

REMARK ON A THEOREM OF  
RIDDLE, SAAB, AND UHL

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ABSTRACT. We give a short proof of the Riddle-Saab-Uhl theorem on Pettis integrability in dual Banach spaces, avoiding the use of Banach space techniques and extending it to certain locally convex spaces.

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**0 Introduction.** In this short note we propose to give a simplified proof of the Riddle-Saab-Uhl theorem [RSU] on Pettis integrability (Corollary 0.3 below). The original proof relies on subtle Banach space theory in combination with measure theoretic results concerning pointwise limits of certain classes of functions. In our approach we only use the latter, thereby avoiding all Banach space techniques. This allows extension of the RSU-theorem to separable metrizable locally convex spaces.

Let  $X$  be a (real) locally convex topological vector space (lcs). As usual, we let  $X^{**}$  denote the dual of  $(X^*, \beta(X^*, X))$ , where  $\beta(X^*, X)$  denotes the strong topology of  $X^*$ . Let  $j : X \rightarrow X^{**}$  denote the natural map. Our main result is:

**THEOREM 0.1.** *Let  $F$  be a countable bounded subset of an lcs  $X$ , let  $x^{**}$  be a weak\*-cluster point of  $jF$  and let  $Q$  be compact Hausdorff. If  $\phi : Q \rightarrow X^*$  is  $\beta(X^*, X)$ -bounded, universally weak\*-integrable and*

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universally scalarly measurable with respect to every weak\*-cluster point of  $jF$ , then

$$\langle x^{**}, \text{weak}^* \int_Q \phi(\sigma) d\mu(\sigma) \rangle = \int_Q \langle x^{**}, \phi(\sigma) \rangle d\mu(\sigma).$$

Let us say that  $X$  has *Property C* if for all  $x^{**} \in X^{**}$  there is a *Countable* bounded set  $F \subset X$  such that  $x^{**}$  is a weak\*-cluster point of  $jF$ . Every metrisable separable lcs has property C. We do not know whether every separable lcs has property C; the problem is that an arbitrary closed subspace of a separable lcs need not be separable [LS].

**COROLLARY 0.2.** *Let  $X$  be an lcs with property C and let  $Q$  be compact Hausdorff. If  $\phi : Q \rightarrow X^*$  is universally scalarly measurable and the convex hull of its range is relatively weak\*-compact, then  $\phi$  is universally Pettis integrable.*

*Proof.* By [S, IV.5.1]  $\phi$  is  $\beta(X^*, X)$ -bounded and by [Si]  $\phi$  is weak\*-integrable. The rest follows from Theorem 0.1.  $\square$

**COROLLARY 0.3** [RIDDLE-SAAB-UHL]. *Let  $X$  be a separable Banach space and let  $Q$  be compact Hausdorff. If  $\phi : Q \rightarrow X^*$  is bounded and universally scalarly measurable, then  $\phi$  is universally Pettis integrable.*

**1 The proof.** Let  $Q$  be compact Hausdorff. If  $x^{**} \in X^{**}$ , then we say that a function  $\phi : Q \rightarrow X^*$  is *universally scalarly measurable with respect to  $x^{**}$*  if  $\langle x^{**}, \phi(\cdot) \rangle$  is  $\mu$ -measurable for every Radon measure  $\mu$  on  $Q$ . We say that  $\phi$  is *Pettis integrable* with respect to  $\mu$  if  $\phi$  is Pettis integrable as a function  $(Q, \mu) \rightarrow (X^*, \beta(X^*, X))$ . By  $\tau_p$  we denote the topology of pointwise convergence.

**LEMMA 1.1.** *Let  $\phi : Q \rightarrow X^*$  be  $\beta(X^*, X)$ -bounded and weak\*-integrable with respect to a Radon measure  $\mu$  and let  $x^{**} \in X^{**}$ . If for every  $\epsilon > 0$  there is a measurable  $Q_\epsilon$  such that  $\mu(Q \setminus Q_\epsilon) < \epsilon$  and*

$$\langle x^{**}, \text{weak}^* \int_{Q_\epsilon} \phi(\sigma) d\mu(\sigma) \rangle = \int_{Q_\epsilon} \langle x^{**}, \phi(\sigma) \rangle d\mu(\sigma),$$

then

$$\langle x^{**}, \text{weak}^* \int_Q \phi(\sigma) d\mu(\sigma) \rangle = \int_Q \langle x^{**}, \phi(\sigma) \rangle d\mu(\sigma).$$

*Proof.* If not, then

$$\left| \langle x^{**}, weak^* \int_Q \phi(\sigma) d\mu(\sigma) \rangle - \int_Q \langle x^{**}, \phi(\sigma) \rangle d\mu(\sigma) \right| = \delta > 0.$$

Choose a bounded set  $F \subset X$  such that  $x^{**}$  is a weak\*-cluster point of  $jF$  (this is always possible: for instance, take the  $\langle E, E^* \rangle$ -polar of the  $\langle E^*, E^{**} \rangle$ -polar of  $\{x^{**}\}$ ) and let  $K := \sup_{q \in Q, x \in F} |\langle \phi(q), x \rangle|$ . This supremum exists since the range of  $\phi$  is  $\beta(X^*, X)$ -bounded. For  $\epsilon = \delta/3K$  choose  $Q_\epsilon$  as indicated. Then

$$\begin{aligned} \delta &= \left| \langle x^{**}, weak^* \int_Q \phi(\sigma) d\mu(\sigma) \rangle - \int_Q \langle x^{**}, \phi(\sigma) \rangle d\mu(\sigma) \right| \\ &= \left| \langle x^{**}, weak^* \int_{Q \setminus Q_\epsilon} \phi(\sigma) d\mu(\sigma) \rangle - \int_{Q \setminus Q_\epsilon} \langle x^{**}, \phi(\sigma) \rangle d\mu(\sigma) \right| \\ &\leq K\epsilon + K\epsilon = \frac{2\delta}{3}. \end{aligned}$$

The estimate is obtained by noting that it holds when  $x^{**}$  is replaced by an arbitrary  $x \in F$  and using that  $x^{**}$  is a weak\*-cluster point of  $jF$ . Thus we have a contradiction.  $\square$

*Proof of Theorem 0.1.* Let  $F = (x_n)$  and define the functions  $f_n, f : Q \rightarrow \mathbb{R}$  by

$$f_n(s) = \langle \phi(s), x_n \rangle, \quad f(s) := \langle x^{**}, \phi(s) \rangle.$$

Let  $\epsilon > 0$  be arbitrary and fix a Radon measure  $\mu$  on  $Q$ . By Lusin's theorem and the regularity of  $\mu$  there is a compact subset  $Q_\epsilon \subset Q$  such that  $\mu(Q \setminus Q_\epsilon) < \epsilon$  and such that  $\langle \phi(\cdot), x_n \rangle$  is continuous on  $Q_\epsilon$  for every  $n$ .

Since  $x^{**}$  is a weak\*-cluster point of  $(x_n)$ , there is a subsequence  $(x_{n_m})$ , depending on  $\epsilon$ , such that

$$\langle x^{**}, weak^* \int_{Q_\epsilon} \phi(\sigma) d\mu(\sigma) \rangle = \lim_{m \rightarrow \infty} \langle weak^* \int_{Q_\epsilon} \phi(\sigma) d\mu(\sigma), x_{n_m} \rangle.$$

Let  $Z := \{f_{n_m} : m \in \mathbb{N}\}$  and note that these functions are uniformly bounded. Let  $K \subset Q_\epsilon$  be compact. We claim that every sequence

in  $Z|_K$  has a subsequence converging pointwise  $\mu|_K$ -a.e. Suppose the contrary. Noting that  $(K, \mu|_K)$  is a perfect measure space, by Fremlin's subsequence theorem [F] there is a sequence in  $Z|_K$ , say  $(f_{n_{m_i}}|_K)$ , all of whose  $\tau_p$ -cluster points in  $\mathbb{R}^K$  are non- $\mu|_K$ -measurable. Let  $(x_{n_{m_i}})$  be the corresponding subsequence of  $(x_{n_m})$  and let  $x_0^{**}$  be a weak\*-cluster point of  $j(x_{n_{m_i}})$  in  $X^{**}$ . Such a cluster point exists since  $jB$  is relatively weak\*-compact for every bounded  $B \subset X$ . Clearly  $x_0^{**}$  is also a weak\*-cluster point of  $(x_n)$ . Moreover, the function  $g(s) := \langle x_0^{**}, \phi(s) \rangle$  is a  $\tau_p$ -cluster point of  $(f_{n_{m_i}})$ , hence non- $\mu|_K$ -measurable as a function on  $K$ . This is a contradiction to the scalar  $\mu$ -measurability of  $\phi$ . This proves the claim.

In particular it follows that if  $\mu$  is Radon on  $Q$  and  $K \subset Q_\epsilon$  is compact, then every sequence in  $Z|_K$  has a  $\mu|_K$ -measurable  $\tau_p$ -cluster point. By Theorem 2.7 (vii) $\Rightarrow$ (ii) of [BFT] and taking  $K = Q_\epsilon$  it follows that every sequence in  $Z|_{Q_\epsilon}$  has a pointwise convergent subsequence. By Theorem 3.11 of [vD],  $Z|_{Q_\epsilon}$  is  $t$ -stable and then Proposition 3.10 of [vD] implies that on  $Q_\epsilon$  the function  $f$  defined above is the pointwise limit of a subsequence  $(f_{n_{m_k}})$  of  $(f_{n_m})$ . Let  $(x_{n_{m_k}})$  be the corresponding subsequence of  $(x_{n_m})$ . Combining everything we see from the dominated convergence theorem that

$$\begin{aligned} \langle x^{**}, \text{weak}^* \int_{Q_\epsilon} \phi(\sigma) d\mu(\sigma) \rangle &= \lim_{k \rightarrow \infty} \int_{Q_\epsilon} \langle \phi(\sigma), x_{n_{m_k}} \rangle d\mu(\sigma) \\ &= \int_{Q_\epsilon} \langle x^{**}, \phi(\sigma) \rangle d\mu(\sigma), \end{aligned}$$

and we can apply the lemma.

Let  $X$  be an arbitrary Banach space. If  $x^{**}$  is the weak\*-limit of some sequence  $(x_n)$  in  $X$ , then it follows immediately from the dominated convergence theorem that  $x^{**}$  commutes with every weak\*-integral. The following corollary shows that it is enough that  $x^{**}$  be a weak\*-cluster point of  $(x_n)$ , provided the integrated function has good measurability properties.

**COROLLARY 1.2.** *Let  $X$  be a Banach space,  $\phi : Q \rightarrow X^*$  be bounded and universally scalarly measurable. Suppose  $x^{**}$  is a weak\*-cluster*

point of a bounded sequence in  $X$ . Then

$$\langle x^{**}, \text{weak}^* \int_Q \phi(\sigma) d\mu(\sigma) \rangle = \int_Q \langle x^{**}, \phi(\sigma) \rangle d\mu(\sigma).$$

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