparation, we first give another simple fact.

THEOREM 8. *For any nonsingular 2×2 matrix A there exist nonzero integer vectors x and y in $\mathbb{R}^2$ such that*

$$\|Ax\|_\infty \cdot \|y^T A^{-1}\|_1 \leqslant \tfrac{1}{2}(\sqrt{2} + 1). \tag{26}$$

Proof. We may assume that $\det A = 1$. Let Λ and Λ^* be the pair of dual lattices

$$\Lambda := \{Ax \mid x \in \mathbb{Z}^2\}, \qquad \Lambda^* := \{y^T A^{-1} \mid y \in \mathbb{Z}^2\}. \tag{27}$$

We may assume that Λ has a basis $b = (b_1, b_2)^T$, $c = (c_1, c_2)^T$ satisfying

$$b_1 \geqslant b_2 \geqslant 0 \qquad \text{and} \qquad c_2 \geqslant -c_1 \geqslant 0. \tag{28}$$

Indeed, let b be a nonzero vector in A minimizing $\|b\|_\infty$. Without loss of generality, $\|b\|_\infty = b_1 \geqslant b_2 \geqslant 0$. Let c be a nonzero vector in A minimizing $|c_2|$ over all nonzero vectors $c \in A$ with $|c_1| \leqslant \|b\|_\infty$. We may assume that $c_2 \geqslant 0$, and that the triangle Δ with vertices 0, b, and c has minimal area. If b and c do not form a basis, Δ would contain another vector c' with the required properties, contradicting the minimality of Δ. Moreover, $\|c\|_\infty = c_2 \geqslant \|b\|_\infty > |c_1|$. If $c_1 > 0$, we can replace c by $c - b$. Thus we obtain b and c satisfying (28).

The arithmetic-geometric inequality $\alpha\beta \leqslant (\frac{1}{2}\alpha + \frac{1}{2}\beta)^2$, applied to $\alpha = (\sqrt{2} - 1)\, b_1 c_2$, $\beta = -(\sqrt{2} + 1)\, b_2 c_1$, and the fact that $b_1 c_2 - b_2 c_1 = \det A = 1$ give

$$-b_1 b_2 c_1 c_2 \leqslant (\tfrac{1}{2}(\sqrt{2} - 1)\, b_1 c_2 - \tfrac{1}{2}(\sqrt{2} + 1)\, b_2 c_1)^2 = (\tfrac{1}{2}(\sqrt{2} + 1) - b_1 c_2)^2. \tag{29}$$

Now $\frac{1}{2}(\sqrt{2} + 1) - b_1 c_2 \geqslant 1 - b_1 c_2 = -b_2 c_1 \geqslant 0$. Hence at least one of $b_2 c_2$ and $-b_1 c_1$ is at most $\frac{1}{2}(\sqrt{2} + 1) - b_1 c_2$. That is, at least one of $(b_1 + b_2)\, c_2$ and $b_1(-c_1 + c_2)$ is at most $\frac{1}{2}(\sqrt{2} + 1)$. Since b and c belong to A and since $(b_2, -b_1)$ and $(c_2, -c_1)$ belong to A^*, we have the required vectors. ▮

In fact, bound $\frac{1}{2}(\sqrt{2} + 1)$ in Theorem 8 is best possible, as is shown by the matrix

$$A = \begin{pmatrix} 1 & 1 - \sqrt{2} \\ \sqrt{2} - 1 & 1 \end{pmatrix}. \tag{30}$$

THEOREM 2. (i) *For any symmetric convex body K in $\mathbb{R}^2$, there exists a nonzero integer vector in K or there exists a nonzero integer vector in $\frac{4}{3} \cdot K^*$.*

(ii) *The factor $\frac{4}{3}$ is best possible.*

Proof. (i) We may assume that K is a polygon. We show that if K contains no nonzero integer vectors in its interior, then $\frac{4}{3}K^*$ contains a nonzero integer vector.

We may assume that each edge of K contains an integer vector in its relative interior (otherwise, we can shift the edge until it contains an integer vector in its relative interior or until the edge "disappears").

If K has four edges, the result directly follows from Theorem 8 (applied to the matrix A with rows the coefficients of the inequalities determining the edges of K (taking for each two parallel edges one of the two)), since $\frac{1}{2}(\sqrt{2} + 1) < \frac{4}{3}$.

If K has at least six edges, let $v_1, ..., v_{2k}$ be the vertices of K (in cyclic order), and let z_i be an integer vector in the relative interior of the edge connecting v_{i-1} and v_i ($i = 1, ..., 2k$, taking indices mod $2k$).

By Minkowski's theorem [13], the volume of K is at most 4. Hence, there exists an $i=1, ..., 2k$, so that the volume V of the quadrangle $(0, z_i, v_i, z_{i+1})$ is at most $4/2k$. As the triangle $(0, z_i, z_{i+1})$ contains no further integer vectors, z_i and z_{i+1} form a basis for the lattice $\mathbb{Z}^2$. So the vector c satisfying $c^{\mathrm{T}}z_i = c^{\mathrm{T}}z_{i+1} = 1$ is an integer vector. Let V_1 and V_2 be the volumes of the triangles $(0, z_i, z_{i+1})$ and (z_i, z_{i+1}, v_i), respectively. So $V_1 = \frac{1}{2}$ and $V_2 = V - V_1$. Moreover, $V_2/V_1 = (c^{\mathrm{T}}v_i - c^{\mathrm{T}}z_i)/c^{\mathrm{T}}z_i$. This implies $c^{\mathrm{T}}v_i = 2V$. Hence

$$\max\{c^{\mathrm{T}}x \mid x \in K\} = c^{\mathrm{T}}v_i = 2V \leqslant 2\frac{4}{2k} \leqslant \frac{4}{3}. \tag{31}$$

(ii) Let K be the convex hull of the vectors $\pm(\frac{2}{3}, \frac{4}{3})$, $\pm(\frac{4}{3}, \frac{2}{3})$, $\pm(-\frac{2}{3}, \frac{2}{3})$. Then K^* is the convex hull of the vectors $\pm(-\frac{1}{2}, 1)$, $\pm(1, -\frac{1}{2})$, $\pm(\frac{1}{2}, \frac{1}{2})$. Since no slight shrinking of K and of $\frac{4}{3} \cdot K^*$ contains any nonzero integer vector, we obtain that $\frac{4}{3}$ is best possible. ∎

Remark 5. In fact, in this proof k cannot exceed 3, as no two of the vectors z_i and z_i' for $i, i' = 1, ..., k$, are equal mod 2 (otherwise $\frac{1}{2}(z_i + z_i')$ would be an integer vector in the interior of K). (This is a special case of a result of Doignon [7] (cf. [2, 17]).)

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