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Sharpness versus robustness of the percolation transition in 2d contact processes

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Abstract

We study versions of the contact process with three states, and with infections occurring at a rate depending on the overall infection density. Motivated by a model described in Kéfi et al. (2007) for vegetation patterns in arid landscapes, we focus on percolation under invariant measures of such processes. We prove that the percolation transition is *sharp* (for one of our models this requires a reasonable assumption). This is shown to contradict a form of 'robust critical behaviour' with power law cluster size distribution for a range of parameter values, as suggested in Kéfi et al. (2007).

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1. Introduction and background

Percolative systems with weak dependences, such as the contact process and its variants, are in the spotlight of recent mathematical research. The present article studies versions of the

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two-dimensional contact process that are motivated by models for vegetation patterns in arid landscapes, as put forward by biologists and agricultural researchers [14]. The central question we address is whether or not the percolation transition for these modified contact processes is *sharp*. This paper demonstrates the applicability of the sharpness techniques for two-dimensional systems, even to quite realistic models, and provides a rigorous mathematical basis for the discussion in the life-science literature [13–15,21,22].

The models and questions we consider are defined precisely in the following subsections. Briefly speaking, however, we consider two main modifications of the 'standard' two-dimensional contact process. Firstly, rather than two states 0 and 1, we allow three states: -1, 0 and 1. Secondly, the transition rates are allowed to vary with the overall density of 1's in the process itself. Contact processes with three states have been considered by several authors before, e.g. [23,24]. We consider two different types of 3-state contact processes, one of which is closely related to the process in [24], the other of which has not previously appeared in the mathematical literature. The second modification has, to the best of our knowledge, not been considered previously in the mathematical literature; we call such processes 'density-driven' (see Definition 1.1) and prove their existence in Section 6.

Our main focus is the question of *percolation* in such processes: whether or not, under an invariant distribution of the process, there can be an unbounded connected set of 1's. (For general information on percolation we refer to [11,3].) For certain parameter values an unbounded connected set of 1's occurs with positive probability and for others not. As the parameters are varied, one obtains in this sense a phase transition which we refer to as the *percolation transition*. In [14] it is suggested, based on numerical simulation, that the type of model we consider may exhibit a form of 'robust critical behaviour', different from the usual 'sharp' phase transition in standard percolation models. We critically discuss this suggestion, based on rigorous results about the percolation transition. Our main results, on sharpness of the transition and lack of 'robustness', are stated in Theorems 2.4, 2.5, and 2.6.

The contact process is one of the most-studied interacting particle systems, see e.g. [19] and references therein. Several multi-type variants have been studied; most of them have been proposed as models in theoretical biology, and focus has typically been on the survival versus extinction of species. See, for example, Cox–Schinazi [7], Durrett–Neuhauser [8], Durrett–Swindle [9], Konno–Schinazi–Tanemura [16], Kuczek [17], Neuhauser [23]. The question of percolation under invariant distributions of the contact process was first studied by Liggett and Steif [20], and a sharpness result for percolation under such distributions was first proved by van den Berg [26], using some of the techniques introduced for Voronoi percolation in [4].

We begin by describing the type of model we consider in more detail.

1.1. Contact processes

The ordinary contact process on $\mathbb{Z}^d$ is a Markov process with state space $\{0,1\}^{\mathbb{Z}^d}$. Elements $x \in \mathbb{Z}^d$ are called 'sites' or 'individuals'. An element of $\{0,1\}^{\mathbb{Z}^d}$ is typically denoted by $\eta = (\eta_x : x \in \mathbb{Z}^d)$ and those $x \in \mathbb{Z}^d$ for which $\eta_x = 1$ are typically called 'infected'. Infected individuals recover at rate κ, independently of each other (often κ is set to 1). Alternatively, a healthy site can become infected by an infected neighbour at rate λ. This occurs independently for different sites and independently of the recoveries.

The main facts about the contact process are the following. For any $\lambda \geq 0$ there exists an *upper invariant measure* $\overline{\nu}$ on $\{0,1\}^{\mathbb{Z}^d}$ which can be obtained as the limiting distribution when initially *all* sites are infected (this follows from standard monotonicity arguments [19]). For each

$\kappa > 0$ there is a critical value $\lambda_c = \lambda_c(\kappa) \in (0, \infty)$ such that if $\lambda \leq \lambda_c$ then the process 'dies out'. This means that the only stationary distribution of the process is the point mass $\delta_\varnothing$ on the configuration consisting of all zeros, or equivalently $\overline{\nu} = \delta_\varnothing$. On the other hand, if $\lambda > \lambda_c$ then there is positive chance that infection is transmitted indefinitely, and hence $\overline{\nu} \neq \delta_\varnothing$. In this regime, there is more than one invariant distribution, each invariant distribution being a convex combination of $\delta_\varnothing$ and $\overline{\nu} \neq \delta_\varnothing$.

We now describe two variations of the ordinary contact process on which this paper focuses. Both processes have three states, meaning that the processes take values in $\{-1, 0, 1\}^{\mathbb{Z}^d}$.

1.1.1. Model A

The first process we consider has parameters $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}$ and $h, \tilde{h}$. The state of a site may change spontaneously from 1 to 0, from 0 to -1, from -1 to 0 or from 0 to 1, at rates $\kappa, \tilde{\kappa}, \tilde{h}, h$ respectively. Alternatively, a site which is in state -1 or 0 may change to state 0 or 1, respectively, at a rate proportional to the number of nearest neighbours which are in state 1, the constants of proportionality being given by $\tilde{\lambda}$ and λ, respectively. These transition rates are informally summarized in the following table:

Spontaneous rates	Neighbour rates
$1 \to 0$ rate κ	$0 \to 1$ rate $\lambda \cdot$#(type 1 nbrs)
$0 \to -1$ rate $\tilde{\kappa}$	$-1 \to 0$ rate $\tilde{\lambda} \cdot$#(type 1 nbrs)
$0 \to 1$ rate h	
$-1 \to 0$ rate $\tilde{h}$	

If $\tilde{\kappa} = \tilde{\lambda} = \tilde{h} = h = 0$ we thus essentially recover the ordinary 2-state contact process. If $\tilde{\kappa} = \tilde{\lambda} = \tilde{h} = 0$ but $h > 0$ we obtain what may be called the 2-state process *with spontaneous infection*.

This 3-state process is closely related to a model proposed to study the desertification of arid regions in [14]. The intuition is that 0 represents a 'vacant' patch of 'good' soil, 1 represents a vegetated patch, and -1 represents a 'bad' patch of soil which must first be improved (to state 0) before vegetation can grow there. Type 1 patches can influence the states of neighbouring patches either by spreading seeds ($0 \to 1$) or improving the soil ($-1 \to 0$), for example by binding the soil better with roots. Much less is known about this 3-state process than about the ordinary contact process with two states. To a large extent this is because the notion of 'path' along which infection spreads is no longer sufficient. In particular, we do not know if there is a unique stationary distribution if all the parameters are strictly positive, as is the case for the 2-state process with spontaneous infection. However, most of our results on the 3-state process are conditional on the assumption that there is a unique stationary distribution $\overline{\nu}$ in this situation. A more precise formulation of the assumption is stated in Assumption 2.2.

1.1.2. Model B

The second process we study is close to a process studied by Remenik [24]. The parameters are $\kappa, \kappa^\star, \lambda, h, \tilde{h}$, and the transitions are summarized in the following table.

Spontaneous rates	Neighbour rates
$1 \to 0$ rate κ	$0 \to 1$ rate $\lambda \cdot$#(type 1 nbrs)
(0 or 1) $\to -1$ rate $\kappa^\star$	
$0 \to 1$ rate h	
$-1 \to 0$ rate $\tilde{h}$	

Thus a site changes state to -1 at rate $\kappa^\star$ *regardless of* the current state, and transitions out of the state -1 occur at a rate independent of the number of type 1 neighbours. In light of this observation, it is possible to interpret Model B as (ordinary) *contact process in a random environment*.

In the case $h = 0$ this process is the one studied in [24]. Remenik puts it forward as a model for the spread of vegetation, with a slightly different interpretation of transitions to state -1 than in Model A. In Remenik's model the interpretation is that "if a site becomes uninhabitable, the particles living there will soon die" (quote from [24]). In Model A, however, transitions to -1 only occur for uninhabited sites (in state 0) with the motivation that they "may undergo further degradation, for example, by processes such as erosion and soil-crust formation" (quote from [14]).

Model B is considerably easier to study than Model A. Indeed, for the case $h = 0$, Remenik interpreted the model as a hidden Markov chain and, building on results by Broman [6], proved strong results such as complete convergence. A key tool to obtaining this result is a *duality* relation, which fails for Model A. For the case $h > 0$, exponential convergence to a unique invariant distribution is stated in Lemma 2.1.

1.2. Density-driven contact processes

It is straightforward to generalize the definitions of the contact processes we consider to allow time-varying infection rates $\lambda(t)$ and $h(t)$ (see Section 4.1 for more on this). Furthermore, in the context of vegetation spread it seems natural to allow the rates governing transitions from state 0 to 1 (λ and h) to depend on the overall density of 1's in the process itself. For example, one may imagine that seeds can be blown over large distances to spread vegetation, and that whether a seed which has landed on a vacant piece of soil indeed becomes a plant may depend on the overall competition of the other plants. Indeed, the model proposed in [14] includes such a mechanism. The model there is defined in discrete time and in a finite region, and it is not immediately obvious that it is possible to define such a process in continuous time and on the infinite graph $\mathbb{Z}^d$. However, in Section 6 we prove the existence of the following class of processes.

Let $X(t)$ be a translation invariant 3-state process (Model A or B), and write $\rho(t) = P(X_0(t) = 1)$ for the *density* of the process.

Definition 1.1 (DDCP)**.** Let the functions $\Lambda, H : [0, 1] \to [0, \infty)$ be given, and let $X(t)$ be a translation-invariant 3-state contact process with parameters κ, $\tilde{\kappa}$, $\tilde{\lambda}$, $\tilde{h}$ and $\lambda(\cdot)$, $h(\cdot)$ in Model A or with parameters κ, $\kappa^\star$, $\tilde{h}$, and $\lambda(\cdot)$, $h(\cdot)$ in Model B. This process is called a *density-driven contact process specified by Λ and H* if λ, h satisfy $\lambda(t) = \Lambda(\rho(t))$ and $h(t) = H(\rho(t))$ for all $t \geq 0$.

We use the abbreviation DDCP for 'density-driven contact process'. Intuitively a DDCP constantly updates its infection rates based on the current prevalence of 1's.

1.3. Outline

In Section 2 we state our main results, which concern on the one hand 'sharpness' and on the other 'lack of robustness'. In Section 3 we prove our results on lack of robustness, deferring the proofs of our sharpness results to Section 5. In Section 4 we describe methods and results from the literature which are needed for the proofs of our main sharpness results. Section 5 contains the proofs of our sharpness result (Theorem 2.6). In Section 6, we prove in general the existence of density-driven processes.

2. Main results

We first formulate a condition about exponentially fast convergence to a unique equilibrium measure for Model A. For Model B, this assumption can be established using standard techniques. Subsequently, we formulate our results on lack of robustness (in Section 2.2) and sharpness (in Section 2.3).

2.1. Convergence to equilibrium

Consider Models A and B with constant parameters. Henceforth, we assume that all parameters are positive, so we assume $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h} > 0$ for Model A and $\kappa, \kappa^\star, \lambda, h, \tilde{h} > 0$ for Model B.

It is well-known (and can be easily proved by a standard coupling argument using the graphical representation, see Section 4.1) that the assumption $h > 0$ implies exponentially fast convergence to equilibrium in Model B. For any $\xi \in \{-1, 0, 1\}^{\mathbb{Z}^d}$ let us write μ_t^ξ for the law of the contact process with initial state ξ. Further, for a finite set $\Lambda \subseteq \mathbb{Z}^d$ let $\mu_{t;\Lambda}^\xi$ denote the restriction of μ_t^ξ to Λ. Similarly, let $\overline{\nu}_\Lambda$ denote the restriction of the upper invariant measure $\overline{\nu}$ to Λ (i.e., marginal of μ_t^ξ on $\{-1, 0, 1\}^\Lambda$).

Lemma 2.1. *For Model B with $h > 0$ and any initial state ξ we have that*

$$d_{\mathrm{tv}}(\mu_{t;\Lambda}^\xi, \overline{\nu}_\Lambda) \leq |\Lambda| e^{-ht}.$$

For Model A we have not been able to establish exponential convergence to equilibrium along the lines of Lemma 2.1. However, it is natural to suppose that such a result should hold when all parameters $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h} > 0$. Our results for Model A rely on such a convergence result, which we now formulate:

Assumption 2.2 (*Exponential Convergence to Equilibrium*)**.** For Model A with strictly positive parameters

1. there is a unique stationary distribution $\overline{\nu}$, and
2. there are constants $C_1, C_2 > 0$ such that for all finite $\Lambda \subseteq \mathbb{Z}^d$ and all initial configurations ξ we have

$$d_{\mathrm{tv}}(\mu_{t;\Lambda}^\xi, \overline{\nu}_\Lambda) \leq C_1 |\Lambda| e^{-C_2 t}.$$

The particular places where Assumption 2.2 is needed are in Lemma 4.4 and in the proof of Theorem 2.6 at the end of Section 5.2.

Here are some heuristic arguments supporting the assumption. First, as already noted the assumption holds straightforwardly in Model B, and it also holds for the two-state contact process *with spontaneous infections* which may be obtained from Model A by setting $\tilde{\kappa} = \tilde{\lambda} = \tilde{h} = 0$. Compared to the 2-state process, the extra state -1 in the 3-state process introduces 'delays' during which particles are insensitive to infection attempts. The delay periods are of random length but with exponential tails, and hence we do not expect the qualitative properties of convergence speed to equilibrium to be different from the 2-state case. Also, by standard general arguments (see [18, Theorem 4.1]), Assumption 2.2 holds for a certain parameter range, namely

when the 'spontaneous rates' are sufficiently large compared with the 'neighbour rates'. Finally, if we couple the system starting with all sites having value -1 with the system starting with all sites having value 1, it seems, intuitively, that the rules of the coupled dynamics provide a stronger tendency to eliminate existing disagreement sites (i.e. sites where the two systems differ in value) than to create new disagreement sites. However, we have not been able to turn this into a proof.

2.2. Percolation and the question of robustness

For any configuration $\eta \in \{-1, 0, 1\}^{\mathbb{Z}^d}$, consider the subgraph induced by sites in state 1 and the nearest-neighbour relation. Let C_0 denote the connected component of the subgraph of 1's containing the origin 0, and write $|C_0|$ for the number of sites in C_0. If ν is a probability measure on $\{-1, 0, 1\}^{\mathbb{Z}^d}$, we say that *percolation occurs* under ν if $\nu(|C_0| = \infty) > 0$. For the rest of this section we fix $d = 2$.

A major focus of this article is to study the phenomenon of percolation when ν is an invariant measure of a 3-state contact process, possibly density-driven. Indeed, one of the main motivations is an intriguing suggestion in [14] concerning a specific version of the density-driven Model A (with explicitly given forms of the functions $h(t)$ and $\lambda(t)$, involving certain parameters). In our context (where the medium is the *infinite* lattice and time is continuous) that version is given by the functions in (2.1). The suggestion in [14] is that this model has a form of 'robust critical behaviour': that there is a non-negligible set of parameter values for which the model has an invariant measure under which the size of an occupied cluster has a power-law distribution.

As the authors of [14] remark, such behaviour is different "from classical critical systems, where power laws only occur at the transition point". Further, the authors suggest that this uncommon behaviour may be explained by strong local positive interactions. (The latter means that the transitions from -1 to 0 and the transitions from 0 to 1 are 'enhanced' by the presence of occupied sites in the neighbourhood.) Later in their paper they argue that an important aspect to explain their 'observed' robust critical behaviour would be that 'disturbances' (transitions to the -1 state) do not affect directly the occupied sites: they first have to change to the 0 state, which "constrains the spatial propagation of the disturbance". In later life-sciences papers the robust criticality is debated [13,15,21,22].

The arguments in [14] and those in the articles mentioned above lack mathematical rigour. Our aim is to contribute by lifting the discussion to a rigorous mathematical level, and by proving mathematical theorems that are relevant for the above mentioned discussion. Our following result, Theorem 2.4, shows (under Assumption 2.2) that in our formulation of the model in [14], criticality is rare, in a strong and well-defined sense. We also show a more general though weaker statement of a similar form (Theorem 2.5).

Definition 2.3. We call a distribution ν on $\{-1, 0, 1\}^{\mathbb{Z}^2}$ *critical* (for percolation) if $\nu(|C_0| \geq n)$ converges to zero subexponentially; that is, $\nu(|C_0| \geq n) \to 0$ as $n \to \infty$, but

$$\liminf_{n\to\infty} \frac{-\log \nu(|C_0| \geq n)}{n} = 0.$$

Calling such a distribution ν 'critical' may be somewhat imprecise, partly as it seems to ignore the possibility of a discontinuous phase transition. However, the name is meant to capture the idea that power law cluster sizes are associated with critical behaviour.

The precise form of the DDCP corresponding to the model in [14] is given by

$$\begin{aligned} \Lambda(\rho) &= \beta \frac{1-\delta}{4} (\varepsilon - g\rho), \\ H(\rho) &= \beta\delta\, \rho\, (\varepsilon - g\rho), \end{aligned} \tag{2.1}$$

where β, ε and g are positive parameters and $\delta \in (0, 1)$. This choice of functions is motivated in the Methods supplement to [14]. Briefly, β represents the seed production rate, δ the fraction of seeds that are spread over long distances, ε the establishment probability of a seed not subject to competition, and g a competitive effect due to the presence of other plants.

For the DDCP where λ and h are density-dependent and given by (2.1) we have the following result:

Theorem 2.4 (*Lack of Robustness*)**.** *Let $d = 2$ and recall Definition 2.3.*

- Model A. *Consider Model A under Assumption 2.2, with $\Lambda(\cdot)$ and $H(\cdot)$ given by (2.1). Then for almost all $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h}$, β, δ, ε and g, the 3-state DDCP does not have a critical invariant measure.*
- Model B. *Similarly, consider Model B with $\Lambda(\cdot)$ and $H(\cdot)$ given by (2.1). Then for almost all $\kappa, \kappa^\star, \tilde{h}$, β, δ, ε and g, the 3-state DDCP does not have a critical invariant measure.*

We also have the following result, which on the one hand holds for much more general Λ, H, but on the other hand has a weaker conclusion. We say that two functions $f, g\colon [0, 1] \to \mathbb{R}$ differ at most ε if $|f(r) - g(r)| < \varepsilon$ for all $r \in [0, 1]$. The result is formulated and proved for Model A, but straightforwardly extends to Model B as well (with Assumption 2.2 replaced by Lemma 2.1).

Theorem 2.5. *Let Λ, H be continuous, strictly positive functions, and suppose the 3-state DDCP with dynamics given by Model A and parameters $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h} > 0$ and $\Lambda, H > 0$ has a critical invariant distribution. Then, under Assumption 2.2, for every $\varepsilon > 0$ there are parameters $\kappa', \tilde{\kappa}', \tilde{\lambda}', \tilde{h}'$ and Λ', H' which each differ at most ε from the original parameters, and for which the corresponding DDCP has no critical invariant measure.*

Theorems 2.4 and 2.5 are proved in Section 3. These results cast considerable doubt on the suggestions in [14] discussed in the beginning of this section.

2.3. Sharpness of percolation transitions

The main step in proving Theorems 2.4 and 2.5 is to establish *sharpness results* for percolation under the invariant measures of contact processes, which we state in this section. Such results are also of independent interest. Given these sharpness results, the proofs of Theorems 2.4 and 2.5 are relatively elementary. For $x, y \in \mathbb{R}^k$ we use the notation $x \prec y$ to indicate that each coordinate of x is strictly smaller than the corresponding coordinate of y.

For Model A, we require Assumption 2.2 (which, for Model B, has been verified in Lemma 2.1). By comparison with Bernoulli percolation it follows immediately that $\overline{\nu}(|C_0| = \infty) > 0$ provided $h, \tilde{h}$ are large enough, or $\tilde{h} > 0$ and h is large enough. In Section 5 we prove the following result.

Theorem 2.6 (*Sharpness for 3-State Contact Process*)**.** *Consider the 3-state contact process with $d = 2$.*

- Model A. *Fix* $\kappa, \tilde{\kappa} > 0$. *Under Assumption 2.2 we have the following. If the parameters* $\lambda, \tilde{\lambda}, h, \tilde{h} > 0$ *are such that* $\overline{\nu}(|C_0| = \infty) = 0$, *then whenever* $(\lambda', \tilde{\lambda}', h', \tilde{h}') \prec (\lambda, \tilde{\lambda}, h, \tilde{h})$, *there is* $c > 0$ *such that* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for all* $n \geq 1$.
- Model B. *Fix* $\kappa, \kappa^\star > 0$. *If the parameters* $\lambda, h, \tilde{h} > 0$ *are such that* $\overline{\nu}(|C_0| = \infty) = 0$, *then whenever* $(\lambda', h', \tilde{h}') \prec (\lambda, h, \tilde{h})$, *there is* $c > 0$ *such that* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for all* $n \geq 1$.

Theorem 2.6 has the following consequence, which will be used in the proof of Theorem 2.4 in Section 3.2. For fixed $\kappa, \tilde{\kappa}/\kappa^\star, \lambda, \tilde{\lambda}, \tilde{h} > 0$ define $h_{\mathrm{p}} = h_{\mathrm{p}}(\lambda) := \inf\{h \geq 0 : \overline{\nu}(|C_0| = \infty) > 0\}$.

Corollary 2.7. *Consider the* 3*-state contact process with* $d = 2$.

- Model A. *Under Assumption 2.2, for all* $\kappa, \tilde{\kappa} > 0$ *and almost all* $\tilde{\lambda}, \tilde{h} > 0$ *the following holds: for all but countably many* $\lambda > 0$, *if* $h < h_{\mathrm{p}}(\lambda)$ *then* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for some* $c > 0$ *and all* $n \geq 1$.
- Model B. *For all* $\kappa, \kappa^\star > 0$ *and almost all* $\tilde{h} > 0$ *the following holds: for all but countably many* $\lambda > 0$, *if* $h < h_{\mathrm{p}}(\lambda)$ *then* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for some* $c > 0$ *and all* $n \geq 1$.

Remark 2.8. The 2-state contact process with spontaneous infection can be obtained from Model B by letting $\tilde{h} = \kappa^\star = 0$ (the state -1 thus plays no role). Although our results are formulated under the assumption that all parameters are positive, it may be seen quite straightforwardly from the proofs that all results of Section 2 apply also to this model.

Remark 2.9. A straightforward modification of the proof of Corollary 2.7 gives the following statement for Model A. Under Assumption 2.2, for all $\kappa, \tilde{\kappa} > 0$, and almost all $\tilde{\lambda}, \tilde{h} > 0$, the following holds for all but countably many $h > 0$: if $\lambda < \lambda_{\mathrm{p}}(h) := \inf\{\lambda \geq 0 : \overline{\nu}(|C_0| = \infty) > 0\}$ then $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ for some $c > 0$ and all $n \geq 1$. In other words, for almost all choices of the parameters $\tilde{\lambda}, h, \tilde{h}, \kappa, \tilde{\kappa}$ there is at most one value of λ for which $\overline{\nu}(|C_0| \geq n)$ goes to 0 slower than exponentially. We may deduce a similar statement for Model B.

Remark 2.10. Our arguments do not, however, allow us to go beyond the "almost all" in Corollary 2.7 (and Remark 2.9). That is, we are not able to prove that the percolation transition in h (respectively, λ) is sharp for *arbitrary* fixed values of the other parameters. More generally, we have not proved the stronger version of Theorem 2.6 where just one (instead of all) the 'good' parameters are decreased.

The essential difficulty is, informally, the following. To obtain such a stronger version of Theorem 2.6, we need to suitably compare the effect of a small change of one parameter with the effect of changes of other parameters. Some comparisons are simple: it is easy to see that the system obtained by increasing h by ε, dominates the system obtained by increasing λ by $\varepsilon/4$. However, it is not obvious how (e.g.) to compare an increase in h with an increase in $\tilde{h}$. Although there is a general approach to such and related problems (see e.g. [1] and Sections 3.2 and 3.3 in [11]) the concrete applicability of that approach depends very much on the details of the model. Moreover, as pointed out in [2], even in some 'classical' percolation models, the technical problems that arise are more delicate than expected earlier.

Therefore, and because our current version of Theorem 2.6 is strong enough to obtain Theorem 2.4 (and the statement of this latter theorem would not essentially benefit from the mentioned stronger version of the former), we do not pursue such improvements in this paper.

3. Proofs of nonrobustness for density-driven processes

In this section we prove Theorems 2.4 and 2.5 on lack of robustness in the model proposed in [14] assuming the sharpness results of Corollary 2.7. We begin with a discussion about the stationary distributions of DDCP.

3.1. Stationary distributions for density-driven processes

We first consider Model A. Let $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h} > 0$ be fixed. Let $X(t)$, $t \geq 0$, be a DDCP for the parameters κ, $\tilde{\kappa}$, $\tilde{\lambda}$, $\tilde{h}$, Λ and H. Suppose that X is stationary, i.e. the distribution of $X(t)$ is constant in t. Denote this distribution by ν. By stationarity, the occupation density $\rho(t)$ of $X(t)$ is constant, say $\rho(t) \equiv \rho$. Writing $\lambda = \Lambda(\rho)$ and $h = H(\rho)$ we thus find that ν is a stationary distribution for the contact process with *constant* parameters λ, h, κ, $\tilde{\kappa}$, $\tilde{\lambda}$, and $\tilde{h}$.

Suppose $\Lambda(\rho), H(\rho) > 0$ for all $\rho \in [0, 1]$. Since by Assumption 2.2 there is only one stationary distribution $\overline{\nu}$, it follows that $\nu = \overline{\nu}$. Let $\overline{\rho}(\lambda, h) = \overline{\nu}(\{\eta : \eta_0 = 1\})$ denote the density of $\overline{\nu}$. It follows that λ and h satisfy the fixed point equations $\lambda = \Lambda(\overline{\rho}(\lambda, h))$ and $h = H(\overline{\rho}(\lambda, h))$, respectively. Conversely, if h and λ satisfy these fixed point equations, then $\overline{\nu}$ is stationary for the DDCP defined by Λ, H. We summarize these findings in the following proposition:

Proposition 3.1. *Let κ, $\tilde{\kappa}$, $\tilde{\lambda}$, $\tilde{h} > 0$ be fixed. Suppose $\Lambda(\rho), H(\rho) > 0$ for all $\rho \in [0, 1]$. Then, under Assumption 2.2, the stationary distributions of the 3-state* DDCP *specified by Λ, H are precisely the measures $\overline{\nu}$ for λ, h satisfying $\lambda = \Lambda(\overline{\rho}(\lambda, h))$ and $h = H(\overline{\rho}(\lambda, h))$.*

The corresponding result is valid for Model B.

3.2. Proof of Theorem 2.4

We prove the theorem using Corollary 2.7. Writing $\gamma_1 = \beta(1 - \delta)/4$ and $\gamma_2 = \beta\delta$, it is sufficient to prove the following claim: *For almost all κ, $\tilde{\kappa}/\kappa^\star$, $\tilde{h}$, γ_1, γ_2, ε and g, the* DDCP *with $\Lambda(\rho) = \gamma_1(\varepsilon - g\rho)$ and $H(\rho) = \gamma_2\rho(\varepsilon - g\rho)$ does not have a critical invariant measure.* In the argument that follows we will frequently use the fact that the product of a measure zero set with any measurable set has measure zero.

We give a proof for Model A only, the proof for Model B is similar. For fixed $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h}$ we call λ *bad* if $\overline{\nu}(|C_0| \geq n)$ does not decay exponentially for all $h < h_{\mathrm{p}}(\lambda)$. For all $\kappa, \tilde{\kappa}$ and almost all $\tilde{\lambda}, \tilde{h}$, Corollary 2.7 implies that the set of bad $\lambda > 0$ is at most countable. We henceforth assume that $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h}$ are fixed and chosen so that the set of bad λ is at most countable.

Suppose the DDCP has a critical invariant measure ν. Note that, since ν is invariant, ρ, λ, h do not vary with t, and that, since ν is critical, ρ, and hence λ and h, are >0. We now consider the two cases, λ 'bad' or not. Here is a brief summary of the argument that follows. In the case when λ is not bad we use the fact that h, and hence also ρ, is then a function of λ only. These additional constraints, together with (2.1), allow us (roughly speaking) to write the remaining parameters in terms of λ and then to deduce the result from the precise form of this expression (see (3.4)). In the case when λ is bad it suffices to show that the set of possible choices of the remaining parameters has measure zero, since there are only countably many such λ. This is done by using (2.1) and Proposition 3.1 to write additional relations among these parameters.

We now turn to the argument proper. If λ is *not* bad then, since ν is critical, h must equal $h_{\mathrm{p}}(\lambda)$. Hence the following two equations hold, where $\rho_{\mathrm{p}}(\lambda) = \rho(\lambda, h_{\mathrm{p}}(\lambda))$ denotes the density

of the upper invariant measure with parameters λ and $h = h_{\mathrm{p}}(\lambda)$:

$$\lambda = \gamma_1(\varepsilon - g\rho_{\mathrm{p}}(\lambda)), \tag{3.1}$$

$$h = h_{\mathrm{p}}(\lambda) = \gamma_2\rho_{\mathrm{p}}(\lambda)(\varepsilon - g\rho_{\mathrm{p}}(\lambda)). \tag{3.2}$$

To prove the claim, fix also g and γ_1. With these parameters fixed, it is clear that for each λ there is at most one ε such that (3.1) holds. Hence the measure of the set of pairs (λ, ε) such that (3.1) holds is zero. It follows from Fubini's theorem that for almost all ε the set $L = L(\varepsilon) = L(\varepsilon, \kappa, g, \gamma_1)$ of those λ for which (3.1) holds has Lebesgue measure 0. (Note that $\rho_{\mathrm{p}}(\lambda)$ is measurable since $\rho(\lambda, h)$ and $h_{\mathrm{p}}(\lambda)$ are measurable.) Now also fix (besides the above mentioned parameters which were already fixed) the parameter ε such that L indeed has measure 0. Note that for each λ there is at most one γ_2 such that (3.2) holds. Let $L' \subset L$ be the set of those $\lambda \in L$ for which there is indeed such a γ_2. Rearranging (3.2) we can write this γ_2 as a function of $\lambda \in L'$:

$$\gamma_2 = \frac{h_{\mathrm{p}}(\lambda)}{\rho_{\mathrm{p}}(\lambda)(\varepsilon - g\rho_{\mathrm{p}}(\lambda))}.$$

Using (3.1), we can 'eliminate' $\rho_{\mathrm{p}}(\lambda)$ from the above expression for γ_2 and get

$$\gamma_2 = \frac{h_{\mathrm{p}}(\lambda)g\gamma_1^2}{\lambda(\varepsilon\gamma_1 - \lambda)}, \quad \lambda \in L'. \tag{3.3}$$

Write the right hand side of (3.3) as a function

$$F(\lambda) := \frac{h_{\mathrm{p}}(\lambda)g\gamma_1^2}{\lambda(\varepsilon\gamma_1 - \lambda)}. \tag{3.4}$$

We want to show that $F(L')$ has measure 0. To do this, we note that $h_{\mathrm{p}}(\lambda)$ is uniformly Lipschitz continuous in λ: for each $\alpha \geq 0$, the process with parameters $\lambda + \alpha$ and h is stochastically dominated by the process with parameters λ and $h + 4\alpha$ (this is intuitively obvious from the description of the dynamics, and can be easily proved using the graphical representation of Section 4.1), so

$$h_{\mathrm{p}}(\lambda) \geq h_{\mathrm{p}}(\lambda + \alpha) \geq h_{\mathrm{p}}(\lambda) - 4\alpha. \tag{3.5}$$

Hence the numerator in the definition of the function F above is locally Lipschitz. It follows that F is locally Lipschitz outside the point $\lambda = \varepsilon\gamma_1$. Hence, since L' has measure 0, $F(L')$ also has measure 0 (any cover of L' with 'small' intervals is mapped under F to a cover of $F(L')$ with comparably small intervals).

We next consider the case when λ is bad. For a fixed bad λ, we can use Proposition 3.1 to write $\rho = \rho(h)$. We can no longer conclude that $h = h_{\mathrm{p}}(\lambda)$, but we still have the relations

$$\lambda = \gamma_1(\varepsilon - g\rho(h)), \quad \text{and} \quad h = \gamma_2\rho(h)(\varepsilon - g\rho(h)). \tag{3.6}$$

We aim to show that, for each fixed bad λ, the set of choices of the parameters $\gamma_1, \gamma_2, \varepsilon, g$ such that (3.6) holds has measure zero. This concludes the proof since a countable union of measure zero sets has measure zero.

If (3.6) holds, we may rearrange to obtain the relations

$$\rho(h) = \frac{\gamma_1}{\lambda\gamma_2}h, \quad \text{and} \quad \varepsilon = \frac{\lambda}{\gamma_1} + g\rho(h) = \frac{\lambda}{\gamma_1} + g\frac{\gamma_1}{\lambda\gamma_2}h. \tag{3.7}$$

We now fix arbitrary $\gamma_2, g > 0$. Then for almost all γ_1, the first relation in (3.7) can only hold for h in a set of measure zero. This follows from Fubini's theorem (using polar coordinates) and the fact that the set $\{(h, \rho(h)) : h > 0\}$ has two-dimensional Lebesgue measure zero. We now fix γ_1 such that the first relation in (3.7) only holds for h in a set of measure zero. It follows that the second relation in (3.7) can only hold for ε in a set of measure zero. Using Fubini's theorem again this concludes the proof. □

3.3. *Proof of Theorem 2.5*

Consider Model A with parameters $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h}$, all strictly positive. Under Assumption 2.2, there is a unique invariant distribution $\overline{\nu}$ for this process, and we have (as stated precisely in the same assumption) exponentially fast convergence to that distribution, from any starting distribution. Recall that $\overline{\rho}$ denotes the density of $\overline{\nu}$ (i.e. the probability under $\overline{\nu}$ that a given vertex has value 1).

Lemma 3.2. *$\overline{\rho}$ is continuous from the right in each of the parameters λ, $\tilde{\lambda}$, h and $\tilde{h}$.*

Proof. Let μ_t denote the distribution at time t if we start the process with all vertices in state 1. By uniqueness, $\overline{\nu}$ is the limit, as $t \to \infty$, of μ_t. From Lemma 4.1 we have that μ_t is stochastically increasing in each of the parameters h, $\tilde{h}$, λ and $\tilde{\lambda}$ (and stochastically decreasing in κ and $\tilde{\kappa}$). Also by obvious monotonicity, μ_t is stochastically decreasing in t. For each $t \geq 0$ the density $\rho(t)$ under μ_t is continuous in each of the parameters λ, $\tilde{\lambda}$, h, $\tilde{h}$, and by the above we have that $\rho(t) \searrow \overline{\rho}$. The result follows since we can interchange the order of any two decreasing limits. □

Proof of Theorem 2.5. Let ν denote the critical invariant measure mentioned in the statement of the theorem. Let ρ denote its density, and let $\lambda = \Lambda(\rho)$ and $h = H(\rho)$. Then, as in Proposition 3.1, ν is invariant under the 3-state contact process dynamics with parameters κ, $\tilde{\kappa}$, λ, $\tilde{\lambda}$, h, $\tilde{h}$. Hence, under Assumption 2.2, ν is the unique measure $\overline{\nu}$ for these parameters. So we have

$$\lambda = \Lambda(\rho(\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h})), \quad \text{and}$$
$$h = H(\rho(\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h})).$$

Now we increase each of the 'good' parameters λ, $\tilde{\lambda}$, h and $\tilde{h}$ by an amount $\in (0, \varepsilon/2)$ so small that

$$\Lambda(\rho(\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h})) \quad \text{and} \quad H(\rho(\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h}))$$

change by at most $\varepsilon/2$. This is possible by the continuity of Λ and H and Lemma 3.2. Denote the new parameters by $\kappa' = \kappa$, $\tilde{\kappa}' = \tilde{\kappa}$, $\lambda', \tilde{\lambda}', h', \tilde{h}'$. From the above it follows that there are continuous functions Λ' and H' which differ at most ε from Λ and H respectively, such that

$$\lambda' = \Lambda'(\rho(\kappa', \tilde{\kappa}', \lambda', \tilde{\lambda}', h', \tilde{h}')), \quad \text{and}$$
$$h' = H'(\rho(\kappa', \tilde{\kappa}', \lambda', \tilde{\lambda}', h', \tilde{h}')).$$

(For example, one may take $\Lambda'(r) = \Lambda(r) + \lambda' - \Lambda(\rho(\kappa', \tilde{\kappa}', \lambda', \tilde{\lambda}', h', \tilde{h}'))$, and take H' in a similar way.) Let ν' be the invariant measure for the contact process with fixed parameters $\kappa', \tilde{\kappa}', \lambda', \tilde{\lambda}', h', \tilde{h}'$. From the above we conclude that ν' is invariant under the DDCP dynamics with parameters $\kappa', \tilde{\kappa}', \tilde{\lambda}', \tilde{h}'$, Λ' and H', and each of these 'new' parameters differs at most ε from the corresponding 'old' one. Moreover, by Theorem 2.6, this ν' is not critical. This completes the proof. □

4. Ingredients from the literature

In this section we discuss a number of results and methods needed for the proofs of our main sharpness result, Theorem 2.6. First we discuss graphical representations for contact processes, then some methods from percolation theory as well as influence results.

4.1. Graphical representations

Central to the study of the contact process is a graphical representation in terms of Poisson processes of 'marks' and 'arrows'. For Model A this is as follows. We write D_1 and D_2 for independent Poisson processes on $\mathbb{Z}^d \times [0,\infty)$ of intensities κ and $\tilde{\kappa}$, which we think of as the processes of 'down' marks ($1 \to 0$ and $0 \to -1$, respectively). Independently of these and of each other we define Poisson processes U_1 and U_2 of 'up' marks ($0 \to 1$ and $-1 \to 0$) of respective intensities h and $\tilde{h}$. Finally, independently of all these and of each other we define Poisson processes A_1 and A_2 of arrows ($0 \to 1$ and $-1 \to 0$, respectively) with respective intensities λ and $\tilde{\lambda}$ on the ordered nearest-neighbour sites xy times $[0,\infty)$. The rates of these processes are summarized in the following table:

Spontaneous transitions (on $\{x\} \times [0,\infty)$)	Neighbour transitions (on $\{xy\} \times [0,\infty)$)
D_1 rate κ	A_1 rate λ
D_2 rate $\tilde{\kappa}$	A_2 rate $\tilde{\lambda}$
U_1 rate h	
U_2 rate $\tilde{h}$	

The interpretation of the Poisson processes is as usual with interacting particle systems: At a point $(x,t) \in U_1$ the site x changes from 0 to 1 (it does not change if it was in state -1 or 1 before), if $(xy,t) \in A_1$ and the process is in state 1 at x and in state 0 at y, then y changes to 1, etc. It will later (in Section 5) be useful to focus on the process of *incoming* arrows on each line $\{y\} \times [0,\infty)$, that is the collection of arrows at points (xy,t) for x a neighbour of y. For all y the incoming arrows form Poisson processes, of intensities $2d\lambda$ and $2d\tilde{\lambda}$ for A_1 and A_2 respectively.

We also consider the three state process with time-varying parameters λ and h. Such process is easily defined via its graphical representation. Let $\lambda(\cdot)$ and $h(\cdot)$ be nonnegative integrable functions, and let A_1 and U_1 be independent Poisson processes of rates $\lambda(\cdot)$ and $h(\cdot)$, respectively.

The process has the following monotonicity in the initial condition and in the graphical representation. Let X denote the 3-state process with initial state $\xi \in \{-1,0,1\}^{\mathbb{Z}^d}$ and graphical representation D_1, D_2, U_1, U_2, A_1 and A_2, and let X' denote the process with initial condition $\xi' \in \{-1,0,1\}^{\mathbb{Z}^d}$ and graphical representation D_1', D_2', U_1', U_2', A_1' and A_2'. If the following hold, then $X'(t) \geq X(t)$ for all $t \geq 0$: $\xi' \geq \xi$, $D_1' \subseteq D_1$, $D_2' \subseteq D_2$, $U_1 \subseteq U_1'$, $U_2 \subseteq U_2'$, $A_1 \subseteq A_1'$ and $A_2 \subseteq A_2'$.

An analogous construction exists for Model B. The only changes are that the Poisson process D_2 has intensity $\kappa^\star$, and represents transition to state -1 irrespective of the previous state. Further, there is no process A_2 for Model B.

The monotonicity statement for these processes reads as follows.

Lemma 4.1 (*Monotonicity*)**.** *Model A is (stochastically) increasing in the initial state and the parameters $\lambda, \tilde{\lambda}, h, \tilde{h}$ and decreasing in $\kappa, \tilde{\kappa}$. Model B is (stochastically) increasing in the initial state and the parameters $\lambda, h, \tilde{h}$ and decreasing in $\kappa, \kappa^\star$.*

This monotonicity property implies for both processes that if the initial state ξ consists of only 1's then the distribution of the process at time t is stochastically decreasing in t. Standard arguments then imply that the process decreases (stochastically) to a limiting distribution, which in both cases we denote by $\overline{\nu}$ and call the *upper invariant measure*.

4.2. Finite-size criterion

Next we present a so-called finite-size criterion for percolation. Its analog for Bernoulli percolation is a well-known classical result which, as pointed out in [26] (see Lemma 2.3 in that paper), can be generalized to the case where the configurations come from the supercritical ordinary contact process. The same arguments as in [26] yield our Lemma 4.2.

Let $d = 2$ and let $H(m, n)$ denote the event that there is a left–right crossing of the rectangle $[0, m] \times [0, n]$ (i.e., that the subgraph of $[0, m] \times [0, n]$ spanned by sites in state 1 contains a path from some $(0, x)$ to some (m, y) where $0 \le x, y \le n$). Let $V(m, n)$ denote the event that there is an up–down crossing of the rectangle $[0, m] \times [0, n]$.

Lemma 4.2 (*For Model A we assume Assumption 2.2 here*)**.** *There is a (universal) constant $\hat{\varepsilon} > 0$ such that the following holds for Model A under Assumption 2.2 and for Model B.*
For all strictly positive values of the parameters, there is $\hat{n}$ (depending on the parameters) such that

(1) *If for some $n \ge \hat{n}$ we have $\overline{\nu}(V(3n, n)) < \hat{\varepsilon}$, then there is $c > 0$ such that $\overline{\nu}(|C_0| \ge k) \le e^{-ck}$ for all $k \ge 0$.*
(2) *If for some $n \ge \hat{n}$ we have $\overline{\nu}(H(3n, n)) > 1 - \hat{\varepsilon}$, then $\overline{\nu}(|C_0| = \infty) > 0$.*

4.3. An influence result

We further need the following combination of the Margulis–Russo formula and an influence result which essentially comes from Talagrand's paper [25] (which in turn is closely related to [12]), where all the p_i's in the description below are equal. For our particular situation we straightforwardly generalized the form (with two different p_i's) in [26, Corollary 2.9]. See also e.g. [10,4,5].

Let $X = (X_{i,j} : 1 \le i \le m, 1 \le j \le n)$ be a collection of independent $\{0, 1\}$-valued random variables such that

$$P(X_{i,j} = 1) = p_i \quad \text{for all } j \in \{1, \dots, n\}.$$

For fixed i, j, let $X^{(i,j)}$ denote the random vector obtained from X by replacing $X_{i,j}$ with $1 - X_{i,j}$ (and keeping all other $X_{i',j'}$ the same). For an event A, define the *influence* of $X_{i,j}$ on A as

$$I_{i,j}(A) = P(\{X \in A\} \triangle \{X^{(i,j)} \in A\}),$$

where $\triangle$ denotes symmetric difference.

Lemma 4.3. *Fix $k \in \{1, \dots, m\}$ and suppose that H is an event which is increasing in the $X_{i,j}$ for $i \le k$, and decreasing in the $X_{i,j}$ for $i \ge k + 1$. Let N denote the number of indices (i, j) such that $I_{i,j}(H)$ is maximal. There is an absolute constant K such that*

$$\sum_{i=1}^{k} \frac{\partial}{\partial p_i} P(H) - \sum_{i=k+1}^{m} \frac{\partial}{\partial p_i} P(H) \ge \frac{P(H)(1 - P(H))}{K \max_i p_i \log(2/\min_i p_i)} \log N.$$

For our application, m represents the number of different types of symbols. We apply it with $m = 6$ and $k = 4$ for Model A, and with $m = 5$ and $k = 3$ for Model B.

4.4. RSW-result

The following result is usually referred to as a RSW-type result as this type of result was pioneered, for Bernoulli percolation, in papers by Russo, Seymour and Welsh. A highly non-trivial extension of (a weak version of) the original RSW-result to a dependent percolation model, namely the random Voronoi model, was proved by Bollobás and Riordan [4] (and modified in [27] to a form which is closer to Lemma 4.4). As pointed out in [4], the result holds under quite mild geometric, positive-association and spatial mixing conditions. In [26] (see Proposition 2.4 in that paper) it is explained that these conditions hold for the supercritical ordinary contact process. The same arguments hold for our models.

Lemma 4.4. *Consider the upper invariant measure $\overline{\nu}$ for Model A under Assumption 2.2 or Model B with $h > 0$. If for some $\alpha > 0$ we have $\limsup_{n\to\infty} \overline{\nu}(H(\alpha n, n)) > 0$ then for all $\alpha > 0$ we have $\limsup_{n\to\infty} \overline{\nu}(H(\alpha n, n)) > 0$.*

5. Proofs of sharpness results

In this section we prove Theorem 2.6 and Corollary 2.7. Here is an outline of the argument that follows. Suppose $\overline{\nu}$ is an invariant measure for which the cluster size $|C_0|$ does *not* have exponential tails. The first part of Lemma 4.2 together with Lemma 4.4 imply that certain crossing probabilities then have uniformly positive probability under $\overline{\nu}$. We want to apply Lemma 4.3 to show that, with an arbitrarily small increase of the relevant parameters, we can 'boost' this to get crossing probabilities close to 1. The second part of Lemma 4.2 then tells us that $|C_0|$ is now infinite with positive probability.

One of the main technical obstacles with carrying out this argument is that Lemma 4.3 applies to events which are defined in terms of a finite number of Bernoulli variables, whereas contact processes are defined in terms of 'continuous' objects (Poisson processes). The first step is therefore a *stability coupling*, a type of coupling which was also used in [4] for the Voronoi model, and later in [26] for the ordinary contact process. It tells us that if we increase the 'good' parameters then we can encode the contact process sufficiently well in terms of Bernoulli variables. This is the topic of Section 5.1. We give a detailed proof of this part of the argument for Model A because it is considerably more complicated than the corresponding one in [26, Lemma 3.2] for the ordinary contact process. (For Model B we give an outline.) The subsequent parts of the proof appear in Section 5.2. Recall that we are only considering the planar case $d = 2$.

We start by pointing out that the monotonicity lemma (Lemma 4.1) implies that Theorem 2.6 follows once we establish the following claim: Let $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h} > 0$ be fixed, and consider the parameterization $\{(\kappa, \tilde{\kappa}, r\lambda, r\tilde{\lambda}, rh, r\tilde{h}) : r \geq 0\}$ for Model A; and let $\kappa, \kappa^\star, \lambda, h, \tilde{h} > 0$ be fixed, and consider the parameterization $\{(\kappa, \kappa^\star, r\lambda, rh, r\tilde{h}) : r \geq 0\}$ for Model B. Define $r_{\mathrm{p}} = \inf\{r \geq 0 : \overline{\nu}(|C_0| = \infty) > 0\}$ as a function of the other parameters. Then the percolation transition is sharp in r, in that if $r < r_{\mathrm{p}}$ then $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ for some $c > 0$.

It is convenient to rescale time so that the total rate of 'events per line' is 1. That is, we assume

$$\begin{aligned} &\kappa + \tilde{\kappa} + 4\lambda + 4\tilde{\lambda} + h + \tilde{h} = 1 \quad \text{(Model A)},\\ &\kappa + \kappa^\star + 4\lambda + h + \tilde{h} = 1 \quad \text{(Model B)}. \end{aligned} \tag{5.1}$$

This clearly leaves the invariant measure $\overline{\nu}$ unchanged. Write

$$q = \begin{cases} 4\lambda + 4\tilde{\lambda} + h + \tilde{h} & \text{(Model A)}, \\ 4\lambda + h + \tilde{h} & \text{(Model B)}. \end{cases}$$

Thus q equals the part of the sum in (5.1) which is decreased in the sharpness theorems. We vary the parameter $q \in [0, 1]$ while keeping the sum (5.1) constant. We are then required to prove, under the relevant assumptions, that if q is such that $\overline{\nu}(|C_0| \geq n)$ goes to 0 as $n \to \infty$, but slower than exponentially, then for any $q'' > q$,

$$\overline{\nu}(|C_0| = \infty) > 0.$$

The objective of the stability coupling is the following. We wish to discretize time into intervals of length $\delta = n^{-\alpha}$ (for a certain $\alpha > 0$), and then apply the influence bound of Lemma 4.3. However, even if we choose n very large, we cannot avoid that there are intervals with more than one symbol. The solution is an intermediate step: for $q' \in (q, q'')$ we couple the processes for values q and q' in such a way that 'essential' symbols have distance at least δ. The existence of such a coupling is stated in Lemma 5.1. Subsequently, we use the influence bound of Lemma 4.3 to conclude that when q' is further increased to q'', then the criterion for percolation in Lemma 4.2 is satisfied.

5.1. Stability coupling

The processes with different values of q can be coupled in a natural way. Since this coupling procedure serves as a 'starting point' for the more complicated coupling in Lemma 5.1, we give a brief sketch here. Let Π be a Poisson point process with unit density on $\mathbb{Z}^2 \times \mathbb{R}$, and write $[\Pi]$ for the support of Π (i.e. the set of points in a realization of this point process). We interpret $(x, t) \in [\Pi]$ as a *symbol* in the graphical representation. In a second step we decide the *type* of the symbol. Types are from the set

$$\mathrm{T} = \begin{cases} \{D_1, D_2, U_1, U_2, A_1^{\uparrow}, A_1^{\downarrow}, A_1^{\leftarrow}, A_1^{\rightarrow}, A_2^{\uparrow}, A_2^{\downarrow}, A_2^{\leftarrow}, A_2^{\rightarrow}\} & \text{(Model A)}, \\ \{D_1, D_2, U_1, U_2, A_1^{\uparrow}, A_1^{\downarrow}, A_1^{\leftarrow}, A_1^{\rightarrow}\} & \text{(Model B)}, \end{cases}$$

corresponding to the notation in Section 4.1, arrow superscripts indicating the direction of (incoming) arrows. From now on we refer to symbols with types D_1 and D_2 as *down* symbols and the remaining as *up* symbols. For each symbol $(x, t) \in [\Pi]$, we consider three independent random variables drawn uniformly from the unit interval, denoted $Q_{(x,t)}$, $B_{(x,t)}$, and $G_{(x,t)}$. These are independent also of all other random variables used. We assign an *up* symbol whenever $Q_{(x,t)} \leq q$ and a *down* symbol whenever $Q_{(x,t)} > q$. For Model A we assign type D_1 if $Q_{(x,t)} > q$ and $B_{(x,t)} \leq \kappa/(\kappa + \tilde{\kappa})$ and we assign type D_2 if $Q_{(x,t)} > q$ and $B_{(x,t)} > \kappa/(\kappa + \tilde{\kappa})$. Similarly, whenever $Q_{(x,t)} \leq q$, we assign an *up* symbol based on the outcome of $G_{(x,t)}$, in such a way that the marginal distributions for the ten different *up* symbols (four different arrows of type A_1, another four of type A_2, and the two types U_1 and U_2) have the desired form. A very similar construction holds for Model B (without the A_2 symbols). We write H^q for the graphical representation thus obtained. So H^q consists of the processes D_1, D_2, A_1, A_2, U_1, U_2 for Model A, and D_1, D_2, A_1, U_1, U_2 for Model B, as in Section 4.1. (Of course, H^q depends not only on q but also on the remaining parameters κ, λ, $\tilde{\kappa}$, $\tilde{\lambda}$ etc; however, we suppress this from the notation.) The reader may convince her-/himself that the marginal distributions coincide with the definition of Section 4.1.

We write $\mathbb{P}$ for the probability measure governing Π, $Q_{(x,t)}$, $B_{(x,t)}$ and $G_{(x,t)}$ as above, and $\mathbb{P}_q$ for the probability measure governing the resulting graphical representation H^q. Thus $\mathbb{P}$ is a coupling of all the $\mathbb{P}_q$'s, $0 \le q \le 1$.

For each $x \in \mathbb{Z}^2$, $q \in [0,1]$ and $n \in \mathbb{N}$, we define the random variable $\eta_x^{(q,n)}$ as the state (0, 1 or -1) at $(x,0)$, subject to the boundary condition assigning state 1 to any point (y,s) with $d(x,y) = \lfloor\sqrt{n}\rfloor$ or $s = -\sqrt{n}$. (Here $d(x,y)$ denotes the usual graph distance.) Note that $\eta_x^{(q,n)}$ is determined by the graphical representation in the space–time region

$$\{(y,s) : d(x,y) \le \lfloor\sqrt{n}\rfloor \text{ and } s \in [-\sqrt{n}, 0]\}. \tag{5.2}$$

More generally, for $x \in \mathbb{Z}^2$, $q \in [0,1]$ and (z,t) in the space–time region (5.2), we define $\eta_z^{(x;q,n)}(t)$ as the state at (z,t) subject to the same boundary condition as in the definition of $\eta_x^{(q,n)}$ above (i.e. the b.c. assigning state 1 to any point (y,s) with $d(x,y) = \lfloor\sqrt{n}\rfloor$ or $s = -\sqrt{n}$). Note that $\eta_x^{(x;q,n)}(0) = \eta_x^{(q,n)}$ and that $\eta_z^{(x;q,n)}(t) = 1$ if $d(x,z) = \lfloor\sqrt{n}\rfloor$ or $t = \sqrt{n}$.

Recall the length $\delta = n^{-\alpha}$ introduced in the paragraph preceding this section. For $v \in \mathbb{Z}^2$ and $k \in \mathbb{N}$, $0 \le k \le \lceil\sqrt{n}/\delta\rceil$, and type $\tau \in \mathrm{T}$, we introduce the indicator functions

$$X_\tau^{(q,k,\delta)}(v) := \mathbb{1}\big\{\exists \text{ symbol of type } \tau \text{ in } \{v\} \times (-k\delta, (-k+1)\delta]\big\}. \tag{5.3}$$

For each δ-interval, these X variables only indicate whether there are symbols of a certain type in the interval, but do not tell us their precise locations or order. However, this information is often enough to conclude the value of $\eta_z^{(x;q,n)}(t)$ defined above: We define $\eta_z^{(x;q,n,\delta)}(t)$ as the maximal $m \in \{-1, 0, 1\}$ for which the values of the elements of

$$\big\{X_\tau^{(q,k,\delta)}(v) \colon \tau \in T,\ v \in \mathbb{Z}^2, \text{ and } k \in \mathbb{N}\big\}$$

imply that $\eta_z^{(x;q,n)}(t) \ge m$. (Because of the boundary condition in the definition of $\eta_z^{(x;q,n)}(t)$ we can, in fact, restrict to v's with $d(v,x) \le \sqrt{n}$ and k's with $0 \le k \le \lceil\sqrt{n}/\delta\rceil$.) Clearly $\eta_z^{(x;q,n,\delta)}(t) \le \eta_z^{(x;q,n)}(t)$ for all $\delta > 0$. From now on we write $\eta_x^{(q,n,\delta)}$ for $\eta_x^{(x;q,n,\delta)}(0)$. (Note that this is the maximal $m \in \{-1, 0, 1\}$ for which the $X_\tau^{(q,k,\delta)}(v)$'s imply that $\eta_x^{(q,n)} \ge m$.)

Let L_n be the box $[n, 5n] \times [n, 2n]$. (The precise choice of this box is not essential.) The following result holds for Model A as well as Model B (subject to the correct interpretation); note that for Model A we do not require Assumption 2.2 for this result.

Lemma 5.1 (*Stability Coupling*). *Let $\alpha > 0$ and, for each n, let $\delta = \delta_n = n^{-\alpha}$. For any $0 < q < q' < 1$, there is a coupling $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}_{q,q',n}$ of $\mathbb{P}_q$ and $\mathbb{P}_{q'}$ such that $\tilde{\mathbb{P}}\big(\forall x \in L_n : \eta_x^{(q,n)} \le \eta_x^{(q',n,\delta)}\big) \to 1$ as $n \to \infty$.*

We give full details for Model A.

Proof for Model A. Let R_n denote the box $[0, 6n] \times [0, 3n]$. Note that $\eta_x^{(q,n)}$ and $\eta_x^{(q',n,\delta)}$, $x \in L_n$, are determined by the graphical representation in the space–time region

$$\mathrm{ST}_n := R_n \times [-\lceil\sqrt{n}\rceil, 0].$$

We let $\delta_1 = \sqrt{\delta} = n^{-\alpha/2}$, and throughout the proof consider intervals I of the form $\{x\} \times [-(k+1)\delta_1, -k\delta_1]$ whose intersection with ST_n is nonempty ($k \in \mathbb{N}_0$). We call such an interval

I occupied whenever $I \cap [\Pi] \neq \varnothing$. Moreover, we call two intervals $I^{(x,k)}$ and $I^{(y,\ell)}$ *neighbours* if either $x = y$ and $|k - \ell| = 1$, or $d(x, y) = 1$ and $k = \ell$. This neighbourhood relation determines the notion of *clusters of neighbouring occupied intervals*, henceforth called *clusters*. We use the notation $\mathcal{C}$ for clusters, and define the *size* $|\mathcal{C}|$ to be the number of *symbols* in $\mathcal{C}$ (not the number of intervals). More precisely, $|\mathcal{C}| = \sum_{I\in\mathcal{C}} |I \cap [\Pi]|$, where $|I \cap [\Pi]|$ is the cardinality of $I \cap [\Pi]$. Note that the clusters depend only on the Poisson process Π, hence the law of the clusters is independent of any of the parameters $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h}$, and the occupation of the intervals is pairwise independent.

Let S_n^α be the event that each occupied cluster in ST_n has size smaller than $\lceil 9/\alpha \rceil$. It follows in an elementary way from properties of the Poisson process (see [26, Eq. (18)]) that

$$\lim_{n\to\infty} \mathbb{P}\big(S_n^\alpha\big) = 1. \tag{5.4}$$

Before proceeding with the argument, here are the main ideas. Ideally, we would like to take α so large that the size of the largest cluster shrinks to 1. However, later on (just before (5.13)) we need to take α rather close to 0; hence the existence of clusters of size ≥ 2 cannot be ruled out, and a result of the form (5.4) is essentially the best bound we get. (The precise value $9/\alpha$ is not important.) We solve the issue by factorizing our probability space $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 determines the clusters of intervals, the relative order (w.r.t. the time coordinates) of the symbols within each cluster, as well as some other information, and Ω_2 is responsible for the 'fine-tuning' (including the precise location of the symbols). We then first sample from Ω_1, which in particular fixes the clusters. For each cluster, when sampling from Ω_2, we use a 'crossover', which sacrifices an unnecessarily 'good' event in order to avoid a 'bad' event (where 'bad' means lack of δ-stability). 'Crossover' techniques have been used earlier for Voronoi percolation in [4, Theorem 6.1], and for percolation in the (ordinary) contact process in [26]. However, it turns out that the model we consider requires a considerably more subtle 'crossover recipe' than in [26].

We now give a detailed description. Outcomes $\omega_1 \in \Omega_1$ contain the following partial information about Π: First of all, for any interval I, ω_1 determines the number of elements of $[\Pi] \cap I$. This identifies the clusters of ST_n. We call an interval $I^{(x,k)} = \{x\} \times [-(k+1)\delta_1, -k\delta_1]$ *vertically isolated* whenever $I^{(x,k)}$ contains precisely one symbol and both $I^{(x,k-1)}$ and $I^{(x,k+1)}$ are not occupied. Further, for any cluster $\mathcal{C}$, we let ω_1 also determine the relative order of symbols in $\mathcal{C}$ (w.r.t. the time coordinates of the symbols), and the value of $G_{(x,t)}$ for all symbols in $\mathcal{C}$. (For the ease of description we here name symbols by (x, t) although the precise time t is not yet determined. Further, recall that $G_{(x,t)}$ tells *which* up type a symbol has *if* its type is up.) Finally, we also let ω_1 determine the value of $B_{(x,t)}$ for symbols in *vertically isolated intervals only*.

Outcomes $\omega_2 \in \Omega_2$ determine the precise location of symbols $(x, t) \in \Pi$, as well as the value $Q_{(x,t)}$ for *all* $(x, t) \in [\Pi]$, and the value of $B_{(x,t)}$ for every (x, t) that is *not contained in a vertically isolated interval*. Write $\mathcal{F}_1$ and $\mathcal{F}_2$ for the corresponding σ-algebras on Ω_1 and Ω_2 respectively.

Following the discussion at the beginning of the section, we can obtain the graphical representation H_n^q of the process on the space–time box ST_n as function of ω_1, ω_2, and q:

$$H_n^q = H_n^q(\omega_1, \omega_2). \tag{5.5}$$

(Actually, H_n^q depends also on the remaining parameters κ, λ, h etc. but we suppress this dependence.) Since $\eta_x^{(q,n)}$ is itself a monotone (in q) function of H_n^q, $\mathbb{P}$ gives a coupling of $\eta_x^{(q,n)}$ and

$\eta_x^{(q',n)}$ such that $\eta_x^{(q,n)} \le \eta_x^{(q',n)}$ for any $q < q'$. The restriction of H_n^q to a cluster $\mathcal{C}$ is denoted $H_{n,\mathcal{C}}^q = H_{n,\mathcal{C}}^q(\omega_1, \omega_2)$.

The 'crossover' referred to above is a mapping $\Omega_2 \to \Omega_2$ which depends on the outcome of $\omega_1 \in \Omega_1$. To this end, fix an instance $\omega_1 \in \Omega_1$ (which, as mentioned above, particularly fixes the clusters). Since $\mathbb{P}(\cdot \mid \mathcal{F}_1)$ acts independently on the different clusters, we can write $\mathbb{P}(\cdot \mid \mathcal{F}_1) = \prod_{\mathcal{C}} \mathbb{P}_{\mathcal{C}}(\cdot \mid \mathcal{F}_1)$. Fix a cluster $\mathcal{C}$ and write $\mathbb{P}_{\mathcal{C}}^{\omega_1}(\cdot) = \mathbb{P}_{\mathcal{C}}(\cdot \mid \mathcal{F}_1)(\omega_1)$. In light of (5.4), we proceed under the assumption that $|\mathcal{C}| < \lceil 9/\alpha \rceil$. On the probability space Ω_2 we now define two events. (The definition of these events involves a cluster $\mathcal{C}$ and hence also ω_1; however, recall that we consider ω_1 as fixed here.) The first event is

$\mathcal{B}$: in $H_{n,\mathcal{C}}^q$ there are two symbols whose time coordinates differ by less than $\delta = n^{-\alpha}$.

The probability of $\mathcal{B}$ is maximized when all $|\mathcal{C}|$ symbols are in one single interval, so that

$$\mathbb{P}_{\mathcal{C}}^{\omega_1}(\mathcal{B}) \le |\mathcal{C}|^2 \, \frac{2\delta}{\delta_1} \le 2\lceil 9/\alpha \rceil^2 \, n^{-\alpha/2}, \quad \text{if } |\mathcal{C}| < \lceil 9/\alpha \rceil, \tag{5.6}$$

which goes to 0 as $n \to \infty$.

Before we state the other event, we need the following notion: A *maximal connected vertical chain* is a union of occupied δ_1-intervals $I^{(x,k)}$, $I^{(x,k+1)}, \ldots, I^{(x,k+m-1)}$, with $k \ge 0$, $m \ge 1$, and where $I^{(x,k+m)}$ and (in case $k \ge 1$) $I^{(x,k-1)}$ are vacant. We call m the length of the chain. Note that a vertically isolated interval (defined earlier) is a maximal connected vertical chain of length 1.

We now define the event $\mathcal{G}$ that:

(1) in $H_{n,\mathcal{C}}^q$ all symbols are *down* symbols (i.e., all symbols in $\mathcal{C}$ have Q-value larger than q),
(2) in $H_{n,\mathcal{C}}^q$, each maximal connected vertical chain of length ≥ 2 has lowest symbol of type D_1 and all other symbols of type D_2, and
(3) in $H_{n,\mathcal{C}}^{q'}$ all symbols are *up* symbols (i.e., all symbols in $\mathcal{C}$ have Q-value smaller than q').

From the above definitions it follows straightforwardly that

$$\mathbb{P}_{\mathcal{C}}^{\omega_1}(\mathcal{G}) \ge (q' - q)^{|\mathcal{C}|} \big(\min\{\kappa/(\kappa + \tilde{\kappa}), \tilde{\kappa}/(\kappa + \tilde{\kappa})\}\big)^{|\mathcal{C}|}. \tag{5.7}$$

By this and (5.6) we thus may choose n sufficiently large (not depending on $\mathcal{C}$ or otherwise on ω_1) such that

$$\mathbb{P}_{\mathcal{C}}^{\omega_1}(\mathcal{G}) \ge \mathbb{P}_{\mathcal{C}}^{\omega_1}(\mathcal{B}) \quad \text{if } |\mathcal{C}| < \lceil 9/\alpha \rceil. \tag{5.8}$$

Write $\mathcal{B}' = \mathcal{B} \setminus \mathcal{G}$. From the above we get, for n sufficiently large, that if $|\mathcal{C}| < \lceil 9/\alpha \rceil$ then there exist a measurable subset $\mathcal{G}' \subset \mathcal{G} \setminus \mathcal{B}$ and a measure-preserving 1–1 map $\psi_{\mathcal{C}}$ on Ω_2 such that

- $\psi_{\mathcal{C}}(\mathcal{B}') = \mathcal{G}'$,
- $\psi_{\mathcal{C}}(\mathcal{G}') = \mathcal{B}'$, and
- $\psi_{\mathcal{C}}(\omega_2) = \omega_2$ whenever $\omega_2 \notin \mathcal{B}' \cup \mathcal{G}'$.

If, on the other hand, $|\mathcal{C}| \ge \lceil 9/\alpha \rceil$, then we let $\psi_{\mathcal{C}}$ be the identity on Ω_2.

The map $\psi_{\mathcal{C}}$ is the crossover mentioned before. Since $\psi_{\mathcal{C}}$ is measure-preserving on Ω_2, we obtain a new coupling of the graphical representations on the cluster $\mathcal{C}$ by considering the graphical representation

$$\tilde{H}_{n,\mathcal{C}}^{q'} := H_{n,\mathcal{C}}^{q'}(\omega_1, \psi_{\mathcal{C}}(\omega_2)),$$

cf. (5.5). The 'overall coupling' is then obtained by constructing $\tilde{H}^{q'}_{n,\mathcal{C}}$ for each cluster $\mathcal{C}$ independently. The resulting graphical representation is denoted $\tilde{H}^{q'}_{n}$. We construct $\eta^{(q,n)}_x$ from the graphical representation $H^q_n = H^q_n(\omega_1, \omega_2)$, and $\eta^{(q',n,\delta)}_x$ from $\tilde{H}^{q'}_n$.

Finally, we check that this coupling has the desired properties. For a given ω_1, let $\mathcal{C}$ be one of the clusters. Recall that $H^{q'}_{n,\mathcal{C}} \geq H^q_{n,\mathcal{C}}$. We have to study $\tilde{H}^{q'}_{n,\mathcal{C}}$ and compare it with $H^q_{n,\mathcal{C}}$. These objects depend on ω_2. There are three cases:

(i) Case $\omega_2 \notin (\mathcal{B} \cup \mathcal{G})$. This case is simple: by the definition of the coupling procedure, we have $\psi_{\mathcal{C}}(\omega_2) = \omega_2$, and $\tilde{H}^{q'}_{n,\mathcal{C}} = H^{q'}_{n,\mathcal{C}}$, which (as we recalled above) dominates $H^q_{n,\mathcal{C}}$. Moreover, since $\omega_2 \notin \mathcal{B}$ we know that $H^q_{n,\mathcal{C}}$, and hence also $\tilde{H}^{q'}_{n,\mathcal{C}}$, does not have two symbols of which the time coordinates differ less than δ. This settles case (i).

(ii) Case $\omega_2 \in \mathcal{B}'$. Then, by the definition of the coupling procedure, $\psi_{\mathcal{C}}(\omega_2) \in \mathcal{G}' \subseteq \mathcal{G} \setminus \mathcal{B}$. By the definition of $\mathcal{G}$, this implies that all symbols in $\tilde{H}^{q'}_{n,\mathcal{C}}$ are up symbols. Since the precise type of an up symbol is determined by ω_1, we get that each symbol in $\tilde{H}^{q'}_{n,\mathcal{C}}$ 'dominates' the corresponding symbol in $H^q_{n,\mathcal{C}}$. Moreover, since $\psi_{\mathcal{C}}(\omega_2)$ is not in $\mathcal{B}$, there are no symbols in $\tilde{H}^{q'}_{n,\mathcal{C}}$ of which the time coordinates differ less than δ. Finally, the *order* (w.r.t. time) of the symbols in $\tilde{H}^{q'}_{n,\mathcal{C}}$ is the same as for $H^q_{n,\mathcal{C}}$ (recall that the order is determined by ω_1). This settles case (ii).

(iii) Case $\omega_2 \in \mathcal{G}$. Then, by the definition of $\mathcal{G}$, the types in $H^q_{n,\mathcal{C}}$ on the maximal connected vertical chains that are not single vertically isolated intervals, are as 'unfavourable' as possible: Consider such a chain and let $I^{(z,k)}$ be its 'highest' (i.e., with largest time index) interval. Since the symbol with smallest time coordinate on the chain has type D_1 and the others D_2, and since there are no incoming arrows, it follows that, for all vertices x for whose state at time 0 this part of space–time is 'relevant' (i.e. for all $x \in L_n$ with $d(x, v) \leq \sqrt{n}$),

$$\eta^{(x;q,n)}_z(-k\delta) = -1.$$

Further, each single vertically isolated interval has (by the definition of $\mathcal{G}$) in $H^q_{n,\mathcal{C}}$ a 'down' symbol. Since the precise type of this down symbol is determined by ω_1, it follows that the corresponding symbol in $\tilde{H}^{q'}_{n,\mathcal{C}}$ is either the same type of down symbol, or an up symbol. From these considerations it follows that, no matter how the symbols in $\tilde{H}^{q'}_{n,\mathcal{C}}$ are located precisely, we have that, if $\mathcal{C}$ would be the only cluster, then each space–time point (z, t) which is the 'highest point' of a maximal connected vertical chain of $\mathcal{C}$, satisfies

$$\eta^{(x;q',n,\delta)}_z(t) \geq \eta^{(x;q,n)}_z(t) \quad \text{for all } x \in L_n \text{ with } d(x, v) \leq \sqrt{n}. \tag{5.9}$$

This settles the last case.

At the end of case (iii) we stated that (5.9) would hold if $\mathcal{C}$ is the only cluster. In fact, by combining this statement with the conclusions concerning case (i) and (ii), and the monotonicity of the contact process dynamics, it follows that (5.9) also holds (for such (z, t)) if there are other clusters (as long as all clusters have size $\leq 9/\alpha$). This completes the proof of Lemma 5.1 for Model A. □

Sketch proof for Model B. The argument for Model B has the same structure, but the details are considerably simpler. One difference is that now the only information represented by ω_1 is the number of symbols in each interval (which in turn defines the clusters) and the values of $U_{(x,t)}$.

All other information (the precise locations of the symbols and the $Q_{(x,t)}$- and $B_{(x,t)}$-values) are represented by ω_2. Another difference is that we modify the definition of the event $\mathcal{G}$ to the following:

(1) in $H^q_{n,\mathcal{C}}$ all symbols are of type D_2;

(2) in $H^{q'}_{n,\mathcal{C}}$ all symbols are *up* symbols.

This implies $Q_{(x,t)} \in (q, q')$ for all symbols in $H^q_{n,\mathcal{C}}$. In particular, no special 'treatment' of maximal connected vertical chains is needed anymore. Eq. (5.7) becomes

$$\mathbb{P}^{\omega_1}_{\mathcal{C}}(\mathcal{G}) \geq (q' - q)^{|\mathcal{C}|}\big(\kappa/(\kappa + \kappa^\star)\big)^{|\mathcal{C}|},$$

so that (5.8) still holds for large enough n. Thus we may define crossover maps $\psi_{\mathcal{C}}$ and the modified graphical representation $\tilde{H}^q_n$ as before. To check that this coupling has the required properties, we distinguish again the three cases (i)–(iii) as for the 3-state model. Indeed, the arguments for cases (i) and (ii) apply verbatim as in the 3-state case. Case (iii) is now considerably simpler than before, because $\omega_2 \in \mathcal{G}$ implies that $H^q_{n,\mathcal{C}}$ has only D_2 symbols and hence $\tilde{H}^{q'}_{n,\mathcal{C}}$ is always 'at least as good'. □

5.2. *Proof of Theorem 2.6*

Let $\overline{\nu}_q$ denote the upper invariant measure for the contact process defined as above with parameter value q. Let $0 < q_1 < 1$ be such that under $\overline{\nu}_{q_1}$ the cluster size $|C_0|$ does *not* have exponential tails. Let $q_2 > q_1$. We will deduce that $\overline{\nu}_{q_2}(|C_0| = \infty) > 0$. This immediately implies Theorem 2.6.

By the first part of Lemmas 4.2 and 4.4 we have that there exist $\varepsilon_1 > 0$ and a sequence $n_i \to \infty$ such that

$$\overline{\nu}_{q_1}(H(4n_i, n_i)) \geq \varepsilon_1 \quad \text{for all } i \geq 1.$$

Recall that L_n denotes the $4n$-by-n rectangle $[n, 5n] \times [n, 2n]$, and write $H_i = H(L_{n_i})$. Recall the definition of $\eta^{(q,n)}$ from the paragraphs preceding and below (5.3). By monotonicity (cf. Lemma 4.1 and the discussion around there) the realization of $\overline{\nu}_q$ on L_n is dominated by $\eta^{(q,n)}$ for all $n \geq 1$. We deduce that

$$\mathbb{P}(\eta^{(q_1,n_i)} \in H_i) \geq \varepsilon_1 \quad \text{for all } i \geq 1. \tag{5.10}$$

Fix $q' \in (q_1, q_2)$. By Lemma 5.1 and (5.10),

$$\mathbb{P}(\eta^{(q',n_i,\delta)} \in H_i) \geq \varepsilon_2 := \varepsilon_1/2$$

for all large enough $i \geq 1$. The latter probability is defined in terms of the Bernoulli variables X of (5.3), so in principle Lemma 4.3 could now be applied. However, we have no good way of bounding the number N of variables with maximal influence. To get around this, we consider a 'symmetrized' version of the event H_i. A similar method was used in e.g. [4,26] and is standard in this type of argument; here we use the 'truncation' implicit in the definition of the $\eta^{(q,n,\delta)}$ and hence, ultimately, the fast convergence of the dynamics (Lemma 2.1 and Assumption 2.2).

Recall that R_n is the box $[0, 6n] \times [0, 3n]$, and consider the 'periodic' set R^{per}_n obtained from R_n by identifying the left and right sides; that is, identifying points $(6n, y)$ and $(0, y)$. We can consider L_n as a subset of R^{per}_n rather than $\mathbb{Z}^2$. Since the variables $\eta^{(q',n_i,\delta)}$ are 'truncated' at distance $\sqrt{n_i}$ the probability that $\eta^{(q',n_i,\delta)} \in H_i$ is (for large enough i) unchanged under this

change of geometry. Let A_i be the event that there is a horizontal crossing of 1's of at least one of the $6n_i - 1$ horizontal translates of L_{n_i} in $R_{n_i}^{\text{per}}$. Thus

$$\pi_i(q') := \mathbb{P}(\eta^{(q',n_i,\delta)} \in A_i) \geq \mathbb{P}(\eta^{(q',n_i,\delta)} \in H_i) \geq \varepsilon_2, \tag{5.11}$$

for all sufficiently large i.

We apply Lemma 4.3 to the event A_i. By symmetry, all $6n_i - 1$ horizontal translates of $X_\tau^{(q',k,\delta)}(v)$ have the same influence, so the number N of Lemma 4.3 satisfies $N \geq 6n_i - 1 \geq n_i$. The number m of that lemma corresponds to the number of different types τ where we do not distinguish between different directions of arrows (because the Poisson intensities do not depend on these directions). Thus $m = 6$ for Model A, and $m = 5$ for Model B. (The number n of variables of each type which appears in that lemma does not figure in the conclusion, so it is irrelevant for us.)

For the next step of the argument, we consider the two models separately. Consider Model A first. We let $p_1, \ldots, p_6$ denote the probabilities that $X_\tau^{(q',k,\delta)}(v)$ equals 1 for $\tau = U_1$, $\tau = U_2$, $\tau \in A_2$, $\tau \in A_1$, $\tau = D_1$, and $\tau = D_2$, respectively. Recall that $\kappa + \tilde{\kappa} + 4\lambda + 4\tilde{\lambda} + \tilde{h} + \tilde{h} = 1$ and that we increase $q = 4\lambda + 4\tilde{\lambda} + h + \tilde{h}$ while keeping $\kappa/\tilde{\kappa}$ and the ratios between any two of $\tilde{\lambda}, \lambda, \tilde{h}, h$ fixed. This implies that there are constants $r_1, \ldots, r_6 \in (0, 1)$ such that $h = r_1 q$, $\tilde{h} = r_2 q$, $\lambda = r_3 q$, $\tilde{\lambda} = r_4 q$, $\kappa = r_5(1 - q)$, and $\tilde{\kappa} = r_6(1 - q)$. Hence p_j equals $1 - e^{-r_j q \delta}$ for $1 \leq j \leq 4$, and $1 - e^{-r_j(1-q)\delta}$ for $j = 5, 6$. It follows that for i large enough

$$\begin{aligned}
\frac{d\pi_i}{dq} &= \sum_{j=1}^{4} \delta r_j e^{-r_j \delta q} \frac{\partial \pi_i}{\partial p_j} - \sum_{j=5}^{6} \delta r_j e^{-r_j \delta (1-q)} \frac{\partial \pi_i}{\partial p_j} \\
&\geq \delta C \Big[\sum_{j=1}^{4} \frac{\partial \pi_i}{\partial p_j} - \sum_{j=5}^{6} \frac{\partial \pi_i}{\partial p_j} \Big] \\
&\geq \delta C \log N \frac{\pi_i(1 - \pi_i)}{K' \delta \log(2/\delta)},
\end{aligned} \tag{5.12}$$

for some constants C, K', and where the last inequality comes from Lemma 4.3.

Let $\varepsilon_3 > 0$ and suppose that $\pi_i(q'') < 1 - \varepsilon_3$ for all $q'' \in (q', q_2)$. Using that $N \geq n_i$ and $\delta = n_i^{-\alpha}$ we deduce from (5.11) and (5.12) that $\pi_i(q_2) \geq C(q_2 - q')\varepsilon_2 \varepsilon_3 / \alpha$. Choosing α sufficiently small we reach the following conclusion:

$$\forall \varepsilon^* > 0 \, \exists \alpha > 0\text{: for large enough } i, \ \pi_i(q_2) \geq 1 - \varepsilon^*. \tag{5.13}$$

For Model B, we obtain (5.13) in literally the same way, except that $\tilde{\lambda} = r_4 = p_4 = 0$ (because there are no A_2 symbols) and $\tilde{\kappa} = \kappa^\star$.

This final argument is the same for both models. Note that the event A_i implies that there is a horizontal crossing of at least one of the following rectangles (regarded as subsets of R_n^{per}):

$$[jn_i, (j+3)n_i (\text{mod } 6n_i)] \times [n_i, 2n_i] \quad 0 \leq j \leq 5.$$

Thus (by using the FKG-inequality) for $\hat{\varepsilon} > 0$ as in Lemma 4.2,

$$\mathbb{P}(\eta^{(q_2,n_i,\delta)} \in H(3n_i, n_i)) \geq 1 - (1 - \mathbb{P}(\eta^{(q_2,n_i,\delta)} \in A_i))^{1/6} \geq 1 - \hat{\varepsilon}/2$$

for all sufficiently large i, where the last inequality comes from (5.13). The family of random variables $(\eta_x^{(q_2,n_i,\delta)} : x \in [n, 4n] \times [n, 2n])$ is clearly stochastically dominated by the family

$(\eta_x^{(q_2,n_i)} : x \in [n,4n] \times [n,2n])$, and the law of this latter family has (by Lemma 2.1 and Assumption 2.2, and using standard arguments) total variation distance at most

$$C_1' \cdot 3n_i^2 \exp\bigl(-C_2'\sqrt{n_i}\bigr)$$

from $\overline{\nu}_{q_2}$. Hence $\overline{\nu}_{q_2}(H(3n_i, n_i)) \geq 1 - \hat{\varepsilon}$ for large enough i, which by Lemma 4.2 implies that $\overline{\nu}_{q_2}(|C_0| = \infty) > 0$. This completes the proof of Theorem 2.6. □

5.3. *Proof of Corollary 2.7*

Recall that for $x, y \in \mathbb{R}^k$ we write $x \prec y$ if each coordinate of x is strictly smaller than the corresponding coordinate of y. We write $\mathbb{R}_+ = (0, \infty)$. When proving Corollary 2.7 we make use of the following fact.

Lemma 5.2. *Let $n \geq 1$ be fixed and let $B \subseteq \mathbb{R}_+^3$ be a measurable set with the following property:*

$$\text{if } a_1, a_2, \ldots, a_m \in B \text{ satisfy } a_1 \prec a_2 \prec \cdots \prec a_m \text{ then } m \leq n. \tag{5.14}$$

For $y \in \mathbb{R}_+^2$ write $B(y) = (\mathbb{R}_+ \times \{y\}) \cap B$ and let Γ denote the set of $y \in \mathbb{R}_+^2$ such that $B(y)$ is uncountable. Then Γ has measure zero.

Proof. The partially ordered set $(B, \prec)$ has height at most n, so by (the dual version of) Dilworth's theorem, B can be partitioned into n antichains. An antichain in this situation is a set satisfying (5.14) with $n = 1$, so it suffices to consider that case.

For $n = 1$, note that if $x < x'$ and both (x, y) and (x', y) belong to B then property (5.14) is preserved if the interval $[x, x'] \times \{y\}$ is added to B. Thus we may assume that B is maximal in the sense that it includes all such intervals. With each $y \in \Gamma$ we may thus associate a rational number $q(y)$ such that $(q(y), y)$ lies in an interval of $B(y)$. We now write $y \in \mathbb{R}_+^2$ in polar coordinates (θ, r) with $\theta \in (0, \pi/2)$ and $r > 0$. Fix θ and $r < r'$ and write $y = (\theta, r)$ and $y' = (\theta, r')$. If $(x, y) \in B(y)$ and $(x', y') \in B(y')$ then $x' \leq x$, by (5.14) with $n = 1$. Thus, if $y, y' \in \Gamma$ then we may choose $q(y') < q(y)$. It follows that for each $\theta \in (0, \pi/2)$ the set of $r > 0$ such that $(\theta, r) \in \Gamma$ is at most countable. By Fubini's theorem (using polar coordinates) it follows that Γ has measure zero. □

Proof of Corollary 2.7 for Model A. Fix arbitrary $\kappa, \tilde{\kappa} > 0$. Recall that h_{p} is decreasing in each of the parameters $\lambda, \tilde{\lambda}, \tilde{h}$. Since $h_{\mathrm{p}}(\lambda, \tilde{\lambda}, \tilde{h}) < \infty$ for $\tilde{h} > 0$ we may restrict $(\lambda, \tilde{\lambda}, \tilde{h})$ to one of the (countably many) sets where h_{p} is bounded above by a fixed integer K. We call a triple $(\lambda, \tilde{\lambda}, \tilde{h})$ 'bad' if there is some $\delta > 0$ such that

$$h_{\mathrm{p}}(\lambda', \tilde{\lambda}', \tilde{h}') \leq h_{\mathrm{p}}(\lambda, \tilde{\lambda}, \tilde{h}) - \delta \quad \text{for all } (\lambda', \tilde{\lambda}', \tilde{h}') \succ (\lambda, \tilde{\lambda}, \tilde{h}). \tag{5.15}$$

If B denotes the set of 'bad' points, then we may write $B = \cup_{n \geq 1} B_n$, where B_n is the set of points such that (5.15) holds with $\delta = K/n$. The set B_n satisfies (5.14), so it follows from Lemma 5.2 that for almost all pairs $(\tilde{\lambda}, \tilde{h})$ the set of $\lambda > 0$ such that $(\lambda, \tilde{\lambda}, \tilde{h})$ is bad is countable.

We are therefore done if we show that if $(\lambda, \tilde{\lambda}, \tilde{h})$ is *not* bad, and $h < h_{\mathrm{p}}(\lambda, \tilde{\lambda}, \tilde{h})$, then the cluster size decays exponentially for the parameter values $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h}$. Writing $\delta = h_{\mathrm{p}}(\lambda, \tilde{\lambda}, \tilde{h}) - h$ we have that there exists $(\lambda', \tilde{\lambda}', \tilde{h}') \succ (\lambda, \tilde{\lambda}, \tilde{h})$ such that

$$h_{\mathrm{p}}(\lambda', \tilde{\lambda}', \tilde{h}') > h_{\mathrm{p}}(\lambda, \tilde{\lambda}, \tilde{h}) - \delta = h.$$

The result now follows from Theorem 2.6. □

Proof of Corollary 2.7 for Model B. Fix $\kappa, \kappa^\star > 0$. The argument for Model B is very similar to the one already given for Model A, using the analog of Lemma 5.2 with $B \subseteq \mathbb{R}^2_+$ (in which case we can actually show that Γ is at most countable). One small difference is that we may now have $h_{\mathrm{p}} = \infty$ for some $(\lambda, \tilde{h})$. We now say that $(\lambda, \tilde{h})$ is *bad* if $h_{\mathrm{p}}(\lambda, \tilde{h}) < \infty$ and there exists $\delta > 0$ such that

$$h_{\mathrm{p}}(\lambda', \tilde{h}') \le h_{\mathrm{p}}(\lambda, \tilde{h}) - \delta \quad \text{for all } (\lambda', \tilde{h}') \succ (\lambda, \tilde{h}), \tag{5.16}$$

and that $(\lambda, \tilde{h})$ is *terrible* if $h_{\mathrm{p}}(\lambda, \tilde{h}) = \infty$ but $h_{\mathrm{p}}(\lambda, \tilde{h}) < \infty$ for all $(\lambda', \tilde{h}') \succ (\lambda, \tilde{h})$. The set B of bad points may be written as

$$B = \bigcup_{K \ge 1} \bigcup_{n \ge 1} B_n^{(K)},$$

where $B_n^{(K)}$ is the set of points $(\lambda, \tilde{h})$ such that $h_{\mathrm{p}}(\lambda, \tilde{h}) \le K$ and (5.16) holds with $\delta = K/n$. Thus $B_n^{(K)}$ satisfies (5.14). The set T of terrible points satisfies (5.14) with $n = 1$. It follows that for almost all $\tilde{h} > 0$, the set of λ such that $(\lambda, \tilde{h}) \in B \cup T$ is at most countable. If $(\lambda, \tilde{h}) \notin B \cup T$ then the result follows in the same way as for Model A. □

6. Existence of density-driven processes

In this section we prove the existence of DDCP (Definition 1.1) using a fixed-point argument. Although this result is strictly speaking not needed for our main results on sharpness and lack of robustness (since, as discussed in Section 3.1, *stationary* DDCP are simply contact processes with constant parameters), we find it interesting in itself.

We consider 3-state processes with constant $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h}$. Recall that we write $\rho(t) = P(X_0(t) = 1)$ for the density of the process. We let L_b^∞ denote the set of measurable $h\colon [0, \infty) \to [0, \infty)$ which are bounded on each compact subinterval.

We prove the following existence result for Model A; a completely analogous result holds for Model B.

Theorem 6.1. *Let $\Lambda, H\colon [0, 1] \to [0, \infty)$ be uniformly Lipschitz continuous. For all $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h} \ge 0$ and each translation-invariant probability measure ν on $\{-1, 0, 1\}^{\mathbb{Z}^d}$, there is a unique pair $(\lambda, h) \in L_b^\infty \times L_b^\infty$ such that the 3-state contact process (Model A) with initial distribution ν and parameters $\kappa, \tilde{\kappa}, \lambda(\cdot), \tilde{\lambda}, h(\cdot), \tilde{h}$ satisfies $\lambda(t) = \Lambda(\rho(t))$ and $h(t) = H(\rho(t))$ for all $t \ge 0$.*

Proof. Let $h, h', \lambda, \lambda' \in L^\infty([0, \infty), [0, \infty))$, and let D_1, D_2, U_2 and A_2 be as in Section 4.1. The intensities of these processes are kept fixed. Let $\underline{U}_1$ be a Poisson process of intensity $h(t) \wedge h'(t)$. Let $\tilde{U}_1^{(h)}$ and $\tilde{U}_1^{(h')}$ denote independent Poisson processes (independent also of $\underline{U}_1$) with intensities $h(t) - (h(t) \wedge h'(t))$ and $h'(t) - (h(t) \wedge h'(t))$, respectively. Write $U_1^{(h)} = \underline{U}_1 \cup \tilde{U}_1^{(h)}$, $U_1^{(h')} = \underline{U}_1 \cup \tilde{U}_1^{(h')}$ and $\overline{U}_1 = \underline{U}_1 \cup \tilde{U}_1^{(h)} \cup \tilde{U}_1^{(h')}$. In the same way (and independently of the Poisson processes above) we define $\underline{A}_1$, $A_1^{(\lambda)}$, $A_1^{(\lambda')}$ and $\overline{A}_1$. Furthermore, let $m := 1 \vee \sup\{\lambda(t) \vee \lambda'(t) : t \ge 0\}$, and let $A_1^{(m)}$ be obtained from $\overline{A}_1$ by appending another independent Poisson process of intensity $m - (\lambda(t) \vee \lambda'(t))$. Note that an element of $A_1^{(m)}$ (at time coordinate t) belongs to $\overline{A}_1 \setminus \underline{A}_1$ with probability $|\lambda'(t) - \lambda(t)|/m \le \|\lambda' - \lambda\|_\infty$.

Let $\underline{X}$ be the contact process with $0 \to 1$ transitions given by $\underline{U}_1$ and $\underline{A}_1$ (and remaining transitions given by D_1, D_2, U_2, A_2). Similarly, $X^{(h,\lambda)}$, $X^{(h',\lambda')}$, and $\overline{X}$ denote the contact processes with $0 \to 1$ transitions given by $U_1^{(h)}$ and $A_1^{(\lambda)}$, with $U_1^{(h')}$ and $A_1^{(\lambda')}$, and with $\overline{U}_1$ and

$\overline{A}_1$, respectively. The construction above is done such that $\underline{X} \le X^{(h,\lambda)}$, $X^{(h',\lambda')} \le \overline{X}$ holds and $\overline{U}_1 \setminus \underline{U}_1$ has rate $|h(t) - h'(t)|$.

Now consider, for each $t \ge 0$, the set $Z_t^{\leftarrow}$, which is defined as the set of space–time points (x, s), $0 \le s \le t$, such that there is a space–time path from (x, s) to $(0, t)$ using arrows from $A_1^{(m)} \cup A_2$. Further, let $|Z_t^{\leftarrow}|$ be the sum of the total Lebesgue measure of all intervals constituting $Z_t^{\leftarrow}$, plus the number of arrows in the space–time paths in the definition of $Z_t^{\leftarrow}$.

Let B_t be the event that none of the arrows in the space–time paths in the definition of $Z_t^{\leftarrow}$ belongs to $\overline{A}_1 \setminus \underline{A}_1$. If B_t occurs and $(\overline{U}_1 \setminus \underline{U}_1) \cap Z_t^{\leftarrow} = \varnothing$, then $\overline{X}_0(t) = \underline{X}_0(t)$ and hence $X_0^{(h,\lambda)}(t) = X_0^{(h',\lambda')}(t)$.

Equip $L^\infty([0,\infty),[0,\infty))^2$ with the norm $\|(\lambda, h)\| = \|\lambda\|_\infty + \|h\|_\infty$, and consider the mapping R from this space to $L^\infty([0,\infty),[0,1])$ given by letting $R(\lambda, h)(t) = P(X_0(t) = 1)$ where X is the 3-state contact process with rates $\kappa, \tilde{\kappa}, \lambda(\cdot), \tilde{\lambda}, h(\cdot), \tilde{h}$. Let $\alpha > 0$. By the above we have, for all $0 \le t \le \alpha$,

$$\begin{aligned} |R(\lambda, h)(t) - R(\lambda', h')(t)| &\le P(B_t^c \text{ occurs or } (\overline{U}_1 \setminus \underline{U}_1) \cap Z_t^{\leftarrow} \ne \varnothing) \\ &\le E|Z_t^{\leftarrow}|(\|\lambda - \lambda'\|_\infty + \|h - h'\|_\infty), \end{aligned}$$

which by obvious monotonicity is at most

$$E|Z_\alpha^{\leftarrow}|(\|\lambda - \lambda'\|_\infty + \|h - h'\|_\infty).$$

Let K_1, K_2 be uniform Lipschitz constants for Λ, H. It follows that

$$\begin{aligned} &\sup_{0\le t\le\alpha} |\Lambda(R(\lambda, h)(t)) - \Lambda(R(\lambda', h')(t))| + \sup_{0\le t\le\alpha} |H(R(\lambda, h)(t)) - H(R(\lambda', h')(t))| \\ &\quad \le (K_1 + K_2)\, E|Z_\alpha^{\leftarrow}|\,(\|\lambda - \lambda'\|_\infty + \|h - h'\|_\infty). \end{aligned}$$

By standard comparison with a branching process, it is easy to see that $E|Z_\alpha^{\leftarrow}|$ is finite for α sufficiently small, and goes to 0 as $\alpha \to 0$. Hence there is an $\alpha_0 > 0$ such that the mapping $\Gamma = \Gamma_{\alpha_0}^{\nu} : (\lambda, h) \mapsto (\Lambda(R(\lambda, h)), H(R(\lambda, h)))$ is a contraction of $L^\infty([0,\alpha_0],[0,\infty))^2$. By Banach's fixed point theorem, this gives the desired result for the time interval $[0, \alpha_0]$. By repeating ('concatenating') this result, it can be extended to $[0, 2\alpha_0]$, $[0, 3\alpha_0]$, etc., which completes the proof. □

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