COMPLETENESS
AND DEFINABILITY

applications of the Ehrenfeucht game
in second-order and intensional logic

Kees Doets
COMPLETENESS
AND DEFINABILITY

applications of the Ehrenfeucht game
in second-order and intensional logic

Kees Doets
COMPLETENESS AND DEFINABILITY

applications of the Ehrenfeucht game
in second-order and intensional logic

ACADEMISCH PROEFSCHRIFT
TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE
WISKUNDE EN NATUURWETENSCHAPPEN AAN DE
UNIVERSITEIT VAN AMSTERDAM OP GEZAG VAN DE
RECTOR MAGNIFICUS, DR S.K. THODEN VAN VELZEN,
HOGLERAAR IN DE FACULTEIT DER GENEESKUNDE, IN
HET OPENBAAR TE VEREIDIGEN IN DE AULA DER
UNIVERSITEIT (ODDE LUTHERSE KERK, SINGEL 411,
HOEK SPUI) OP

WOENSDAG 6 MEI 1987 TE 16.00 UUR

door

HANS CORNELIS DOETS

GEBOREN TE VLAARDINGEN
promotor: Prof. Dr J.F.A.K. van Benthem
co-promotor: Prof. Dr A.S. Troelstra
dedicated to Liesbeth
Preface

Part I: Definability

1. Fraïssé-Ehrenfeucht theory for $L_{\omega_1 \omega}$ and some of its fragments.
   1.0 introduction p. 1
   1.1 notation and terminology 5
   1.2 $\alpha$-equivalence 6
   1.3 ordinal-bounded Ehrenfeucht games 7
   1.4 Fraïssé-Karp sequences 12
   1.5 logic 13
   1.6 Scott sentences 15
   1.7 the finite case 17
   1.8 the unbounded case 18
   1.9 basis results 20

2. On $n$-equivalence of binary trees.
   2.1 introduction 23
   2.2 playing in trees 24
   2.3 characterizing $n$-equivalence of $B_m$ 28
   2.4 finiteness of binary trees is not $\Sigma^1_1$ 32

Part II: Completeness

3. Monadic $\Pi^1_1$-theories of $\Pi^1_1$-properties: linear orderings.
   3.1 Introduction: $\omega$ and finite orderings 36
   3.2 Monadic $\Pi^1_1$-theory of scattered ordering 44
   3.3 Monadic $\Pi^1_1$-theory of complete orderings, of well-orderings and of the reals 47
   3.4 Appendix: strengthening 3.2.4 and 3.3.4 55
4. Monadic $\Pi^1_1$-theory of well-founded trees.  

Part III: Applications to intensional and intuitionistic logic

5. Fine structure of modal correspondence theory.  

6. Game theory for intensional logics, exact universal Kripke models and normal forms.  

7. Completeness for $\mathcal{Z}$-time.  


Appendix A: Can time be directional?  

Appendix B: Reduction of higher-order logic.  

references  

samenvatting
This dissertation contains results on classical first- and second-order logic (parts I and II) and their intensional colleagues: modal- tense- and intuitionistic (propositional) logic (part III).

One underlying theme is Ehrenfeucht's game and some of its variants.

Chapter 1 is an introduction to Ehrenfeucht game theory and its relation with (quantifierrank-) $\alpha$-equivalence in (infinitary) logic. Section 1.0 intends to whet the appetite for the finite Ehrenfeucht game.

In chapter 2 the game is played on binary trees. A characterisation is obtained of all trees $n$-equivalent with the binary tree $B_m$ all of whose branches have length $m$. In particular, it follows that $B_m$ has infinite $n$-equivalents when $m \geq 2^n - 1$. This has been applied by Rodenburg [1986] to solve a problem in intuitionistic correspondence theory; the story is told in chapter 8.

Part II shows how to axiomatize certain monadic $\Pi^1_1$-theories, most of them dealing with well-foundedness. Chapter 3 is on linear orderings. One of the nicer results is in 3.3 where the effects of the Suslin property for the monadic $\Pi^1_1$-theory of $\mathbb{R}$ are isolated. Chapter 4 generalizes the method of 3 to the case of trees.

In part III, chapter 5 discusses Löwenheim–Skolem type problems in modal correspondence theory. It is shown that most examples of non-first-order definable modal formulas already cannot be first-order defined on finite frames. On the other hand, an example is given of a non-first-order definable formula which is first-order definable on all countable frames.

Chapter 6 modifies the Ehrenfeucht game for use in intensional logic; exact Kripke models are constructed universal with respect to finite partially ordered Kripke models.
Chapter 7 presents our version of Z-completeness.

In chapter 9, games and the universal-exact Kripke model appropriate for one-variable intuitionistic formulas are applied to solve some problems in intuitionistic correspondence theory left open by Rodenburg [1982].

Appendix A constructs asymmetric linear orderings with lots of homogeneity-properties in each uncountable cardinal.

Appendix B reduces all of higher-order logic to monadic second-order logic — indicating the expressive possibilities of modal logic in the Kripke semantics.

To help the reader find his way, here is an indication what can be omitted without loss of understanding of the rest. In chapter 1, sections 4, 8 and 9 are not needed for the other chapters. Also, not much will be lost if, in the discussion of the α-game, the reader always assumes α to be finite. Section 2.4 can be read independently from the rest of chapter 2. Section 3.2 may be omitted. In part III, all chapters can be read individually (except for a couple of references where this is indicated.)

I am obliged to several people for different reasons; in particular I wish to thank here prof. Specker for a lecture featuring Ehrenfeucht games; Piet Rodenburg for the communication of his problems to which chapters 2/8/9 are devoted and the elimination of numerous mistakes in a previous version of this text; Anne Troelstra and Dick de Jongh for scientific support and the software used to produce this text on the Macintosh Plus.

However, above all, my gratitude concerns my thesis-advisor Johan van Benthem whose determination and persuasiveness eventually turned out to be irresistible.
Part I: DEFINABILITY

It were not best that we should all think alike; it is difference of opinion that makes horse-races. – *Pudd’nhead Wilson’s Calendar*

1. Fraïssé–Ehrenfeucht theory for $L_{\infty\omega}$ and some of its fragments.

1.0 Introduction.

This chapter introduces five guises of $\alpha$-equivalence between models, where $\alpha$ is an arbitrary ordinal.

For $\alpha=\omega$, this relation (called *elementary equivalence* and denoted by $\equiv$) is a basic one in model theory. For models of the same finite similarity-type, $A \equiv B$ just means that $A$ and $B$ have the same true (first-order) sentences. However, there are some uses for refinements, as is argued below.

$\alpha$-Equivalence for finite $\alpha$ is explained game-theoretically as follows. Suppose $A$ and $B$ are models (of the same similarity type) and $n \in \mathbb{N}$. The $n$-game on $A$ and $B$, $G(A, B, n)$, has two players, $I$ and $II$. They move alternately. $I$ is allowed the first move; each player is allowed $n$ moves. A
Therefore, a move of player $I$ and the following counter-move of player $II$ form an ordered pair in $A \times B$ (where $A$ and $B$ are the universes of $A$ and $B$ respectively).

When the game is over, the set of ordered pairs of moves is an at most $n$-element relation $h \subseteq A \times B$. $II$ has won the play by definition if $h$ is a partial isomorphism between $A$ and $B$, that is, if $h$ is an injection on its domain which preserves the structure of the models. Of course, the larger $n$, the better $I$’s chances to defeat $II$.

Here is an example: $A$ consists of the neighbour-relation $R$ on the set of integers $\mathbb{Z}$ defined by $nRm$ iff $|n−m|=1$; $B$ is the quotient of this structure modulo $(9)$ - this can be visualized as a circle with 9 points. Now, player $II$ can win each game of length 3 between these models; however, $I$ has a way to win the 4-game: his first three moves are chosen at equal distances in $B$; his fourth is in $\mathbb{Z}$ not meeting any of $II$’s three moves there. Now whatever the answer of $II$ in $B=\mathbb{Z}/(9)$, he cannot avoid one of the moves of $I$.

To explain the “meaning” of the game somewhat, let me remark that player $I$, in order to win, tries to point to differences between the models involved; while player $II$ on the other hand tries to argue that the models are somehow similar.

Finally, $A$ and $B$ are called $n$-equivalent if $II$ has a winning strategy for $G(A, B, n)$, that is, a method by which he can beat $I$ no matter the choice of moves by $I$. So, in the example above, $(\mathbb{Z}, R)$ and $(\mathbb{Z}, R)/(9)$ are 3-equivalent but not 4-equivalent.

An easy case where $II$ has such a winning strategy no matter the length of the game is when $A$ and $B$ are isomorphic: $II$ then simply applies the isomorphism (or its inverse) to the moves of $I$.

1. Fraissé–Ehrenfeucht theory
$n$-Equivalence of $A$ and $B$ is denoted by $A \equiv^n B$.

The following well-known example will be used later on.

1.0.1 Proposition. Finite linear orderings $A$ and $B$ are $n$-equivalent iff $|A|=|B|$ or $|A|, |B| \geq 2^n - 1$.

To prove 1.0.1, we employ a simple

1.0.2. Lemma. Suppose that $A = (A, <)$ and $B = (B, <)$ are linear orderings. Then $A \equiv^{n+1} B$ iff ("back") for all $b \in B$ there exists $a \in A$ such that $a \downarrow \equiv^n b \downarrow$ and $a \uparrow \equiv^n b \uparrow$ and ("forth") for all $a \in A$ there exists $b \in B$ such that $a \downarrow \equiv^n b \downarrow$ and $a \uparrow \equiv^n b \uparrow$.

(Here, $a \downarrow = \{a' \in A | a' < a\}$ and $a \uparrow = \{a' \in A | a < a'\}$ are identified with the ordered models on these sets.)

Notice that, for the lemma to make sense in general, empty models must be allowed explicitly.

Proof of 1.0.1. Induction on $n$. For $n=0$ there is nothing to prove. Assume the result for $n$ and suppose that $|A|, |B| \geq 2^{n+1} - 1$. Use 1.0.2. Suppose $a \in A$. Now, distinguish three cases. (i) $a \downarrow < 2^n - 1$. Pick $b \in B$ such that $|b\downarrow|=|a\downarrow|$. Then $a \downarrow \equiv^n b \downarrow$. Furthermore, we must have $|a\uparrow|, |b\uparrow| \geq 2^n - 1$; hence $a \uparrow \equiv^n b \uparrow$ by the induction hypothesis. (ii) $a \uparrow < 2^n - 1$. This is similar to (i). (iii) $|a\downarrow|, |a\uparrow| \geq 2^n - 1$. Pick $b \in B$ such that $|b\downarrow|, |b\uparrow| \geq 2^n - 1$ (such a $b$ necessarily exists) and apply the inductive hypothesis. The other parts of the result are left for the reader. ☐
predecessors. $\omega^*$ is the order-type of $(\omega, >)$ and $\zeta = \omega^* + \omega$ is the order type of the integers.

1.0.3 Example. (i) If $m \geq 2^n - 1$, then: $m \equiv^n \omega + \omega^*.$

(ii) For all $n$: $\omega \equiv^n \omega + \zeta.$

Proof. Induction on $n$, using 1.0.2. For (i), use 1.0.1. (ii) follows from (i). □

Below we shall encounter several situations in which either $\equiv$ is too fine or ordinary model theory is of no avail. In such cases, game theory provides a way out.

As to the first type, in part II we shall be confronted with models which have $n$-equivalents with a certain property for all $n$, while elementary equivalents with that property do not exist. As to the second, much of part III is concerned with model theory for finite models. In such a non-elementary domain, compactness and Löwenheim–Skolem theorems cannot be much of a help.

Chapter 1 is organized as follows. Section 1.1 fixes terminology. 1.2 defines $\alpha$-equivalence for general $\alpha$ in the simplest of ways. (cf. Poizat's admirable [1985] logic-introduction for a treatment based on such an approach.) 1.3 is about games. For $\alpha > \omega$, this is due to Barwise [1973]. 1.4 is Karp's [1965]-characterisation generalizing Fraïssé's [1955] (one of the oldest references in the subject, save for Fraïssé's thesis). 1.5 gives the connection with (infinitary) logic and 1.6 finally defines Scott-sentences coding up the game-theoretic information on a sequence of elements. Remarkably, these were introduced at first for infinitary logic only, cf.

1. Fraïssé-Ehrenfeucht theory
Chang [1968]. 1.7 specializes the discussion to (finite) first-order logic. In 1.8, the ordinal parameter \( \alpha \) approaches absolute infinity. In game theory, this corresponds with not putting an upper bound on the length of the game, which seems to be the obvious way to "infinitize" games; however, 1.3 showed this to be a very crude step indeed. 1.9 finally collects some information on when \( =^\alpha \) equals \( =^\infty \).

1.1 Notation and terminology.

A **model** is a complex \( A=(A,\ldots) \) consisting of a set \( A \) (which, contrary to usual logical convention, often is allowed to be empty) together with any number of ("finitary") relations. Thus, functions (and, often, constants as well) are excluded from models.

A **language** (or similarity-type) is a set of relation-symbols, together with a specification of the number of arguments (the arity) for each symbol in the set.

If \( L \) is a language, an \( L \)-model is a model together with a surjection of \( L \) onto its set of relations such that arities are preserved. Thus, an \( L \)-model can be considered as a couple \( A=(A,^\ast) \) such that for each \( R\in L \), if \( R \) is \( n \)-ary then \( R^\ast \subset A^n \).

\( h: A \to B \) is an isomorphism between the \( L \)-models \( A=(A,^\ast) \) and \( B=(B,^\circ) \) if it is bijective and preserves corresponding relations, i.e., if for each \( R\in L \), if \( R \) is \( n \)-ary and \( a_0,\ldots,a_{n-1} \in A \) then

\[
R^\ast(a_0,\ldots,a_{n-1}) \iff R^\circ(ha_0,\ldots,ha_{n-1}).
\]

If \( A=(A,^\ast) \) is an \( L \)-model and \( B \subset A \) then \( B=A|B \) is the \( L \)-model \( (B,^\circ) \).
$h: A \rightarrow B$ is a partial isomorphism between the $L$-models $A = (A, \cdot)$ and $B = (B, \circ)$ if $\text{Dom} \ h$ is finite and $h$ is an isomorphism between $A|\text{Dom} \ h$ and $B|\text{Ran} \ h$. Notice that each finite part of an isomorphism is a partial isomorphism but not every partial isomorphism is part of an isomorphism.

The next definition is basic for this chapter.

1.2.1 Definition. For $L$-models $A, B$ and ordinals $\alpha$, $I_\alpha(A, B)$ is a set of partial isomorphisms between $A$ and $B$ defined as follows:

(i) $I_0(A, B)$ consists of all partial isomorphisms between $A$ and $B$;
(ii) $h \in I_{\alpha+1}(A, B)$ iff ("back") for all $b \in B$ there is an $a \in A$ such that $h \cup \{(a, b)\} \in I_\alpha(A, B)$ and ("forth") for all $a \in A$ there is a $b \in B$ such that $h \cup \{(a, b)\} \in I_\alpha(A, B)$;
(iii) for $\alpha$ a limit: $I_\alpha(A, B) = \bigcap_{\xi < \alpha} I_\xi(A, B)$.

Suppose $a = (a_0, \ldots, a_{k-1}) \in A^k$ and $b = (b_0, \ldots, b_{k-1}) \in B^k$. $a$ and $b$ are called $\alpha$-equivalent (notation: $a^\alpha = b$ or, more explicitly, $(a, a) \equiv (B, b)$) iff the correspondence $(a, b) = \{(a_j, b_j) | i < k\}$ is in $I_\alpha(A, B)$. (\(\omega\)-equivalence usually is called elementary equivalence.)

The main purpose of this chapter is to present four or five characterisations of $\alpha$-equivalence, most of them connecting it with $L_{\infty\omega}$ and one in terms of the Ehrenfeucht game. Before we introduce the last one however, we give a simple lemma.

1.2.2 Lemma. If $\alpha < \beta$ then $I_\beta(A, B) \subseteq I_\alpha(A, B)$.

Proof. Induction on $\beta$. For limit $\beta$, this is obvious, so assume $\beta = \gamma + 1$. Let
\( h \in I_\delta(A, B) \); we prove \( h \in I_\alpha(A, B) \) using induction on \( \alpha \). For \( \alpha = 0 \) and limit \( \alpha \) this is clear. Thus, assume \( \alpha = \delta + 1 \) and \( h \in I_\delta(A, B) \). To see that \( h \in I_{\delta + 1}(A, B) \) we check the forth-condition only: let \( a \in A \) be arbitrary. Since \( h \in I_{\delta + 1}(A, B) \), there exists \( b \in B \) such that \( h \cup \{(a, b)\} \in I_\delta(A, B) \). Now \( \alpha = \delta + 1 \leq \gamma \), hence, by induction-hypothesis on \( \gamma \), \( h \cup \{(a, b)\} \in I_\delta(A, B) \).

1.3 Ordinal-bounded Ehrenfeucht games.

We now turn to the first characterisation of \( \alpha \)-equivalence. This is the imaginative reformulation in terms of games, due to Ehrenfeucht, which, in practical applications, seems to work best.

Suppose that \( A \) and \( B \) are models for the same language, that \( h \in I_0(A, B) \) and that \( \alpha \) is some ordinal. \( G(A, B, h, \alpha) \) is the following game for two players \( I \) and \( II \): \( I \) and \( II \) make moves alternately as follows. \( I \) begins. His first move consists of three things: (i) an ordinal \( \alpha_0 < \alpha \); (ii) one of the models \( A, B \); (iii) one element of the model chosen under (ii). \( II \) now is allowed, as a counter-move, to choose one element from the model not chosen by \( I \). Next, it is \( I \)'s turn again. His second move again consists of three things: (i) an ordinal \( \alpha_1 < \alpha_0 \); (ii) one of the models, and (iii) an element of the model chosen. Again, \( II \) picks an element from the other model. This goes on with the proviso that the sequence of ordinals picked by \( I \) must be strictly descending. Of course, after a finite number of moves and counter-moves in this fashion, player \( I \) must pick the ordinal 0 eventually. Player \( II \) is allowed one last counter-move, and this is where the play of the game ends. Suppose the moves of \( I \) have been as follows:
$d_i \in A$ if $C_i = B$ and $d_i \in B$ if $C_i = A$). Let $(a_i, b_i) = (c_j, d_j)$ in case $c_j \in A$ and $d_j \in B$; $(a_i, b_i) = (d_i, c_i)$ otherwise ($i < k$). We shall say that player II has won the play of the game when $\mu \{(a_i, b_i) \mid i < k\} \in I_0(A, B)$. Otherwise, player I has won.

This description of the game is inadequate when one of the models is empty, or if $\alpha = 0$. Therefore we agree that II wins the play when $\alpha = 0$ or if both models are empty; and I wins if II has no answer since the model he has to choose from is empty.

Again, the "meaning of the game" is explained somewhat by noting that it is the "intention" of I to show that $A$ and $B$ (more accurately, $(A, a)_{a \in \text{Dom}(h)}$ and $(B, ha)_{a \in \text{Dom}(h)}$) are different in some sense, while it is the "intention" of II to argue that they are not.

Clearly, the larger $\alpha$, the easier the task of I, since the play can be somehow longer. Though each play necessarily must be finite, I can make it last as long as he wants without telling II - at least some of the time. Only if $\alpha$ is finite it is clear beforehand that the play will end after $\alpha$ moves for both players at worst. If $\alpha = \omega$, the first move of I betrays his intentions as to the length of the play. But if, say, $\alpha = \omega + 3$, I has three moves to go for free before he must inform II how many more he'll use, etc.

Example: 1.0.3(ii) shows that $\omega \equiv \omega^\omega + \zeta$. However, $\omega \not\equiv \omega + 1 + \omega + \zeta$. To see this, let I start with the move $(\omega, \omega + \zeta, 0, \zeta)$ (where by $0, \zeta$ I mean the 0 of the $\zeta$-part of $\omega + \zeta$). Now if II answers with $n \in \omega$ then I chooses $(n + 1, \omega + \zeta, -1, \zeta)$, starting a decreasing sequence of length $n + 1$ in $\omega + \zeta$.

For finite $\alpha$, the game was invented by Ehrenfeucht [1961] as a reformulation of the Fraïssé characterisation (cf. below); for general $\alpha$, cf. Barwise [1973].

1. Fraïssé-Ehrenfeucht theory
1.3.1 Example. An ordinal is identified with the well-ordered set of its predecessors. If $\beta < \alpha$ then player $I$ has a method to win each play of $G(\alpha, \beta, \phi, \alpha)$. Proof: the first move of $I$ is $(\beta, \alpha, \beta)$. If $II$ answers with $\gamma \in \beta$, $I$ copies this in his second move as $(\gamma, \alpha, \gamma)$. As $\gamma < \beta$, $II$ will answer with some $\delta < \gamma$ in $\beta$ which is copied by $I$ in his third move, and so on. Eventually, $II$ will pick $0$ (his sequence of ordinals must be strictly descending if he is not to lose: remember $II$ has to take care that the alternate moves build a partial isomorphism). This is copied by $I$ - and $II$ now is without answer.

A method for one of the players to win each play of the game is called a winning strategy for that player. More precisely, a winning strategy for $II$ is a function $\sigma$ defined on sequences of odd length such that each play $(a_0, b_0, \ldots, a_{k-1}, b_{k-1})$ of the game (where the successive moves of $I$ are $a_0, \ldots, a_{k-1}$; those of $II$ being $b_0, \ldots, b_{k-1}$) with the property that for all $i<k$: $b_i = \sigma(a_0, b_0, \ldots, a_i)$, is a win for $II$. Winning strategies for $I$ are defined similarly.

1.3.2 Theorem. In each game $G(A, B, h, \alpha)$, one of the players has a winning strategy.

Proof. Immediate from the Gale-Stewart theorem, since all plays have finite length. However, here is the argument:

First, we define a position in $G(A, B, h, \alpha)$ to be a finite set $g \subset A \times B$ together with some $\beta < \alpha$. By the phrase "$I$ and $II$ have reached the position $(g, \beta)$ after $k$ moves each in their play" we shall understand that $II$ is in the position defined by which side is winning. To begin with,
order) and that (ii) $\beta$ is the last ordinal played by $I$.

A position $(g, \beta)$ now is defined to be a win for $I$ obviously in case he has a winning strategy for $G(A, B, h \cup g, \beta)$.

Now, we can prove 1.3.2 using these notions as follows. Suppose that $I$ has no winning strategy for $G(A, B, h, \alpha)$. This clearly means that the initial position $(\emptyset, \alpha)$ is no win for $I$. Let $II$ play according to the following method: if possible he picks his moves in such a way that after each of his moves a position $(g, \beta)$ remains which is no win for $I$. By induction, we verify that $II$ always has a counter move which just performs this: first, since $(\emptyset, \alpha)$ is no win for $I$, whatever the first move of $I$, $II$ has an answer leaving no win for $I$. Similarly, if in the play a position $(g, \beta)$ is reached which is no win for $I$ then, whatever $I$ does next, $II$ can counter leaving no win for $I$. Eventually (and this is where the finiteness of the game comes into play), this sequence of moves comes to an end with a position $(f, 0)$. Being no win for $I$, we must have $f \cup h \in I_0(A, B)$. Hence, the method described is winning for $II$. 

Now comes the first characterisation of $\alpha$-equivalence.

1.3.3 Theorem. For $h \in I_0(A, B)$ and $\alpha$ an ordinal the following are equivalent:

(i) $h \in I_\alpha(A, B)$;

(ii) $II$ has a winning strategy for $G(A, B, h, \alpha)$.

Proof.

(i) $\Rightarrow$ (ii). Assume $h \in I_\alpha(A, B)$. The strategy of $II$ consists in $II$ trying to
pick his moves in such a fashion that in each position \((g, \beta)\) of the play he has \(hu g \in I_\beta(A,B)\). This is satisfied at the starting-position \((\emptyset, \alpha)\) by assumption. If the play ends with \((f,0)\) and \(hu f \in I_0(A,B)\), then \(II\) has won by definition. Lastly, \(II\) can keep this requirement during the play: Suppose a position \((g, \beta)\) has been reached with \(hu g \in I_\beta(A,B)\) and \(I\) chooses, say, \((\gamma, A, a)\) as his next move, where \(\gamma < \beta\). Then \(\gamma + 1 \leq \beta\), hence \(hu g \in I_{\gamma + 1}(A,B)\) by 1.2.2. Therefore, \(b \in B\) exists such that \(hu g \cup \{(a,b)\} \in I_\gamma(A,B)\) — and this will be the answer of \(II\), of course.

(ii) \(\Rightarrow\) (i). Let \(\sigma\) be a winning strategy for \(II\) in \(G(A,B,h,\alpha)\); we show \(h \in I_\alpha(A,B)\) by induction on \(\alpha\):

If \(\alpha = 0\) there is nothing to prove since \(h \in I_0(A,B)\) by hypothesis. If \(\alpha\) is a limit, \(\sigma\) wins every play in \(G(A,B,h,\beta)\) with \(\beta < \alpha\) for \(II\), therefore \(h \in \bigcap_{\beta \in \alpha} I_\beta(A,B) = I_\alpha(A,B)\). Finally, assume \(\alpha = \beta + 1\). We check the forth-condition: let \(a \in A\) be arbitrary. Consider \((\beta, A, a)\) as a first move of \(I\). Suppose \(\sigma\) provides the answer \(b \in B\) for \(II\). Clearly, \((a)\) is a winning strategy for \(II\) in \(G(A,B,hu\{(a,b)\} , \beta)\). By induction hypothesis then, \(hu\{(a,b)\} \in I_\beta(A,B)\). \(\Box\)

1.3.4 Example. In 3.4.1 below, we re-prove Ehrenfeucht's result that \(\omega^\omega\) is \(\omega\)-equivalent with the ordered structure \(\Omega\) of all ordinals. However, \(\omega^\omega\) is not \((\omega + 1)\)-equivalent with this structure, which can be seen as follows.

According to Rosenstein [1982] corollary 6.19 p.106, if \(\alpha < \omega^{n+1}\) then \(\alpha \not\equiv 2n+2 \omega^{n+1}\).(*) Now, let \(I\) play \((\omega,\Omega,\omega^\omega)\) as a first move. Suppose \(II\) counters with \(\alpha < \omega^{n+1}\). Then, by (*), \((2n+2,\Omega,\omega^{n+1})\) is a winning move for \(I\).

Hence $=^\alpha$ will never coincide with isomorphism on the ordinals as there always are $\delta \geq \alpha$ (of the same power as $\alpha$) such that $\omega^\delta = \delta$.

1.4 Fraïssé–Karp sequences.

A Karp sequence for $A, B, h, \alpha$ is a sequence $\langle I_\xi \mid \xi \leq \alpha \rangle$ of length $\alpha+1$ such that the following hold:

1. (i) $I_\xi \subseteq I_0(A,B)$ for all $\xi \leq \alpha$;
    (ii) $\xi < \delta \leq \alpha \Rightarrow I_\delta \subseteq I_\xi$;
    (iii) $h \in I_\alpha$;
2. for all $\xi < \alpha$ and $g \in I_{\xi+1}$:
   ("back") for all $b \in B$ there exists $a \in A$ such that $g \cup \{(a,b)\} \in I_\xi$;
   ("forth") for all $a \in A$ there exists $b \in B$ such that $g \cup \{(a,b)\} \in I_\xi$.

Karp sequences for finite $\alpha$ are due to Fraïssé [1955] which is one of the oldest references in the subject. For general $\alpha$, they are due to Karp [1965]. They provide a second characterisation:

1.4.1 Theorem. $h \in I_\alpha(A,B)$ iff there is a Karp sequence for $A, B, h, \alpha$.

Proof. If $h \in I_\alpha(A,B)$ then obviously $\langle I_\xi(A,B) \mid \xi \leq \alpha \rangle$ is the required Karp sequence, by 1.2.2. For the converse, assume $\langle I_\xi \mid \xi \leq \alpha \rangle$ is a sequence as required. Now, copy the proof of 1.3.3.(i) $\Rightarrow$ (ii), replacing $I_\xi(A,B)$ by $I_\xi(\xi \leq \alpha)$ (notice that 1.2.2 has been built into the definition of Karp sequence). This shows II to have a winning strategy in

1. Fraïssé–Ehrenfeucht theory
G(A, B, h, α), which suffices by the other half of 1.3.3. □

1.5 Logic.

Now, it is time for logic.

Let $L$ be any language. Fix a countably infinite set of variables. Atomic formulas of $L$ are all identities $x_0 = x_1$ and expressions $R(x_0, \ldots, x_{k-1})$ where $R \in L$ is $k$-ary and $x_0, \ldots, x_{k-1}$ are variables. The set $L_{\omega \omega}$ of (infinitary) formulas of $L$ is the least one such that

1. all atomic formulas are in $L_{\omega \omega}$;
2. if $\varphi \in L_{\omega \omega}$ then $\neg \varphi \in L_{\omega \omega}$;
3. if $\Phi \subset L_{\omega \omega}$ is any set then $\bigwedge \Phi$ and $\bigvee \Phi$ are in $L_{\omega \omega}$;
4. if $\varphi \in L_{\omega \omega}$ and $x$ is a variable then $\forall x \varphi$ and $\exists x \varphi$ are in $L_{\omega \omega}$.

I assume known the basic notions and semantics concerning these

objects.

The quantifier rank $qr(\varphi)$ of $\varphi \in L_{\omega \omega}$ is an ordinal recursively defined as follows:

1. $qr(\varphi) = 0$ if $\varphi$ is atomic;
2. $qr(\neg \varphi) = qr(\varphi)$;
3. $qr(\bigwedge \Phi) = qr(\bigvee \Phi) = \sup \{ qr(\varphi) \mid \varphi \in \Phi \}$;
4. $qr(\forall x \varphi) = qr(\exists x \varphi) = qr(\varphi) + 1$.

Here is the logical characterisation of $\alpha$-equivalence:
and every valuation $f$ of the free variables of $\psi$ into $\text{Dom } h$: $A \models \psi[f]$ iff $B \models \psi[h \circ f]$ (i.e., $h$ preserves satisfaction of quantifier-rank $\leq \alpha$ - formulas).

Proof. I only give one half here - the other has to wait for the next (and last) characterisations.

Suppose $\psi$ has $\text{qr}(\psi) \leq \alpha$ and $\psi, h$ do not satisfy the equivalence. Say, $A \models \psi[f]$, but $B \models \neg \psi[h \circ f]$. If $\psi$ is atomic, then $h \not\in I_0(A, B)$. If $\psi = \bigwedge \Psi$, there is a $\psi \in \Psi$ such that $A \models \psi[f]$ and $B \models \neg \psi[h \circ f]$. The same goes when $\psi = \bigvee \Psi$. If $\psi = \neg \psi$, then $A \models \neg \psi[f]$ and $B \models \psi[h \circ f]$. Repeating this, we find a quantified subformula, $\forall x \psi$ say, of $\psi$, satisfied by $f$ resp. $h \circ f$ in exactly one of the models, $A$ say. Hence, for some $b \in B$ $B \models \neg \psi[(h \circ f) \cup \{(x, b)\}]$; on the other hand, for all $a \in A$ $A \models \psi[f \cup \{(x, a)\}]$.

Now we can give the first move in a winning strategy for $I$ in $G(A, B, h, \alpha)$: $I$ plays the triple $(\text{qr}(\psi), B, b)$. Whatever $II$'s counter-move $a \in A$, $I$ proceeds to find a quantified subformula of $\psi$ satisfied by $f$ and $a$ (resp. $h \circ f$ and $b$) in exactly one of the models and repeats the procedure.

Ultimately, an atomic formula and a position $(0, g)$ are reached such that, necessarily, $h \cup g \not\in I_0(A, B)$.

Summarizing: $I$ can use $\psi$ as a guide to find a winning sequence of moves.

This gives one half of the theorem by 1.3.3. $\Box$

1. Fraïssé-Ehrenfeucht theory
1.6 Scott-sentences.

For the last couple of characterisations of $\alpha$-equivalence a definition is needed which is due to Scott (cf. Chang [1966]).

1.6.1 Definition. Fix an enumeration $v_0, v_1, v_2, \ldots$ of all variables.

For $A = (A, \ldots)$, $a = (a_0, \ldots, a_{k-1}) \in A^k$ and $\alpha$ an ordinal, define the formula $[a]^\alpha = [(A, a)]^\alpha$ (the $\alpha$-characteristic of $a$ in $A$) as follows:

1. $[a]^0$ is the conjunction of all atomic or negated atomic formulas in $v_0, \ldots, v_{k-1}$ satisfied by $a$ in $A$;
2. $[a]^\alpha+1 = \bigwedge_{a \in A} \exists v_k [aa]^\alpha \land \forall v_k \bigvee_{a \in A} [aa]^\alpha$;
3. $[a]^\alpha = \bigwedge_{\xi < \alpha} [a]^\xi$ when $\alpha$ is a limit.

(Here, juxtaposition denotes prolongation: $aa = (a_0, \ldots, a_{k-1}, a)$.)

To appreciate this definition, notice that it is (in a sense to be explained) a formalisation of the definition of $\alpha$-equivalence, 1.2.1.

1.6.2 Lemma. $[a]^\alpha$ has quantifier rank $\alpha$, free variables $v_0, \ldots, v_{k-1}$ and it is satisfied by $a$ in $A$.

The following completes the sequence of theorems 1.3.3, 1.4.1, 1.5.1:

1.6.3 Theorem. For $a \in A^k$ and $b \in B^k$ the following are equivalent:

1. $a =^\alpha b$;
2. $B \models [a]^\alpha [b]$;
3. $[b]^\alpha ( = [(B, b)]^\alpha ) = [a]^\alpha$.

Proof, and proof second half of 1.5.1.
then: \( A \models \varphi [a] \iff B \models \varphi [b] \).

We are going to show 1 \(\Rightarrow\) 0 \(\Rightarrow\) 2 \(\Rightarrow\) 3 \(\Rightarrow\) 1.

Using the Ehrenfeucht game-characterisation, 1 \(\Rightarrow\) 0 has been proved above.

2 \(\Rightarrow\) 3: induction on \(\alpha\). For \(\alpha = 0\) and limit \(\alpha\), this is clear. Assume the implication for \(\alpha\) and all \(a\) and \(b\). Now, let \( B \models [a]^{\alpha+1} [b] \). (\(\ast\)) In order that \([a]^{\alpha+1} = [b]^{\alpha+1}\) it suffices that \(\{ [a a]^a a \in A \} = \{ [b b]^a b \in B \}\).

2 \(\Rightarrow\) 3: let \( a \in A \). By (\(\ast\)), \( b \in B \) exists such that \( B \models [a a]^a [b b] \). By induction hypothesis, \([a a]^a = [b b]^a\). Hence, \([a a]^a \in \{ [b b]^a \mid b \in B \}\). The \(\Rightarrow\)-part is similar.

3 \(\Rightarrow\) 1: induction on \(\alpha\). For \(\alpha = 0\) this is clear. For limit \(\alpha\), notice that if \([a]^\xi = [b]^\xi\) then \([a]^\xi = [b]^\xi\) for all \(\xi < \alpha\) by 1.6.2; now, apply the induction-hypothesis.

Lastly, assume 3 \(\Rightarrow\) 1 for \(\alpha\) and all \(a, b\) and let \([a]^{\alpha+1} = [b]^{\alpha+1}\). To see that \( a \models^{\alpha+1} b \), we check the forth-condition:

Pick \( a \in A \). Since \([a a]^a\) is a quantifier rank-\(\alpha\) subformula of \([a]^{\alpha+1}\), it must occur in \([b]^{\alpha+1}\) as well; hence \([a a]^a = [b b]^a\) for some \( b \in B \). By induction hypothesis then, \( a a \models^{a} b b \). \(\Box\)

Remark. The equivalent 1.6.3.3 shows that the \(\alpha\)-characteristics are canonical objects associated with the \(\alpha\)-equivalence classes. If one does not care about them being \(L_{\infty \omega}\)-formulas at the same time, their definition can be simplified, leaving out logic almost completely, by

1.6.1.2'. \([a]^n = \{ [a a]^n \mid a \in A \}\).

This remark is due to Shelah [1975]. Of course, this can be extended into the transfinite by putting, for limits \(\gamma\),

3'. \([a]^\gamma = \bigcup_{\xi < \gamma} [a]^\xi\).

1. Fraïssé-Ehrenfeucht theory
1.7 The finite case

Specializing to the finite fragment $L_{\omega_1}$ of $L_{\omega_1\omega}$, first-order logic, there is a slight problem. Notice that, though each finite formula has finite quantifier rank, not every finite quantifier rank $L_{\omega_1\omega}$-formula is finite. However, we have the (in the sequel, often used)

1.7.1 Lemma. If the language of $A$ is finite then, for all $k, n \in \mathbb{N}$, there are only finitely many $n$-characteristics belonging to sequences of length $k$.

Proof. Induction on $n$. When we have a finite language, there are only finitely many atomic formulas in $\nu_0, \ldots, \nu_{k-1}$ for each $k$. If their number is $s_k$, it is clear that there are at most $2^{s_k}$ possible formulas $[a]^n$ for $a$ of length $k$. Hence, there are $\leq \exp \exp s_{k+1}$ formulas $[a]^{n+1}$, and so on.

Therefore, we can specialize 1.5.1 to

1.7.2 Theorem. For models $A, B$ of the same finite language, when $a \in A^k$, $b \in B^k$, $n \in \omega$, the following are equivalent:

1. $a \equiv^n b$

2. for all finite formulas $\varphi$ of quantifier-rank $\leq n$ in the appropriate number of free variables: $A \models \varphi[a] \iff B \models \varphi[b]$.

Proof. $1 \Rightarrow 2$ : by 1.5.1. $2 \Rightarrow 1$ : if 2. holds then, by 1.6.3 and 1.7.1, $B \models [a]^n[b]$. The result follows by 1.6.3.
Suppose that $A$ and $B$ are models. Let us construct a Karp sequence for $A, B$ which is as long as possible: $I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_\alpha \supset \ldots$; for instance, we might have $I_\alpha = I_\alpha(A, B)$. By a cardinality-argument (cf. 1.9.1), the sequence must become stationary at some place $\alpha$, i.e., $I_\alpha = \bigcap \limits_\kappa I_\kappa$. Now, $I_\alpha$ either is empty, or it isn’t. Either way, it has the following back-and-forth property: if $h \in I_\alpha$ then (back) for all $b \in B$ there is $a \in A$ such that $h \cup \{(a, b)\} \in I_\alpha$, and (forth) for all $a \in A$ there is $b \in B$ such that $h \cup \{(a, b)\} \in I_\alpha$. This simply is due to the fact that $I_\alpha = I_{\alpha + 1}$.

$I_\alpha$ can be looked at as the greatest fixed point of the monotone operator defined on subsets of $I_0(A, B)$ which associates with each $X \subset I_0(A, B)$ the set \{ $g \mid \forall a \in A \exists b \in B (g \cup \{(a, b)\} \in X) \land \forall b \in B \exists a \in A (g \cup \{(a, b)\} \in X)$ \}. Now, $A$ and $B$ are called partially isomorphic if a non-empty set $I$ of partial isomorphisms exists with this back-and-forth property.

In the games $G(A, B, h, \alpha)$, the ordinal $\alpha$ is a means to preserve finiteness of all plays, without giving a fixed upper bound on their lengths (at least when $\alpha \geq \omega$). Somehow, the larger $\alpha$, the longer player $I$ can keep the play going (and the better his chances to defeat $II$). The natural limit is the game $G(A, B, h) = G(A, B, h, \Omega)$ where $I$ can go on forever, and plays have length $\omega$. Of course, if $I$ is to win, this becomes apparent after a finite number of moves already. Hence, the Gale-Stewart determinacy theorem still applies. We now have the following “limiting case” of our characterisations, which is presented without parameters for simplicity.

1. Fraissé-Ehrenfeucht theory
1.8.1 Theorem. The following are equivalent:

1. $\bigcap_\alpha I_\alpha(A,B) \neq \emptyset$;

2. $\text{II}$ has a winning strategy for $G(A, B, \emptyset)$;

3. $A$ and $B$ are partially isomorphic;

4. $A \equiv_{\omega_1} B$ (i.e., they have the same $L_{\omega_1\omega}$-theory).

Proof.

1 $\Rightarrow$ 2. Suppose $I_{\alpha+1}(A,B) = I_\alpha(A,B) \neq \emptyset$. $\text{II}$ plays in such a way that only positions $h$ are reached for which some $g \in I_\alpha(A,B)$ exists with $h \subseteq g$.

2 $\Rightarrow$ 3. The required set is the one of all positions $h$ reached during every possible play in which $\text{II}$ uses his winning strategy.

3 $\Rightarrow$ 4. If $I$ is a set witnessing 3., the constant sequence $I, I, I, \ldots$ of length $\alpha$ is a Karp sequence. Apply 1.4.1/1.5.1.

4 $\Rightarrow$ 1. Let $\alpha$ be such that $I_\alpha(A,B) = \bigcap_\xi I_\xi(A,B)$. By 1.5.1, $\emptyset \in I_\alpha(A,B)$. $\Box$

1.8.2 Corollary. (Barwise [1973]) Countable partially isomorphic models are isomorphic.

Proof. In $G(A, B, \emptyset)$, let $I$ enumerate all elements of $A \cup B$. If $\text{II}$ uses a winning strategy, the result of the play is an isomorphism as required. $\Box$

1.8.3 Example. On the ordinals, partial isomorphism coincides with isomorphism.
As part of the above, we saw that, for all $A$ and $B$, there is an $\alpha$ such that $A \equiv_{\omega} B$ iff $A \equiv^{\alpha} B$. As to how large such an $\alpha$ must be

1.9.1 Theorem. 

1. If $|A| = n < \omega$ then $A \equiv_{\omega} B$ iff $A \equiv B$ iff $A \equiv^{n+1} B$;

2. If $A$ and $B$ are infinite then $A \equiv_{\omega} B$ iff $A \equiv^{\alpha} B$, where $\alpha$ is the least ordinal of power $> |A|, |B|$.

Proof. 1. In $G(A, B, n+1)$, let $I$ first enumerate $A$. If $II$ uses a winning strategy, the position reached must be surjective (otherwise, let $I$ play a new element of $B$). 2. Notice that $|I_0(A, B)| = \max (|A|, |B|)$. Furthermore, the first $\alpha$ for which $I_{\alpha}(A, B) = I_{\alpha+1}(A, B)$ has $|\alpha| \leq |I_0(A, B)|$, hence is less than the first one of power $> |A|, |B|$. $\Box$

Of course, 1.9.1.2 is quite weak. A much better result is Nadel's basis theorem from Nadel [1974]. Here, an admissible set is a transitive model of the theory KP, which is a weak version of ZF at least sufficient to carry out the proof of 1.9.2; cf. Barwise [1975]. To see that the next theorem strengthens 1.9.1, note that, by the Löwenheim-Skolem theorem, each transitive infinite set is a member of lots of admissibles of the same power.

1.9.2 Theorem. Let $A$ be an admissible set such that $L, A, B \in A$ and let $\alpha = A \cap OR$ be the set of ordinals in $A$.

If $A \equiv^\alpha B$ then $A \equiv_{\omega} B$.

Proof. Since $\equiv_{\omega}$ can be defined inductively, this is immediate from

1. Fraïssé-Ehrenfeucht theory
Gandy's theorem (see for instance Barwise [1975] chapter VI). However, here is a direct proof:

Suppose that $\varphi_0$ is any $L_{\omega^\omega}$-sentence such that, say, $A \models \varphi_0$ but $B \not\models \neg \varphi_0$. Let $\beta = qr(\varphi_0)$. We know that $I$ has a winning strategy $\sigma$ for $G(A, B, \varphi, \beta)$. Let $T$ be the set of all positions $((a, b), \xi) (\xi \leq \beta)$ which can be reached in some play where $I$ uses his strategy $\sigma$.

Claim: for every $((a, b), \xi) \in T$ there exists $\psi \in \mathcal{A} \cap L_{\omega^\omega}$ in the appropriate number of free variables such that $A \models \psi[a]$ and $B \models \neg \psi[b]$.

As $((\varphi, \varphi), \beta) \in T$ to begin with, this claim establishes the result. It is proved by induction on $\xi$ as follows:

If $((a, b), 0) \in T$, we cannot have $a^0 \neq b$, since $I$ was using a winning strategy. Therefore there is a formula $\psi$ as required which is either atomic or negated atomic. Now, let $((a, b), \xi) \in T$ and $\xi > 0$. ($a \in A^k$) Suppose $\sigma$ prescribes the move $(\delta, A, a)$ for $I$ in this position. Then $\delta < \xi$,

and, by induction hypothesis,

$$\forall b \in B \exists \psi \in \mathcal{A} \left[ A \models \psi[a, a] \land B \models \neg \psi[b, b] \right].$$

By $\Sigma$-collection (cf. Barwise [1975] thm.4.4 p.17) on $A$, obtain $\Phi \in \mathcal{A}$ such that

1. $\forall b \in B \exists \psi \in \mathcal{A} \left[ A \models \psi[a, a] \land B \models \neg \psi[b, b] \right]$, and

2. $\forall \psi \in \Phi \exists b \in B \left[ A \models \psi[a, a] \land B \models \neg \psi[b, b] \right]$.

Now, $\varphi = \exists x_k \land \Phi$ is the formula required. (When $\sigma$ prescribes a move in $B$, a similar argument finds $\varphi$ of the form $\forall x_k \lor \Phi$.) $\Box$

The Scott-rank $\text{sr}(A)$ of $A$ is the least ordinal $\alpha$ such that for all $k$ and $a, b \in A^k$: if $a \equiv^\alpha b$ then $a \equiv^{\alpha + 1} b$. (Thus, $A$ is $\omega$-homogeneous iff $\text{sr}(A) \leq \omega$.)
(In particular, $sr(A)$ exists.)

Proof. Apply 1.9.2 to $(A, a), (A, b) \in A$. ∎

For $\alpha = sr(A)$, the Scott-sentence of $A$ is

$$\sigma_A = [\emptyset]^\alpha \land \bigwedge \forall x ([a]^\alpha \to [a]^\alpha+1).$$

A one-sided version of 1.9.1 is part 3 of the theorem below:

1.9.4 Theorem. 1. $qr(\sigma_A) = sr(A) + \omega$;

2. $A \vDash \sigma_A$;

3. $B \vDash \sigma_A$ iff $B \equiv_{\omega} A$.

Proof of 3.: $\Leftarrow$ by 1, 2. $\Rightarrow$: If $B \vDash \sigma_A$ then in particular $B \vDash [\emptyset]^\alpha$, hence $B \equiv\alpha A$. (⋆) But also, $B \vDash \bigwedge \forall x ([a]^\alpha \to [a]^\alpha+1)$; therefore $II$ can win $G(A, B, \emptyset, x)$ for each $x$ as follows: he simply moves in such a way that only positions $((a, b), x)$ are reached with $a \equiv\alpha b$. By (⋆), this is right at the initial position. If $((a, b), x)$ is reached and $a \equiv\alpha b$, then $B \vDash [a]^\alpha [b]$ and hence $B \vDash [a]^\alpha+1 [b]$ by assumption. Now if $I$ plays, say, $a \in A$, there is $b \in B$ with $B \vDash [a a]^\alpha [bb]$, etc. ∎

1. Fraïssé-Ehrenfeucht theory
2. On \( n \)-equivalence of binary trees.\(^1\)

2.1 Introduction

For \( m > 0 \), \( B_m \) is the binary tree all of whose branches have length \( m \).

This chapter characterizes the binary trees \( n \)-equivalent with \( B_m \) (for each pair \( n, m \)). It grew out of corollary 2.2.4 below which says that \textit{finiteness} of binary trees is not first-order expressible. This fact is used in Rodenburg [1986] to solve a problem in correspondence theory for intuitionistic propositional calculus, cf. chapter 8 below. Section 2.4 contains the much stronger result that \textit{finiteness} of binary trees is not even \( \Sigma^1_1 \) - be it with a less informative proof. For an application of the Ehrenfeucht game played on \textit{structures of trees}, cf. Tiuryn [1984].

Playing on trees is appreciably less easy than it was on finite linear orderings, due to the fact that a handy decomposition lemma like 1.0.2 is not available: decomposing a tree yields much more than just two

\(^1\) Z is reproduced from Rodenburg [1986] with slight improvements.
Let me define the notions involved.

A tree is a non-empty partially ordered set $T= (T, <)$ where, for each $a \in T$, the set $a \downarrow = \{ x \in T | x < a \}$ is linearly ordered by $<$; in this chapter, moreover, it is required always that the sets $a \downarrow$ are finite.

A least element of a tree is called its root. The tree $(T, <)$ is binary if it has a root and each non-empty set $a \uparrow = \{ x \in T | a < x \}$ $(a \in T)$ has exactly two minimal elements, in other words, each non-maximal $a \in T$ has exactly two immediate successors. A branch through $T$ is a maximal linearly ordered subset. A branch above $a \in T$ is a branch in the subtree $a \uparrow$ of $T$. The length of a branch is its order type; the order-type of $a \downarrow$, denoted by $h(a)$, is called the height of $a$ in $T$.

2.2 Playing in trees.

For each $n \geq 1$, define the class $Q(n)$ of binary trees as follows.

2.2.1 Definition. Let $n \geq 1$. The binary tree $T$ satisfies $Q(n)$ iff the following conditions are met:

Q.1($n$) if $n=2$ then $T$ has a maximal element; if $n \geq 3$, there is a maximal element above any given one.

Q.2($n$) every branch through $T$ has length $\geq 2^n - 2$.

Q.3($n$) some branch through $T$ has length $\geq 2^n - 1$.

Q.4($n$) for all $x \in T$ and $m < 2^{n-1} - 1$: if some branch above $x$ has length $m$ then every branch above $x$ has length $m$.

2. $n$-equivalence of binary trees
Notice that every binary tree satisfies Q(1): Q.1(n)–Q.3(n) only demand something if \( n \geq 2 \); and Q.4(n) is non-trivial for \( n \geq 3 \) only.

The next theorem shows that Q(n) is included in an \( n \)-equivalence class. In section 2.3 it is shown that Q(n) actually can be expressed by a quantifier rank-\( n \) sentence; whence it can be used to characterize \( n \)-equivalence with \( B_m \) for any \( m \).

2.2.2 Theorem. If the binary trees \( T^1 \) and \( T^2 \) satisfy Q(n), then \( T^1 \equiv^n T^2 \).

Proof. Induction on \( n \).

The case \( n = 1 \) is clear. Assuming 2.2.2 to hold for \( n \) we check it for \( n+1 \) using Ehrenfeucht-games. Thus, suppose \( t \in T^1 \) is the first move of player \( I \) in the \( (n+1) \)-game between \( T^1 \) and \( T^2 \). There are three cases to consider.

Case 1. \( h(t) < 2^n - 1 \).

Decompose \( T^1 \) in:

(i) two top-subtrees \( t^1 \) and \( t^2 \), final sections of \( T^1 \), the roots of which are the two minimal elements of \( t \uparrow \);
(ii) the linear ordering \( t \downarrow \) of type \( h(t) \);
(iii) the trees \( t_a \) (where \( a < t \), final sections of \( T^1 \); the root of \( t_a \) being the immediate successor of \( a \in t \downarrow \) which is not \( \leq t \).

(The items under (ii) and (iii) are absent in case \( t \) is the root of \( T^1 \).)

Since \( T^1 \) satisfies Q(\( n+1 \)), it is clear that all trees in this decomposition satisfy Q(\( n \)). For instance, Q.1(\( n+1 \)) is inherited from Q.1(\( n+1 \)) by final sections (this is true even if \( n = 1 \) or \( n = 2 \)). Q.4(\( n+1 \)) implies Q.4(\( n \)); and if \( T^1 \) satisfies Q.4(\( n+1 \)) then so do his final sections. By Q.2(\( n+1 \)), each branch in, say, \( t^1 \) has length \( \geq 2^{n+1} - 2 - h(t) - 1 > 2^{n+1} - 2 - (2^n - 1) - 1 = 2^n - 2 \) by assumption, and hence is a \( (n+1) \)-game.
Now player II answers $t$ with some $s \in T^2$ for which $h(s) = h(t)$.

Let $j$ be the isomorphism between $t \downarrow$ and $s \downarrow$. $s$ induces a decomposition of $T^2$ similar to the one described for $t$ in which all trees satisfy $Q(n)$. By induction hypothesis, corresponding trees in the decompositions are $n$-equivalent. Therefore, II can win the remaining $n$-game using the following strategy: above $t$ or $s$ he uses winning strategies between $t^i$ and $s^i$ ($i = 1, 2$). Below $t$ or $s$ he answers using the isomorphism $j$. Finally, a move in some $t_a$ ($a < t$) by $I$ is answered using a winning strategy between $t_a$ and $s_{j(a)}$ and vice versa.

This strategy is clearly winning for II since the union of partial isomorphisms between corresponding substructures in the decompositions is a partial isomorphism between $T^1$ and $T^2$.

**Case 2.** There is no branch of length $\geq 2^n - 1$ above $t$.

By Q.4($n+1$) there exists $u \leq t$ such that all branches above $u$ have length $2^n - 2$. Hence, $u$ is the root of a final section $B_u$ of $T^1$ in which all branches have length $2^n - 1$.

Since $T^2$ satisfies Q.1($n+1$) (for, $n+1 \geq 2$) and Q.4($n+1$), there exists $v \in T^2$ which is the root of a final section $B_v$ of $T^2$ isomorphic with $B_u$. (1)

By Q.2($n+1$), $u \downarrow$ and $v \downarrow$ have order types $\geq 2^{n+1} - 2 - (2^n - 1) = 2^n - 1$; hence $u \downarrow \equiv^n v \downarrow$ by proposition 1.0.1. (2)

If $a < u$, branches above $a$ containing $u$ have length $\geq 2^n - 1$; by Q.4($n+1$) therefore, all branches above $a$ have length $\geq 2^n - 1$; in particular, all branches through $u_a$ have length $\geq 2^n - 1$. Thus, $u_a$ (defined as under case 1.(iii)) satisfies $Q(n)$. The same goes for the $v_b$ ($b < v$).

By induction hypothesis, $u_a \equiv^n v_b$ whenever $a < u$ and $b < v$. (3)

Now II uses the following strategy. First, he answers $t$ using the
isomorphism (1). The remaining \(n\)-game is dealt with as follows. Between \(B_u\) and \(B_v\), \(II\) goes on using the isomorphism (1). Below \(u\) or \(v\) he uses the winning strategy (2). If \(I\) makes a move \(x\) in some \(u_a (a < u)\) for the first time while \(a\) has not been played yet, \(I\) is granted the extra move \(a\) as well. Then \(II\) answers \(a\) by some \(b < v\) using (2) and next answers \(x\) by some \(y \in v_b\) using (3).

Of course, if \(a\) has been played before, \(b\) has been fixed already and no extra move is granted (this occurs in particular when \(x\) is not the first move in \(u_a\) by either player).

**Case 3.** \(h(t) \geq 2^n - 1\) and some branch above \(t\) has length \(\geq 2^n - 1\).

By Q.4(\(n+1\)) then, all branches above \(t\) have length \(\geq 2^n - 1\). Hence, in the decomposition described under 1. above, \(t^1\) and \(t^2\) satisfy Q(\(n\)). If \(a < t\), branches above \(a\) containing \(t\) and, hence, all branches above \(a\), have length \(\geq 2^n - 1\); thus \(t_a\) satisfies Q(\(n\)).

Since \(T^2\) satisfies Q.3(\(n+1\)) and \(2^{n+1} - 1 = 2(2^n - 1) + 1\), \(II\) can find \(s \in T^2\) such that \(h(s) = 2^n - 1\) while some branch above \(s\) has length \(\geq 2^n - 1\). It follows that \(s^1\), \(s^2\) and all \(s_b (b < s)\) satisfy Q(\(n\)).

For the remaining \(n\)-game, \(II\) uses a strategy similar to the one used under 2. above; except that above \(s\) or \(t\) he uses that \(s^i \equiv^n t^i\) (\(i = 1, 2\)).  

2.2.3 Examples. The following trees satisfy Q(\(n\)).

1. The binary tree \(B_m\) all of whose branches have length \(m \geq 2^n - 1\).

2. **Infinite** binary trees provided that, along every infinite branch,
   (i) there occur infinitely many side-trees, and (ii) all finite side-trees occurring are of type 1 (i.e., are of the form \(B_m\) for some \(m \geq 2^n - 1\)).
These examples make manifest the following

2.2.4 Corollary. Finiteness of trees is not a first-order property on the class of all binary trees.

Noteworthy also is

2.2.5 Corollary. "Every branch has length $\geq 2^n - 1$" and its negation some branch has length $\leq 2^n - 2$ ($n > 1$) cannot be expressed by first-order sentences of quantifier-rank $n$ on the class of (finite) binary trees.

2.3 Characterizing $n$-equivalence with $B_m$.

By 2.2.3, $B_m$ satisfies $Q(n)$ whenever $m \geq 2^n - 1$; hence 2.2.2 gives one half of the following

2.3.1 Theorem. Let $m \geq 2^n - 1$. A binary tree $T$ satisfies $Q(n)$ iff $T \equiv^n B_m$.

The other half is established by propositions 2.3.2-4 below.

These results show that $Q(n)$ can be expressed by a first-order sentence of quantifier rank $n$. And this can be used to construct a simple quantifier rank-$n$ logical equivalent of $\llbracket B_m \rrbracket^n$ for $m \geq 2^n - 1$: the theory of binary trees has a straightforward quantifier rank-4 axiomatisation.

2. $n$-equivalence of binary trees
(instead of finiteness of the $x\downarrow$, we only require discreteness with first and last element of these orderings - to see this works, use chapter 4).

Notice that $Q.1(n)$ can be expressed by a first-order sentence of quantifier rank $\leq n$. $Q.2(n)$-$Q.4(n)$ are dealt with by 2.3.4, 2.3.2 and 2.3.3, respectively.

In the sequel, $\varphi^<_x$ and $\varphi^>_x$ denote the formulas obtained from $\varphi$ by restricting quantifiers to the sets $\{y | y < x\}$ resp. $\{y | x < y\}$ (and changing bound variables when necessary).

2.3.2 Proposition. Define the sentences $\varphi_n$ by:

- $\varphi_1$ is $\exists x (x = x)$
- $\varphi_{n+1}$ is $\exists x (\varphi_n^<_x \land \varphi_n^>_x)$.

Then $\varphi_n$ has quantifier rank $n$ and it holds in a tree iff there is a branch of length $\geq 2^n - 1$.

Proof. Obvious. $\blacklozenge$

In view of 2.2.5, the next result is not entirely trivial.

2.3.3 Proposition. Let $k$ be any integer $\geq 1$ and $T$ a binary tree such that $T \equiv_{n+1} B_k$. Then $T$ satisfies $Q.4(n+1)$.

Proof. We may assume that $n > 1$ since otherwise $Q.4(n+1)$ is trivially satisfied.

Suppose that $T$ fails to satisfy $Q.4(n+1)$. Let $m < 2^n - 1$ be minimal such that for some $t \in T$ and branches $\alpha$ and $\beta$ above $t$ we have $|\alpha| = m$ and $|\beta| > m$.

By minimality of $m$, $\alpha \cap \beta = \emptyset$.

($\Rightarrow$) if $\alpha \cup \beta = T$, then $T \equiv_{n+2} B_k$.
the root have length \( m \).

Let \( t^2 \) be the other top-tree of \( T \) through which \( \beta \) is a branch. Again by minimality of \( m \), all branches through \( t^2 \) have length \( \geq m \). Finally, we may as well assume \( \beta \) to have finite length since \( n+1 \geq 3 \) and Q.1(3) is a quantifier rank-3-sentence true in \( B_k \) and hence valid in \( T \).

Now, choose \( x \in B_k \) with \( (T, u) \equiv \beta_{B_k}^n \beta_k(x) \) (1).

By (a variation on) proposition 1.0.1 it follows that all branches through \( B_x = \{ x' \in B_k | x \leq x' \} \) have length \( m \). Let \( y \in B_k \) be the element \( \neq x \) with \( y \perp = x \perp \). Then all branches through \( B_y = \{ y' \in B_k | y \leq y' \} \) have length \( m \) too.

Notice that if \( s \in \beta \) and \( (T, u, s) \equiv \beta_{B_k}^n \beta_k(x, z) \) then \( z \geq y \), since \( n-1 \geq 1 \) and and \( \neg s \leq u \wedge \forall w < u (w < s) \) holds and has quantifier rank 1.

The proof is finished by indicating how \( I \) can defeat \( II \) in the \( n \)-game between \( (T, u) \) and \( (B_k, x) \), contradicting (1).

If \( m < 2^{n-1} \) then \( I \), by picking the largest element \( s \) of \( \beta \), wins the \( n \)-game: \( II \) has to answer with a maximal element \( z \geq y \), whence there remain \( m-1 < 2^{n-1}-1 \) elements in \( \{ w < z | y \leq w \} \) = \( \{ w < z | \neg w \leq z \} \) and \( I \) can defeat \( II \) in \( n-1 \) more moves by playing on \( \beta \) below \( s \) (use 1.0.1).

If \( 2^{n-1} \leq m \) then \( 2^{n-1} < |\beta| \) and \( I \) picks \( s \in \beta \) such that \( \{ v \in \beta | s < v \} \) has \( 2^{n-1}-1 \) elements. On penalty of losing (cf. 2.3.2) \( II \) must answer with a \( z \geq y \) above which there are branches of length \( \geq 2^{n-1}-1 \). But then \( \{ w < z | y \leq w \} \) has \( \leq m-2^{n-1} < 2^{n-1}-2^{n-1} = 2^{n-1}-1 \) elements left and \( I \) needs only \( n-1 \) more moves on \( \beta \) below \( s \) to defeat \( II \). \( \Box \)

2.3.4 Proposition. Suppose that \( k \geq 2^{n+1} - 2 \). Let \( T \) be a binary tree such that \( T \equiv \beta_{B_k}^{n+1} \). Then \( T \) satisfies Q.2(n+1).

Proof. Suppose that some branch \( \alpha \) through \( T \) has length \( m < 2^{n+1} - 2 \).
Since the quantifier rank-\((n+1)\)-sentence
\[ \forall x \ ( \neg \varphi_n^x \rightarrow \exists y \ (x^y) ) \]
\((\varphi_n^x\text{ defined in 2.3.2})\) holds in \(B_k\) (for, \(2^n \leq 2^{n+1} - 2\)), \(\alpha\) has an element \(t\) of height \(2^n - 2\).

Now \(\{s \in \alpha \mid t < s\}\) is a branch above \(t\) of length \(m - (2^n - 1) < 2^{n+1} - 2 - (2^n - 1) = 2^n - 1\), hence, by 2.3.3, every branch above \(t\) has length \(m - (2^n - 1)\).

But, the quantifier rank-\((n+1)\)-sentence \(\forall x ( \varphi_n^x \lor \varphi_n^{>x} )\) is satisfied in \(B_k\); on the other hand, \(x = t\) is a counter-example in \(T\). \(\Box\)

2.3.5 Proposition. For each \(m < 2^n - 1\) there is a sentence of quantifier rank \(\leq n\) which is satisfied in a tree iff all of its branches have length \(m\).

**Proof.** Left to the reader. \(\Box\)

2.3.6 Corollary. If \(m < 2^n - 1\) and \(T\) is a binary tree such that \(T \equiv B_m^n\), then \(T \equiv B_m\).

For the next corollary, compare 1.0.1.

2.3.7 Corollary. \(B_m \equiv B_k^n\) iff \(m = k\) or \(m \geq 2^n - 1\).

Hence, summarizing,

2.3.8 Theorem. The binary tree \((T, <)\) is \(n\)-equivalent with \(B_m\) iff either \(m < 2^n - 1\) and \(T \equiv B_m\) or \(m \geq 2^n - 1\) and \(T\) satisfies \(Q(n)\).
Corollary 2.2.4 says that finiteness of binary trees is not a first-order property. Obviously, it is (monadically) $\Pi^1_1$: by König’s lemma, a binary tree is infinite iff it has a branch without endpoint. This section is devoted to a proof of the

2.4.1 Theorem. Finiteness of binary trees is not monadically $\Sigma^1_1$-definable.

To prove this, you need a simple

2.4.2 Lemma. For all $k, n \in \mathbb{N}$ there are $p, q \in \mathbb{N}$ such that (i) $p < q$ and (ii) for all $X_1, ..., X_k \subseteq B_p$ there are $Y_1, ..., Y_k \subseteq B_q$ with

$$(B_p, X_1, ..., X_k) \equiv^n (B_q, Y_1, ..., Y_k).$$

Proof. Let $\Sigma$ be the (by 1.7.1, finite) set of $n$-characteristics of models $(T, <, X_1, ..., X_k)$ where $(T, <)$ is any finite binary tree and define $h: \mathbb{N} \to \mathcal{P}(\Sigma)$ by $h(p) = \{ \tau \in \Sigma | B_p \models \exists X_1 ... \exists X_k \tau \}$.

By the pigeon-hole principle, there are $\Delta \subseteq \Sigma$ and an infinite $A \subseteq \mathbb{N}$ such that each $p \in A$ has $h(p) = \Delta$. Now, choose $p < q$ in $A$. $\blacklozenge$

Proof of 2.4.1. Suppose that the $\Sigma^1_1$-sentence $\exists X_1 ... \exists X_k \tau$ defines finiteness of binary trees. Let $n$ be the quantifier rank of $\tau$. Let $p < q$ be given by 2.4.2. Choose $X_1, ..., X_k \subseteq B_q$ such that $(B_q, X_1, ..., X_k) \models \tau$.

I shall construct an infinite $n$-equivalent of $(B_q, X_1, ..., X_k)$, contradicting the assumption.

The construction involves a series of pairs $(T_0, Q_0), (T_1, Q_1), (T_2, Q_2), ...$
where $T_i$ is a finite binary tree (with $k$ unary relations) and $Q_i \subset T_i$ is an upward closed subset of $T_i$, the ordering on which is an isomorph of $B_q$.

This is done as follows: first, $T_0 = (B_q, X_1, \ldots, X_k)$; $Q_0 = T_0$. If $(T_i, Q_i)$ has been constructed, choose $p_{i+1} \in Q_i$ such that the tree on $P_{i+1} = \{ x \in T_i | p_{i+1} \leq x \}$ is an isomorph of $B_p$. By 2.4.2, choose an $n$-equivalent $Q_{i+1}$ of $P_{i+1}$, the tree on which is an isomorph of $B_q$. $T_{i+1}$ is obtained from $T_i$ by replacing $P_{i+1}$ by $Q_{i+1}$.

Put $F_{i+1} = T_i \setminus P_{i+1} = T_{i+1} \setminus Q_{i+1}$. Look at the picture to see how you get as far as $T_2$:

I shall need the following three simple facts about this construction.

**Claim 1.** If all elements of the sequence $a$ belong to $F_{i+1}$ then $(T_i, a) \equiv_n (T_{i+1}, a)$.

**Proof.** $T_{i+1}$ is obtained from $T_i$ by exchanging $P_{i+1}$ for $Q_{i+1}$. Now, $Q_{i+1} \equiv_n P_{i+1}$; let $\sigma$ be a winning strategy for II in the $n$-game between these models. Consider the $n$-game between $(T_i, a)$ and $(T_{i+1}, a)$. II can win this by using $\sigma$ when I plays in $P_{i+1}$ or $Q_{i+1}$ and copying I on $F_{i+1}$. □
Claim 3. If $j \leq i$, $a$ belongs to $F_j$ and $(Q_j, a) \equiv^{n-1} (T_i \setminus F_j, b)$ then 
$$(T_j, a, a) \equiv^{n-1} (T_i, a, b).$$

Proof. $T_i$ can be considered as obtained from $T_j$ by exchanging $Q_j$ for $T_i \setminus F_j$. Argue again as in the proof of claim 1. \Box

To make the construction work, the selection of $p_{i+1}$ in $Q_i$ requires some care. To explain how this is done, let $\Sigma$ be the (finite) set of $(n-1)$-characteristics of models $(A, <, Y_1, \ldots, Y_k, a)$ where $(A, <)$ is a finite tree. Choose an enumeration $f: \mathbb{N} \to \Sigma \times \mathbb{N}$ such that for all $(\sigma, j) \in \Sigma \times \mathbb{N}$ there is an $i \geq j$ with $f(i) = (\sigma, j)$. (This happens for instance if $f$ assumes each value $(\sigma, j)$ an infinite number of times.) We now require that

$$[\#_i] \text{ if } f(i) = (\sigma, j), j \leq i \text{ and for some } a^1 \in T_i \setminus F_j: \sigma = \square(T_i \setminus F_j, a^1) \equiv^{n-1}, \text{ then } p_{i+1} \text{ has been picked such that for some } a^2 \in T_i \setminus F_j: \sigma = 
\square(T_i \setminus F_j, a^2) \equiv^{n-1} \text{ and } \neg a^2 \equiv p_{i+1} \text{ - i.e., } a^2 \in F_{i+1}.$$

(To see that this can be effected, act as follows. If $a^1 \in F_i$ or if $a^1 \in Q_i$ and \{ $y \in Q_i \mid a^1 \leq y$ \} has height $> p$, then any choice of $p_{i+1}$ will do. If $a^1 \in Q_i$ and \{ $y \in Q_i \mid a^1 \leq y$ \} has height $\leq p$, pick $x < a^1$ in $Q_i$ such that \{ $y \mid x \leq y$ \} has height $p+1$. Let $p_{i+1}$ be the immediate successor of $x$ which is not $\leq a^1$.)

Now, let $T = \bigcup_i F_i$. Then $T$ is an infinite binary tree. I claim that $T \equiv^n T_0$.

To see this, consider the Ehrenfeucht $n$-game on these models. For II to win, it suffices that he chooses his counter moves in such a way that after his $m$-th move a position $(a, b)$ is reached such that, if the elements of $a$ all belong to $F_j$, then

2. $n$-equivalence of binary trees
\((T_j, a) \equiv^{\eta - m}(T_0, b)\).

This obviously is correct for the initial position \((0, 0)\) of the game, since there, \(j = 0\).

Now, suppose that after \(m\) moves for each player a position \((a, b)\) has been reached such that \([\ast]_j\) holds. Since the game is played between \(T\) and \(T_0\), player II can choose his next move in either one of these models.

(i). Suppose that II plays a next move \(a \in T\). Since the \(F_j\) are cumulative, we may assume, without restricting the generality of the argument, that \(a \in F_i\) for some \(i \geq j\). Now, by claim 2, \([\ast]_j\) holds as well. Therefore, II can find \(b \in T_0\) such that \((T_i, a, a) \equiv^{\eta - m - 1}(T_0, b, b)\) and \([\ast]_i\) has been secured.

(ii). Now, suppose I plays \(b \in T_0\). By \([\ast]_j\), \(a \in T_j\) exists with \((T_j, a, a) \equiv^{\eta - m - 1}(T_0, b, b)\). If \(a \in F_j\), then II can use \(a\) as his answer, obtaining \([\ast]_j\) for the new position. So, suppose that \(a \in T_j \setminus F_j = Q_j\). Let \(\sigma = I(Q_j, a)^{-1}\). Pick \(i \geq j\) with \(f(i) = (\sigma, j)\).

By construction, \(Q_j = T_i \setminus F_j\); hence, \(a^1 \in T_i \setminus F_j\) exists with \(\sigma = I(T_i \setminus F_j, a_1)^{-1}\). By \([\neq]_i\), there is an \(a_2 \in F_{i+1}\) such that \(\sigma = I(T_i \setminus F_j, a_2)^{-1}\).

Now, notice that, by claim 1, \((T_{i+1}, a, a_2) \equiv^{\eta}(T_i, a, a_2)\) and, by claim 3, \((T_i, a, a_2) \equiv^{\eta - 1}(T_j, a, a)\). Therefore, combining, \((T_{i+1}, a, a_2) \equiv^{\eta - 1}(T_j, a, a) \equiv^{\eta - m - 1}(T_0, b, b)\) (by choice of \(a\) above). Thus II, by playing \(a_2\), establishes \([\ast]_{i+1}\) for the new position. \(\blacksquare\)
3. Monadic $\Pi^1_1$-theories of $\Pi^1_1$-properties: linear orderings.

3.1 Introduction; $\omega$ and finite orderings.

Natural axioms of a number of theories are of the second-order ($\Pi^1_1$-) form $\forall R \varphi(R)$, where $\varphi$ is first-order and $R$ is a second-order variable. For instance, the induction principle of arithmetic, completeness of the reals, Zermelo’s Aussonderungsaxiom and the Fraenkel-Skolem replacement-axiom in set theory are of this type.

As to first-order versions of these principles, the natural option is to require $\varphi(R)$ not for all $R$ but for parametrically first-order definable $R$ only, thus replacing the second-order axiom by its corresponding first-order schema.

Obviously, the new theory will have models not allowed by the old one.
(by the Löwenheim–Skolem–Tarski theorem for instance) and hence it may turn out to be strictly weaker than its second-order companion. For instance, second-order arithmetic is categorical, hence it implies first-order sentences beyond the scope of the first-order induction schema.

On the other hand, if the language is restricted sufficiently, conservation may occur. This chapter contains a number of examples. They all concern theories of linear orderings (but see chapter 4 below where one of our examples is generalized to trees); conservation is proved with respect to monadic $\Pi^1_1$-sentences. The method of proof consists in showing how to transfer counter-examples to a $\Pi^1_1$-sentence on a "non-standard" model to a standard model. To illustrate this method, I present the simplest case, the ordering of the natural numbers $\omega$, in what follows.

It is clear what it means for an ordered set to satisfy complete induction when there is a least element and every element has an immediate successor. Definable induction requires that every definable set containing the least element and closed under immediate successors contains every element. Complete induction is the usual $\Pi^1_1$-instrument transforming a suitable set of first-order quantifier rank $\leq 3$ axioms into a categorical description of the order type $\omega$. Definable induction doesn't come close to this (cf. example 1.0.3(ii)), but it suffices for the monadic $\Pi^1_1$-theory:

3.1.1 Theorem. If $1. (M, <) \equiv^3 (\omega, <)$ and

2. $M = (M, <, x_1, \ldots, x_n)$ satisfies definable induction

then this monotonicity for inductive statements.
countable. Define \( X = \{ a \in M \mid \forall b < a \ (\{ b, a \} \text{ has a finite } n\text{-equivalent }) \} \).
(Notice that a subset is identified here with the submodel with that universe.) Now, \( X \) is a definable set: \( [b, a) \) has a finite \( n\)-equivalent iff it satisfies an \( n\)-characteristic belonging to a finite model; and of these, there are only finitely many (see 1.7.1). Hence, \( X \) is defined by the formula
\[
\forall y < x \forall \{ \tau^{[y,x]} | \tau \in \Sigma \},
\]
where \( \Sigma \) is the set of such characteristics and \( \tau^{[y,x]} \) denotes relativisation of quantifiers in \( \tau \) to the interval \([y, x)\). Trivially, \( X \) contains the least element of \( M \). Also, \( X \) is closed under immediate successors: if \( S \) is a finite \( n\)-equivalent of \([b, a)\) and \( c \) is the immediate successor of \( a \) then it is clear that the ordered sum \( S + \{ a \} \) is the required finite \( n\)-equivalent of \([b, c)\). By definable induction then, \( X = M \). Let \( a_0 \) be the least element of \( M \) and choose \( a_0 < a_1 < a_2 < \ldots \) cofinal in \( M \) (which we have assumed to be countable!). Choose a finite \( n\)-equivalent \( S_i \) of \([a_i, a_{i+1})\) for each \( i \). Then \( S = \Sigma_i S_i \) is the required \( n\)-equivalent of order type \( \omega \). (For the handling of ordered sums, cf. below, in particular 3.1.7.) ∎

Virtually the same proof works for the class of finite ordered models.

Notice that a linear ordering \((M, <)\) is finite if it contains a least and a greatest element, every non-maximal element has an immediate successor and restricted induction is satisfied, which says that every set containing the least element and closed under immediate successors (insofar as they exist) contains the greatest element as well. (Of course, other characterisations work as well.) Restricted induction brings along its first-order companion: definable restricted induction.

3. monadic \( \Pi^1_1 \) -theories: linear orderings
3.1.2 Theorem. If the linearly ordered model $M$:

1. has least and greatest element and every non-maximal element has an immediate successor;
2. satisfies definable restricted induction

then $M$ has finite $n$-equivalents for all $n$.

Proof. Begin as in the proof of 3.1.1. Definable restricted induction now shows $X$ to contain the greatest element $b$ of $M$. Thus, $[a, b)$ has a finite $n$-equivalent and so does $[a, b] = M$, as required. $\Box$

3.1.3 Examples. The following models show that we cannot strengthen the conclusions of 3.1.1–2 to: $M$ has an elementary equivalent (i.e., a model $n$-equivalent with $M$ for all $n$ simultaneously) which has order-type $\omega$ (resp. which is finite). For the second, let $M$ have order type $\omega + \omega^*$ and let the $X_i$ be empty (use 1.0.3(i) to see that this satisfies definable restricted induction). For the first, consider $M+N$ where $M$ is the previous model and $N$ has order type $\omega$ - but $X_0 = N$ this time (again, use 1.0.3(i)). The bigger $n$, the longer an $n$-equivalent of $M$ has to be (namely, at least $2^n-1$ by 1.0.1.) and hence the larger the first element of $X_0$ in an $n$-equivalent of $M+N$.

3.1.4 Remark. The direct method of proof of 3.1.1–2 works for the class of well-ordered models too; instead, I shall derive that result from the corresponding one for the order-complete models in section 3.3.

In the following basic theorem, $\Sigma$ is a set of first-order sentences in a language $L$ and $\forall R \varphi (R)$ is a $\Pi^1_1$-sentence over $L$.

Let $X_1, \ldots, X_k$ be new unary relation-symbols and $L_k = L \cup \{X_1, \ldots, X_k\}$. 
$L_k$-formulas obtained from $\varphi$ by replacing each occurrence $R(t_1, \ldots, t_m)$ by some fixed $L_k$-formula $\eta(t_1, \ldots, t_m)$ (taking measures against clash of variables). Thus, $L_k$-definably-$\varphi$ intuitively requires $\varphi(R)$ only when $R$ is (parametrically) first-order definable in the language $L_k$.

The union over all $k$ of these schemata is the first-order schema corresponding to $\forall R \varphi(R)$.

3.1.5 Theorem. The following two conditions are equivalent:

(i) for each first-order formula $\psi = \psi(X_1, \ldots, X_k)$ in the language $L_k$: if $\Sigma + \forall R \varphi(R) \models \forall X_1 \ldots \forall X_k \psi$, then $\Sigma + L_k$-definably-$\varphi \models \psi(X_1, \ldots, X_k)$;

(ii) each model $(M, U_1, \ldots, U_k)$ of $\Sigma + L_k$-definably-$\varphi$ has an $n$-equivalent satisfying $\Sigma + \forall R \varphi(R)$ for each $n$.

Proof. (i) $\Rightarrow$ (ii): let $(M, U_1, \ldots, U_k) \models \Sigma + L_k$-definably-$\varphi$ have the $n$-characteristic $\mathbb{E}[\varphi]^n = \tau(X_1, \ldots, X_k)$. We want a model of $\Sigma + \forall R \varphi(R) + \exists X_1 \ldots \exists X_k \tau$. If such a model does not exist, then $\Sigma + \forall R \varphi(R) \models \forall X_1 \ldots \forall X_k \neg \tau$; hence by (i) $\Sigma + L_k$-definably-$\varphi \models \neg \tau(X_1, \ldots, X_k)$, contradicting the assumptions on $(M, U_1, \ldots, U_k)$.

(ii) $\Rightarrow$ (i): assume $\Sigma + \forall R \varphi(R) \models \forall X_1 \ldots \forall X_k \psi$ and let $(M, U_1, \ldots, U_k)$ be a model of $\Sigma + L_k$-definably-$\varphi$. By (ii), there is an $n$-equivalent satisfying $\forall R \varphi(R) + \Sigma$ where we take $n$ to be the quantifier rank of $\psi$. By assumption, this $n$-equivalent satisfies $\forall X_1 \ldots \forall X_k \psi$, hence $\psi(X_1, \ldots, X_k)$ is also satisfied, so $(M, U_1, \ldots, U_k)$ must satisfy this formula as well.

Below, the $\Sigma$ of the theorem will always be finite. Therefore we may require that the $n$-equivalent of (ii) satisfies $\forall R \varphi(R)$ only, without

3. monadic $\Pi^1_1$-theories: linear orderings
invalidating the truth of (ii) $\Rightarrow$ (i): simply let $n$ be at least the maximum of quantifier ranks of formulas in $\Sigma$.

In what follows, results of type (ii) are proved. According to the theorem, this shows that, in the context of $\Sigma$, the first-order schema corresponding to $\forall R \varphi(R)$ suffices to prove all monadic $\Pi^1_1$-consequences of this second-order statement. (Actually, theorem 3.3.9 below does a little better.)

All models encountered here will have the form $M = (M, <, U_1, \ldots, U_k)$, where $<$ linearly orders $M$ and $U_1, \ldots, U_k \subseteq M$. If $X \subseteq M$, then $X$ (and sometimes $X$ as well) denotes the submodel of $M$ with universe $X$, i.e., $X = M_X$. $I \subseteq M$ is an interval if $x < y < z$ and $x, z \in I$ imply $y \in I$; notations like $(x, z)$ and $[x, z)$ denote specific intervals as usual.

If $I$ is an ordered set and $m$ a function on $I$ associating a model $m(i)$ with every $i \in I$, we may form the ordered sum $\sum_{i \in I} m(i)$, being the model obtained from the $m(i)$ by glueing (disjoint copies of) them one after the other according to the ordering of $I$. Formally, $\sum_{i \in I} m(i)$ can be defined as the model with universe $\bigcup_{i \in I} (m(i) \times \{i\})$ with the ordering defined by: $(a, i) < (b, j)$ iff $i < j$, or: $i = j$ and $a < b$ (here, $<_i$ and $<_I$ denote the orderings of $m(i)$ resp. $I$); and if $U^i_n$ is the $n$-th unary relation of $m(i)$ ($1 \leq n \leq k$) then $U^i_n = \bigcup_{i \in I} (U^i_n \times \{i\})$ is the corresponding one of the ordered sum.

A condensation of an ordered model $M$ is a partition of $M$ into intervals. Any condensation $P$ of $M$ inherits an ordering from $M$ by putting, for $p, q \in P$: $p < q$ iff for some $a \in p$ and $b \in q$ (equivalently, iff for all $a \in p$ and $b \in q$: $a < b$. Hence, a condensation $P$ of $M$ is nothing but a way to write $M$ as an ordered sum $M = \sum_{p \in P} p$. 
3.1.6 Lemma. Let \( R \) be any transitive binary relation on the ordered model \( M \). Define \( \sim = \sim_R \) by: \( a \sim b \) iff one of the following holds:

(i) \( a = b \)

(ii) \( a < b \) and for all \( c, d \) such that \( a \leq c < d \leq b \): \( c \sim d \)

(iii) \( b < a \) and for all \( c, d \) such that \( b \leq c < d \leq a \): \( c \sim d \).

Then \( \sim \) induces a condensation.

Proof. Straightforward. \( \Box \)

All condensations used in the sequel are defined in this fashion.

3.1.7 Lemma. If for all \( i \in I \) \( m(i) \equiv^n m'(i) \), then \( \Sigma_{i \in I} m(i) \equiv^n \Sigma_{i \in I} m'(i) \).

Proof. It is straightforward to describe a winning strategy for the second player in the Ehrenfeucht \( n \)-game between these sums under the condition given. \( \Box \)

The following generalisation of 3.1.7 is needed in section 3.3.

3.1.8 Lemma. Suppose that \( I \) and \( J \) are ordered sets and that \( m \) and \( m' \) associate ordered models \( m(i) \) resp. \( m'(j) \) to each \( i \in I \) resp. \( j \in J \) such that:

\( (*) \quad (I, \{ i \mid m(i) \models \sigma \})_{\sigma \in \Sigma} \equiv^n (J, \{ j \mid m'(j) \models \sigma \})_{\sigma \in \Sigma} \)

where \( \Sigma \) is the set of \( n \)-characteristics.

Then \( \Sigma_{i \in I} m(i) \equiv^n \Sigma_{j \in J} m'(j) \).

3. monadic \( \Pi^1_1 \)-theories: linear orderings
Proof. Use the Ehrenfeucht game-technique. If the first player chooses, say, \( a \in \Sigma, m(i) \), the second player locates the \( i \in I \) for which \( a \in m(i) \), then uses (*) to find a \( j \) corresponding to \( i \); in particular, \( m'(j) =^a m(i) \), and a counter move is readily found; etc. \( \Box \)

3.1.9 Examples where 3.1.5(ii) fails.

1. (van Benthem.) Consider the \( \Pi^1_1 \)-statement \( \forall X \varphi(X) \) in the language of \( < \) where \( \varphi(X) \) says: \( X \) and its complement cannot both be cofinal. Obviously, every ordered model of \( \forall X \varphi(X) \) has a greatest element. On the other hand, the first-order schema corresponding to \( \varphi \) does not imply this. A counter-model is \( (\omega, <) \): notice that each definable set here is either finite or cofinite. (Proof: Use 1.0.3(ii). Let \( \psi(x) \) be any formula in the free variable \( x \). If no \( a \in \zeta \) satisfies \( \psi \) in \( \omega + \zeta \) then \( \exists y \forall x (y < x \rightarrow \neg \psi) \) holds in \( \omega + \zeta \); therefore it holds in \( \omega \) by 1.0.3(ii) and the set defined by \( \psi \) in \( \omega \) must be finite. On the other hand, if some \( a \in \zeta \) satisfies \( \psi \) in \( \omega + \zeta \) then every \( a \in \zeta \) satisfies \( \psi \) in \( \omega + \zeta \) - this is because for each pair \( a, b \in \zeta \) there is an automorphism \( h \) of \( \omega + \zeta \) such that \( ha = b \). Hence, \( \exists y \forall x (y < x \rightarrow \psi) \) holds in \( \omega + \zeta \); therefore, it holds in \( \omega \) and the set defined by \( \psi \) in \( \omega \) must be cofinite. \( \Box \))

2. In theories defining a pairing, the restriction to monadic languages is only apparent and results like ours can fail badly. We mentioned the case of arithmetic; also, each model of set theory certainly is definably well-founded, nevertheless such models need not have a well-founded \( n \)-equivalent for \( n \) large enough: well-founded models have standard integers, therefore they are arithmetically correct; but, Gödel-sentences are arithmetical.
satisfies \textit{definable well-foundedness} (cf. chapter 4). Must it have a
well-founded $n$-equivalent for each $n$? (The method of chapter 4 does not
suffice; the objection of 3.1.9.2 does not apply.)

3.2 Monadic $\Pi^1_1$-theory of scattered orderings.

A linear ordering $M=(M, \prec)$ is called \textit{scattered} if it does not embed
the ordering $(\mathbb{Q}, \prec)$ of the rationals.

$\mathbb{Q}$ embeds \textit{every} countable ordering; in particular, it embeds $\omega^*$. It
follows that every well-ordering is scattered. I shall need the

3.2.1 Lemma. A \textit{scattered ordered sum of scattered orderings is}
scattered.

\textit{Proof.} Suppose $\mathbb{Q} \subseteq \sum_{i \in I} m(i)$. If some $\mathbb{Q} \cap m(i)$ contains at least \textit{two}
rationals, it contains the interval between them and so $m(i)$ cannot be
scattered. Hence, sending $p \in \mathbb{Q}$ to the $i \in I$ for which $p \in m(i)$ embeds $\mathbb{Q}$ in $I$, a
contradiction. $\square$

There is more than one way to formalize scatteredness into a
$\Pi^1_1$-statement and not every formalisation is a good one from our point of
view.

3.2.2 Example. Let $\delta$ express that $\prec$ is a dense ordering containing at
least two elements. $\varphi(X)$ is the formula obtained from $\neg \delta$ by relativizing
quantifiers to membership in (the set) X. Clearly, M is scattered iff it satisfies \( \forall X \varphi(X) \). Here is an example of a model \( M=\langle M, <, X, Y, Z \rangle \) which is definably-\( \varphi \) but has no scattered 3-equivalent. Partition \( Q \) into dense subsets \( R, S, T \) and put \( M=\Sigma_{q \in Q} M_q \), where \( M_q=\langle \mathbb{Z}, <, X^q, Y^q, Z^q \rangle \) and \( X^q = \mathbb{Z} \) if \( q \in R \); \( X^q \) is empty otherwise; similarly, \( Y^q = \mathbb{Z} \) or \( =\varnothing \) depending on whether \( q \in S \) or not; and \( Z^q = \mathbb{Z} \) or \( =\varnothing \) depending on whether \( q \in T \). Notice that each interval \( M_q \) of \( M \) is a set of indiscernibles of \( M \) (use automorphisms of \( M_q \)) hence, if \( A \) is a definable set of \( M \), either \( A \cap M_q = \varnothing \) or \( M_q \subset A \). Therefore, no non-empty definable set of \( M \) is densely ordered and it follows that \( M \) is definably-\( \varphi \). On the other hand, the fact that \( M \) satisfies sentences such as \( \forall x \in X \forall y \in Y (x < y \rightarrow \exists z \in Z (x < z < y)) \) shows that no 3-equivalent of \( M \) can be scattered. 

A "good" formalisation of scatteredness should avoid this counter-example.

3.2.3 Lemma. An ordering is scattered iff it has no densely ordered condensation.

Proof. Only if: use the axiom of choice. If: suppose that \( Q \subset M \). Define \( \sim \) by way of 3.1.6 where \( aRb \) iff \( a < b \) and \( (a, b) \cap Q \) is finite. It is easy to see that \( \sim \) induces a dense condensation.

The (dyadic!) \( \Pi^1_1 \)-characterisation of scatteredness contained in the lemma is a "good" one according to the following theorem, where we call a model definably scattered if no definable equivalence partitions \( M \) into a dense ordering of intervals. Notice that the model of 3.2.2 is not
Proof. I use what Rosenstein [1982] calls a condensation-argument, originating with Hausdorff. Define \( \sim \) in the fashion of 3.1.6 with \( aRb \) meaning that \((a, b)\) has a scattered \( n \)-equivalent (if \( a < b \)). By 3.2.1, \( R \) is transitive. Hence, \( \sim \) induces a condensation by 3.1.6.

Also, \( \sim \) is definable (compare the proof of 3.1.1): there are only finitely many \( n \)-characteristics; let \( \Gamma \) be the (finite) set of \( n \)-characteristics belonging to scattered models. Then \((c, d)\) has a scattered \( n \)-equivalent iff \( M \models \bigvee_{\tau \in \Gamma} \tau^{(c, d)} \) where \( \tau^{(c, d)} \) is obtained from \( \tau \) by relativizing quantifiers to membership in \((c, d)\). It is now clear that \( \sim \) can be defined as well.

Claim 1: each equivalence class has a scattered \( n \)-equivalent.

Proof: let \( I \) be an equivalence class and \( a \in I \).

(i) \( I \) has a greatest element \( b \). Then \( a \sim b \) and \( I^{>a} = \{ x \in I | a \leq x \} = [a, b] \) has a scattered \( n \)-equivalent by definition.

(ii) If not, choose a sequence \( a_0 = a < a_1 < \ldots < a_{\xi} < \ldots (\xi < \alpha) \) cofinal in \( I \). Each \([a_{\xi}, a_{\xi+1})\) and, hence, each \([a_{\xi}, a_{\xi+1})\) has a scattered \( n \)-equivalent \( A_{\xi} \). Hence \( I^{>a} = \Sigma_{\xi<\alpha} [a_{\xi}, a_{\xi+1}) \) has the \( n \)-equivalent \( \Sigma_{\xi<\alpha} A_{\xi} \) by 3.1.7 which, by 3.2.1, is scattered.

Argue similarly for \( I^{<a} = \{ x \in I | x < a \} \); so, \( I = I^{<a} + I^{>a} \) has a scattered \( n \)-equivalent. \( \Box \)

Claim 2: the induced ordering of the equivalence classes is dense.

Proof: suppose \( I < J \) are equivalence classes and no equivalence class is between \( I \) and \( J \). Let \( a \in I \) and \( b \in J \); suppose that \( a < c < d < b \). Then \((c, d)\) has

3. monadic \( \Pi^1_1 \)-theories: linear orderings
a scattered $n$-equivalent: if $c, d \in I$ or $c, d \in J$ this is clear; and if $c \in I$ and $d \in J$ we know from the argument above that $I^{>c}$ and $J^{<d}$ have scattered $n$-equivalents; but, $(c, d) = I^{>c} \cup J^{<d}$. Therefore, $a \sim b$, a contradiction. ∎

Since $M$ is definably scattered, $\sim$ cannot have more than one equivalence class: $M$ itself. Consequently, $M$ must have a scattered $n$-equivalent by the first claim. ∎

3.2.5 Remark. By 3.2.2 and 3.2.3, we have two $\Pi^1_1$-formalisations of scatteredness; however, the first-order schema corresponding to the second one (definable scatteredness) is strictly stronger than the first-order schema belonging to the first.

3.3. Monadic $\Pi^1_1$-theory of complete orderings, of well-orderings and of the reals.

The ordering $(M, \leq)$ is complete if each non-empty set with an upper bound has a least upper bound (a sup). Hence, $M$ is called definably complete if this holds for definable sets.

3.3.1 Theorem. If $M$ is definably complete, it has complete $n$-equivalents for each $n$.

Before proving 3.3.1, here is an example and a corollary.
the conclusion of 3.3.1 to requiring an elementary equivalent of $M$.

Choose rationals $q_0 < q_1 < q_2 < \ldots$ and $r_0 > r_1 > r_2 > \ldots$ such that $\lim q_i = \lim r_i$ is irrational; take $A = \{ q_i \mid i \in \mathbb{N} \} \cup \{ r_i \mid i \in \mathbb{N} \}$ and consider $M = (\mathbb{Q}, <, A)$. For each $n$, the models $(\mathbb{R}, <, \{1, \ldots, m\})$ for $m \geq 2^n - 1$ are $n$-equivalents of $M$ (use 1.0.1). On the other hand, suppose that $(N, <, B)$ is a complete elementary equivalent of $M$. It follows that $B$ has order-type $\omega + \alpha$ for some $\alpha$. $N$ must contain a sup of the first $\omega$ elements of $B$. However, $M$ lacks an element which is a limit of $A$'s - a contradiction. ❇️

$M$ is (definably) well-ordered if each non-empty subset of $M$ (which is parametrically first-order definable on $M$) has a least element.

The following trivial lemma may look surprising, as completeness usually is considered only in the context of dense orderings.

3.3.3 Lemma. $M$ is (definably) well-ordered iff it is (definably) complete, has a least element, and every non-maximal element has an immediate successor.

Proof. Suppose $\emptyset \neq X \subseteq M$ and $X$ has no minimum. Put $Y = \{ y \in M \mid \forall x \in X (y < x) \}$. $Y$ is definable if $X$ is definable. Since the least element of $M$ must be in $Y$, $Y$ is non-empty; moreover, every $x \in X$ is an upper bound of $Y$. Thus, $Y$ has a sup $y$. If $y \in Y$, the immediate successor of $y$ is minimal in $X$. Hence, $y \notin Y$. But then, $y$ must be minimal in $X$, a contradiction. ❇️

3.3.4 Corollary. If $M$ is definably well-ordered, it has well-ordered $n$-equivalents for each $n$.

3. monadic $\Pi^1_1$ -theories: linear orderings
Proof. Notice that 3.3.3 defines well-order as completeness plus a quantifier rank 3-statement. By 3.3.3, $M$ is definably complete. Thus, let $m = \max(n, 3)$ and take $N$ to be a complete $m$-equivalent of $M$ by 3.3.1. By 3.3.3 again, $N$ is the model required. 

We say that the sum $\Sigma_{i \in I} m(i)$ is completely ordered if the ordering of $I$ is complete.

3.3.5 Lemma. (i) Completely ordered sums of complete orderings with endpoints are complete.

(ii) Well-ordered sums of complete orderings with least elements are complete.

Proof. (i): Let $X \subseteq \Sigma_{i \in I} m(i)$ have an upper bound in $m(i_0)$. Then $J = \{ j \mid X \cap m(j) \neq \emptyset \}$ has the upper bound $i_0$. Let $j = \sup J$. (a) $j \in J$. Then $\max m(j)$ is an upper bound for $X \cap m(j)$ and $\sup X = \sup (X \cap m(j))$. (b) $j \notin J$. Then $\sup X = \min m(j)$. (ii): similar. 

Proof of 3.3.1. Define $\sim$ in the fashion of 3.1.6 with $aRb$ meaning: $a < b$ and $(a, b)$ has a complete $n$-equivalent.

Notice that $R$ is transitive. Hence, $\sim$ induces a condensation by 3.1.6. (N.B. this would not have been so obvious in case we would have defined $x \sim y$ to mean that $(x, y)$ had a complete $n$-equivalent only.)

Furthermore, $\sim$ is definable: compare the proof of 3.2.4. Hence, the equivalence classes are definable as well.

Claim 1: each equivalence class with an upper (lower) bound has a
Proof: let $I$ be an equivalence class and $a \varepsilon I$.

If $I$ has no upper bound, choose $a_0 = a < a_1 < a_2 < \ldots < a_\xi < \ldots$ (for $\xi < \alpha$) cofinal in $I$. Choose a complete $n$-equivalent $N_\xi$ of $[a_\xi, a_{\xi+1})$ for each $\xi < \alpha$. Then $\sum_{\xi < \alpha} N_\xi$ is a complete $n$-equivalent of $\sum_{\xi < \alpha} [a_\xi, a_{\xi+1}) = \{ x \varepsilon I | a \leq x \} = I \succ a$. If $I$ has an upper bound, it must have a sup $s$ by definable completeness. I claim that $s \varepsilon I$ (and so, $s$ is the maximum of $I$, $I \succ a = [a, s]$ and hence $(a, s)$ and, therefore, $I \succ a$ as well, have a complete $n$-equivalent by definition). For if not, choose $a_0 = a < a_1 < a_2 < \ldots < a_\xi < \ldots$ cofinal in $I$ again to show that $(a, s)$ has a complete $n$-equivalent as before.

Much the same goes for the other half $I \prec a = \{ x \varepsilon I | x < a \}$ of $I$, and so the claim has been proved. $\Box$

Claim 2: the induced ordering on the class $M/\sim$ of equivalence classes is dense.

Proof: suppose $I \prec J$ are neighbours in $M/\sim$.

Then $a = \sup I$ and $b = \inf J$ are neighbours in $M$; moreover, $a \varepsilon I$ and $b \varepsilon J$.

Hence, $(a, b)$ is empty; therefore, $a \sim b$ – a contradiction. $\Box$

If there is but one equivalence class, we are done. So, assume not. The rest of the proof works towards a contradiction.

The following argument is taken from Rosenstein [1982] (thm. 7.17, p. 117). Choose a complete $n$-equivalent $\tau(I)$ for each $I$ with $I \varepsilon M/\sim$ in such a way that $T = \{ \tau(I) | I \varepsilon M/\sim \}$ is finite (this is possible by 1.7.1). Now, if for some $\sigma \varepsilon T$, $\{ I \varepsilon M/\sim | \tau(I) = \sigma \}$ is not dense in the ordering of $M/\sim$, there must be a proper interval $C_0 \subset M/\sim$ such that no $I \varepsilon C_0$ has $\tau(I) = \sigma$. Repeating this argument (first with $C_0$ and $T \backslash \{ \sigma \}$ etc.) using induction on the finite cardinal $|T|$, one ultimately arrives at the following
Claim 3: there is a proper (open) interval $D$ of $M/\sim$ and a set $\Sigma \subset T$ such that

(i) every $I \in D$ has $\tau(I) \in \Sigma$, and

(ii) if $\sigma \in \Sigma$ then $\{I \in D | \tau(I) = \sigma\}$

is dense in $D$.

The contradiction aimed for is contained in the next

Claim 4: $D$ has but one element.

Proof: suppose that $a, b \in \bigcup D$ and $a < b$. We need to show that $(a, b)$ has a complete $n$-equivalent. Suppose that $a \in I, b \in J$. If $I = J$, there is nothing to prove. Let $E$ be the interval $(I, J)$ in $D$. Now, $(a, b) = I^> + \bigcup E + J^<$; therefore it suffices to show that these components have complete $n$-equivalents. For $I^>$ and $J^<$, this is known already (cf. the proof of claim 1). Therefore, it remains to show that $\bigcup E$ has such an $n$-equivalent as well.

First, notice that claim 3 remains valid if we replace $D$ by $E$.

Now, construct a complete $n$-equivalent $N$ of the submodel $\bigcup E = \sum_{I \in E} I$ of $M$ as follows: let $h: \mathbb{R} \rightarrow \Sigma$ be any partition of $\mathbb{R}$ into $|\Sigma|$ classes $\{x \in \mathbb{R}| h(x) = \sigma\}$ ($\sigma \in \Sigma$) each of which is dense in $\mathbb{R}$ and put $N = \sum_{x \in \mathbb{R}} h(x)$.

By 3.3.5 and claim 1, $N$ is completely ordered. It remains to show that $N$ is $n$-equivalent to $\bigcup E$.

First, notice that the models $(E, <, \{I \in E | \tau(I) = \sigma\})_{\sigma \in \Sigma}$ and $(\mathbb{R}, <, \{x \in \mathbb{R}| h(x) = \sigma\})_{\sigma \in \Sigma}$ (with $|\Sigma|$ unary relations each) are partially isomorphic and a fortiori $n$-equivalent. (The argument for dense orderings is well-known; the extra structure involved here — partitions into $|\Sigma|$-many dense sets — doesn’t complicate it terribly much.) The result now follows from 3.1.8. \[ \square \]
type of the set of reals. The following example shows that we cannot strengthen the conclusion of 3.3.1 by requiring the \( n \)-equivalent to be of type \( \lambda \) under the assumption that the ordering of \( M \) is dense.

3.3.6 Example. For \( x \in \mathbb{R} \), let \( m(x) = ([0,1], <, \varnothing) \) if \( x \) is rational, and \( m(x) = ([0,1], <, [0,1]) \) otherwise. Consider \( M = \sum_{x \in \mathbb{R}} m(x) \).

\( M \) has the complete order type \( (1 + \lambda + 1) \cdot \lambda \) (cf. 3.3.5) - so it certainly is definably complete. On the other hand, the proof of lemma 3.3.8 below shows that it lacks a \( 5 \)-equivalent of order-type \( \lambda \): each complete \( 5 \)-equivalent of \( M \) has a definable equivalence splitting the model in an uncountable number of proper intervals - contradicting the Suslin property of \( \mathbb{R} \). \( \Box \)

Hence, the Suslin property of \( \mathbb{R} \) contributes to its monadic \( \Pi^1_1 \)-theory. The following definition, suggested by 3.3.6, isolates this contribution:

3.3.7 Definition. \( M \) has property I if each densely ordered condensation of \( M \) has a dense set of singletons.

3.3.8 Lemma. Models of order type \( \lambda \) and, more generally, all complete orderings with the Suslin property have property I.

Proof. Suppose that \( P \) is a densely ordered condensation of a Suslin ordering. Suppose that \( p < q \) in \( P \) but \( (p, q) \) does not contain a singleton. By Suslinity, \( (p, q) \) must be countable; hence, it has the order type of the rationals. Therefore, \( (p, q) \) has as many bounded sets without \( \sup \) as there are irrationals. Let \( K \) be such a set. Then \( \bigcup K \) is a bounded set in the

\[ \text{3. monadic } \Pi^1_1 \text{-theories: linear orderings} \]
original ordering without sup. □

3.3.9 Theorem. If M is definably-I, definably complete and densely ordered without endpoints, then it has n-equivalents of order type $\lambda$ for each n.

Proof. First, we follow the proof of 3.3.1 with some slight modifications. To begin with, we may assume by the Löwenheim–Skolem theorem that M is only countable.

Now, define $\sim$ by the scheme of 3.1.6 with aRb meaning: $a < b$ and $(a, b)$ has an n-equivalent of order type $\lambda$. Again, $R$ is transitive, so $\sim$ induces a condensation by 3.1.6.

Claim 1: each equivalence class has an n-equivalent of one of the following types: $1, \lambda+1$ (if it begins $M$), $1+\lambda$ (if it ends $M$), $\lambda$ (if it does both) or $1+\lambda+1$.

Proof: much as before. Notice that we need to form countable sums only since $M$ is countable, thus preserving separability of the models involved – thereby guaranteeing one of the order types required. □

Claim 2: $M/\sim$ is densely ordered.

Proof: use that the ordering of $M$ is dense! □

Again, it suffices to show that $M$ is the only class in $M/\sim$. Suppose it is not.
models of order type either 1 or 1+\lambda+1 such that

(i) every \(I \in D\) has an \(n\)-equivalent in \(\Sigma\); and

(ii) if \(\alpha \in \Sigma\) then \(\{I \in D | I \equiv^n \alpha\}\) is dense in \(D\).

Proof: as before. \(\Box\)

In order to reach the desired contradiction, and stepping over some obvious details (cf. the proof of 3.3.1, claim 4), construct an \(n\)-equivalent \(N\) of \(\bigcup D\) of order type \(\lambda\) as follows:

Since \(M\) is definably-I, \(M/\sim\) has a dense set of singletons; hence \(\Sigma\) contains a singleton model \(\tau_0\). Take \(h: \mathbb{R} \to \Sigma\) partitioning \(\mathbb{R}\) into \(|\Sigma|\) classes \(\{x \in \mathbb{R} | h(x) = \alpha\}\) (\(\alpha \in \Sigma\)) each of which is dense and such that \(\{x \in \mathbb{R} | h(x) = \tau_0\}\) happens to be the set of irrationals.

Put \(N = \Sigma_{x \in \mathbb{R}} h(x)\).

By 3.3.5, \(N\) is complete as before and it is easy to see that \(N\) has a countable dense set this time - whence \(N\) has the order type \(\lambda\).

That \(N \equiv^n \bigcup D\) follows as before, using 3.1.8. \(\Box\)

3.3.10 Corollary. Every ordering which has I, is complete and is densely ordered without endpoints satisfies the monadic \(\Pi^1_1\)-theory of \(\mathbb{R}\).

3.3.11 Example. For each ordinal \(\alpha\), \(\lambda+(1+\lambda) \cdot \alpha\) has the required properties (and differs from \(\lambda\) for \(\alpha \geq \omega_1\), since \(\omega_1\) is not \(\leq \lambda\)).

3.3.12 Remark. Kunen [1966] §17 contains a \(\Pi^1_1\)-characterization of \((\mathbb{R}, <)\).

3. monadic \(\Pi^1_1\)-theories: linear orderings
3.4. Appendix: strengthening 3.2.4 and 3.3.4.

Let $M_0$ be the smallest class of order types such that

1. $1 \in M_0$
2. $\alpha, \beta \in M_0 \Rightarrow \alpha + \beta \in M_0$
3. $\alpha \in M_0 \Rightarrow \alpha \cdot \omega, \alpha \cdot \omega^* \in M_0$.

By 3.2.1, all types in $M_0$ are scattered.

By a theorem of Laëuchli and Leonard (cf. Rosenstein [1982] thm.7.9 p.115) $M_0$ contains $n$-equivalents for each scattered ordering and for all $n$. (Shelah [1975] notes that almost the same proof shows this to be the case relative to the monadic second-order language as well.)

Their method of proof shows that the extra unary relations of our models do not spoil this situation:

3.2.4' Theorem. If $M$ is definably scattered, it has (scattered) $n$-equivalents with order type in $M_0$ for each $n$.

Proof. On $M$, define $\sim$ by way of 3.1.6 with $aRb$ meaning: $a < b$ and $(a, b)$ has an $n$-equivalent with order type in $M_0$. By (1), (2), $R$ is transitive, so $\sim$ induces a condensation.

Claim 1: each equivalence class has an $n$-equivalent with order type in $M_0$.

Proof: if the class $I$ is unbounded, choose $a < a_0 < a_1 < a_2 < \ldots$ cofinal in $I$ (by Löwenheim-Skolem, assume $M$ is countable). For $i < j$, let $h(i, j)$ be the $n$-characteristic of $[a_i, a_j)$. By Ramsey's theorem, there is an infinite set
$N_0 + N \cdot \omega$ and this model has order type in $M_0$ by (2), (3). The same goes for $I^a$, etc. 

As before, $M/\sim$ must either be dense or consist of one class only; since the first alternative cannot obtain, the proof is finished. ☐

Next, let $K$ be the smallest class of order types such that

1. $1 \in K$
2. $\alpha, \beta \in K \Rightarrow \alpha + \beta \in K$
3. $\alpha \in K \Rightarrow \alpha \cdot \omega \in K$.

Clearly, $K \subseteq M_0$. All types in $K$ are well-ordered and it is easy to see (using Cantor normal forms) that $\alpha \in K$ iff $0 < \alpha < \omega^\omega$.

$K$ contains $n$-equivalents for each well-ordering and for all $n$ (it is easy to see that no smaller class has this property).

Again, extra unary relations do not change this state of affairs:

3.3.4: Theorem. If $M$ is definably well-ordered, it has (well-ordered) $n$-equivalents with order-type in $K$ for each $n$.

Proof. Define $X = \{ a | \forall b < a [b, a) \text{ has an } n\text{-equivalent with order type in } K \}$. $X$ is definable, hence if $X \neq M$ then $M \setminus X$ has a least element $a$. Pick $b < a$ such that $[b, a)$ has no $n$-equivalent with type in $K$. By (1), (2), $a$ cannot be a successor. Now, choose $b < b_0 < b_1 < \ldots$ cofinal in $[b, a)$ and argue as in the previous proof. ☐

3. Monadic $\Pi^1_1$-theories: linear orderings

\[ \omega^\omega \equiv (\text{OR}, <) \] (where OR is the class of all ordinals).

Proof. If I starts an \((n+1)\)-game with a move \(\alpha \in \text{OR}\), II answers with an \(n\)-equivalent \(\beta \in \omega^\omega\). (Notice that \(\alpha \uparrow \equiv \text{OR}\) and \(\beta \uparrow \equiv \omega^\omega\).) \(\Box\)

The classes \(M_0\) and \(K\) were inductively defined by closure properties obtained by looking at what it takes to prove 3.2.4 and 3.3.4. In the same way, one may find, e.g., a class \(C\) of order types such that each completely and densely ordered model without endpoints has \(n\)-equivalents with types in \(C\); closure properties needed here are

1. \(\lambda \in C\)
2. \(\alpha, \beta \in C \Rightarrow \alpha + 1 + \beta \in C\)
3. \(\alpha \in C \Rightarrow (\alpha + 1) \cdot \omega, (1 + \alpha) \cdot \omega^* \in C\)
4. If \(h: \mathbb{R} \rightarrow C\) has \(h(\mathbb{R})\) finite and all \(\{x \in \mathbb{R} | h(x) = \sigma\} (\sigma \in h(\mathbb{R}))\) dense in \(\mathbb{R}\) then \(\sum_{x \in \mathbb{R}} (1 + h(x) + 1) \in C\).
4. Monadic $\Pi^1_1$-theory of well-founded trees.

The previous chapter dealt with linearly ordered models only; the scope is widened here somewhat to the notion of a tree. A partially ordered set $M = (M, \prec)$ is called a tree iff, for each $m \in M$, the set $\downarrow m = \{ m' \in M | m' \prec m \}$ is linearly ordered.

The $\Pi^1_1$-property considered here is well-foundedness: $M$ is well-founded iff each non-empty set has a minimal element; equivalently, when $M$ is a tree: iff each $\downarrow m$ is well-ordered.

Definable well-foundedness, of course, restricts this to definable sets.

The proof of theorem 4.1 below can be considered as a paradigm for a method applicable in a variety of situations where the models considered belong to certain types of partial orderings (trees being the simplest
example) and the $\Pi^1_1$-property involved can be either well-foundedness, converse well-foundedness, or, more generally, some kind of completeness as in section 3.3. It did not seem useful however to aim for this greater generality here, as a most general result probably does not exist and the generalizations obtained of the result below all appeared to be rather arbitrary.

4.1 Theorem. If $M$ is a tree with finitely many extra unary relations which is definably well-founded, then it has well-founded $n$-equivalents for all $n$.

Before embarking on the proof, we need some surgical terminology on trees and three lemmas.

A component of $M$ is a maximal connected subset. An element of $M$ is minimal iff it is the least element of its component. In particular, components are (first-order parametrically) definable. Therefore, if $M$ is definably well-founded, so are its components. I do not know whether the converse of this holds, though - and this makes for some complications in the formulation of the lemmas below.

If $X \subset M$ is downward closed (i.e., if $a \leq b \in X$ implies $a \in X$) then $M \setminus X$ is upward closed, and vice versa.

I shall use the following notations. If $X \subset M$ then $(M, X)$ denotes the expansion of $M$ obtained by adding $X$ as a new unary relation.

**Warning.** If $a \in M$ then $a^\uparrow$ equals $\{ c \in M \mid a \leq c \}$ in this chapter. Thus, $a \in a^\uparrow$, but $a \notin a^\downarrow$. 
If \( a < b \) in \( M \) then, for each \( n \), \(((a↑)\setminus(b↑),[a,b])\) has an \( n \)-equivalent \((N,\beta)\) such that \( \beta \) is well-ordered and all components of \( N\setminus\beta \) are definably well-founded.

Thus, \( N \) can be used as a substitute (within \( n \)-equivalence) of the part \((a↑)\setminus(b↑)\) of \( M \), thereby exchanging \([a,b]\) for the well-ordered \( \beta \) and preserving definable well-foundedness of the rest – component-wise. The other two lemmas are similar in spirit. The proof of 4.1 finally will show how to carry out such substitutions repeatedly, thereby eventually arriving at the desired well-founded \( n \)-equivalent. To see that such substitutions actually work, the following remark is needed.

**Remark.** In what follows, a lot of cutting and pasting of trees has to be performed. To see that in each case \( n \)-equivalence is preserved, the Ehrenfeucht-game technique can be applied in much the same way as in the proof of claim 1 in the proof of theorem 2.4.1. The general procedure is as follows. Suppose that \( M' \) is obtained from \( M \) by exchanging some part \( N \) by an \( n \)-equivalent \( N' \). In all cases occurring, it will be clear how this exchange-process has to be performed since the way \( N \) is "attached" to \( M\setminus N \) will be particularly simple. Let \( f \) be the identity-map on \( M\setminus N \). Now,
suppose that, for each partial isomorphism \( h \) between \( N \) and \( N' \), the union \( f \cup h \) is a partial isomorphism between \( M \) and \( M' \). (In applications it always will be rather obvious that this condition is satisfied.) Then it will be the case that \( M \equiv_n M' \). \textbf{Proof:} Consider the \( n \)-game between \( M \) and \( M' \). Player II wins this if, to answer moves by I in either \( N \) or \( N' \), he uses a winning strategy for the \( n \)-game between these models, and if he copies the moves of player II on \( M \setminus N \).

\textbf{Proof of 4.2.} Let \( X \) be the set of \( b \in M \) such that for all \( a < b \), \((a \uparrow b \downarrow, [a, b])\) has an \( n \)-equivalent of the type desired. The lemma asserts that \( X = M \).

Suppose that \( X \not= M \). Observe that \( X \) is definable: by 1.7.1, there are only finitely many \( n \)-characteristics of models \((N, \beta)\) such that \( \beta \) is well-ordered and all components of \( N \setminus \beta \) are definably well-founded; moreover, that \((a \uparrow b \downarrow, [a, b])\) satisfies a given characteristic is a first-order property of \((M, a, b)\). By definable well-foundedness, \( M \setminus X \), assumed to be non-empty, has a minimal element \( b \). Suppose \( a < b \) is such that a corresponding \( n \)-equivalent of the required type doesn't exist. Obviously, \( b \) cannot have an immediate predecessor. Choose \( a_0 = a < a_1 < \ldots < a_\xi < \ldots (\xi < \alpha) \) cofinal in \( b \downarrow \). By minimality of \( b \), choose \((N_\xi, \beta_\xi) \equiv ^n (a_\xi \uparrow a_{\xi+1} \downarrow, [a_\xi, a_{\xi+1}])\) such that \( \beta_\xi \) is well-ordered and all components of \( N_\xi \setminus \beta_\xi \) are definably well-founded for each \( \xi < \alpha \).
The model \( \Sigma_{\xi<\alpha} (N_{\xi}, \beta_\xi) \), obtained by glueing the \( \beta_\xi \) one after the other now forms a counter-example to the choice of \( a \) and \( b : \) to see that this is an \( n \)-equivalent of \( (a \uparrow \backslash b \uparrow, [a, b]) \), apply the remark to 
\( (a \uparrow \backslash b \uparrow, [a, b]) = \bigcup_{\xi<\alpha} (a_{\xi} \uparrow \backslash a_{\xi+1} \uparrow, [a_{\xi}, a_{\xi+1}]) \) and \( \Sigma_{\xi<\alpha} (N_{\xi}, \beta_\xi) \). \( \Box \)

4.3 Corollary. Suppose the tree \( M \) is definably well-founded and \( b \in M \). Then, for each \( n \), \((M, b)\) has an \( n \)-equivalent \((M', b')\) such that \( b' \downarrow \) is well-ordered and all components of \( M' \backslash (b' \downarrow) \) are definably well-founded.

Proof. If \( b \) is minimal in \( M \), then \((M, b)\) itself satisfies the stipulations. Otherwise, let \( a \) be the least element of \( b \downarrow \). (For a picture: look at the one for 4.2; now, put \( a \) in the root of the tree.) By 4.2, \((a \uparrow \backslash b \uparrow, [a, b])\) has an \( n \)-equivalent \((N, \beta)\) with \( \beta \) well-ordered and all components of \( N \backslash \beta \) definably well-founded. Replace \((a \uparrow \backslash b \uparrow, [a, b])\) in \( M \) by \((N, \beta)\); the result is \( M' \). In \( M' \), \( b \downarrow = \beta \). Thus, putting \( b' = b \) makes \( b' \downarrow \) well-ordered. The components of \( M' \backslash (b' \downarrow) \) are the ones of \( N \backslash \beta \) plus \( b \uparrow \) plus the \( M \)-components different from the one containing \( b \) (if any); these are all definably well-founded. Finally, \((M', b') \equiv^n (M, b)\) follows from the remark above. \( \Box \)
The next lemma is the version of 4.3 with finitely many b's at the same time:

4.4 Lemma. Suppose that the tree $M$ is definably well-founded and $A \subseteq M$ is finite. Then, for each $n$, $(M, a)_{a \in A}$ has an $n$-equivalent $(M', a')_{a \in A}$ such that each $a' \downarrow (a \in A)$ is well-ordered and all components of $M' \setminus \bigcup_{a \in A} a' \downarrow$ are definably well-founded.

Proof. Induction on the number of elements of $A$.

To start with, we have 4.3. For the induction-step, choose $a \in A$ and put $B = A \setminus \{a\}$. Apply the inductive hypothesis to $(M, a)$ and $B$ to obtain $(M', a', b')_{b \in B}$ with all $b' \downarrow (b \in B)$ well-ordered and $M' \setminus \bigcup_{b \in B} b' \downarrow$ definably well-founded - component-wise.

(i) Suppose that for some $b \in B$, $a < b$. Then $a' < b'$, $a' \downarrow$ is well-ordered, $M' \setminus \bigcup_{a \in A} a' \downarrow = M \setminus \bigcup_{b \in B} b' \downarrow$, and we are done.

(ii) If not, let $C$ be the component of $M' \setminus \bigcup_{b \in B} b' \downarrow$ containing $a'$. By 4.3, obtain $(C', a'') =^n (C, a')$ with $a'' \downarrow$ well-ordered and $C' \setminus (a'') \downarrow$ definably well-founded, component-wise. Replace $(C, a')$ in $M'$ by $(C', a'')$ to obtain the desired model $(M'', a'', b')_{b \in B}$. ✷

We are now ready for the

Proof of 4.1.

Define a sequence of models $M_0, M_1, M_2, \ldots$ and sets $T^0, T^1, T^2, \ldots$ such that:

1. $M_0 = M$; $T^0 = \emptyset$;
2. $T^i$ is a well-founded downward-closed part of $M_i$ and every component
(considered as a submodel of $M_i$) (i.e., $T^i \subseteq T^{i+1}$, and for $a, b \in T^{i+1}$, if $b \in T^i$ and $a < b$ then $a \in T^i$);

4. $(M_i, t)_{t \in T^i} \equiv^n (M_{i+1}, t)_{t \in T^i}$;

5. for all $a \in M_i \setminus T^i$ there is $b \in T^{i+1}$ such that $(M_i, t, a)_{t \in T^i} \equiv^{n-1} (M_{i+1}, t, b)_{t \in T^i}$.

$M_{i+1}$ and $T^{i+1}$ will be obtained from $M_i$ and $T^i$ by replacing $M_i \setminus T^i$ in $M_i$ by an $n$-equivalent with a well-founded initial part (namely, $T^{i+1} \setminus T^i$) preserving definable well-foundedness component-wise. This will take care of 2-4. However, $T^{i+1}$ must be big enough so as to satisfy 5. This is achieved in the following manner:

Let $C$ be a component of $M_i \setminus T^i$. Choose $A \subset C$ such that for each $c \in C$ there is an $a \in A$ with $(C, a) \equiv^{n-1} (C, c)$ and such that $A$ is finite – this can be done according to 1.7.1. By 4.4, $(C, a)_{a \in A}$ has an $n$-equivalent $(C', a')_{a \in A}$ with every $a' \downarrow$ well-ordered and $C' \setminus \bigcup_{a \in A} a' \downarrow$ definably well-founded, component-wise.

$M_{i+1}$ is obtained from $M_i$ by exchanging $C$ for $C'$ and making similar replacements for every other component of $M_i \setminus T^i$. $T^{i+1}$ is $T^i$ plus all the $\bigcup_{a \in A} (a' \downarrow \cup \{a\})$ so encountered.

It is now obvious that 2.-5. are satisfied.

Now, put $N = \bigcup_{i} T^i$.

By 2-3, $N$ is well-founded. I claim that $N$ is an $n$-equivalent of $M$.

Consider the Ehrenfeucht-$n$-game between these models. Notice that, as $T^i \subset M_i$, in order to win, it suffices for the second player to choose his moves in such a way that after his $k$-th move sequences $a_0, \ldots, a_{k-1} \in M$ and $t_0, \ldots, t_{k-1} \in N$ have been played such that for all $i$, if $a_0, \ldots, a_{k-1} \in T^i$ then

$$[\ast], (M, a_0, \ldots, a_{k-1}) \equiv^n (M_i, t_0, \ldots, t_{k-1}).$$

4. monadic $\Pi^1_1$-theory of well-founded trees
Notice that \( i \leq j \) and \([\ast]_i\) imply \([\ast]_j\) by condition 4. above.

Now, the second player can keep up with this requirement:

First, if \( k=0 \) then \([\ast]_0\) holds since \( M_0=M \).

Next, suppose the players have arrived at a position where \([\ast]_i\) still is satisfied.

(a) Let the first player choose \( t_k \in \mathbb{N}, \) say, \( t_k \in T^j \). If \( j \leq i \) then \([\ast]_i\) provides the second player with an \( a_k \in M \) such that \( (M, a_0, \ldots, a_k) \equiv_{n-k-1} (M_i, t_0, \ldots, t_k) \). If \( i < j \), simply use \([\ast]_j\).

(b) Assume the first player chooses \( a_k \in M \). By \([\ast]_i\), there is a \( u \in M_i \) such that \( (M, a_0, \ldots, a_k) \equiv_{n-k-1} (M_i, t_0, \ldots, t_{k-1}, u) \). If, by a stroke of luck, \( u \in T^i \), we are done. If not, by condition 5. there is \( t_k \in T^{i+1} \) such that \( (M_{i+1}, t, t_k) \equiv_{T^i} (M_i, t, u) \) in particular, \( (M_{i+1}, t_0, \ldots, t_k) \equiv_{n-k-1} (M_i, t_0, \ldots, t_{k-1}, u) \). Hence, \( (M, a_0, \ldots, a_k) \equiv_{n-k-1} (M_{i+1}, t_0, \ldots, t_k) \); so the second player chooses \( t_k \) – thereby ensuring \([\ast]_{i+1}\) to hold for the resulting sequences. \( \Box \)
III Applications to intensional and intuitionistic logic

Few things are harder to put up with than the annoyance of a good example. – Pudd’nhead Wilson’s Calendar

5. Fine structure of modal correspondence theory.

For a given formula \( \varphi \) of modal propositional logic, the condition on frames \( \mathcal{W} = (W, R) \) that \( \varphi \) is valid on \( \mathcal{W} \) is monadic \( \Pi^1_1 \). (See below for the necessary definitions.) Correspondence theory asks (among other things) whether it can be expressed in first-order terms as well, i.e., whether it is first-order definable. This chapter presents some more information about modal formulas known to be not first-order definable.

To begin with, a couple of examples of van Benthem [1985] is examined. To prove first-order undefinability, use was made there of compactness- and Löwenheim–Skolem theorems. These results are strengthened here by showing non-first-order definability on finite frames in all cases. (In one

5. modal correspondence theory
case, we show that even a monadic $\Sigma^1_1$-definition on finite frames does not exist.) Of course, on the class of finite frames, other methods have to be used, and so the Ehrenfeucht game has to come to the rescue once more. Secondly, an example is given of a non-first-order definable formula which, however, is so definable on countable frames.

For readers unfamiliar with modal logic, let me recall the definitions of the notions involved. The formulas of modal propositional logic are generated from a set of variables by means of the ordinary connectives plus the (unary) modal operators $\Box$ (necessity) and $\Diamond$ (possibility). The **Kripke semantics** for these objects relative to frames, i.e., models $W=(W,R)$ where $R \subseteq W^2$, works as follows. Suppose that $V$ is a valuation, that is, a map assigning subsets of $W$ to variables. The triple $(W,R,V)$ is called a **(Kripke) model**. The **forcing relation** $\models$ connects elements of $W$ with formulas and is inductively defined by the clauses: (i) $w \models \varphi$ iff $w \in V(\varphi)$ when $\varphi$ is a variable; (ii) $w \models \neg \varphi$ iff $w \not\in V(\varphi)$; (iii) $w \models \varphi \land \psi$ iff $w \models \varphi$ and $w \models \psi$; (iv) $w \models \Box \varphi$ iff for all $v$ such that $wRv$: $v \models \varphi$; (v) $w \models \Diamond \varphi$ iff for some $v$ such that $wRv$: $v \models \varphi$. We say that $\varphi$ is valid on $W$ iff $\varphi$ is forced at every $w \in W$ under all possible valuations.

5.1 **Theorem.** Validity of $\Box(p \rightarrow \Box p) \rightarrow (\Diamond p \rightarrow \Box p)$ is not first-order on finite frames.

**Proof.** This may be seen as a modification of the proof of van Benthem [1985] lemma 10.1. Consider the frame $C_k=\langle\{0,\ldots,k-1\}, R\rangle$ where $R$ is defined by: $iRj \equiv |i-j|=1 \ (\text{mod} \ k)$. $C^w_k$ is obtained from this by the addition of one new element $w$ which is $R$-connected with every $i<k$. Look at the
Claim: $\Box(p \rightarrow \Box p) \rightarrow (\Diamond p \rightarrow \Box p)$ is valid in every $C^w_k$.

Proof of claim: suppose $\Diamond p$ and $\Box(p \rightarrow \Box p)$ are forced at $i < k$. By the first, either (i) $w \models p$ or (ii) $i-1 \models p$ or (iii) $i+1 \models p$. By the second assumption then, either (i) $w \models \Box p$ or (ii) $i-1 \models \Box p$ or (iii) $i+1 \models \Box p$. In case (ii) it follows that $w \models p$, hence again $w \models \Box p$, and similarly in case (iii). Thus, $w \models \Box p$ in all cases, whence $i \models \Box p$ follows. Next, suppose $\Diamond p$ and $\Box(p \rightarrow \Box p)$ are forced at $w$. Then $i \models p$ for some $i < k$; therefore, $i-1 \models p$, $i+1 \models p$, $i-2 \models p$, $i+2 \models p$ (everything mod $k$) and so on – therefore, $p$ is forced at every $i < k$; whence $w \models \Box p$. $\Box$

Next, let $2C^w_k$ be the frame obtained from the disjoint union of two copies of $C^w_k$ by identifying the respective nodes $w$.

Claim: $\Box(p \rightarrow \Box p) \rightarrow (\Diamond p \rightarrow \Box p)$ is valid in no $2C^w_k$.

For, let $w(p)$ be one of the $C^w_k$. Then $\Box(p \rightarrow \Box p)$ and $\Diamond p$ are forced at $w$, but $\Box p$ is not. $\Box$

Now, if $\Box(p \rightarrow \Box p) \rightarrow (\Diamond p \rightarrow \Box p)$ had a first-order equivalent, this would have some quantifier rank $n$. Therefore, the following lemma establishes the result. $\Box$

5.2 Lemma. If $k \geq 2^n$ then: $C^w_k \equiv^n 2C^w_k$.

5. modal correspondence theory
The lemma is the result of a series of observations 5.3-5. Recall that 
\[ \zeta = \omega^* + \omega. \]

5.3 Lemma. \( \zeta \equiv \zeta + \zeta. \)

**Proof.** By induction on \( n \), check that \( \zeta \equiv^n \zeta + \zeta \) for all \( n \), using 1.0.2, 1.0.3(ii) and its consequence, \( \omega^* \equiv \zeta + \omega^*. \)

Let \( \varphi \) be a quantifier rank-\( m \) formula in the free variables \( x_0, \ldots, x_{k-1} \). In each frame \( A \), \( \varphi \) defines a relation \( \varphi^A = \{ a \in A^k | A \models \varphi[a] \} \subset A^k \) which can serve as the interpretation of a new \( k \)-ary relation symbol. Now:

5.4 Lemma. If \( A \equiv^m B \), then: \( (A, \varphi^A) \equiv^n (B, \varphi^B). \)

**Proof.** Replacing the new \( k \)-ary relation symbol by its definition \( \varphi \) in a formula raises the quantifier rank of that formula with at most \( m \).

Now, let \( \eta \) be the formula

\[ [x < y \land \neg \exists z(x < z \land z < y)] \lor [y < x \land \neg \exists z(y < z \land z < x)]. \]

This is a quantifier rank-1 formula defining *immediate neighbourhood* in linear orderings. If \( A = (A, <) \) is a linear ordering, let \( A^\eta = (A, \eta^A) \). By 5.4, in particular, if \( A \equiv^{n+1} B \) then \( A^\eta \equiv^n B^\eta \).

In the following lemma, \( C_m + C_I \) is the disjoint union of \( C_m \) and \( C_I \).

5.5 Lemma. 1. If \( k, m \geq 2^n \) then \( C_k \equiv^n C_m \); 2. if \( k \geq 2^n \) then \( C_k \equiv^n \zeta^n \); 3. if \( k, m, i \geq 2^n \) then \( C_k \equiv^n C_m + C_I \).
$n$ and suppose now that $k, m, I \geq 2^{n+1}$.

1. A first move $i \in C_k$ by $I$ is answered with an arbitrary $j \in C_m$ by $II$. Now, notice the following. $k-1, m-1 \geq 2^{n+1} - 1$, hence $k-1 \equiv^{n+1} m-1$ by 1.0.1. Therefore, $(k+1, 0, k) \equiv^{n+1} (m+1, 0, m)$ as well and hence $((k+1)^n, 0, k) \equiv^n ((m+1)^n, 0, m)$. Now, identification of $0$ and $k$ in $(k+1)^n$ produces an isomorph of $(C_k, i)$ and the same goes for $0$, $m$, $(m+1)^n$ and $C_m$. Therefore, a trick like the one of 5.4 (identification of $0$ and $k$ just means re-interpreting $=\equiv$ as: $x = y \lor (x = 0 \land y = k) \lor (x = k \land y = 0)$ – and this doesn’t need quantifiers) shows that $(C_k, i) \equiv^n (C_m, j)$.

2. If $k \geq 2^{n+1}$ then $k-1 \geq 2^{n+1} - 1$, hence $k-1 \equiv^{n+1} \omega + \omega^*$ by 1.0.3(i). Therefore, $(k-1)^n \equiv^n (\omega + \omega^*)^n$. Adding endpoints, $((k+1)^n, 0, k) \equiv^n ((\omega + \omega^*)^n, 0, 0^*)$ (where $0^*$ is the greatest element of $\omega^*$). Identifying $0$ and $k$ in $(k+1)^n$ produces $(C_k, i)$ for any $i < k$; likewise, identifying $0$ and $0^*$ in $(\omega + \omega^*)^n$ produces $(\zeta^n, 0)$. Hence, $(C_k, i) \equiv^n (\zeta^n, 0)$, etc.

3. $C_k \equiv^n \zeta^n \equiv^n \zeta^n + \xi^n \equiv^n C_m + C_j$ by 2. and 5.3. $lacksquare$

Remark. The bound $2^n$ in 5.5 is not sharp. For instance, if $k, m \geq 7$ then $C_k \equiv^3 C_m$.

Finally, 5.2 now follows from 5.5.3 noting that the addition of $w$ cannot spoil the winning strategy of $II$.

5.6 Theorem. The McKinsey formula $\Box \Diamond p \rightarrow \Diamond \Box p$ is not first-order definable on finite frames.

Proof. (Compare van Benthem [1985] Thm.10.2.) $PZ_k$ is the following frame.
Its universe is \( \{w\} \cup \{0, \ldots, k-1\} \cup \{0', \ldots, (k-1)\}'\); its relation \( R \) is defined by: \( aRb \) iff either \( a=w \) and \( b \in \{0, \ldots, k-1\} \) or \( a \in \{0, \ldots, k-1\} \) and \( b \) is either \( a' \) or \( (a+1)' \) (mod \( k \)) or \( a = b \in \{0', \ldots, (k-1)\}'\). Truth of \( p \rightarrow p \) is unproblematic at all nodes except \( w \). There, it depends on whether \( k \) is even or odd. Truth of \( p \) at \( w \) means that for each \( i \leq k \) at least one of \( i' \) and \( (i+1)' \) (mod \( k \)) is in \( \mathcal{V}(p) \). If \( k \) is odd, then for some \( i < k \) both \( i' \) and \( (i+1)' \) must be in \( \mathcal{V}(p) \), hence \( p \) is forced at \( i \) and \( p \) is forced at \( w \).

However, when \( k \) is even, set \( \mathcal{V}(p) = \{ i' | i \text{ even} \} \) and the McKinsey formula clearly is not forced at \( w \). If there is a first-order equivalent, this must have a certain quantifier rank. Therefore, the result is an immediate consequence of the following lemma.

\[ \square \]

5.7 Lemma. If \( k, m \geq 2^n \), then \( PZ_k \equiv^n PZ_m \).

Proof. Let \( C'_k \) be the asymmetric version of \( C_k \), i.e., it is the frame \( (\{0, \ldots, k-1\}, R) \) where \( iRj \) iff \( j = i+1 \) (mod \( k \)). Just like in 5.5.1, we have \( C'_k \equiv^n C'_m \) for \( k, m \geq 2^n \) (apply 5.4 using the left disjunct of the formula \( \eta \)). Any winning strategy \( \sigma \) for \( II \) in \( G(C'_k, C'_m, n) \) can now be transformed into a valid winning strategy \( \sigma' \) in \( G(C_k, C_m, n) \) in either frame is answered by
Similar considerations yield the following results. Here, a formula \( \varphi \) is called **locally defined** by \( \psi(x) \) if in each frame \( W \), for each \( w \in W: w \models \varphi \) iff \( W \models \psi[w] \). (Definability in 5.1 and 5.6 also is called **global definability**.)

**5.6 Theorem.** The following formulas are **not locally first-order definable** on finite frames:

1. \( \Box(\Box p \lor p) \rightarrow \Diamond (\Box p \land p) \) (cf. van Benthem [1985] 10.3)
2. \( \Box(p \lor q) \rightarrow \Diamond (\Box p \lor \Box q) \) (l.c. 10.4)
3. \( \Diamond (\Box p \lor \Box \neg p) \)
4. \( \Box(\Box p \lor p) \rightarrow \Diamond (\Diamond p \lor p) \) (l.c. 10.6).

**Remark.** Several results on non-first-order definability are rather unsatisfying in that they show first-order expressibility to be so forbiddingly weak in the poor language given. In such circumstances, it would be quite natural to look for a wider type of definability. A possibility that suggests itself would be definability using an infinitary language. Another one would be higher-order definability. Now, modal formulas are \( \Pi^1_1 \)-definable by nature. If such a principle is \( \Sigma^1_1 \) as well, it must be first-order by the interpolation theorem. However, on non-elementary model classes, this need not to be so. For instance, which of the previous examples is \( \Delta^1_1 \) on the finite models? The models used in 5.6 cannot be used to prove non-\( \Delta^1_1 \)-definability here since they can be
Δ₁¹-distinguished. The same goes for the models used to prove 5.8: they too employ the even/odd-indistinguishability by first-order means. On the other hand, this is not the case in 5.1, as the following theorem shows. Therefore, that result can be strengthened to non-monadic Σ₁¹-definability.

5.9 Theorem. Any monadic Σ₁¹-sentence valid in each Cₚ holds on some Cₜ+Cᵣ as well.

Proof. Suppose that ∃X₁...∃Xₖσ holds on each Cₚ. Let σ have quantifier-rank n. Choose a finite set Σ of finite linearly ordered models α=(A, <, X₁,...,Xₖ) such that each such model has an (n+1)-equivalent in Σ. By the finite Ramsey-theorem, there is a natural number p such that for each p-element set P and each map h:[P]²→Σ there exists a 2ⁿ⁺¹-element set Q⊆P homogeneous for h, i.e., for some α∈Σ we have h(x,y)=α for all x,y∈Q.

Now, consider a model (Cₚ,X₁,...,Xₖ)≡σ. Cut it open at some place; this produces a linearly ordered p-element model. We may as well assume this to have universe p={0,...,p−1}, where the elements have their natural order. For i<j<p, let h(i,j) be an (n+1)-equivalent of [i,j) in Σ - here, [i,j) denotes the submodel of p with universe {m<p | i≤m<j}. By choice of p, there is a 2ⁿ⁺¹-element Q⊆p homogeneous for h; say, h(i,j)=α for all i<j in Q. Put x=min Q and y=max Q. Then [x,y)≡ⁿ⁺¹α and, hence, p≡ⁿ⁺¹(←,x)+α+[y,→). Since Q has 2ⁿ⁺¹ elements, it divides [x,y) into 2ⁿ⁺¹−1 intervals, each (n+1)-equivalent with α.

Therefore: α≡ⁿ⁺¹α⋅(2ⁿ⁺¹−1)
≡ⁿ⁺¹α⋅(ω+ω*) (for, 2ⁿ⁺¹−1≡ⁿ⁺¹ω+ω* by 1.0.3(i) - use 3.1.8)
ordering between $\omega$, $\omega^-$ and $\zeta$ doesn't matter any longer; therefore, $\alpha^n \equiv^n \alpha^n + [\alpha \cdot \zeta]^n$.

It follows that the original model $(C_p, X_1, \ldots, X_k)$ is $n$-equivalent to its disjoint sum with $[\alpha \cdot \zeta]^n$. However, for $m = 2^n$, $\zeta^n \equiv^n C_m$ (by 5.5.2). Hence, $[\alpha \cdot \zeta]^n$ is $n$-equivalent with a model $(C_{am}, Y_1, \ldots, Y_k)$ where $a$ is the number of elements of $\alpha$. Therefore, $\exists X_1 \ldots \exists X_k \phi$ holds in $C_p + C_{am}$. $\square$

The examples given suggest the pattern: if a modal formula is not first-order definable, then it is not first-order on finite frames already; conversely: if it is first-order on finite frames then it is first-order generally.

However, this is not true.

5.10 Example. Löb's axiom $\Box(\Box p \rightarrow p) \rightarrow \Box p$ is first-order on finite models but is not so generally: it is not first-order on countable models. For, the axiom is true in a frame $W = (W, R)$ iff $R$ is transitive and conversely well-founded (if $aRb, bRc$ but not $a Rc$, put $V(p) = W \setminus \{b, c\}$ and notice that $\Box(\Box p \rightarrow p)$ holds at $a$ but $\Box p$ doesn't; if $R$ is transitive, truth of Löb's axiom just means that $R$ is conversely well-founded on each set \( \{b \mid a R b\} \) - and so it is well-founded on $W$ - and this property is known to be not first-order on countable models: a standard consequence of the compactness and downward Löwenheim–Skolem theorems. However, the axiom is first-order on finite frames since there, (converse) well-foundedness reduces to mere irreflexivity. $\square$

This example triggers the question: if a modal formula is first-order on countable frames, must it be first-order generally?

Again, the answer is negative.

5. modal correspondence theory
5.11 Example. Consider Fine's axiom $\Phi$: $\Box(\Box(p \lor q) \rightarrow \Box(\Box p \lor \Box q))$.

$\psi(x)$ is the first-order condition on frames $(W, \prec)$ that $\forall y \ni x \exists y' \ni x[\forall z \ni y'(z \ni y) \land \forall z, z' \ni y'(z = z')]$ (i.e., for each $\prec$-successor $y$ of $x$ there is another one $y'$ which has at most one $\prec$-successor; moreover, each $z \ni y'$ succeeds $y$ as well). By the way, $\forall x \psi(x)$ is the first-order condition with respect to which Fine's axiom is complete, i.e., a modal formula is modally derivable from $\Phi$ iff it is valid on all frames satisfying $\forall x \psi$ (see van Benthem [1985] lemma 8.10).

Now, notice:

(i). For every frame $W$ and $x \in W$: if $W \models \psi(x)$ then $\Phi$ is true at $x$.

For, suppose that $x \prec y \models \Box(p \lor q)$. Since $W \models \psi(x)$, there is an $y' \ni x$ with at most one successor such that $y' \prec z \Rightarrow y \prec z$. Clearly then, either $y' \models \Box p$ or $y' \models \Box q$, therefore, $y' \models \Box p \lor \Box q$ and $x \not\models \Box(\Box p \lor \Box q)$. #

(ii). For every countable $W$ and $x \in W$: if $\Phi$ is true at $x$ then $W \models \psi(x)$.

To see this, put $X_y = \{ z \in W \mid y \prec z \}$, $U = \{ X_y \mid x \prec y \}$ and $X = \bigcup U = \bigcup_{x \prec y} X_y$. Assume $x \models \Phi$; we check that $W \models \psi(x)$: let $x \prec y$. There are two cases.

(a) There is a finite $X_{y'} \subseteq X_y$ in $U$. Let $X_{y'}$ be one with a minimal number of elements. If it is empty, the rest is trivial. If $X_{y'} = \{ z \}$, we are done as well. So it suffices to show that $X_{y'}$ cannot have more than one element. If it has, choose $V(p)$ and $V(q)$ non-empty disjoint such that $X_{y'} = V(p) \cup V(q)$. Now, $y' \models \Box(p \lor q)$, hence $x \models \Box(\Box p \lor q)$. As $x \models \Phi$ by assumption, $y'$ exists such that $x \prec y'$ and $X_{y'} \subseteq V(p)$ or $X_{y'} \subseteq V(q)$, contradicting the minimality of $X_{y'}$.

(b) Each $X_{y'} \subseteq X_y$ in $U$ is infinite. By the countability-assumption, let $X_0, X_1, X_2, \ldots$ be an enumeration of $\{ X \in U \mid X \subseteq X_y \}$. Inductively, choose $a_i, b_i \in X_i$ all different from one another. Put $V(p) = \{ a_i \mid i \in \mathbb{N} \}$ and $V(q) = X_y \setminus V(p)$. Then $y' \models \Box(p \lor q)$, $x \not\models \Box(p \lor q)$ and hence $y'$ exists such that $x \prec y'$, $y' \models \Box p \lor \Box q$; contradicting $x \models \Phi$.
(iii). If $\Phi$ would be (locally) first-order, $\Psi$ would be a first-order equivalent of $\Phi$: being an equivalent on countable frames by (i) and (ii), it must be an equivalent on all frames, by the Löwenheim–Skolem theorem. However:

(iv). There are uncountable frames $W$ such that $\Phi$ holds at some $x \in W$ but $W \not\models \Psi[x]$. For instance, let $U$ be the collection of all infinite sets of natural numbers; put $W = \{U\} \cup U \cup \mathbb{N}$ and $x < y$ iff $y \in x$. It is easy to see that $W = (W, <) \not\models \Psi[U]$, because $U$ does not contain singletons or the empty set.

Next, choose any valuation $V(p)$, $V(q)$ of $p$ and $q$. If $U \models \Box (p \lor q)$, this means that for some $x \in U$, $x \subseteq V(p) \cup V(q)$, hence $x = (x \cap V(p)) \cup (x \cap V(q))$. $x$ is infinite, therefore one of $x \cap V(p)$ and $x \cap V(q)$ must be infinite as well. If this is $x \cap V(p)$ then $x \cap V(p) \in U$, $x \cap V(p) \models \Box p$, $x \cap V(p) \models \Box p \lor \Box q$ and $U \models \Box (\Box p \lor \Box q)$. ⋄

(v). The above answers the question for global first-order definability as well. For clearly, $\forall x \Psi$ globally defines $\Phi$ on countable models, and the counter-example from (iv) can be transformed into a global one by making $<$ reflexive on $\mathbb{N}$. ⋄

(vi). (van Benthem) This example can be transformed into one for strict partial orderings. Let $\Phi$ be $\Diamond(\Diamond \top \land \Box (p \lor q)) \rightarrow \Diamond(\Diamond \top \land (\Box p \lor \Box q))$ and $\Psi(x)$: $\forall y \exists x(\exists z(y < z) \rightarrow \exists y' > x(y'$ has exactly one $<$-successor; besides, this successor succeeds $y$ as well)]. Almost the same proofs show (i)-(iii) to be the case for these formulas; the transitive closure of the frame of (iv) shows that $\Phi$ can hold in an uncountable frame where $\Psi$ is false.

5.12 Problem. Determine the least cardinal $\mu$ such that: if a modal formula is first-order definable on frames of power $\leq \mu$, then it is so generally.
Here are some partial answers on this problem and its variant for tense logic. Restricting to linear (irreflexive) orderings, van Benthem obtained the following

5.13 Theorem. If a modal formula is first-order on the class of all countable linear orderings then it is first-order on all linear orderings.

The theorem follows from the

5.14 Lemma. If the linearly ordered frame $F$ is an elementary submodel of $G$ then each modal formula true on $G$ holds on $F$ as well.

Proof that 5.14 implies 5.13. Suppose that $\varphi$ is a modal formula not first-order on all linear orderings. By (a slight modification of) van Benthem [1985] p.91 Theorem B.6 (with the same proof), there are elementary equivalent linear orderings $G_1$ and $G_2$ such that $\varphi$ holds on $G_2$ but is false on $G_1$. Since $\varphi$ is $\Pi^1_1$, $G_1$ may be taken to be countable, by the downward Löwenheim-Skolem theorem. Let $F \prec G_2$ be countable. If $\varphi$ is first-order on countable linear orderings, it must be false on $F$; according to 5.14, this is a contradiction. 

Proof of 5.14. Suppose that $F=(F,<)\prec G=(G,\prec)$ and $V$ is a valuation on $F$ falsifying the modal formula $\varphi$. It suffices to construct a $\varphi$-falsifying valuation $W$ on $G$. This is done by extending $V$ to $G$ as follows. Let $n$ be the modal rank of $\varphi$. The object is to define $W$ such that $(F,V) \prec_n (G,W)$, by which I mean that, for each modal formula $\psi$ of rank $\leq n$: (i) for all $a \in F$: a forces $\psi$ in $(F,V)$ iff $a$ forces $\psi$ in $(G,W)$ and (ii) if $\psi$ is forced somewhere
Define an equivalence ~ on F by x~y iff ∃a≤x,y≥a x,y: [a]ⁿ = [b]ⁿ.
Obviously, ~ is a condensation splitting F in a finite number of intervals
(since the number of n-characteristics [a]ⁿ is finite). Let u ∈ GF. To
decide how we shall define W on u, notice that there are the following six
possibilities as regards the location of u compared to the condensation ~.
1. u occurs in an interval, i.e., there are a < u and b > u with [a]ⁿ = [b]ⁿ.
2. u occurs between neighbouring intervals c₁ < u and c₂ > u in F/ ~.
   Notice that either c₁ has no maximum or c₂ has no minimum, since F < G.
   So, we have the subcases 2.1–2.3:
   2.1. c₁ has a maximum; c₂ has no minimum.
   2.2. c₁ has no maximum; c₂ has a minimum.
   2.3. c₁ has no maximum; c₂ has no minimum.
3. u is smaller than all x ∈ F. Notice that F cannot have a least element in
   this case, since F < G.
4. Similarly, u is greater than all x ∈ F.
Call u,v ∈ GF equivalent iff they are not separated by some x ∈ F. There
may be infinitely many equivalence classes; I shall extend V step by step
to each equivalence class. To do so needs a possibly transfinite iteration
which is handled by the following version of the elementary chain lemma:

5.15 Lemma. Let <Aₖ|ₖ<α> be a sequence of Kripke models such that
   Aₖ <ₗₖ Aₖ whenever k < φ < α. Let A = ∪ₖ<α Aₖ; then Aₖ <ₗₖ A for all k < α.

Proof. That for all ψ of rank ≤ n, all k < α and all a ∈ Aₖ: all-ψ (in Aₖ) iff
all-ψ (in A) is proved using induction on the rank of ψ; the rest of the
theorem is an immediate consequence of this. □

5. modal correspondence theory
Now, construct the sequence $<A_\xi|\xi<\alpha>$ as follows. $A_0$ is the Kripke model $(F,V)$. Take unions at limits - this works by 5.15. $A_{\xi+1}$ is obtained from $A_\xi$ by the addition of one equivalence class of $G\setminus F$ and extending the valuation in the suitable fashion as indicated below, depending on whether we have case 1, 2.1–2.3, 3 or 4.

Case 1. Suppose the equivalence class $c$ of $G\setminus F$ has $a<c$ and $b>c$ in $F$ with $[a]^n = [b]^n$. Extend the valuation on the $uec$ by copying its behaviour on $a$ (which is the same on $b$). Let $A = A_\xi$ (hence, $[a]^n = [A,a]^n$) and $B = Auc = A_{\xi+1}$ with the valuation as indicated.

Claim 1. If $uec$ and $m \leq n$ then $[B,u]^m = [A,a]^m$.

Proof. Induction on $m$. For $m=0$, this is immediate from the definition of the valuation on $B$. For $m+1$: (i) if $a'>a$ realizes the $m$-characteristic $\sigma$ in $A$, there must be $b'>b$ realizing $\sigma$ as well. But, $u < b'$. (ii) if $v > u$ realizes the $m$-characteristic $\sigma$ in $B$ and $v \in A$ then, by induction hypothesis, $v$ realizes $\sigma$ in $A$ as well and $a < v$. And if $v \in c$ then, by induction hypothesis, $[A,b]^m = [A,a]^m = [B,v]^m$, so $b > a$ realizes $\sigma$ in $A$. \(\Box\)

Claim 2. If $x \in A$ then $[A,x]^n = [B,x]^n$.

Proof. Immediate from 1. \(\Box\)

Case 2.1. $c_1 < c < c_2$; $c_1,c_2 \in F\setminus \sim$ are neighbours; $c_1$ has a maximum, $c_2$ has no minimum, $c$ is an equivalence class of $G\setminus F$. Again, let $A$ be the model $A_\xi$ constructed so far, $B = Auc = A_{\xi+1}$. Choose an $n$-characteristic $\sigma$ which is realized in $A$ by elements occurring initial in $c_2$; extend the valuation on $c$ by copying from those elements.
$m$-characteristic of any element having $n$-characteristic $o$).

**Proof.** Induction on $m$. ☐

Again, we have a second claim as in case 1 as a consequence. Notice that this argument also works for case 3.

Case 2.2. $c_1 < c < c_2$; $c_1, c_2 \in F/\sim$ are neighbours; $c_1$ has no maximum but $c_2$ has a minimum; $c$ is an equivalence class of $G/\sim$. Let $\text{VAR}_\varphi$ be the set of variables of $\varphi$; $a \in A$ has shape $X \subseteq \text{VAR}_\varphi$ when $X = \{ x \in \text{VAR}_\varphi | a \in V(x) \}$. Now define $W$ on $c$ such that each shape occurs cofinal in $c_1$ iff it occurs cofinal in $c$ (since $F < G$, $c$ cannot have a maximum). Employing the Ehrenfeucht game characterization, it is clear that we obtain the proper type of extension (i.e., $n$-characteristics are being preserved and no new $n$-characteristic is realized).

Case 2.3. $c_1 < c < c_2$; $c_1, c_2 \in F/\sim$ are neighbours; $c_1$ has no maximum and $c_2$ has no minimum; $c$ is an equivalence class of $G/\sim$. If $c$ has no maximum, the method of 2.2 may be employed. However, the method of 2.1 works in all cases.

Case 3. This is handled the same way as 2.1.

Case 4. Use the method of 2.2.

5.16 Remark. (van Benthem) The same procedure can be used for the logic of time – except that there no solution can be obtained for the extension-pattern 2.3 above. But, in some cases, we *know* that pattern

5. modal correspondence theory
cannot occur. For instance, if a tense logical formula is first-order on all countable discrete linear orderings, it must be first-order on all discrete orderings.
6. Game theory for intensional logics, exact-universal Kripke models and normal forms.

The Ehrenfeucht game technique can be modified for use in investigations of intensional logics. Here, the case of modal logic is considered only; the modifications needed for tense and intuitionistic logic below are then more or less clear. Using the game characterization, exact-universal models are built which can be used to construct normal forms. This will be used extensively in a comparatively simple case in chapter 9.

6.1. Suppose then that $A=(A,R)$ ($R \subseteq A^2$) is any frame and $V: \text{VAR} \rightarrow \mathcal{P}(A)$ is a valuation mapping the set \text{VAR} of propositional variables onto subsets of $A$. The pair $(A,V)$ is called a Kripke model. For a modal formula $\varphi$ and $a \in A$, $\varphi$ is forced at $a$ (all-$\varphi$) iff $a$ satisfies the standard interpretation $\text{ST}(\varphi)$ of $\varphi$ in the associated model $(A,V(p))_{p \in \text{VAR}}$, where $\text{ST}(\varphi)$ is defined

6. game-theory for intensional logics
by the following clauses:

1. \( \text{ST}(p) = p(v_0) \)

(here, \( p \in \text{VAR} \) on the right-hand side of this equation is used as a unary relation-symbol interpreted by \( V(p) \) in \( (A, V(p))_{p \in \text{VAR}} \))

2. (i) \( \text{ST}(\neg \varphi) = \neg \text{ST}(\varphi) \)

   (ii) \( \text{ST}(\varphi \land \psi) = \text{ST}(\varphi) \land \text{ST}(\psi) \)

   (and similarly for the other connectives if present)

3. (i) \( \text{ST}(\Box \varphi) = \forall v_1 (R(v_0, v_1) \rightarrow \text{ST}(\varphi)^+) \)

   (ii) \( \text{ST}(\Diamond \varphi) = \exists v_1 (R(v_0, v_1) \land \text{ST}(\varphi)^+) \)

Here, \( \psi^+ \) is obtained from \( \psi \) by raising indices of all variables in \( \psi \) by 1.

Clearly, if the modal formula \( \varphi \) has modal rank \( n \) (defined in the obvious way) then \( \text{ST}(\varphi) \) is an \( R \)-restricted first-order formula with \( v_0 \) as only free variable which has quantifier rank \( n \).

6.2. The restricted Ehrenfeucht game of length \( n \) on Kripke models \( (A, V) \) and \( (B, W) \) (\( A=(A, R), B=(B, S) \)) with initial position \( (a_0, b_0) \in A \times B \) is played by \( I \) and \( II \) as follows. First, \( I \) chooses either \( a_1 \in A \) such that \( a_0 Ra_1 \) or \( b_1 \in B \) such that \( b_0 Sb_1 \). In the first case, \( II \) answers with some \( b_1 \in B \) such that \( b_0 Sb_1 \). In the second, \( II \) chooses \( a_1 \in A \) such that \( a_0 Ra_1 \). A position \( (a_1, b_1) \in A \times B \) results and the procedure is repeated until each player has had \( n \) moves. In so doing, they have set up an \( n \)-element sequence \( \langle (a_i, b_i) | i < n \rangle \) in \( A \times B \); and we shall say that \( II \) has won the play iff for each \( i < n \) and \( p \in \text{VAR} \): \( a_i \in V(p) \) iff \( b_i \in V(p) \) – otherwise \( I \) has won.

Of course, this game has its ordinal-bounded version. But somehow, intensional logic never is considered in the context of an infinitary language. (But see dynamic logic.)
For \((A, V)\) a Kripke model, \(a \in A\) and \(n \in \mathbb{N}\), define the modal formula 
\(\llbracket a \rrbracket^n = \llbracket (A, V, a) \rrbracket^n\) as follows:

1. \(\llbracket a \rrbracket^0 = \bigwedge \{ \{ p \in \text{VAR} | a \in V(p) \} \cup \{ \neg p | p \in \text{VAR} \land a \notin V(p) \} \};\)
2. \(\llbracket a \rrbracket^{n+1} = \llbracket a \rrbracket^n \land \bigwedge_{aRa} \llbracket a' \rrbracket^n \land \bigwedge_{aRa} \Box \llbracket a' \rrbracket^n.\)

Clearly, \(\llbracket a \rrbracket^n\) is a formula of modal rank \(n\) forced at \(a\) in \((A, V)\). A modification of the proofs of 1.5.1/1.6.3 will show that

6.4 Theorem. For \((B, W)\) a Kripke model, \(b \in B\) and \(\llbracket b \rrbracket^n = \llbracket (B, W, b) \rrbracket^n\), the following are equivalent:

1. \(II\) has a winning strategy for the restricted Ehrenfeucht game of length \(n\) on \((A, V)\), \((B, W)\) with initial position \((a, b)\);
2. for each modal formula \(\varphi\) of rank \(\leq n\): \(a \vDash \varphi\) iff \(b \vDash \varphi\);
3. \(b \vDash \llbracket a \rrbracket^n;\)
4. \(\llbracket b \rrbracket^n = \llbracket a \rrbracket^n.\)

Suppose now that \(K\) is a class of Kripke models.

6.5 Definition. The Kripke model \((A, V)\) is called

(i) \(K\)-universal if for each \((B, W) \in K\) and \(b \in B\) there is an \(a \in A\) such that for all \(n \in \mathbb{N}\): \(\llbracket a \rrbracket^n = \llbracket b \rrbracket^n.\)

(ii) exact if for all \(a_1, a_2 \in A\): if for all \(n \in \mathbb{N}\), \(\llbracket a_1 \rrbracket^n = \llbracket a_2 \rrbracket^n\), then \(a_1 = a_2.\)

I discuss one important case only.
6.6 Theorem. There exists an (obviously: unique within isomorphism) exact Kripke model which is universal with respect to all Kripke models over finite (reflexive) partial orderings.

Proof. Let us denote the model to be constructed by \((U, \leq, V)\).

It will turn out that \(>\) is well-founded. Let \(\varphi_k\) be the rank of \(k \in U\) relative to \(>\) (i.e., \(\varphi_k = \sup \{ \varphi_l + 1 \mid l > k \}\)). It will turn out that all ranks are finite.

Define \(U_n = \{ k \in U \mid \varphi_k = n \} \) \((n \in \omega)\) and \(\sigma_k = \{ p \in \text{VAR} \mid k \in V(p) \}\). It is now easy to build the \(U_n\) one after the other explicitly by recursion on \(n\), defining \(\leq\) and \(\sigma\) along the way, by simply imagining what possibilities may occur in a finite Kripke model over a partially ordered set.

First, \(U_0 = \mathcal{P} (\text{VAR})\); \(\sigma \mid U_0\) is defined by: \(\sigma k = k\).

Next, assume the \(U_i\) constructed up to and including \(U_n\); \(\sigma\) and \(\leq\) defined on \(\bigcup_{i \leq n} U_i\). \(U_{n+1}\) now consists of the following two types of objects:

1. Each \(k \in U_n\) has predecessors \((k, j) \in U_{n+1}\) for each \(j \in U_0\) such that \(j \neq \sigma k\);
   \(\sigma\) is defined on these by \(\sigma(k, j) = j\).

2. For each anti-chain \(A \subset \bigcup_{i \leq n} U_i\) which intersects \(U_n\) and has at least two elements and each \(j \in U_0\) there is an element \((A, j) \in U_{n+1}\) which precedes every \(k \in A\); \(\sigma\) is defined on these by \(\sigma(A, j) = j\).

The picture gives a tiny top-part of the model when \(\text{VAR} = \{ p \}\) is a

\[
\begin{align*}
\text{VAR} &= \{ p \} \\
U_0 &: \varnothing \quad \{ p \} \\
U_1 &: (\varnothing, \{ p \}) \quad (\{ \varnothing, \{ p \}, \varnothing \}) \quad (\{ \varnothing, \{ p \}, \{ p \} \}) \quad (\{ p \}, \varnothing)
\end{align*}
\]

\[\begin{align*}
U_2 &: (\{ \varnothing \}, \varnothing) \quad (\{ \varnothing \}, \{ \varnothing \}) \quad (\varnothing) \quad (\{ \varnothing \}, \{ p \}) \quad (\{ \varnothing \}, \{ p \})
\end{align*}\]
elements of $U_2$ shown. The picture which goes with thm.9.1 below represents an important submodel $U'$ of $U$ for $\text{VAR} = \{ p \}$. This completes the description of $(U, V)$. That it works is seen below in 6.7/8. □

From now on, $K$ is the class of finite Kripke models over partially ordered sets.

6.7 Theorem. For each $(B, W) \in K$, there is a canonical map $h: B \rightarrow U$ such that

1. $a \leq b \Rightarrow ha \leq hb$
2. $ha \leq k \Rightarrow \exists b \geq a (hb = k)$
   
   $(h$ is a $p$-morphism)
3. for all $b \in B$, $n \in \mathbb{N}$: $[b]^n = [hb]^n$.
   
   $(hence, (U, V) is K-universal)$

Proof. $hb$ is defined by recursion on $q b$, the rank of $b \in B$ with respect to the relation $>$, where $B = (B, <)$.

If $q b = 0$, $hb = \sigma b ( = \{ p | b \in W(p) \})$.

If $q b > 0$, let $C = \{ hb' | b < b' \}$; let $A$ be the set of minimal elements of $C$. Then $A$ is an anti-chain with at least one element of rank $q b - 1$. Distinguish three possibilities.

1. $A = \{ k \}$, $\sigma b = \sigma k$. Put $hb = k$.
2. $A = \{ k \}$, $\sigma b \neq \sigma k$. Put $hb = (k, \sigma b)$.
3. $|A| \geq 2$. Put $hb = (A, \sigma b)$.

Now, (i)-(iii) are clear. □

6.8 Lemma. $(U, V)$ is exact.

6. game-theory for intensional logics
Proof. Suppose that $k, l \in U$ and for all $n \in \mathbb{N}$, $[k]^n = [l]^n$. Apply induction on $k$. Let $A = \{ x \in U | k < x \}$ and $B = \{ x \in U | l < x \}$. By induction hypothesis and 6.9 below, it follows that $A = B$. Now, $k = l$ is clear. \qed

6.9 Lemma. If $a, b$ are elements of finite Kripke models in $K$ and $[a]^{2^a+1} = [b]^{2^b+1}$ then $[a]^n = [b]^n$ for all $n \in \mathbb{N}$.

Proof. Induction on $a$. First, if $a = 0$, $a$ is maximal and the result is obvious. Next, assume $a = n > 0$, $[a]^{2^n+1} = [b]^{2^n+1}$ and $m$ is minimal such that $[a]^m \neq [b]^m$. Let $I$ use a winning strategy in the $m$-game on $(a, b)$ which always picks elements of minimal possible rank. If, using this strategy, $I$ starts picking $a$ or $b$, $II$ answers $b$ resp. $a$ and $I$ "looses a tempo": there are $m-1$ moves left for either player and so $II$ can win by choice of $m$. If $I$ starts with $a' > a$, $II$ picks $b' > b$ with $[b']^{2^n} = [a']^{2^n}$ and wins by the inductive hypothesis. If $I$ starts with $b' > b$, $II$ picks $a' \geq a$ such that $[a']^{2^n} = [b']^{2^n}$. There are two cases to distinguish.

(i) $a' \neq a$. Then $qa' < qa$ and $II$ wins by the inductive hypothesis.

(ii) $a' = a$. Consider the second move of $I$. This cannot be $a'$ or $b'$ for tempo-loss will result. Also, it cannot be $b'' > b'$ since $I$'s strategy picks elements of least rank, so it would have choosen $b''$ as a first move already. Therefore, it will be some $a'' > a$ and $II$ wins by the inductive hypothesis. \qed

6.10 Problem. For each $k \in U$, we know by 6.9 that $[k]^{2^k+1}$ defines $k$ in the sense that $k$ is the only element of $U$ at which the formula is forced. Determine for each $k \in U$ the least $n$ such that $[k]^n$ defines $k$ (and give a more manageable equivalent of $[k]^n$). (The construction of Sehtman [1978]
6.11 Lemma. For all \( k \in U \) and \( n \in \mathbb{N} \) there is an \( l \in U \) such that

1. \([l]^n = [k]^n\);

2. \( q_l \leq n \).

Proof. Similar to the one of 6.7. Induction on \( n \). For \( n = 0 \), this is clear. Next, let \( C \) be the set of \( l \in \bigcup_{i \leq n} U_i \) such that for some \( k' > k \), \([l]^n = [k]^n\). Let \( A \) be the set of minimal elements of \( C \). The required \( l \) is constructed from \( A \) and \( ok \).

We now have the following corollary:

6.12 On normal forms. Let \( \varphi \) be a formula of modal rank \( n \). On Kripke models in \( K \), \( \varphi \) is equivalent with \( \bigvee \{ [k]^n | qk \leq n \land k \vdash \varphi \} \).
7. Completeness for $\mathbb{Z}$-time.

The theorem of this chapter, asserting completeness of a certain system of tense logic with respect to $\mathbb{Z}$-time, is due to Segerberg [1970]. A different proof is in van Benthem [1983] (cf. II.2.3.15) and another, relatively simple one is in de Jongh et al. [1986]. Our proof is related to the method of chapter 3; however, the relationship is not that exact, due to the fact that the tense logical formalism lacks first-order possibilities such as quantifier relativization. This weakness also is responsible for the fact that the Suslin property of $\mathbb{R}$ has no influence on the theory of $\mathbb{R}$-time; contrast this with 3.3.6/9. Nevertheless, we shall put to good use tense logical versions of $n$-characteristics.

The logic of time has operators $G$ and $F$ with the same semantics as the modal ones $\Box$ and $\Diamond$. Next to these, there is a dual pair: $H$ $(\forall H \phi$ iff $\forall H \exists H' \phi)$ and $K$ $(\forall K \phi$ iff $\forall K H \phi)$. The case $H$ is of particular interest.
downward as well as upward in the ordering (which represents the time
structure) and II has to follow I in this respect. Also, there are the
\( n \)-characteristics \( [a]^n \) coding the game-theoretic behaviour of \( a \) in the
\( n \)-game with respect to a finite set of variables.

**Theorem.** The tense logical theory of \( Z \) (**integer time**) is axiomatized
by the following principles:

\[
\begin{align*}
\text{trans} & : \quad Gp \rightarrow GGp \\
\text{succ} & : \quad FT; PT \quad (T \text{ the constant for true}) \\
r\text{-lin} & : \quad Fp \rightarrow G(Fp \lor p \lor Pp) \\
l\text{-lin} & : \quad Pp \rightarrow H(Pp \lor p \lor Fp) \\
\text{modified L"ob} & : \quad G(Gp \rightarrow p) \rightarrow (FGp \rightarrow Gp) \\
& \quad H(Hp \rightarrow p) \rightarrow (PHp \rightarrow Hp).
\end{align*}
\]

For a precise definition of tense logical derivability, cf. van Benthem

**Proof.** Suppose that the formula \( \chi \) cannot be derived using these principles.
I shall show how to construct a valuation \( V \) on \( Z \) such that
\( (Z, <, V) \models \neg \chi[n] \) for some \( n \in Z \).

As a first step, we need the Henkin construction for tense logic (cf. van
Benthem l.c. pp.170-173). This produces a model \( (M, R, V) \) such that

1. the axioms given all hold (universally) in the model;
2. for some \( m \in M \), \( (M, R, V) \models \neg \chi[m] \).

In fact, \( M \) consists of all sets of formulas maximal consistent with the
given axioms, and \( R \) is defined by

3. \( xRy \) iff for all \( \varphi \), if \( G \varphi \vDash x \) then \( \varphi \vDash y \).

\[ \text{7. completeness for } Z\text{-time} \]
The basic tense logical axioms now allow one to prove the following truth lemma:

4. \((M, R, V) \vdash \varphi[x] \textit{ iff } \varphi \in x\)
on the basis of the following definition of \(V\)

5. \(x \in V(p) \textit{ iff } p \in x\).

By assumption, \(\neg \chi\) will be in some \(m \vDash M\), so 2. follows from 4. Also, 1. follows; as substitution is one of the derivation rules, substitution instances of the axioms are satisfied as well. 6-8 now investigate the effect the axioms trans up to 1-1in have on the structure of \((M, R)\); this is standard procedure.

6. \(R\) is transitive.

Proof: suppose \(x R y R z\). If \(G \varphi \vDash x\) then by 4., \(x\) satisfies \(G \varphi\) and hence \(G G \varphi\) (use trans). Hence, \(G G \varphi \vDash x\) by 4. again. Applying 3. twice, this gives \(G \varphi \vDash y\) and \(\varphi \vDash z\). Therefore, \(x R z\) by 3. \(\Box\)

7. \(M\) has no \(R\)-minimum or \(R\)-maximum.

Proof: immediate from succ. \(\Box\)

8. Every two elements with a common upper bound (resp. lower bound) are comparable.

Proof: suppose that \(x, y R z, x \neq y, \neg x R y, \neg y R x\). For instance, \(\varphi \vDash x \setminus y, G \varphi \vDash x, \varphi \vDash y, G y \vDash \eta, \eta \vDash x\). Now, \(z\) satisfies \(P(\varphi \wedge G \varphi \wedge \eta)\). Hence, by 1-1in, it satisfies \(H[F(\varphi \wedge G \varphi \wedge \eta) \vee (\varphi \wedge G \varphi \wedge \eta) \vee P(\varphi \wedge G \varphi \wedge \eta)]\) as well. Therefore, \(y\) satisfies one of \(F(\varphi \wedge G \varphi \wedge \eta), \varphi \wedge G \varphi \wedge \eta, P(\varphi \wedge G \varphi \wedge \eta)\). But the first alternative contradicts \(y\) satisfying \(G \eta\), the second \(\varphi \vDash y\) and the third \(\varphi \vDash y\). \(\Box\)

Define \(\sim\) on \(M\) by

\(x \sim y \textit{ iff } x = y\) or: both \(x R y\) and \(y R x\).
\( |x| R/ \sim |y| \) iff \( x R y \).

Let \( m \) be the \( \chi \)-falsifying element (2.) then restricting to equivalence classes \( |x| \) with \( x R m \) or \( x = m \) or \( m R x \) produces a linear ordering by \( \sim \); since \( R \) is transitive, the model-theoretic properties of \( (M, R, V) \) won't change by restricting to such \( x \) - hence we may assume \( x R m \) or \( x = m \) or \( m R x \) for all \( x \in M \) to begin with.

Let \( \text{VAR}_\chi \) be the (finite) set of variables in \( \chi \). \( \mathcal{P}(\text{VAR}_\chi) \) is the set of shapes; \( x \) has \( S \subset \text{VAR}_\chi \) (\( S \) occurs at \( x \)) iff \( S = x \cap \text{VAR}_\chi \).

Let \( A \) be an equivalence class of \( M \) under \( \sim \). If \( A = \{a\} \) and \( \neg a R a \) then \( A^* = A \). In all other cases, \( A^* \) denotes a model \( (\mathbb{Z}, \prec, V_A) \) of order-type \( \zeta \) such that

(i) each shape occurring in it occurs in \( A \);

(ii) if \( S \) occurs in \( A \) then the set \( \{n \in \mathbb{Z} \mid n \text{ has } S\} \) has neither lower nor upper bound.

Now, define \( N = \Sigma_{A \in M/\sim} A^* \).

9. Suppose that \( x \in A \in M/\sim \) and \( n \in A^* \) have the same shape. Then for all formulas \( \varphi \) over \( \text{VAR}_\chi \): \( M \models \varphi[x] \) iff \( N \models \varphi[n] \).

**Proof.** Use the Ehrenfeucht game appropriate to tense logic. Notice that II can always take care to leave a position \( (y, m) \) for I for which (i) for all \( A \in M/\sim : y \in A \) iff \( m \in A^* \) and (ii) \( y \) and \( m \) have the same shape. \( \Box \)

Therefore, we now have a counter-model to \( \chi \) of an order type which is a sum of \( \zeta \)'s and 1's. To finally transform this into a counter-model of type \( \zeta \), I use the modified Löb-axioms.

---

7. completeness for \( \mathbb{Z} \)-time
10. Suppose that \( \varphi \) is a formula over \( \text{VAR}_\chi \) such that 
\( \varphi^N = \{ n \in N \mid N \models \varphi(n) \} \) is non-empty and upward (downward) 
bounded. Then \( \varphi^N \) has a maximum (minimum).

Proof. Let \( N \models \varphi(n) \) and \( m < n \). Then \( m \) satisfies \( F \varphi \) and \( FG \neg \varphi \). Since \( F \varphi \) 
amounts to \( \neg G \neg \varphi \), by the first modified Löb-axiom (with \( \neg \varphi \) 
substituted for \( p \)): \( N \models \neg G(G \neg \varphi \to \neg \varphi)[m] \). Choose \( k > m \) such that \( k \) 
satisfies \( G \neg \varphi \) and \( \varphi \); \( k \) is the required maximum. \( \Box \)

Let \( k \) be the rank of \( \chi \). Put \( T = \{ [x]^k \mid x \in N \} \) (\( [x]^k \) codes the "behaviour" 
of \( x \) in the game of length \( k \)). Define \( T^+ = \{ \tau \in T \mid \{ x \in N \mid \tau = [x]^k \} \) has an 
upper bound \} and \( T^- = \{ \tau \in T \mid \{ x \in N \mid \tau = [x]^k \} \) has a lower bound \}. By 10., to 
each \( \tau \in T^+ \) there is a maximal \( x = x_\tau \) with \( [x]^k = \tau \) and similarly for \( T^- \). Let 
\( A_0 \) be the set \( \{ x_\tau \mid \tau \in T^+ \cup T^- \} \). Choose \( A^+ \subset N \) of order type \( \omega \) such that 
\( A_0 < A^+ \) and such that for all \( \tau \in T \setminus T^+ \), \( \{ x \in A^+ \mid [x]^k = \tau \} \) is infinite. 
Similarly, choose \( A^- \subset N \) of order type \( \omega^* \) such that \( A^- < A_0 \) and each 
\( \{ x \in A^- \mid [x]^k = \tau \} \) for \( \tau \in T \setminus T^- \) is infinite. Finally, \( A \) is the submodel of \( N \) 
obtained by restricting to \( A^- \cup A_0 \cup A^+ \). 
So, \( A \) has order type \( \zeta \); it suffices to prove

11. If the formula \( \psi \) over \( \text{VAR}_\chi \) has rank \( \leq k \) and \( x \in A \) then 
\( A \models \psi[x] \) iff \( N \models \psi[x] \).

Proof. Induction on \( \psi \). There is but one interesting case. Suppose that 
\( N \models F \psi[x] \). Then \( y > x \) exists such that \( N \models \psi[y] \). Let \( \tau = [y]^k \). By 
construction, there is a \( z > x \) in \( A \) such that \( [z]^k = \tau \). But then, \( N \models \psi[z] \) as 
well. By induction hypothesis \( A \models \psi[z] \). Hence, \( A \models F \psi[x] \). \( \Box \)

8.1 Introduction. By a suggestion of Troelstra, Rodenburg [1982] continued the investigation of intuitionistic correspondence theory which was begun by van Benthem (in a preliminary version of van Benthem [1984]). (For the latest developments, cf. Rodenburg [1986].) Part of this asks for which intuitionistic formulas \( \varphi \), validity of \( \varphi \) on intuitionistic (i.e., partially ordered) frames - which is a \( \Pi^1_1 \)-condition to begin with - can be defined in first-order terms. This question is part of its modal companion: by the Gödel translation, intuitionistic formulas can be translated into modal terms, and this means that the intuitionistic correspondence problem considers rather special modal formulas only. But also, the class of frames is restricted to partial orderings, and in the \( \Pi^1_1 \)-condition, the second-order quantifiers range over upward closed sets only. And this makes for the rather different flavor of the subject. In
1983, I solved two problems left open in Rodenburg [1982]. The first, on whether validity of a certain formula \( B_2 \) (called \( SP_2 \) in Rodenburg [1986]) is first-order definable on the class of (binary) trees, is dealt with in this chapter. The second is the same problem for intuitionistic formulas in one variable and the class of finite partial orderings. This is the content of chapter 9.

8.2 Preliminaries. Consider a propositional language over a set \( \text{VAR} \) of propositional variables with a propositional constant \( \bot \) for falsehood and operations \( \rightarrow, \land \) and \( \lor \). Negation \( \neg \) can either be taken as a primitive or else be explained as \( \neg \varphi = (\varphi \rightarrow \bot) \). Let \( A = (A, \leq) \) be a partially ordered set and \( \mathcal{V}: \text{VAR} \rightarrow \mathcal{P}(A) \) a valuation; then \((A, \mathcal{V})\) is the corresponding Kripke model. The intuitionistic semantics of the formulas of the above language on such a model can be explained simply in modal terms as follows: for \( a \in A \), \( \varphi \) a formula, define \( a \vdash \varphi \) (intuitionistically) iff \( a \vdash \varphi^m \) (modally), where the modal translation \( \varphi^m \) of \( \varphi \) is defined as follows:

(i) for \( p \in \text{VAR} \), \( p^m = p \); \( \bot^m = \bot \);
(ii) \( m \) preserves \( \land \) and \( \lor \);
(iii) \( (\varphi \rightarrow \psi)^m = \Box(\varphi^m \rightarrow \psi^m) \)

(hence, \( (\neg \varphi)^m = \Box \neg \varphi^m \)).

In Kripke models \((A, \mathcal{V})\) for the intuitionistic language, \( A \) is always a partially ordered set and \( \mathcal{V} \) is such that each \( \mathcal{V}(p) \) (\( p \in \text{VAR} \)) is upward closed, i.e., \( a \in \mathcal{V}(p) \) and \( a \leq b \) imply \( b \in \mathcal{V}(p) \).

8.3. Rodenburg [1982] considers the intuitionistic formula
\[ \psi = p \land \neg q \]
\[ \chi = \neg r \land q, \]

and puts the question whether it is first-order definable on the class of "rooted trees". The binary trees of 2.1 form an elementary subclass of this collection. Therefore, a negative answer would be implied by showing \( B_2 \) not to be first-order on binary trees. Now, Rodenburg's arguments show that

8.4 Lemma. For binary trees \( T \), \( B_2 \) is valid on \( T \) iff \( T \) is finite.

Hence, his problem is answered negatively by corollary 2.2.4 above. Notice, however, that 2.4.1 even shows \( B_2 \) not to be \( \Sigma^1_1 \) on binary trees.

Proof of 8.4. First, let \( T \) be infinite. By König's lemma, it has an infinite branch \( \alpha \). Suppose \( \alpha = \{t_0, t_1, t_2, \ldots \} \) where \( t_0 < t_1 < t_2 < \ldots \) and let \( s_i \) be the immediate successor of \( t_i \) different from \( t_{i+1} \). Define the following valuation on \( T \):

\[ V(p) = \{ t \in T \mid \exists i (s_{3i} \leq t \lor s_{3i+1} \leq t) \} \]
\[ V(q) = \{ t \in T \mid \exists i (s_{3i+1} \leq t \lor s_{3i+2} \leq t) \} . \]

Clearly,
\[ s_{3i} \leq t \implies t \models \psi \]
\[ s_{3i+1} \leq t \implies t \models \neg \psi \]
\[ s_{3i+2} \leq t \implies t \models \neg \chi ; \]

hence,
\[ t \in T \setminus \alpha \implies t \models \neg \psi \lor \psi \lor \chi \quad (1) . \]

Also, no \( t \in \alpha \) forces one of \( \neg \psi, \neg \psi \lor \psi \lor \chi \), i.e.,
\[ t \in \alpha \implies t \not\models \neg \psi \lor \neg \psi \lor \neg \chi \quad (2) . \]

In particular, \( t_0 \) does not force \( \neg \psi \lor \neg \psi \lor \neg \chi \). Thus, to see that \( B_2 \) is not valid on \( (T, V) \), it suffices to see that \( t_0 \models \neg \psi \lor \neg \psi \lor \neg \chi \to \psi \lor \psi \lor \chi \). Hence,
suppose \( t_0 \leq u \models \neg \varphi \lor \neg \psi \lor \neg \chi \). By (2), \( u \models \alpha \), therefore, by (1), \( u \models \varphi \lor \psi \lor \chi \), and we’re done.

Conversely, assume that \( T \) is finite, but \( T \not\models B_2 \). So, there are a valuation \( V \) and \( u \in T \) such that \( u \not\models B_2 \). For instance, \( u \leq w_0 \models \neg \varphi \lor \neg \psi \lor \neg \chi \Rightarrow \varphi \lor \psi \lor \chi \) but \( w_0 \not\models \neg \varphi \lor \neg \psi \lor \neg \chi \). Since \( T \) is finite, we may assume \( w_0 \) is a maximal element with these properties. Since \( w_0 \not\models \neg \varphi \), there exists \( w_1 \geq w_0 \) with \( w_1 \not\models \varphi \); similarly, there are \( w_2, w_3 \geq w_0 \) such that \( w_2 \not\models \psi \), \( w_3 \not\models \chi \). Now, \( \neg w_1 \leq w_2 \), for if not, then \( w_2 \not\models \varphi \) (a formula forced at some place \( w_1 \) remains forced at every \( w_2 \geq w_1 \)), hence \( w_2 \not\models \varphi \), contradicting \( w_2 \not\models \varphi \).

In the same way, it is seen that no two of \( w_1, w_2, w_3 \) are comparable; in particular, they are all \( > w_0 \). Let \( w \) be the greatest lower bound of \( w_1 \) and \( w_2 \). If \( w_0 < w \) then \( w \not\models \neg \varphi \lor \neg \psi \lor \neg \chi \Rightarrow \varphi \lor \psi \lor \chi \) (since this is forced by \( w_0 \)) and \( w \not\models \neg \varphi \lor \neg \psi \lor \neg \chi \) (by maximality of \( w_0 \)) thus \( w \not\models \varphi \lor \psi \lor \chi \). However, \( w \not\models \varphi \) (if not, then \( w_2 \not\models \varphi \), but \( w_2 \not\models \psi \)), \( w \not\models \psi \) (else \( w_1 \not\models \psi \)) and \( w \not\models \chi \) (otherwise, \( w_1 \not\models \chi \)) — a contradiction. Therefore, \( w = w_0 \). But a similar argument shows that \( w \) is greatest lower bound of each two of \( w_1, w_2, w_3 \) — an impossibility in a binary tree. \( \Box \)

By contrast, Rodenburg [1982] theorem 6.1.7 says that one-variable formulas all are first-order on (general) trees. Compare this also with the results discussed in the next chapter.
9. Formulas in one variable.

A problem left open by Rodenburg [1982] was to determine which intuitionistic formulas in one variable are first-order definable on finite partial orderings. This is solved below. First, however, I construct universal-exact Kripke models for intuitionistic logic and present a logical analysis of the one-variable model using Ehrenfeucht's game. This is applied subsequently to the problem of first-order definability.

In the proof of 6.6, an exact Kripke model \((U, \mathcal{V})\) universal with respect to Kripke models (for modal logic) over finite partial orderings was explicitly constructed. In this model, no \(\mathcal{V}(p)\) is upward closed. (Remember this is required of intuitionistic models.) However, \(U\) has a maximal upward closed subset \(U^i\) such that \(\mathcal{V}^i(p) = \mathcal{V}(p) \cap U^i\) is upward closed, this is

\[
U^i = \{k \in U | \forall p \in \text{VAR} \forall m \in U [ k \leq m \land k \in \mathcal{V}(p) \Rightarrow m \in \mathcal{V}(p)] \}.
\]
9.1 Theorem. 1. \((U^i, V^i)\) is (modally) exact;

2. \((U^i, V^i)\) is universal with respect to finite
   intuitionistic Kripke models.

Proof. 1. Obvious. 2. Modify the proof of 6.6. \(\Box\)

The structure of \((U^i, V^i)\) for \(\text{VAR}=(p)\) having one element is quite simple
and has been known for a long time. Let us have a short look at it.

[Diagram]

The elements in the picture are named by their codes from chapter 6 as
well as by natural numbers, the reason for which is explained by 9.2 below.

There is but one element in \(V(p)\): \(0=(p)\). Also, there is but one element
\((k, j)\) with \(k \in U^i\): \(2=(0,1)=(0,\emptyset)\). An anti-chain has at most two elements;
since \(V(p)=\{0\}\), this means that every other non-maximal element has the
form \((A, 1)\) with \(|A|=2\). Problem 6.10 for this model is solved as follows.
In particular, $n$ is the least number $i$ such that \( \psi \) defines $n$ in \((U_i, V_i)\).

**Proof.** Let us agree to denote the partial ordering of $U_i$ by $\leq *$; $<$ then can be used for the usual ordering of $\mathbb{N}$ (as is done in 9.2). Hence, $n \leq m$ iff $m+2 \leq n$.

If, in the $k$-game on a position $(i, j)$, player I plays some $l \geq i$, $j$, II can win by just copying the moves of I. Therefore, the **optimal** strategy of I always is to pick $i-1 \geq j$ (if $i < j$) resp. $j-1 \geq i$ (if $j < i$): he must get at the top of the model as fast as he can without losing, for, in order to win, he has to take care that exactly one of the players has to arrive at 0, the only element in $V(p)$. As long as II cannot copy I, his optimal strategy consists in repeating one of the previous moves (namely, $i$ if I has chosen $i-1$ in position $(i, j)$ etc.): this surely is the slowest way to drag him to the top of the model where 0 is lurking. Therefore, if $n \leq m$, the play will go like this if both players do their utmost; as long as $k \leq n$:

\[
\begin{align*}
I: & \quad n-1 \quad n-2 \quad n-3 \quad \ldots \quad n-k \\
II: & \quad n \quad n-1 \quad n-2 \quad \ldots \quad n-(k-1).
\end{align*}
\]

Obviously then, I wins iff $k \geq n$. $\Box$

As to defining (**modal**) formulas:

**9.3 Lemma.** (i) $\psi_0 = p$ defines 0;
(ii) $\psi_1 = \Box \neg p$ defines 1;
(iii) $\psi_2 = \neg p \land \Box \Diamond p$ defines 2;
(iv) if $\psi_n$, $\psi_{n+1}$, $\psi_{n+2}$ define resp. $n$, $n+1$, $n+2$, then $\psi_{n+3} = \Diamond \psi_n \land \Diamond \psi_{n+1} \land \neg \Diamond \psi_{n+2}$ defines $n+3$ ($n \geq 0$).

9. one-variable formulas
Proof. Clear from the picture. □

It is easy to show by induction that for intuitionistic formulas \( \varphi \), the set \( \{ k \in U^i | k \vdash - \varphi \} \) defined by \( \varphi \) always will be upward closed. Now, \( U^i \) has the following types of upward closed sets:

1. \( \emptyset \);
2. \( 0 \uparrow = \{ 0 \}; 1 \uparrow = \{ 1 \}; 2 \uparrow = \{ 0, 2 \}; \)
3. \( (n+3) \uparrow = \{ 0, \ldots, n+1, n+3 \} \ (n \geq 0) \);
4. \( n \uparrow \cup (n+1) \uparrow = \{ 0, \ldots, n+1 \} \ (n \geq 0) \);
5. \( U^i = \mathbb{N} \).

9.4 Lemma on normal forms.

1. \( \bot \) defines \( \emptyset \);
2. (i) \( \psi_0 = p \) defines \( 0 \uparrow \);
   (ii) \( \psi_1 = \neg p \) defines \( 1 \uparrow \);
   (iii) \( \psi_2 = \neg \neg p \) defines \( 2 \uparrow \);
3. if \( \psi_n, \psi_{n+1}, \psi_{n+2} \) define resp. \( n \uparrow, (n+1) \uparrow \) and \( (n+2) \uparrow \) then:
4. \( \psi_{n+3} = \psi_{n+2} \rightarrow \psi_n \lor \psi_{n+1} \) defines \( (n+3) \uparrow \), and
5. \( \psi_n \lor \psi_{n+1} \) defines \( n \uparrow \cup (n+1) \uparrow \);
6. \( p \rightarrow p \) defines \( \mathbb{N} \).

Proof. Notice that intuitionistic formulas are intended here. Therefore, a given formula \( \psi \) defines \( X \subset U^i \) intuitionistically iff \( \psi^m \) (see 8.2) defines \( X \) modally, and this is easily checked in all cases. □

Remark. Notice that the \( \varphi_n \) from 9.3 has modal rank \( n \) – the least possible rank for an \( n \)-defining formula according to 9.2. Also, \( \psi_n^m \) has rank \( n \). The
iff $\neg \Box \varphi_n$ is.

**Proof.** Semantically, this is clear on $(U^i, V^i)$. But a syntactic proof also is possible. $\blacksquare$

9.6 Lemma. Let $\varphi$, $\psi$ be intuitionistic one-variable formulas and $k$ the maximum of the modal ranks of $\varphi^m$ and $\psi^m$. Then $\varphi$ and $\psi$ are equivalent (i.e., true on the same frames) iff

$$\{ m \leq k+1 | m \vDash \varphi \} = \{ m \leq k+1 | m \vDash \psi \}.$$

**Proof.** Suppose this condition holds. By 9.2, $[n]^k = [n']^k$ when $n' > n > k$. Let $a$ be any element of any model $(F, V)$. Then $[a]^k$ must be one of $[0]^k, ..., [k+1]^k$; say, $[a]^k = [m]^k$ with $m \leq k+1$. Hence, $a \vDash \varphi$ iff $m \vDash \varphi$ iff $m \vDash \psi$ iff $a \vDash \psi$. $\blacksquare$

9.7 Remark. It is easy to extend the one-variable model to one that is exact and universal with respect to all Kripke models: simply add a least element $\infty$. To see that this suffices, let $(A, V)$ be any Kripke model and $a \in A$. If $a \vDash \varphi_n$ for some $n$, then for all $k$, $[a]^k = [n]^k$. And, if no $\varphi_n$ is forced at $a$ then, by induction on $n$, it follows that $a \vDash \Box \varphi_n$ for all $n$. But then obviously, for all $k$, $[a]^k = [\infty]^k$.

Of course, one can do modal and intuitionistic propositional logic with infinitary connectives $\land$ and $\lor$ and define formulas $[a]^\alpha$ for all ordinals $\alpha$ relative to an element $a$ in a Kripke model. The above amended model is still universal with respect to such an extended formalism - by the same proof; this is because the players essentially have one optimal strategy each (cf. 9.2) not depending on the bound $\alpha$ of the game. (cf. also de Jongh

9. one-variable formulas
I now turn to correspondence theory. According to 9.4, each intuitionistic one-variable formula is (in the Kripke model semantics) equivalent with a formula of one of the following shapes: $\bot$, $p \rightarrow p$, $\psi_n$, $\psi_n \lor \psi_{n+1}$.

Rodenburg [1982] has a complete classification of these formulas as to first-order definability on arbitrary partial orderings as follows: $\bot$, $p \rightarrow p$, $\psi_n$ for $n \leq 4$ and $\psi_n \lor \psi_{n+1}$ for $n \leq 3$ are first-order; the rest is not.

He then put the question: what about first-order definability of these formulas on finite partially ordered sets? Of course, only cases $n > 4$ for the $\psi_n$ and $n \geq 3$ for the $\psi_n \lor \psi_{n+1}$ remain to be investigated.

The answer is given by the

**9.6 Theorem.** On finite partial orderings, $\psi_5$, $\psi_n$ for $n \geq 7$ and $\psi_n \lor \psi_{n+1}$ for $n \geq 4$ are not first-order (in fact, they are not monadically $\Sigma^1_1$; however, $\psi_5$ is.

**Proof.** By 6.7, each finite model is canonically mapped into $U^i$. $h$ always will denote this map; so all $\psi_n$ iff $ha = n$. A model realizes $n$ iff $ha = n$ for some $a$; $n$ is realizable in a frame iff it is realized under a suitable valuation on that frame. As to the $\psi_n$ for $n = 5$ and $n \geq 7$, notice that $\psi_n$ does not hold in a given model iff $n - 1$ or some $m \geq n + 1$ is realizable in it and this is true iff $n - 1$ is realizable. In particular, $\psi_5$ does not hold iff 4 is realizable. I shall now show that 4-realizability is not first-order on
"finitisation" of Rodenburg's proof for arbitrary partial orderings.

Consider the frame $PZ_k$ of theorem 5.6. Make it into a partial ordering $TPZ_k$ by replacing the relation of $PZ_k$ by its transitive closure.

Claim 1. 4 is not realizable in any $TPZ_k$.

Proof. Suppose 4 is realized under some valuation. If at all, this can happen at the least element $w$ only. There must be top-elements realizing 0 and 1; hence there are $b > w$ and $c, d > b$ with $hc = 0$ and $hd = 1$. But then, $hb = 3$, contradicting $w < b$ and $hw = 4$. ☒

Next, consider the frame $2TPZ_k$ obtained from two copies of $TPZ_k$ by identifying their least elements.

Claim 2. 4 is realizable in each $2TPZ_k$.

Proof. Just let $V(p)$ be the set of top-elements of one of the $TPZ_k$-copies in $2TPZ_k$. ☒

Therefore, in order to see that 4-realizability is not first-order on finite frames, we need

Claim 3. If $k \geq 2^n$ then $TPZ_k \equiv^n 2TPZ_k$.

Proof. Compare the proof of 5.2. ☒

Showing that $\psi_6$ is first-order definable on finite frames amounts to showing that 5-realizability is so definable. This is the content of the following

9. one-variable formulas
Claim 4.5 is realizable in a finite frame A iff there are \(x, x_0, y_0, x_1, x_2\) and \(x_3\) such that:

1. \(x < x_2 < x_0; x < x_3 < y_0, x_1\)
2. \(\neg x_2 \leq x_3\)
3. \(\forall y \,(x < y \text{comp} x_2 \rightarrow y \lor x_1)\)
4. \(\forall y \,(x_3 < y \text{comp} y_0 \rightarrow y \lor x_1)\).

Here, \(x \text{comp} y\) means: \(x \leq y \lor y \leq x\); and \(x \lor y\) stands for \(\neg \exists z \,(x \leq z \land y \leq z)\).

Proof. To see this, first suppose that \(\mathcal{V}(p) \subseteq A\) and \(x \in A\) are such that \(hx = 5\). We may assume that \(x\) is a maximal element with this property. Choose \(x_3 > x\) maximal with \(hx_3 = 3\). Finally, choose \(x_0, y_0, x_1, x_2\) with \(h\)-values 0, 0, 1 and 2 respectively such that (1) holds. Now, (2) holds as well. As to (3): if \(x < y \text{comp} x_2\) then, by maximality of \(x\), \(hy = 2\) or \(= 0\); hence \(y \lor x_1\). As to (4): if \(x_3 < y \text{comp} y_0\) then, by maximality of \(x_3\), \(hy = 0\); hence \(y \lor x_1\).

Conversely, assume that \(x, x_0, y_0, x_1, x_2, x_3\) satisfy (1)-(4).

Define \(\mathcal{V}(p) = \{ a \in A \mid \neg a \leq x_2 \land \neg a \leq x_3 \land a \lor x_1 \}\).

Notice that \(\mathcal{V}(p)\) is upward closed. Now:

- \(hx_0 = 0\): \(\neg x_0 \leq x_2; \neg x_2 \leq x_3\) and \(x_2 < x_0\) hence \(\neg x_0 \leq x_3\); by (3), \(x_0 \lor x_1\). So \(x_0 \in \mathcal{V}(p)\), i.e., \(hx_0 = 0\).
- \(hy_0 = 0\): \(\neg y_0 \leq x_3; \neg x_3 \leq x_2\) and \(x_3 < y_0\), hence \(\neg y_0 \leq x_2\); by (4), \(y_0 \lor x_1\). So, \(y_0 \in \mathcal{V}(p)\).

- \(hx_1 = 1\): Suppose \(x_1 \leq z\). Obviously, \(\neg z \lor x_1\). Thus, \(z \notin \mathcal{V}(p)\).

- \(hx_2 = 2\): \(x_2 \notin \mathcal{V}(p)\) is clear. Suppose \(x_2 \leq y\). If \(y = x_2\) then \(y < x_0 \in \mathcal{V}(p)\). If \(x_2 < y\) then by (3), \(y \lor x_1\); furthermore, \(\neg y \leq x_3\), since if \(y \leq x_3\) then \(x_2 \leq x_3\). Hence, \(y \in \mathcal{V}(p)\).

- \(hx_3 = 3\): First, \(x_3 < y_0, x_1\). Second, suppose \(x_3 < z\) and \(hz = 2\). Then \(z \notin \mathcal{V}(p)\), \(\neg z \leq x_3, \neg z \leq x_2\), thus by definition of \(\mathcal{V}(p)\), \(\neg z \lor x_1\). But then, \(hz = 2\) is impossible.
For the rest of the proof, I need a

Trivial remark. Suppose that \( V \) is any valuation on \( U^i \) different from \( V^i \). Then under it, no \( a \in U^i \) obtains a canonical value \( > 3 \) in \((U^i, V^i)\).

Now, modify the frame \( U^i \) by disconnecting the element 4 from its successors 0, 1 and 2. Consider the frames obtained from this by identifying the now successor-less element 4 with the least node \( w \) of a model \( TPZ_k \) or \( 2TPZ_k \); cf. the picture below where \( TPZ_4 \) is attached at the \( U^i \)-node nr. 4.
By the trivial remark and by what we know about the (2)TPZ$^k$, for $n \geq 6$, the resulting frame can realize the canonical value $n$ iff it realizes this value at $n$ iff a model 2TPZ$^k$ is attached at 4. Therefore, final segments of these models (from $n$ upward) can be used to show that $\psi_n$ is not first-order definable on finite partial orderings for $n \geq 7$.

As to the $\psi_n \lor \psi_{n+1}$, finally, a frame can be made into a counter-example to this formula iff under a suitable valuation a canonical value $\geq n+2$ is realized, and this happens iff one of $n+2$, $n+3$ is realized. To see this is not a first-order condition when $n \geq 4$, the model classes constructed for $\psi_{n+4}$ can be used.

For instance, take the case $n=4$. Look at the partial ordering used to construct the $\psi_6$-model classes. Whatever the valuation on such a model, $h5=4$ is impossible by the trivial remark. Therefore, the model realizes a value $n \geq 6$ iff it does so at element 7 iff $h4=4$. (In fact, if $h4=4$ then $h7=7$ iff $h5=5$ and $h7=6$ iff $h5=3$.) But we know this to be not first-order on the given partial orderings. This finishes the proof of 9.8. 

To contrast the $\psi_n \lor \psi_{n+1}$-case with that of the $\psi_n$, let us call $A \subset \mathbb{N}$ realizable on a model iff some element obtains a value in $A$ under a suitable valuation. Then notice that by Rodenburg [1982], realizability for {4, 5} and {5, 6} is first-order definable while this is not the case for 4, 5 and 6 separately (and it is so for 5 only if we restrict attention to finite partial orderings).
Appendix A: can time be directional?

Van Benthem [1983] contains a list of properties which the time-structure \((T, \leq)\) may have:

**SYM**: \((T, \leq) \cong (T, \geq)\).

**SYM**: each \(t \in T\) splits \(T\) into order-isomorphic halves \((\leftarrow, t) = \{ t' \in T \mid t' < t \}\) and \((t, \rightarrow) = \{ t' \in T \mid t < t' \}\).

**HOM**: for all \(t, t' \in T\) there is an automorphism moving \(t\) to \(t'\).

**HOM**: for all \(t_1 < t_2, t_3 < t_4\), if \(|(t_1, t_2)| = |(t_3, t_4)|\) then some automorphism maps \((t_1, t_2)\) onto \((t_3, t_4)\).

**REF**: each interval \((t_1, t_2)\) is isomorphic with \((T, \leq)\).

On p.45, the question is posed as to whether REF implies SYM. Below, I answer this negatively by constructing, for each uncountable power, a
linear ordering of that power which has every property listed above - except the first one.

Another solution to this problem is given by Droste [198?].

Given an uncountable initial number, let \( \Omega \) be a well-ordering of that type. Define the increasing sequence of linear orderings \( A_0 \subset A_1 \subset A_2 \subset \ldots \) as follows:

(i) \( A_0 = \Omega \);

(ii) \( A_1 \) is obtained from \( A_0 \) by inserting, for each \( a \in A_0 \), a copy \( \Omega(a) \) of \( \Omega \) between \( a \) and its predecessors in \( A_0 \);

(iii) \( A_{i+2} \) is obtained from \( A_{i+1} \) by inserting a copy \( \Omega(a) \) of \( \Omega \) between \( a \) and its predecessors for each \( a \in A_{i+1} \setminus A_i \) \( (i \geq 0) \).

\( \Omega_\infty \) is the union \( \bigcup_n A_n \).

Clearly, \( \Omega_\infty \) has the same power as \( \Omega \).

By induction on \( n \) it follows that each \( A_n \) is well-ordered (it has the order type \( \Omega^{n+1} \)). Suppose \( X \subset \Omega_\infty \) has order type \( \Omega^* \) (the reversal of \( \Omega \)). Then each set \( X \cap A_n \), being both well-ordered and conversely well-ordered, must be finite. Therefore, \( X = \bigcup_n (X \cap A_n) \) is at most countable - a contradiction. Since \( \Omega \subset \Omega_\infty \), \( \Omega_\infty \) does not satisfy SYM.

We are going to prove the

**Lemma.** If \( a \in \Omega_\infty \) then \( \langle \leftarrow, a \rangle \) and \( \langle a, \rightarrow \rangle \) are both isomorphic with \( \Omega_\infty \).

From this, SYM* and HOM easily follow. As to REF, if \( t_1 < t_2 \) then \( (t_1, \rightarrow) \) is isomorphic with \( \Omega_\infty \) by the lemma. Since \( \Omega_\infty \) is isomorphic with each of its initials \( \langle \leftarrow, a \rangle \), so is \( (t_1, \rightarrow) \). In particular then, \( (t_1, t_2) \cong (t_1, \rightarrow) \cong \Omega_\infty \). HOM* now is a trivial consequence.
I use the following notations. \( \Omega_n = A_n \setminus \bigcup_{i<n} A_i \). \( a \leftarrow b \) means that \( a \) is an element of the copy \( \Omega(b) \) of \( \Omega \) inserted immediately in front of \( b \). For \( \chi \subset \Omega_\infty \), \( C(\chi) = \chi \cup \bigcup_{x \in \chi} \Omega(x) \). \( C^n \) is the \( n \)-th iterate of \( C \). Hence, \( A_n = C^n(\Omega) \).

Now, suppose that \( a \in \Omega_\infty \). Say, \( a \in \Omega_n \).

For \( i \leq n \), let \( a_i \) be the least element of \( (a, \rightarrow) \cap \Omega_i \).

Then \( a_n \leftarrow a_{n-1} \leftarrow \ldots \leftarrow a_0 \) and \( (a, a_i) \cap \Omega_i = \emptyset \); if \( n > 0 \) then \( a \leftarrow a_{n-1} \) as well.

Dealing with the second half of the lemma first: since \( \Omega \) has initial type, \( (a_0, \rightarrow) \cap \Omega \cong \Omega \). Let \( h \) be the isomorphism.

\( h \) extends to an isomorphism \( h_1 : [a_1, \rightarrow) \cap C(\Omega) \to C(\Omega) \) which maps \( [a_1, \rightarrow) \cap \Omega(a_0) \) onto \( \Omega(ha_0) \) and, for \( a_0 < b \), \( \Omega(b) \) onto \( \Omega(hb) \).

Repeating this extension-procedure, ultimately an isomorphism \( h_n \) between \( [a_n, \rightarrow) \cap C^n(\Omega) \) and \( C^n(\Omega) \) is obtained.

Going further one more step but this time using \( \Omega(a_n) \cong \Omega(b) \) (\( b \) least element of \( C^n(\Omega) \)), we get \( (a, \rightarrow) \cap C^{n+1}(\Omega) \cong C^{n+1}(\Omega) \).

Proceeding, we finally find \( (a, \rightarrow) \cong \Omega_\infty \), as desired.

Turning to \( (\leftarrow, a) \), define

\[
X_0 = (\leftarrow, a) \cap \Omega_0 \\
X_{i+1} = (\leftarrow, a) \cap \Omega(a_i) \quad (i < n)
\]

Then \( X_i \subset \Omega_i \) and \( X_0 < X_1 < X_2 < \ldots < X_n < \Omega(a) < a \).

### Situation for \( a \in \Omega_2 \):

- \( \Omega_0: \quad X_0 \quad [\quad \text{empty} \quad] \quad a_0 \)
- \( \Omega_1: \quad \Omega(a_0): X_1 \quad [\quad \text{empty} \quad] \quad a_1 \ldots \)
- \( \Omega_2: \quad \Omega(a_1): X_2 \quad [\text{empty}] \quad a \quad a_2 \ldots \)
- \( \Omega_3: \quad \Omega(a) \)

---

*App.A: can time be directional?*
Because $X_0$ is bounded by $a$ in $\Omega$ and $X_i+1$ is bounded by $a_i+1$ in $\Omega(a_i)$ ($i < n$), these well-ordered sets all have cardinal $< \Omega$. Therefore, $\bigcup_{i \leq n} X_i$ has cardinal $< \Omega$; hence $\mathcal{X} = \bigcup_{i \leq n} X_i \cup \Omega(a) \cong \Omega$ by some isomorphism $h$. Extend $h$ to an isomorphism: $\mathcal{C}(\mathcal{X}) \to \mathcal{C}(\Omega)$, using $\Omega(x) \cong \Omega(hx)$ for each $x \in \mathcal{X}$, and repeat this procedure; eventually, obtain $\bigcup_n \mathcal{C}(\mathcal{X}) \cong \Omega_\omega$.

It remains to show that $(\leftarrow, a) = \bigcup_n \mathcal{C}(\mathcal{X})$.

One inclusion is clear. So, assume $b < a$. Suppose that $b = b_m \leftarrow b_{m-1} \leftarrow \ldots \leftarrow b_0 \in \Omega_0$.

Now, $a_0 < b_0$ is impossible since then $a_0 < b_i$ for all $i$, in particular, $a_0 < b$. If $b_0 < a_0$ then, since $(a, a_0) \cap \Omega_0 = \emptyset$, $b_0 \in X_0$, and we are done. The remaining possibility is $b_0 = a_0$. But then, $m > 0$. Consider $b_1$. $a_1 < b_1$ is impossible and $b_1 < a_1$ leads to $b_1 < a$ and $b_1 \in X_1$ since $(a, a_1) \cap \Omega_1 = \emptyset$. So, assume $b_1 = a_1$. Then $m > 1$. Repeating this argument, we arrive at the case $b_m = a_n$ and $m > n$. But then, $a < b_{n+1}$, hence $a < b$ - a contradiction. \[\Box\]
Appendix B: 

reduction of higher-order logic.

In 1975, S.K. Thomason gave a reduction of second-order logic to modal logic. Van Benthem [1985] pp.23-24 added interest to this by explaining how all of higher-order logic could then be reduced as well. For the reduction of higher-order to second-order logic, he referred to an old unpublished note of mine. However, it appeared that such a reduction was given by at least two people already, viz. Hintikka [1955] and Montague [1965].

These authors use rather different methods. Hintikka employed a rather straightforward direct translation into monadic second-order logic, of which a modern presentation occurs in van Benthem and Doets [1983]. Montague's paper was based on a translation into the language of set theory, as was my note. For completeness' sake, this is presented below.

App.B: reduction of higher-order logic
Montague goes on to transfinite type theory and has applications to spectra besides.)

B.1 Definition. The (ZF) model $A=(A,E)$ is called absolute for powers if for each $a \in A$ and $X \subseteq \{ x \in A \mid x E a \}$ there exists $b \in A$ such that $X = \{ x \in A \mid x E b \}$.

Power absoluteness obviously is a $\Pi^1_1$-property which is expressed by the $\Pi^1_1$-form of Zermelo's Aussonterung-axiom. (Notice however that it is first-order on finite models.)

Let $V_\alpha = \bigcup_{\beta < \alpha} P(V_\beta)$ be the $\alpha$-th level of the cumulative hierarchy. Natural models have the form $(V_\alpha, \in)$. Clearly, these are power absolute. Conversely:

B.2 Lemma. If $A=(A,E)$ is a power absolute model of ZF, then $E$ is well-founded and the isomorph $(B,\epsilon)$ of $A$ where $B$ is transitive is a natural model.

Proof. Suppose that $X \subseteq A$ is a non-empty set without $E$-minimal element. Since ZF proves the existence of transitive closures, there is an $a \in A$ such that $X \cap \{ x \in A \mid x E a \}$ is non-empty and has no $E$-minimal element (simply let $a$ be the "transitive closure" of an element of $X$). By power absoluteness, choose $b \in A$ such that $\{ x \in A \mid x E b \} = X \cap \{ x \in A \mid x E a \}$. Now, $b$ contradicts the regularity axiom of ZF. Next, let $B=(B,\epsilon)$ be the isomorph of $A$ with $B$ transitive. Suppose that $\alpha$ is the set of ordinals in $B$. By transfinite induction on $\beta \in \alpha$, it follows that $V_\beta^B = B \cap V_\beta = V_\beta$, using power absoluteness. Hence, $B = \bigcup_{\beta < \alpha} V_\beta^B = \bigcup_{\beta < \alpha} V_\beta = V_\alpha$. □
type theory over $L$. Without loss of generality we may suppose that $L^\omega \subseteq V_\omega$. Let $^\gamma$ be an effective operation mapping $L^\omega$ into the set of ZF-terms such that for each transitive ZF-model $B=(B, \epsilon)$ and each $\varphi \in L^\omega$:

$\ ^\gamma \varphi^B = \varphi$. (Such an operation exists if the identification of formulas with hereditarily finite sets is a reasonable one.) Let $T(x,y)$ be a set-theoretic formula obtained as a straightforward translation of the definition of "$\varphi$ is true in $A$" ($\varphi \in L^\omega$, $A$ any $L$-model) in set-theoretic terms.

**B.3 Lemma.** For any natural ZF-model $(V_\alpha, \epsilon)$, if $M \in V_\alpha$ is an $L$-model and $\varphi \in L^\omega$, then: $(V_\alpha, \epsilon) \models T(\ ^\gamma \varphi^\gamma, M) \iff M \models \varphi$.

**Proof.** Being a ZF-model, $V_\alpha$ must contain the ("full") type-structure $M^\omega$ over the model $M$ as soon as it contains $M$. Now $T$ can be read as expressing ordinary (many-sorted) first-order truth with respect to $M^\omega$. Any advanced set theory book (e.g., Drake [1974]) explains how this is done in a way absolute for transitive models.  

The above holds for power absolute transitive models of Zermelo set theory as well. If $B=(B, \epsilon)$ is such a model and $M \in B$, then $M^\omega$ need not be in $B$ of course; however, any particular $\varphi$ refers only to types below a certain level $n=n_\varphi$ and the type-structure on $M$ up to and including $n$ will be in $B$.

Now, let $L$-mod$(y)$ be a set-theoretic formula expressing being an $L$-model which is absolute for transitive models. The reduction is given now by the following

**B.4 Theorem.** For $\varphi \in L^\omega$: $\varphi$ is valid iff for each power absolute ZF-model $A$: $A \models \forall y [L$-mod$(y) \rightarrow T(\ ^\gamma \varphi^\gamma, y)]$.

*App.B: reduction of higher-order logic*
Proof. (**=**) If \( M \) is an \( L \)-model, let \( A \) be a natural model containing \( M \) and use B.3. (**⇒**) Let \( A\models(A,E) \) be power absolute. By B.2, we may suppose it is natural. Let \( M \in A \) be such that \( A \models L - \text{mod}(M) \). Hence, \( M \) actually is an \( L \)-model, whence \( M \models \varphi \) by assumption. Therefore, \( A \models T(\forall \varphi \forall M) \) by B.3. \( \Box \)

Lastly, since power absoluteness is a monadic \( \Pi^1_1 \)-condition, B.4 reduces higher-order truth to truth for monadic \( \Sigma^1_1 \)-sentences in a language with one binary relation (\( \in \)).


Let \( A \) be any set, \( \Gamma \subseteq \mathcal{P}(A) \), \( W=\{ \Gamma \cup \Gamma \cup \{A \times \{0,1\}\} \} \), \( \prec \) is the relation on \( W \) given by van Benthem [1984] p.197 in somewhat different terms:

(i) \( \Gamma \prec B \) for all \( B \in \Gamma \)
(ii) \( B \prec (a,i) \) iff \( a \in B \) and \( i=0 \), or: \( a \notin B \) and \( i=1 \)
(iii) \( \Gamma \prec a \) for all \( a \in A \)
(iv) \( a \prec (a,i) \) for all \( a \in A \), \( i \leq 2 \)
(v) \( (a,i) \prec (a,i) \) for all \( a \in A \), \( i \leq 2 \).

Van Benthem l.c. shows that, for \( \Gamma = \mathcal{P}(A) \), the McKinsey axiom \( \Box \Diamond p \rightarrow \Diamond \Box p \) holds in \( (W, \prec) \) but (if \( A \) is infinite) it does not hold in any proper elementary submodel. Now actually, he proves something better: for general \( \Gamma \subseteq \mathcal{P}(A) \), the axiom holds in the corresponding model \( \text{iff} \ \Gamma = \mathcal{P}(A) \). Therefore, it can be used to express power absoluteness, and the previous theorem indicates what this may mean.

Van Benthem [1985] has more examples of this phenomenon.
Barwise, J.

van Benthem, J.F.A.K.

van Benthem, J.F.A.K. and H.C. Doets

Chang, C.C.

Doets, H.C.

Drake, F.
Amsterdam.

Droste, M.

Ehrenfeucht, A.
Fraïssé, R.

Hintikka, K.J.J.

de Jongh, D.H.J.

de Jongh, D., F. Veltman and R. Verbrugge

Karp, C.

Kunen, K.

Montague, R.M.

Nadel, M.

Poizat, B.

Rodenburg, P.H.
[1982] *Classical first-order definability of intuitionistic formulas*. Report 82-08, Mathematisch
Rodenburg, P.H.  

Rosenstein, J.G.  

Segerberg, K.  

Sehtman, V.B.  

Shelah, S.  

Tiuryn, J.  

Twain, Mark  
Samenvatting

Dit proefschrift bevat resultaten over eerste- en tweede-orde logica (delen I en II) en modale- tijds- en intuïtionistische (propositie-) logica (deel III).

Een verbindend thema is Ehrenfeucht's spel en enkele varianten daarvan.

Hoofdstuk 1 is een inleiding in Ehrenfeucht-speltheorie en zijn relatie met (quantorrang-) α-equivalentie in (infinitaire) logica. Sectie 1.0 bedoelt een smaakmaker voor het eindige spel te zijn.

In hoofdstuk 2 wordt het spel gespeeld op binaire bomen. Er wordt een karakterisering verkregen van alle bomen die n-equivalent zijn met de binaire boom $B_m$ waarvan alle takken lengte $m$ hebben. In het bizonder volgt dat $B_m$ oneindige $n$-equivalenten heeft als $m \geq 2^n - 1$. Dit werd toegepast door Rodenburg [1986] bij het oplossen van een probleem uit de intuïtionistische correspondentie-theorie; het verhaal staat in hoofdstuk 8.

Deel II laat zien hoe sommige monadische $\Pi^1_1$-theorieën kunnen worden geaxiomatiseerd (de meesten gaan over gefundeerdheid). Hoofdstuk 3 heeft betrekking op lineaire ordeningen. Sectie 3.3 isoleert het effect van de Suslin-eigenschap op de monadische $\Pi^1_1$-theorie van $\mathbb{R}$. Hoofdstuk 4 generaliseert de methode van 3 naar het geval van bomen.

In deel III bespreekt hoofdstuk 5 Löwenheim-Skolem-type problemen in modale correspondentie-theorie. Aangetoond wordt dat de meeste voorbeelden van niet-eerste-orde-definieerbare formules al niet eerste-orde zijn op eindige结构en. Aan de andere kant wordt een voorbeeld gegeven van een niet-eerste-orde definieerbare formule die wel eerste-orde is op alle aftelbare structuren.

Hoofdstuk 6 modificereert het Ehrenfeucht-spel voor gebruik in
intensionele logica; exacte Kripke-modellen worden geconstrueerd die universeel zijn met betrekking tot eindige partieel geordende Kripke-modellen.

Hoofdstuk 7 geeft onze versie van het bewijs van de $\mathcal{Z}$-volledigheidssystelling.

In hoofdstuk 9 worden spelen en het universeel-exacte Kripke-model adequaat voor één-variabele-intuïtionistische formules toegepast bij het oplossen van enkele problemen in intuïtionistische correspondentietheorie die open bleven in Rodenburg [1982].

Appendix A construeert asymmetrische lineaire ordeningen met veel homogeniteitseigenschappen in ieder overaftelbaar kardinaalgetal.

Appendix B reduceert hogere-orde logica tot monadische tweede-orde logica – dit geeft een indruk van de expressie-mogelijkheden van modale logica onder de Kripke-semantiek.