On a class of two-dimensional nearest-neighbour random walks

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Abstract

For positive recurrent nearest-neighbour, semi-homogeneous random walks on the lattice $\{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$ the bivariate generating function of the stationary distribution is analysed for the case where one-step transitions to the north, north-east and east at interior points of the state space all have zero probability. It is shown that this generating function can be represented by meromorphic functions. The construction of this representation is exposed for a variety of one-step transition vectors at the boundary points of the state space.

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1. Introduction

Since the early 1970s random walks on the lattice $\{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$ have received increasing attention in the literature, mainly due to their use in modelling traffic flow patterns in telecommunication and computer networks. For details concerning their applications see for example Takagi [17]. For the performance analysis of such networks, information concerning the stationary distribution of these random walks is of prime importance. Around 1980 it became obvious that the determination of the stationary distribution could be formulated as a boundary value problem, cf. Fayolle and Iasnogorodsky [7], Cohen and Boxma [6], Cohen [5]; for a review paper, see [4], where the problem formulation as a singular Fredholm integral equation is also discussed.

Nearest-neighbour random walks are a special but important subclass. There is a study by Groeneveld, dating from the early 1960s; unfortunately, it has never been published. By using uniformisation he solved the functional equation for the 'shortest queue' model and showed that the solution can be expressed in terms of elliptic functions; Malyshev's approach [14] is of a similar character.

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Among the nearest-neighbour models, the 'shortest queue' is a much studied one. Its basic analysis is due to Kingman [13] and Flatto and McKean [10]. In the context of boundary value problems this model has been analysed in [7] and [6].

In a nearest-neighbour random walk the one-step transition from an interior point $(i,j) \in \{1,2,\dots\} \times \{1,2,\dots\}$ of the state space leads with probability 1 to a neighbouring point $(i+h, j+k)$, $(h,k) \in \{-1,0,1\} \times \{-1,0,1\} \setminus \{(0,0)\}$. In the shortest queue model the one-step transitions with $(h,k) \in \{(1,1), (1,0), (0,1)\}$ all have probability 0. In Hofri [11] and Jaffe [12], models with this feature are also discussed almost along the same lines as in [10], and as with the 'shortest queue' model the bivariate generating functions of the stationary distributions are meromorphic, i.e. their only singularities are isolated poles. These random walks are featured by the absence of one-step transitions to the north, north-east and east, (N, NE, E), see also Adan [1]; from the results obtained in these studies the conjecture arises whether for such random walks the generating function of the stationary distribution, if it exists, is always meromorphic. In the present study we show that under some mild conditions this conjecture is true. Questions concerning the algebraic character of the generating function have also recently been studied by Fayolle et al. [8]. Flatto and Hahn [9] provide a model with an algebraic generating function, in which the one-step transition to the north-east has a non-zero probability; see also Wright [18].

The character of the generating function of the stationary distribution of a nearest-neighbour random walk is determined by the number of branch points of the zeros of the so-called kernel

$$p_1 p_2 - \phi_3(p_1, p_2);$$

where $\phi_3(p_1, p_2)/(p_1 p_2)$ is the bivariate generating function of the distribution of the one-step transition from an interior point of the state space. The kernel is in general a biquadratic in p_1 and p_2, and its zeros, e.g. p_1 as a function of p_2, generally have four branching points, two within and two outside the unit disk if the drifts $\mu_3 - 1$ and $\nu_3 - 1$ are negative, as in (2.5) below. In the case with no N, NE and E one-step transitions there are only two finite branching points, both inside the unit disk. The branching points inside the unit disk play an essential role in the analysis of the functional equation for the bivariate generating function of the stationary distribution. The branching points outside the unit disk play a decisive role in the analytic continuation of the bivariate generating function into the domain outside the torus generated by the two unit disks.

In the present study we consider the nearest-neighbour walk without one-step transitions to the N, NE and E at interior points of the state space; it is assumed that the process is positive recurrent, cf. Assumption 2.1. It is shown that the generating function of the stationary distribution can be described in terms of meromorphic functions. The construction of these functions is outlined; some weak restrictions have been made, cf. Assumption 4.1. Even with these restrictions quite a number of variants have to be considered.

The organisation of the present study is now described. In Section 2 the model of the nearest-neighbour random walk is defined by its one-step transition vectors at interior and at the boundary points of the state space $\{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$. The random walk is semi-homogeneous, i.e. the distribution of the transition vector at interior points is independent of the position of such a point, similarly for the boundary points on the positive horizontal axis and the positive vertical axis, cf. (2.1) and (2.2). In this section the functional equation to be solved is formulated on the zero set of the kernel. By using the analytic properties of these zeros the functional equation is replaced by two equations with two unknown functions $\Omega_1(p)$, $\Omega_2(p)$, defined for $|p| \leq 1$; the two branching points of a zero located inside the unit disk are instrumental here, see Section 3. From the structure of the coefficients in these two equations it is shown that $\Omega_1(p)$ and $\Omega_2(p)$ can be continued analytically into $|p| > 1$. In Section 4 it is shown that the only singularities of these analytic continuations are poles; for the determination of the residues at these pole sets recursive linear equations are derived. The pole sets of $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are generated by the zeros in $|p| > 1$ of some of the coefficients in the two equations for $\Omega_1(\cdot)$, $\Omega_2(\cdot)$; the number of such zeros depends on the character of the transition vectors at the boundary points of the state space. From the results so obtained it is seen that the solutions of the two functional equations for $\Omega_1(p)$ and $\Omega_2(p)$, with $\Omega_1(p)$, $\Omega_2(p)$ both regular in $|p| < 1$ and continuous in $|p| \leq 1$, are meromorphic functions with known pole sets and recursively defined residues in $|p| > 1$. In Section 5, $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are both expressed as the sum of a polynomial and a number of meromorphic functions with given poles and residues. These meromorphic functions are, apart from a factor, explicitly known; their construction follows from the results in Section 4. The polynomials and their degrees still have to be determined. Substitution of the expression for $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ into the two equations for these functions (see Section 6) leads to the determination of the degrees and the coefficients of these polynomials, and a set of linear equations for the unknown factors in the meromorphic functions remains. It is shown that the equations have a solution and so $\Omega_1(p)$, $\Omega_2(p)$ are determined. Finally, it is shown that the solution so constructed leads to the unique solution of the functional equation for the bivariate generating function of the stationary distribution. In order not to interrupt the exposition of the construction of the solution, all the algebraic computations are given in Appendixes A, B and C.

The construction of the generating function of the stationary distribution may be also formulated as a boundary value problem, cf. [6], and as such it can be solved completely, even when one-step transitions to the N, NE and E occur. If they do not occur then the present approach is simpler, because it avoids the explicit calculation of a conformal mapping.

The present investigation has been initiated after reading the studies by Adan [1] and Adan et al. [2] on nearest-neighbour random walks without N, NE and E one-step transitions. In his search for a direct derivation of explicit expressions for all state probabilities of the two-dimensional stationary distribution, Adan starts from the equilibrium equations for these probabilities. For the general equation of this set, i.e. the equation containing no state probabilities of the boundary points of the state space, he constructs a class of solutions. By choosing suitable linear combinations of these solutions Adan tries to satisfy the boundary conditions, i.e. the equations containing boundary states. Using an iterative procedure, he succeeds in constructing a sequence which, whenever it converges absolutely at all points of the state space, provides in the limit the solution of the equilibrium equations and the norming condition. This aspect of absolute convergence is actually the problem of choosing the exponents of convergence in constructing the meromorphic functions, cf. Section 5 and Appendix C. Adan's approach leads to an attractive algorithm for the numerical evaluation of the various state probabilities. From his analysis it may be shown that the generating function of these state probabilities is indeed meromorphic.

2. Description of the model

We consider the two-dimensional stochastic process $\{z_n, n = 0, 1, 2, \ldots\}$ with state space $\mathscr{S}$,

$$z_n \equiv (x_n, y_n) \in \mathscr{S} := \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}.$$

For the characterisation of the structure of the z_n-process we introduce the following four sequences of stochastic vectors:

(i) for every fixed $k = 0, 1, 2, 3,$

$$(\xi_n^{(k)}, \eta_n^{(k)}), n = 0, 1, 2, \ldots, \text{is a sequence of i.i.d. stochastic vectors with } (\xi_n^{(k)}, \eta_n^{(k)}) \in \mathscr{S}; \tag{2.1}$$

(ii) the four families $\{(\xi_n^{(k)}, \eta_n^{(k)}),\ n = 0, 1, 2, \ldots\}$ are independent families.

The structure of the z_n-process is defined by the following recursive relations:

$$\begin{cases} \text{(i) } z_0 \equiv (x_0, y_0) \in \mathscr{S} \text{ is the starting point;} \\ \text{(ii) } x_{n+1} = [x_n - 1]^+ + \xi_n^{(k)}, \\ \qquad y_{n+1} = [y_n - 1]^+ + \eta_n^{(k)}, \end{cases} \tag{2.2}$$

with

$$\begin{aligned} k &= 3 \quad \text{for} \quad x_n > 0, \quad y_n > 0, \\ k &= 2 \quad \text{for} \quad x_n = 0, \quad y_n > 0, \\ k &= 1 \quad \text{for} \quad x_n > 0, \quad y_n = 0, \\ k &= 0 \quad \text{for} \quad x_n = 0, \quad y_n = 0, \end{aligned}$$

and

$$a^+ := \max(0, a) \text{ for } a \text{ real.}$$

We introduce the following notation and definitions.

(i) (ξ_k, η_k) indicates a stochastic vector with the same state space and the same distribution as $(\xi_n^{(k)}, \eta_n^{(k)})$, i.e.

$$(\xi_k, \eta_k) \sim (\xi_n^{(k)}, \eta_n^{(k)}), \quad k = 0, 1, 2, 3; \tag{2.3}$$

(ii) $\phi_k \equiv \phi_k(p_1, p_2) := E\{p_1^{\xi_k} p_2^{\eta_k}\}, \qquad |p_1| \le 1, |p_2| \le 1,$

$$\mu_k := E\{\xi_k\}, \qquad \nu_k := E\{\eta_k\}.$$

From (2.1) and (2.2) it is seen that the z_n-process is a discrete-time parameter Markov chain.

The class of nearest-neighbour random walks to be analysed in the present study is specified by taking

$$\begin{cases} \phi_0(p_1, p_2) = b_{10} p_1 + b_{01} p_2 + b_{11} p_1 p_2, \\ \phi_1(p_1, p_2) = h_{11} p_1^2 p_2 + h_{01} p_1 p_2 + h_{-1,1} p_2 + h_{10} p_1^2 + h_{-1,0}, \\ \phi_2(p_1, p_2) = w_{11} p_1 p_2^2 + w_{10} p_1 p_2 + w_{1,-1} p_1 + w_{01} p_2^2 + w_{0,-1}, \\ \phi_3(p_1, p_2) = r_{-1,1} p_2^2 + r_{-1,0} p_2 + r_{-1,-1} + r_{0,-1} p_1 + r_{1,-1} p_1^2, \end{cases} \tag{2.4}$$

with

$$\phi_k(1, 1) = 1, \qquad k = 0, 1, 2, 3,$$

and all coefficients in (2.4) non-negative.

Note that (2.2) and (2.4) imply that from a point (x, y) with $x > 0$, $y > 0$, no one-step transition can occur to the north, the north-east and the east.

We make the following assumptions.

Assumptions 2.1

$$\begin{cases} \text{(i)} & \mu_3 - 1 := r_{1,-1} - \{r_{-1,1} + r_{-1,0} + r_{-1,-1}\} < 0, \\ & \nu_3 - 1 := r_{-1,1} - \{r_{1,-1} + r_{0,-1} + r_{-1,-1}\} < 0; \\ \text{(ii)} & 0 < 4 r_{-1,1} r_{1,-1} < 1; \\ \text{(iii)} & \mu_1 - 1 - \nu_1 \dfrac{1 - \mu_3}{1 - \nu_3} < 1, \\ & \nu_1 - 1 - \mu_2 \dfrac{1 - \nu_3}{1 - \mu_3} < 1; \\ \text{(iv)} & h_{11} + h_{01} > 0 \text{ or } w_{11} + w_{10} > 0; \\ \text{(v)} & h_{01} > 0 \text{ whenever } h_{11} = h_{01} = 0, \\ & w_{10} > 0 \text{ whenever } w_{11} = w_{10} = 0. \end{cases} \tag{2.5}$$

Remark 2.1. Concerning (2.5) (ii) it is noted that $4 r_{-1,1} r_{1,-1} = 1$ implies $r_{-1,1} = r_{1,-1} = \frac{1}{2}$ since $0 \le r_{-1,1} \le 1$, $0 \le r_{1,-1} \le 1$, $r_{-1,1} + r_{1,-1} \le 1$; and so the second inequality in (2.5) (ii) is implied by (2.5) (i). Although the case $r_{1,-1} = r_{-1,1} = \frac{1}{2}$ is an interesting case, we shall discard it because its analysis requires a slightly different approach, similarly if $r_{-1,1} = 0$ or $r_{1,-1} = 0$. The condition (2.5) (iv) has been introduced to guarantee that any two states of $\mathscr{S}$ can be reached from each other with positive probability, so that the state space $\mathscr{S}$ is irreducible. The conditions (2.5) (v) have been introduced to restrict the number of variants which have to be considered in the analysis, see the derivations in Appendix C. However, if (2.5) (v) is not assumed, the required analysis does not change essentially.

Whenever the conditions (2.5) (i) hold and the state space $\mathscr{S}$ is irreducible then the conditions (2.5) (iii) are necessary and sufficient for the z_n-process to be positive recurrent, cf. Cohen [5], Section II.2.6, and hence they imply that the z_n-process possesses a unique stationary distribution. If (2.5) (i) is not introduced it is still possible that the z_n-process is positive recurrent, cf. [5]; however, we shall not discuss such cases here.

In the present study our interest lies in the study of the functional equation for the stationary distribution of the z_n-process. To formulate this equation let $(\boldsymbol{x}, \boldsymbol{y})$ be a stochastic vector with distribution the stationary distribution of the z_n-process.

Put

$$\Phi(p_1, p_2) := \boldsymbol{E}\{p_1^{\boldsymbol{x}} p_2^{\boldsymbol{y}}\}, \qquad |p_1| \leq 1, |p_2| \leq 1.$$

It is then easily derived (cf. [5], (II.4.1.3)) that for $|p_1| \leq 1$, $|p_2| \leq 1$,

$$\begin{aligned}(p_1 p_2 - \phi_3)\Phi(p_1, p_2)/\Phi(0,0) &= p_1 p_2 \phi_0 - \phi_3 + (p_2\phi_1 - \phi_3) p_1 \Omega_1(p_1) \\ &\quad + (p_1\phi_2 - \phi_3) p_2 \Omega_2(p_2),\end{aligned} \tag{2.6}$$

with

$$\begin{aligned}\Omega_1(p_1) &:= \frac{1}{p_1}\{\Phi(p_1, 0) - \Phi(0,0)\}/\Phi(0,0), \qquad |p_1| \leq 1, \\ \Omega_2(p_2) &:= \frac{1}{p_2}\{\Phi(0, p_2) - \Phi(0,0)\}/\Phi(0,0), \qquad |p_2| \leq 1, \\ \phi_j &\equiv \phi_j(p_1, p_2), \qquad j = 0, 1, 2, 3.\end{aligned} \tag{2.7}$$

From (2.6) and (2.7) it follows that for $j = 1, 2$,

(2.8) *$\Omega_j(p)$ is regular for $|p| < 1$, continuous for $|p| \leq 1$, and $0 \leq \Omega_j(1) \leq 1$.*

Put

$$\begin{aligned}\omega_2(p_1, p_2) &:= p_2 - \phi_2(p_1, p_2), \\ \omega_1(p_1, p_2) &:= p_1 - \phi_1(p_1, p_2), \\ \omega_0(p_1, p_2) &:= 1 - \phi_0(p_1, p_2).\end{aligned} \tag{2.9}$$

By $(\hat{p}_1, \hat{p}_2)$ we shall denote a zero of the kernel

$$Z(p_1, p_2) := \phi_3(p_1, p_2) - p_1 p_2, \tag{2.10}$$

so

$$Z(\hat{p}_1, \hat{p}_2) = 0.$$

Writing

$$\hat{\omega}_j := \omega_j(\hat{p}_1, \hat{p}_2), \qquad j = 0, 1, 2, \tag{2.11}$$

we obtain from (2.6) the functional equation

$$\hat{\omega}_1 \Omega_1(\hat{p}_1) + \hat{\omega}_2 \Omega_2(\hat{p}_2) + \hat{\omega}_0 = 0, \tag{2.12}$$

because by definition

$$|\Phi(\hat{p}_1, \hat{p}_2)| \leq 1, \qquad |\hat{p}_1| \leq 1, \quad |\hat{p}_2| \leq 1. \tag{2.13}$$

Further, this functional equation applies for every zero $(\hat{p}_1, \hat{p}_2)$ of $Z(p_1, p_2)$ with $|p_1| \leq 1$, $|p_2| \leq 1$, cf. (2.10).

The analysis of the functional equation (2.13) for the conditions (2.8) is the main goal of the present study.

3. On the analysis of the functional equation

In this section we derive some properties of the solution of the functional equation (2.12) which satisfies (2.8). Zeros $(\hat{p}_1, \hat{p}_2)$ of the kernel $Z(p_1, p_2)$ are analysed in Appendix A, cf. Lemma A.2. For every $\hat{p}_2$ and $\hat{p}_1$ given by

$$\hat{p}_1 = \frac{1}{2r_{1,-1}} \{\hat{p}_2 - r_{0,-1} \pm (1 - 4r_{1,-1}r_{-1,1})^{\frac{1}{2}} \sqrt{(\hat{p}_2 - \delta_{21})(\hat{p}_2 - \delta_{22})}\}, \tag{3.1}$$

$(\hat{p}_1, \hat{p}_2)$ is a zero of $Z(p_1, p_2)$; here the two signs correspond with $\hat{p}_1 = P_{12}(\hat{p}_2)$, $\hat{p}_1 = P_{11}(\hat{p}_2)$, cf. (A.8) of Appendix A, and δ_{21}, δ_{22} are the branch points of $\hat{p}_1$ as a function of $\hat{p}_2$. Lemma A.2 states that for $|\hat{p}_2| = 1$ the zero $P_{12}(\hat{p}_2)$ lies in $|p_2| \leq 1$, the other one $P_{11}(\hat{p}_2)$ is in $|p_1| > 1$. Note that $-1 < \delta_{21} < \delta_{22} < 1$, cf. (A.5). Put

$$\mathscr{G} := \{p : \delta_{21} \leq p \leq \delta_{22}\}, \qquad \mathscr{H} := \{p : |p| \leq 1\} \backslash \mathscr{G}, \tag{3.2}$$

and note that each of the functions in (3.1) is regular in $\mathscr{H}$ and continuous in the closure $\bar{\mathscr{H}}$ of $\mathscr{H}$. Note also that $\omega_j(p_1, p_2), j = 0, 1, 2$, cf. (2.9), are polynomials in p_1 and p_2, and hence $\omega_j(\hat{p}_1, \hat{p}_2)$ is regular in $\hat{p}_2 \in \mathscr{H}$, continuous in $\hat{p}_2 \in \bar{\mathscr{H}}$.

For $|\hat{p}_2| = 1$ and $\hat{p}_1 = P_{12}(\hat{p}_2)$, so $|P_{12}(\hat{p}_2)| \leq 1$, cf. (A.8), the functional equation (2.12) reads

$$\hat{\omega}_1 \Omega_1(P_{12}(\hat{p}_2)) + \hat{\omega}_2 \Omega_2(\hat{p}_2) + \hat{\omega}_0 = 0. \tag{3.3}$$

In (3.3) $\hat{\omega}_j \equiv \omega_j(P_{12}(\hat{p}_2), \hat{p}_2)$ and $\Omega_2(\hat{p}_2)$ are regular in $\hat{p}_2 \in \mathscr{H}$, and continuous in $\hat{p}_2 \in \bar{\mathscr{H}}$, cf. (2.8). Consequently it follows from (3.3) that $\Omega_1(P_{12}(\hat{p}_2))$ can be continued analytically from $|\hat{p}_2| = 1$ into $\hat{p}_2 \in \mathscr{H}$, and this continuation is continuous in $\bar{\mathscr{H}}$. Because the coefficients in $\omega_j(p_1,p_2)$ are all real, cf. (2.4) and (2.9), and the coefficients in the series expansions of $\Omega_1(p)$, and of $\Omega_2(p)$ are all non-negative, cf. (2.7), and have a sum bounded by 1, it follows that (3.3) may be rewritten for $\hat{p}_2 \in \bar{\mathscr{H}}$ as

$$\psi_1(\hat{p}_2) + \psi_2(\hat{p}_2)\sqrt{(\hat{p}_2 - \delta_{21})(\hat{p}_2 - \delta_{22})} = 0, \tag{3.4}$$

with $\psi_1(\hat{p}_2)$, $\psi_2(\hat{p}_2)$ convergent power series in $\hat{p}_2$ with real coefficients.

For $\hat{p}_2 \in (\delta_{21}, \delta_{22})$ the square root in (3.4) is purely imaginary and since for such $\hat{p}_2$, $\psi_1(\hat{p}_2)$ and $\psi_2(\hat{p}_2)$ are both real, because they are power series with real coefficients, it follows by continuity from (3.4) that

$$\psi_1(p) = 0, \quad \psi_2(p) = 0 \quad \textit{for} \quad p \in (\delta_{21}, \delta_{22}). \tag{3.5}$$

From (3.1) it is seen that $P_{12}(\hat{p}_2)$ and $P_{11}(\hat{p}_2)$ are each other's complex conjugates for $\hat{p}_2 \in (\delta_{21}, \delta_{22})$ and so (3.5) implies that (3.4) also holds for $\hat{p}_2 \in (\delta_{21}, \delta_{22})$ with the plus sign replaced by a minus sign, that is we have for $\hat{p}_2 \in (\delta_{21}, \delta_{22})$,

$$\hat{\omega}_1\Omega_1(P_{11}(\hat{p}_2)) + \hat{\omega}_2\Omega_2(\hat{p}_2) + \hat{\omega}_0 = 0, \tag{3.6}$$

with

$$\hat{\omega}_j = \omega_j(P_{11}(\hat{p}_2), \hat{p}_2), \quad j = 0, 1, 2.$$

Using the same arguments as above it is seen that (3.6) can be continued analytically from $\hat{p}_2 \in (\delta_{21}, \delta_{22})$ into $\mathscr{H}$. It follows that the relations (3.5) are equivalent, for $|p_2| \leq 1$, to

$$\omega_1(P_{12}(p_2), p_2)\Omega_1(P_{12}(p_2)) + \omega_2(P_{12}(p_2), p_2)\Omega_2(p_2) + \omega_0(P_{12}(p_2), p_2) = 0, \tag{3.7}$$

$$\omega_1(P_{11}(p_2), p_2)\Omega_1(P_{11}(p_2)) + \omega_2(P_{11}(p_2), p_2)\Omega_2(p_2) + \omega_0(P_{11}(p_2), p_2) = 0. \tag{3.8}$$

Analogously, we have, cf. Remark A.1, for $|p_1| \leq 1$,

$$\omega_1(p_1, P_{21}(p_1))\Omega_1(p_1) + \omega_2(p_1, P_{21}(p_1))\Omega_2(P_{21}(p_1)) + \omega_0(p_1, P_{21}(p_1)) = 0, \tag{3.9}$$

$$\omega_1(p_1, P_{22}(p_1))\Omega_1(p_1) + \omega_2(p_1, P_{22}(p_1))\Omega_2(P_{22}(p_1)) + \omega_0(p_1, P_{22}(p_1)) = 0. \tag{3.10}$$

Because $\omega_j(p_1,p_2)$, $j = 0$, 1, 2, are polynomials in p_1 and p_2, and $P_{12}(p)$ and $P_{11}(p)$ are regular in the entire finite p-plane slit along $\mathscr{G}$, it is seen that all the coefficients in (3.7) and (3.8) are regular in this slit p-plane. From Lemma A.2 we have $P_{11}(1) > 1$ and since all coefficients in (3.8) are real for $p_2 = 1$ and $\Omega_1(p)$ is a power series in p with non-negative coefficients, it follows that $\Omega_1(p_1)$ has an analytic continuation in $|p_1| < P_{11}(1)$. Analogously, $\Omega_2(p_2)$ has such a continuation in $|p_2| < P_{21}(1)$, cf. (A.11), and $\Omega_1(P_{11}(1))$, $\Omega_2(P_{21}(1))$, are both finite. So the relations (3.7) and (3.8) hold for $|p_2| \leqq P_{21}(1)$, and (3.9) and (3.10) hold for $|p_1| \leq P_{11}(1)$, i.e. the domain of validity of (3.7) and (3.8) has been extended by analytic continuation; similarly for (3.9) and (3.10). Actually, the relations (3.7) and (3.8) are linear in $\Omega_1(\cdot)$, $\Omega_2(\cdot)$, and their coefficients are all regular in the p_2-plane slit along $\mathscr{G}$, and so by analytic continuation it is seen as above that $\Omega_1(p)$ and $\Omega_2(p)$ possess analytic continuations for $|p| > 1$, except possibly at those points p where the coefficients of $\Omega_2(p)$ in (3.7) and $\Omega_1(p)$ in (3.10) are zero; at such points $\Omega_1(\cdot)$ and/or $\Omega_2(\cdot)$ may have poles, and these poles may generate other poles via (3.8) and (3.10); see Section 4. Branching points cannot occur, since $P_{12}(p)$, $P_{11}(p)$, are regular in the slit p-plane and the relations (3.7) and (3.8) are linear in $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$. Consequently, it follows that $\Omega_1(p)$ and $\Omega_2(p)$ which are regular for $|p| < 1$, cf. (2.8), are meromorphic functions for $|p| > 1$, if their singularities, i.e. their poles, do not have a finite accumulation point, and this is actually the case, as will be shown, cf. (C.2) of Appendix C and (6.5). Note that a function is meromorphic if it is regular in the finite complex plane except for, at most, a finite number of singularities in every bounded domain, these singularities being simple or multiple poles.

Remark 3.1. Because the z_n-process is positive recurrent and the state space $\mathscr{S}$ is irreducible (cf. Remark 2.1), $\Omega_1(p)$ and $\Omega_2(p)$ cannot be polynomials.

Remark 3.2. From the derivations above it is readily seen that the set of relations (3.7) and (3.8) is equivalent to the set (3.9) and (3.10). These sets are obtained from each other by analytic continuation.

4. On the determination of the poles and residues

In the preceding section it has been shown that $\Omega_1(p)$ and $\Omega_2(p)$ should be meromorphic functions with poles located in $|p| > 1$, cf. (2.8). In this section we discuss the location of these poles and derive relations for the residues at these poles.

Because $\Omega_1(p)$ and $\Omega_2(p)$ are meromorphic and the coefficients in (3.7) and (3.8) are regular for $|p| \geq 1$, the principle of permanence, cf. [3], p. 106, implies that (3.7) and (3.8) hold for $|p| \geq 1$; i.e. for $|p| \geq 1$,

(4.1) $$\omega_1(P_{12}(p),p)\Omega_1(P_{12}(p)) + \omega_2(P_{12}(p),p)\Omega_2(p) + \omega_0(P_{12}(p),p) = 0,$$

(4.2) $$\omega_1(P_{11}(p),p)\Omega_1(P_{11}(p)) + \omega_2(P_{11}(p),p)\Omega_2(p) + \omega_0(P_{11}(p),p) = 0.$$

For the present, we consider the case, cf. (2.5) (iv),

(4.3) $$w_{11} + w_{01} > 0;$$

the discussion of the other case of (2.5) (iv) is similar. Hence Lemma B.1 (ii) guarantees the existence of a zero $\sigma^{(1)}$ such that

(4.4) $$\omega_2(P_{12}(\sigma^{(1)}),\sigma^{(1)}) = 0, \qquad |\sigma^{(1)}| > 1.$$

This zero $\sigma^{(1)}$ generates recursively the two sequences, cf. Remarks A.2 and A.3,

(4.5) $$\{\sigma^{(i)}, i=1,2,\ldots\} \quad \text{and} \quad \{\tau^{(i)}, i=1,2,\ldots\},$$

with

$$\tau^{(i)} := P_{11}(\sigma^{(i)}) \quad \text{and} \quad \sigma^{(i+1)} := P_{21}(\tau^{(i)}).$$

From (A.8), (A.11) and Remarks A.2 and A.3, it is seen that

$$1 < |\sigma^{(1)}| < |\tau^{(1)}| < |\sigma^{(2)}| < \cdots < |\sigma^{(i)}| < |\tau^{(i+1)}| < |\sigma^{(i+1)}| < \ldots,$$

(4.6) $$\sigma^{(i)} = P_{22}(\tau^{(i)}), \quad \tau^{(i)} = P_{12}(\sigma^{(i+1)}),$$

$\sigma^{(i)}$ *and* $\tau^{(i)}$ *are all positive or are all negative.*

Put

(4.7) $$A_2(p) := \omega_1(P_{12}(p), p)\Omega_1(P_{12}(p)) + \omega_0(P_{12}(p), p),$$

and suppose for the present (cf. Remarks 4.1 and 6.5 below) that

(4.8) $$0 \neq |A_2(\sigma^{(1)})| < \infty.$$

It then follows from (4.1) and (4.8) since $\sigma^{(1)}$ is a simple zero (cf. (4.4) and Lemma B.1 (ii)), that

(4.9) $$p = \sigma^{(1)} \text{ is a simple pole of } \Omega_2(p).$$

Consequently, if

(4.10) $$\omega_1(P_{11}(p), p) \neq 0 \quad \text{for} \quad p = \sigma^{(1)},$$

then (4.2) implies that

(4.11) $$p = \tau^{(1)} = P_{11}(\sigma^{(1)}) \text{ is a simple pole of } \Omega_1(p).$$

If, however

(4.12) $$\omega_1(P_{11}(\sigma^{(1)}, \sigma^{(1)})) \equiv \omega_1(\tau^{(1)}, P_{22}(\tau^{(1)})) = 0,$$

then $p = \tau^{(1)}$ is a pole with multiplicity 2 of $\Omega_1(p)$.

Suppose that (4.10) holds and that

(4.13) $$\omega_1(P_{12}(p), p) \neq 0 \quad \text{for} \quad p = \sigma^{(2)},$$

then, cf. (4.1),

(4.14) $$\Omega_2(p) \text{ has a simple pole at } p = \sigma^{(2)},$$

since

(4.15) $$\omega_2(P_{12}(\sigma^{(2)}), \sigma^{(2)}) \neq 0.$$

This relation (4.15) holds, because (4.3) and Lemma B.1 (ii) imply that $\omega_2(P_{12}(p),p)$ has only one zero in $|p|>1$ if $w_{11}=0$, whereas if $w_{11}>0$ it has two such zeros but with different signs and this contradicts (4.6).

If

$$\omega_1(P_{12}(\sigma^{(2)}),\sigma^{(2)})=0, \tag{4.16}$$

then

$$\Omega_2(p) \text{ has no pole at } p=\sigma^{(2)}. \tag{4.17}$$

It is seen that if $\Omega_2(p)$ has a pole at $\sigma^{(2)}$ then, starting from (4.11) with $p=\sigma^{(2)}$ instead of $p=\sigma^{(1)}$, analogous conclusions follow. So by starting with $\sigma^{(1)}$ it is seen that $\sigma^{(1)}$ may generate sequences of poles of $\Omega_1(p)$ and $\Omega_2(p)$, however, we have to consider (4.8), (4.10) and (4.13) in more detail. See the following remark.

Remark 4.1. As noted in Remark A.3 we may complete the sequences (4.5) by $\tau^{(0)}$, $\sigma^{(0)}$, $\tau^{(-1)}$, $\sigma^{(-1)}, \ldots$, so that

$$|\tau^{(-n)}|<1<|\sigma^{(-n+1)}|<\cdots<|\tau^{(-1)}|<|\sigma^{(0)}|<|\tau^{(0)}|<|\sigma^{(1)}|<|\tau^{(1)}|<\ldots, \tag{4.18}$$

with n finite, supposing that a τ-element is the first one which becomes less than 1 in absolute value; if it is a σ-element only minor changes are needed in the following considerations.

If $n=0$ so that

$$\Omega_1(P_{12}(\sigma^{(1)}))=\Omega_1(\tau^{(0)}), \qquad |\tau^{(0)}|<1,$$

then, cf. (2.8), $|\Omega_1(\tau^{(0)})|<1$, i.e. $A_2(\sigma^{(1)})$ is finite, and so $\Omega_2(p_2)$ has a simple pole at $p_2=\sigma^{(1)}$ if $A_2(\sigma^{(1)})\neq 0$; otherwise, if $A_2(\sigma^{(1)})=0$, $\sigma^{(1)}$ does not generate a pole of $\Omega_1(p)$.

If $n>1$ then consider (4.1) for $p=\sigma^{(0)},\ldots,\sigma^{(-n+1)}$. Whenever for these p the coefficient in (4.1)

$$\omega_1(P_{12}(p),p)\neq 0, \tag{4.19}$$

then $\Omega_1(P_{12}(p_2))$ and $\Omega_2(p_2)$ have no poles for these values of p and so $A_2(\sigma^{(1)})$ is finite; note that $\omega_2(P_{12}(p),p)\neq 0$ for those p, cf. the proof of (4.15).

Further if

$$\omega_1(P_{12}(p),p)\neq 0 \quad \text{for all } p=\sigma^{(i+1)}, \qquad i=1,2,\ldots, \tag{4.20}$$

then, since $\omega_2(P_{12}(p),p)\neq 0$ for $p=\sigma^{(i+1)}$, $i=1,2,\ldots$, cf. (4.15), it follows from (4.1) that if $A_2(\sigma^{(1)})=0$ then all $\Omega_1(P_{12}(p_2))$ and all $\Omega_2(p_2)$ are finite for $p=\sigma^{(i+1)}$, $i=1,2,\ldots$, and so $\sigma^{(1)}$ does not generate sequences of poles for $\Omega_1(p)$ and $\Omega_2(p)$. Concerning $A_2(\sigma^{(1)})\neq 0$, see Remark 6.5 below.

The analysis in the following sections is based on the following assumption.

Assumption 4.1. For $i = -n+1,\ -n+2, \dots, 0, 1, 2, \dots,$ cf. (4.5) and (4.18), assume that

$$\text{(4.21)}\qquad \begin{aligned} &\text{(i)}\quad \omega_1(\tau^{(i)}, P_{22}(\tau^{(i)})) \equiv \omega_1(P_{11}(\sigma^{(i)}), \sigma^{(i)}) \neq 0,\\ &\text{(ii)}\quad \omega_1(\tau^{(i)}, P_{21}(\tau^{(i)})) \equiv \omega_1(P_{12}(\sigma^{(i+1)}), \sigma^{(i+1)}) \neq 0. \end{aligned}$$

Remark 4.2. It is readily seen that (4.21) (i) excludes the case where poles with multiplicity larger than 1 do occur, cf. (4.12), whereas (4.21) (ii) disregards the case with a finite number of poles generated by $\sigma^{(1)}$, cf. (4.16). From the definition of the $\sigma^{(i)}$ and $\tau^{(i)}$ it is readily seen that, in general, Assumption 4.1 will hold.

From the discussion above it follows for the case (4.3) if $A(\sigma_1) \neq 0$, cf. (4.7) and Remark 4.1, that

$$\begin{aligned} &\Omega_2(p) \text{ has a simple pole at } p = \sigma^{(1)}, &&\text{cf. (4.9)},\\ &\Omega_1(p) \text{ has a simple pole at } p = \tau^{(1)}, &&\text{cf. (4.11)},\\ &\Omega_2(p) \text{ has a simple pole at } p = \sigma^{(2)}, &&\text{cf. (4.14)}, \end{aligned}$$

and, generally,

$$\text{(4.22)}\qquad \begin{cases} \Omega_2(p) \text{ has a simple pole at } p = \sigma^{(i)}, & i = 1, 2, \dots,\\ \Omega_1(p) \text{ has a simple pole at } p = \tau^{(i)}, & i = 1, 2, \dots; \end{cases}$$

Assumption 4.1 excludes the case of poles with multiplicity larger than 1.

Next we start with the determination of the residues at the various poles.

For $i = 1, 2, \dots,$ put

$$\text{(4.23)}\qquad \begin{aligned} R_1(\tau^{(i)}) &:= \lim_{p \to \tau^{(i)}} (p - \tau^{(i)})\Omega_1(p),\\ R_2(\sigma^{(i)}) &:= \lim_{p \to \sigma^{(i)}} (p - \sigma^{(i)})\Omega_2(p). \end{aligned}$$

It follows from (4.1) with $A_2(\sigma^{(1)}) \neq 0$, note that $|A_2(\sigma^{(1)})| < \infty$ (cf. Remark 4.1), that since $\sigma^{(1)}$ is a simple pole of $\Omega_2(p)$,

$$\text{(4.24)}\qquad R_2(\sigma^{(1)}) = -A_2(\sigma^{(1)}) \left[\frac{d}{dp}\omega_2(P_{12}(p), p)\right]^{-1}_{p=\sigma^{(1)}}.$$

Further, from (4.2),

$$\Omega_1(P_{11}(p)) = -\frac{\omega_2(P_{11}(p), p)}{\omega_1(P_{11}(p), p)}\Omega_2(p) - \frac{\omega_0(P_{11}(p), p)}{\omega_1(P_{11}(p), p)},$$

and so

$$\text{(4.25)}\qquad R_1(\tau^{(1)}) = -\left\{\frac{\omega_2(P_{11}(p), p)}{\omega_1(P_{11}(p), p)}\left[\frac{d}{dp}P_{11}(p)\right]\right\}_{p=\sigma^{(1)}} R_2(\sigma^{(1)}).$$

Analogously, it follows from (4.1) that

$$R_2(\sigma^{(2)}) = -\left\{\frac{\omega_1(P_{12}(p),p)}{\omega_2(P_{12}(p),p)}\left[\frac{d}{dp}P_{12}(p)\right]^{-1}\right\}_{p=\sigma^{(2)}} R_1(\tau^{(1)}). \tag{4.26}$$

Generally, we have for $i = 1,2,\ldots,$

$$\begin{aligned} R_1(\tau^{(i)}) &= -\left\{\frac{\omega_2(P_{11}(p),p)}{\omega_1(P_{11}(p),p)}\left[\frac{d}{dp}P_{11}(p)\right]\right\}_{p=\sigma^{(i)}} R_2(\sigma^{(i)}), \\ R_2(\sigma^{(i+1)}) &= -\left\{\frac{\omega_1(P_{12}(p),p)}{\omega_2(P_{12}(p),p)}\left[\frac{d}{dp}P_{12}(p)\right]^{-1}\right\}_{p=\sigma^{(i+1)}} R_1(\tau^{(i)}). \end{aligned} \tag{4.27}$$

Remark 4.3. Because $P_{12}(p)$ and $P_{11}(p)$ are regular functions of p for $|p| > 1$, it is seen from (4.5) that the derivatives in (4.26) and (4.27) are all non-zero and finite.

5. Definition of meromorphic functions

Let σ with $|\sigma| > 1$ be a zero of

$$\omega_2(P_{12}(p),p), \tag{5.1}$$

assuming that such a zero exists, cf. Lemma B.1. Denote by $S_2(\sigma)$ the sequence

$$S_2(\sigma) := \{\sigma^{(1)},\tau^{(1)},\sigma^{(2)},\tau^{(2)},\ldots\}, \tag{5.2}$$

with, cf. (4.5),

$$\sigma^{(1)} = \sigma, \qquad \tau^{(i)} := P_{11}(\sigma^{(i)}), \qquad \sigma^{(i+1)} := P_{21}(\tau^{(i)}), \qquad i = 1,2,\ldots. \tag{5.3}$$

Analogously, let τ, $|\tau| > 1$, be a zero of

$$\omega_1(p, P_{22}(p)), \tag{5.4}$$

and define the sequence

$$S_1(\tau) := \{\tau^{(1)},\sigma^{(1)},\tau^{(2)},\sigma^{(2)},\ldots\}, \tag{5.5}$$

with

$$\tau^{(1)} = \tau, \qquad \sigma^{(i)} := P_{21}(\tau^{(i)}), \qquad \tau^{(i+1)} := P_{11}(\sigma^{(i)}), \qquad i = 1,2,\ldots. \tag{5.6}$$

The sequence $S_2(\sigma)$ generates the meromorphic functions

$$M_2(p|\sigma) := \sum_{i=1}^{\infty}\frac{R_{2\sigma}(\sigma^{(i)})}{p-\sigma^{(i)}}\left[\frac{p}{\sigma^{(i)}}\right]^{m_{2\sigma}}, \tag{5.7}$$

$$M_1(p|\sigma) := \sum_{i=1}^{\infty}\frac{R_{1\sigma}(\tau^{(i)})}{p-\tau^{(i)}}\left[\frac{p}{\tau^{(i)}}\right]^{m_{1\sigma}}, \tag{5.8}$$

(5.9)
$$\begin{cases} \text{(i)} & \text{where for each } i,\ R_{2\sigma}(\sigma^{(i)}) \text{ and } R_{1\sigma}(\tau^{(i)}) \text{ are defined as } R_2(\sigma^{(i)}) \text{ and } R_1(\tau^{(i)}) \text{ in (4.27), the index } \sigma \text{ has been incorporated in the notation to indicate that the series in (5.7) and (5.8) are generated by the sequence } S_2(\sigma), \text{ cf. (5.2); and} \\ \text{(ii)} & m_{2\sigma} \text{ and } m_{1\sigma} \text{ are the smallest non-negative integers for which the series} \end{cases}$$

(5.10)
$$\sum_{i=1}^{\infty}\frac{R_{2\sigma}(\sigma^{(i)})}{[\sigma^{(i)}]^{m_{2\sigma}+1}} \quad \text{and} \quad \sum_{i=1}^{\infty}\frac{R_{1\sigma}(\tau^{(i)})}{[\tau^{(i)}]^{m_{1\sigma}+1}}, \text{ respectively,}$$

converge absolutely.

Because of (C.2), (C.3) and the existence of the limits in (C.11) and (C.12) it is readily seen that $m_{2\sigma}$ and $m_{1\sigma}$ are always well defined for the sequence $S_2(\sigma)$; for details see Remark 6.1 below.

Analogously, the sequence $S_1(\tau)$, cf. (5.5), generates the meromorphic functions

(5.11)
$$M_1(p|\tau) := \sum_{i=1}^{\infty}\frac{R_{1\tau}(\tau^{(i)})}{p-\tau^{(i)}}\left[\frac{p}{\tau^{(i)}}\right]^{m_{1\tau}},$$

(5.12)
$$M_2(p|\tau) := \sum_{i=1}^{\infty}\frac{R_{2\tau}(\sigma^{(i)})}{p-\sigma^{(i)}}\left[\frac{p}{\sigma^{(i)}}\right]^{m_{2\tau}}.$$

Note that here the $\tau^{(i)}$ and $\sigma^{(i)}$ are different from those in (5.7), (5.8).

The functions in (5.7), (5.8), (5.11) and (5.12) are well defined in the sense that they converge uniformly and absolutely in every finite circle $|p| \le R$, with $R > 1$, whenever the terms with poles inside the circle with radius R are deleted from the sum, cf. [16], p. 309, [3], p. 219.

In Appendix C it is shown, cf. (C.18), that

(5.13)
$$m_{2\sigma} = m_{1\sigma}, \qquad m_{2\tau} = m_{1\tau},$$

and so we can delete the indices 2 and 1 and write

(5.14)
$$m_\sigma := m_{2\sigma} = m_{1\sigma}, \qquad m_\tau := m_{2\tau} = m_{1\tau}.$$

6. Solution of the functional equation

In this section we construct the solution of the functional equation (2.12), i.e. we show how $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are determined.

From Lemma B.1 it is seen that $\omega_2(P_{12}(p),p)$ has at most two zeros in $|p| > 1$, say, σ_{21} and σ_{22}, and, similarly, $\omega_1(p, P_{22}(p))$ has at most two zeros in $|p| > 1$, say, τ_{11} and τ_{12}. Assumption (2.5) (iv) guarantees that at least one of these possible four zeros exists. In the subsequent analysis we shall always use these four zeros;

if, however, a zero does not exist then all symbols referring to that non-existing zero should be *deleted* from the text, cf. Remark 6.4 below.

For the present, assume that

(6.1) $$A_2(\sigma_{21}) \neq 0, A_2(\sigma_{22}) \neq 0 \quad \text{and} \quad A_1(\tau_{11}) \neq 0, A_1(\tau_{12}) \neq 0,$$

cf. (4.8) and Remark 4.1, with

(6.2) $$A_1(p) := \omega_1(p, P_{22}(p))\Omega_2(P_{22}(p)) + \omega_0(p, P_{22}(p)).$$

Remark 6.1. For the relevant alterations to be made in the subsequent analysis if (6.1) does not hold, see Remark 6.5 below.

Each of the zeros σ_{21} and σ_{22} generates a sequence of the type defined in (5.2), and similarly so do the zeros τ_{11} and τ_{12}, cf. (5.5). Denote these sequences by

(6.3) $$S_2(\sigma_{21}), S_2(\sigma_{22}) \quad \text{and} \quad S_1(\tau_{11}), S_2(\tau_{12});$$

and assume that (4.21) applies for the elements of $S_2(\sigma_{21})$ and those of $S_2(\sigma_{22})$. The analogous assumption is made for the elements of $S_1(\tau_{11})$, as well as for those of $S_1(\tau_{12})$.

For each of these four sequences we construct a pair of meromorphic functions, cf. (5.7) and (5.8) for σ_{21} and σ_{22}, and (5.11) and (5.12) for τ_{11} and τ_{12}; note that here (6.1) is used, cf. Remark 4.1. These pairs of meromorphic functions are denoted by

(6.4) $$\begin{aligned} &\{M_2(p|\sigma_{21}), M_1(p|\sigma_{21})\}, \qquad \{M_2(p|\sigma_{22}), M_1(p|\sigma_{22})\}, \\ &\{M_1(p|\tau_{11}), M_2(p|\tau_{11})\}, \qquad \{M_1(p|\tau_{12}), M_2(p|\tau_{12})\}. \end{aligned}$$

Put

(6.5) $$\begin{aligned} \Omega_2(p) &:= Q_2(p) + M_2(p), \\ \Omega_1(p) &:= Q_1(p) + M_1(p), \end{aligned}$$

where

(6.6) $$\begin{cases} \text{(i)} & M_2(p) := M_2(p|\sigma_{21}) + M_2(p|\sigma_{22}) + M_2(p|\tau_{11}) + M_2(p|\tau_{12}), \\ & M_1(p) := M_1(p|\sigma_{21}) + M_1(p|\sigma_{22}) + M_1(p|\tau_{11}) + M_1(p|\tau_{12}); \\ \text{(ii)} & Q_2(p) \text{ and } Q_1(p) \text{ are both polynomials in } p \text{ of degree } n_2 \text{ and } n_1 \\ & \text{respectively; these degrees will be specified below.} \end{cases}$$

Substitution of (6.5) into the functional equations (4.1) and (4.2) yields, for $|p| \geq 1$,

$$\omega_1(P_{1j}(p), p)Q_1(P_{1j}(p)) + \omega_2(P_{1j}(p), p)Q_2(p) + \omega_0(P_{1j}(p), p) = I_{1j}(p), \quad j = 1, 2,$$
(6.7)

where

$$(6.8)\qquad -I_{1j}(p) := \omega_1(P_{1j}(p),p)M_1(P_{1j}(p)) + \omega_2(P_{1j}(p),p)M_2(p), \qquad j=1,2.$$

Remark 6.2. Consider one of the sequences in (6.3), say $S_2(\sigma_{21})$ and put

$$(6.9)\qquad \begin{aligned} &\tau_0 := P_{12}(\sigma_{21}^{(1)}) \quad \text{with} \quad \sigma_{21}^{(1)} = \sigma_{21}, \\ &\tau_i := \tau_{21}^{(i)}, \qquad \sigma_i := \sigma_{21}^{(i)}, \qquad i = 1,2,\ldots . \end{aligned}$$

It then follows from (6.7) for $i = 1,2,\ldots,$

$$(6.10)\qquad \begin{aligned} &\omega_1(\tau_{i-1},\sigma_i)Q_1(\tau_{i-1}) + \omega_2(\tau_{i-1},\sigma_i)Q_2(\sigma_i) + \omega_0(\tau_{i-1},\sigma_i) = I_{12}(\sigma_i), \\ &\omega_1(\tau_i,\sigma_i)Q_1(\tau_i) + \omega_2(\tau_i,\sigma_i)Q_2(\sigma_i) + \omega_0(\tau_i,\sigma_i) = I_{11}(\sigma_i), \end{aligned}$$

which represents a set of linear (recursive) equations for the elements of the sequences

$$(6.11)\qquad \{Q_1(\tau_i),\quad i=0,1,\ldots\} \quad \text{and} \quad \{Q_2(\sigma_i),\quad i=1,2,\ldots\}.$$

It is seen that these sequences are uniquely determined when $I_{12}(\sigma_i)$, $I_{11}(\sigma_i)$ and $Q_2(\sigma_1)$ are known.

Put, cf. (2.4),

$$(6.12)\qquad \begin{aligned} s_1 &:= 3 \quad \text{for} \quad h_{11} > 0, \\ &:= 2 \quad \text{for} \quad h_{11} = 0, \qquad h_{01} + h_{10} > 0, \\ &:= 1 \quad \text{for} \quad h_{11} = 0, \qquad h_{01} + h_{10} = 0; \\ s_2 &:= 3 \quad \text{for} \quad w_{11} > 0, \\ &:= 2 \quad \text{for} \quad w_{11} = 0, \qquad w_{10} + w_{01} > 0, \\ &:= 1 \quad \text{for} \quad w_{11} = 0, \qquad w_{10} + w_{01} = 0; \\ s_0 &:= 2 \quad \text{for} \quad b_{11} > 0, \\ &:= 1 \quad \text{for} \quad b_{11} = 0. \end{aligned}$$

It is then readily seen that the following limits exist, cf. (2.4), (2.5), (2.9) and Lemma A.3,

$$\delta_{j,12} := \lim_{p\to\infty} p^{-s_j}\omega_j(P_{12}(p),p), \qquad \delta_{j,11} := \lim_{p\to\infty} p^{-s_j},\omega_j(P_{11}(p),p), \qquad j=0,1,2,$$
(6.13)

and their values are finite and non-zero, except possibly for $s_1 = 1$ and $s_2 = 1$ where they may be zero in special cases; note that (2.5) (iv) excludes $s_1 = s_2 = 1$.

Again we consider the sequence $S_2(\sigma_{21})$ and with the abbreviated notation introduced in (6.9) put, cf. (5.9) and (5.14),

(6.14) $$R_1(\cdot) := R_{1\sigma_{21}}(\cdot), \qquad R_2(\cdot) := R_{2\sigma_{21}}(\cdot), \qquad m := m_{\sigma_{21}},$$

and

$$-I_{12}^{(i)} p|\sigma_{21}) := \omega_2(P_{12}(p),p)\frac{R_2(\sigma_1)}{p-\sigma_1}\left[\frac{p}{\sigma_1}\right]^m, \qquad \text{for } i=1,$$

$$:= \omega_1(P_{12}(p),p)\frac{R_1(\tau_{i-1})}{P_{12}(p)-\tau_{i-1}}\left[\frac{P_{12}(p)}{\tau_{i-1}}\right]^m + \omega_2(P_{12}(p),p)\frac{R_2(\sigma_i)}{p-\sigma_i}\left[\frac{p}{\sigma_i}\right]^m$$

$$\text{for } i=2,3,\dots,$$

$$-I_{11}^{(i)}(p|\sigma_{21}) := \omega_1(P_{11}(p),p)\frac{R_1(\tau_i)}{P_{11}(p)-\tau_i}\left[\frac{P_{11}(p)}{\tau_i}\right]^m + \quad \omega_2(P_{11}(p),p)\frac{R_2(\sigma_i)}{p-\sigma_i}\left[\frac{p}{\sigma_i}\right]^m$$

$$\text{for } i=1,2,\dots .$$

(6.15)

Further we introduce

$$-I_{12}(p|\sigma_{21}) := -\sum_{i=1}^{\infty} I_{12}^{(i)}(p|\sigma_{21})$$

$$= \omega_1(P_{12}(p),p)M_1(P_{12}(p)|\sigma_{21}) + \omega_2(P_{12}(p),p)M_2(P_{12}(p)|\sigma_{21}),$$

$$-I_{11}(p|\sigma_{21}) := -\sum_{i=1}^{\infty} I_{11}^{(i)}(p|\sigma_{21})$$

(6.16) $$= \omega_1(P_{11}(p),p)M_1(P_{11}(p)|\sigma_{21}) + \omega_2(P_{11}(p),p)M_2(P_{11}(p)|\sigma_{21}).$$

The form (6.15) may be rewritten as:

$$\begin{cases} \text{(i)} \; I_{12}^{(i)}(p|\sigma_{21}) = \dfrac{-1}{p-\sigma_i}\left\{\dfrac{\omega_1(P_{12}(p),p)}{p^{s_1}}\dfrac{R_1(\tau_{i-1})}{\tau_{i-1}^m}\left[\dfrac{P_{12}(p)}{p}\right]^m \dfrac{p-\sigma_i}{P_{12}(p)-\tau_{i-1}}p^{m+s_1} \right. \\ \qquad \left. + \dfrac{\omega_2(P_{12}(p),p)}{p^{s_2}}\dfrac{R_2(\sigma_i)}{\sigma_i^m}p^{m+s_2}\right\}, \qquad i=2,3,\dots, \\ \text{(ii)} \quad I_{11}^{(i)}(p|\sigma_{21}) = \dfrac{-1}{p-\sigma_i}\left\{\dfrac{\omega_1(P_{11}(p),p)}{p^{s_1}}\dfrac{R_1(\tau_i)}{\tau_i^m}\left[\dfrac{P_{11}(p)}{p}\right]^m \dfrac{p-\sigma_i}{P_{11}(p)-\tau_i}p^{m+s_1} \right. \\ \qquad \left. + \dfrac{\omega_2(P_{11}(p),p)}{p^{s_2}}\dfrac{R_2(\sigma_i)}{\sigma_i^m}p^{m+s_2}\right\}, \qquad i=1,2,\dots, \\ \text{(iii)} \qquad I_{12}^{(1)}(p|\sigma_{21}) = -\dfrac{R_2(\sigma_i)}{\sigma_1^m}\dfrac{\omega_2(P_{12}(p),p)}{(p-\sigma_1)p^{s_2}}p^{m+s_2}. \end{cases}$$

(6.17)

Consider the term in parentheses in the first relation of (6.17). Because $P_{12}(p)$ is regular for $|p| > 1$, this term is also regular; note $\tau_{i-1} = P_{12}(\sigma_i)$. Consider the series expansion of this term in a neighbourhood of σ_i; note $|\sigma_i| > 1$. The second relation of (4.27) has actually been obtained from (4.1) by multiplying (4.1) by $p - \sigma_i$ and then letting $p \to \sigma_i$. Hence the second relation of (4.27) implies that this series expansion should contain the factor $p - \sigma_i$, i.e. $p = \sigma_i$ is a zero of the term inside parentheses of (6.17) (i). Hence, by using Assumption 4.1, Lemma A.3 and (6.13), it follows that the following limit exists and

$$(6.18)\qquad 0 < |\lim_{p\to\infty} p^{-n_{\sigma_{21}}} I_{12}^{(i)}(p|\sigma_{21})| < \infty, \qquad i = 2, 3, \dots,$$

where, cf. (6.14),

$$(6.19)\qquad n_{\sigma_{21}} := \max(m + s_1 - 1, m + s_2 - 1).$$

Similarly, from (6.17) (ii) and (iii),

$$(6.20)\qquad \begin{aligned} &0 < |\lim_{p\to\infty} p^{-n_{\sigma_{21}}} I_{11}^{(i)}(p|\sigma_{21})| < \infty, \qquad i = 1, 2, \dots, \\ &0 < |\lim_{p\to\infty} p^{-(m+s_2-1)} I_{12}^{(i)}(p|\sigma_{21})| < \infty. \end{aligned}$$

Note that the existence of the zero σ_{21} implies that $s_2 \geq 2$, cf. (6.13) and Lemma B.1.

It further follows that the functions defined in (6.15) are regular for all finite p with $|p| > 1$, so that the definition of m, cf. (6.14), implies for every finite $R > 1$, cf. (6.16),

$$(6.21)\qquad p^{-n_{\sigma_{21}}} I_{12}(p|\sigma_{21}) = -p^{-n_{\sigma_{21}}} \lim_{N\to\infty} \sum_{i=1}^{N} I_{12}^{(i)}(p|\sigma_{21}), \qquad |p| < R,$$

where the sum in (6.21) converges absolutely and uniformly in p for $1 < |p| < R$, to a finite limit for $N \to \infty$, which is uniformly bounded for $|p| < R$ (cf. the discussion in Appendix C below (C.13)). An analogous result applies for $I_{11}(p|\sigma_{21})$. Consequently,

(6.22) *$I_{12}(p|\sigma_{21})$ has a pole of order $n_{\sigma_{21}}$ at infinity ($p = \infty$), and similarly for $I_{11}(p|\sigma_{21})$.*

Lemma 6.1. *$I_{12}(p|\sigma_{21})$ and $I_{11}(p|\sigma_{21})$ are polynomials of degree $n_{\sigma_{21}}$.*

Proof. From the conclusions concerning the sum in (6.21) and the regularity of its terms, it is seen that $I_{12}(p|\sigma_{21})$ is regular for finite p and hence from (6.22) and Liouville's theorem it follows that $I_{12}(p|\sigma_{21})$ is a polynomial of degree $n_{\sigma_{21}}$; an analogous result applies for $I_{11}(p|\sigma_{21})$.

An analysis similar to that leading to (6.22) for the sequence $S_2(\sigma_{21})$, leads to analogous results for the other sequences in (6.3). Put

$$(6.23)\qquad \begin{cases} \text{(i)} & n_u := \max\,(m_u + s_1 - 1, m_u + s_2 - 1 | u \in (\sigma_{21}, \sigma_{22}, \tau_{11}, \tau_{12})), \\ \text{(ii)} & n(s_1, s_2) := \max\,(n_{\sigma_{21}}, n_{\sigma_{22}}, n_{\tau_{11}}, n_{\tau_{12}}, s_0), \end{cases}$$

the index u being that one for which the max occurs in (i). Further, cf. (6.6), (6.12), (6.16), (6.22),

$$\gamma_{1j} := \lim_{p\to\infty} p^{-n(s_1,s_2)}[\omega_1(P_{1j}(p),p)M_1(P_{1j}(p)) + \omega_2(P_{1j}(p),p)M_2(p) + \omega_0(P_{1j}(p),p)],$$
(6.24)

for $j = 1, 2$; note that these limits are finite and that (6.23) implies

$$(6.25)\qquad n(s_1, s_2) \geq s_j \geq 1, \qquad j = 1, 2.$$

Hence we obtain from the functional equations (6.7) by using (6.13), (6.24) and (6.25); for $|p| \to \infty$,

$$\delta_{j,11}Q_1(\alpha_{1j}p)p^{s_1} + \delta_{j,12}Q_2(p)p^{s_2} = \gamma_{1j}p^{n(s_1,s_2)} + O(p^{n(s_1,s_2)-1}), \qquad \text{for } j = 1, 2.$$
(6.26)

Note that (2.5) (iv) excludes the case $s_1 = s_2 = 1$, cf. (6.12).

From (6.25) and (6.26) it follows that the degrees n_2 and n_1 of the polynomials $Q_2(p)$ and $Q_1(p)$, cf. (6.6) (ii), are determined by

$$(6.27)\qquad n_j = n(s_1, s_2) - s_j, \qquad j = 1, 2.$$

Remark 6.3. In Remark C.2 it has been pointed out that if in (C.21) m is replaced by $m + h$, $h = 1, 2, \ldots$, then the convergence is maintained; and so in the definitions (5.7) and (5.8) we may also take as exponents $m_\sigma + h$ with a non-negative integer. In doing so it is readily seen that the degrees of Q_2 and Q_1 then become larger. Such a change in exponents in (5.7) and (5.8) implies that in the representation (6.5) a polynomial is substracted from $M_2(p)$ and added to $Q_2(p)$.

The relations (5.7), (5.8), (5.11) together with (6.5) and (6.6) characterise the structure of the functions $\Omega_1(p)$ and $\Omega_2(p)$; they also determine these functions uniquely, as will be shown below.

Because the degrees of the polynomials $Q_2(\cdot)$ and $Q_1(\cdot)$ have been determined, we need for the explicit determination of the coefficients of these polynomials a total of $n_1 + n_2 + 2$ linear equations. In Remark 6.2 it has been shown, by using the sequence $S_2(\sigma_{21})$ which generates the sets of the poles of the meromorphic functions $M_2(p|\sigma_{21})$, $M_1(p|\sigma_{21})$, that the values of the polynomial $Q_1(p)$ at the τ-points and those of $Q_2(p)$ at the σ-points of the sequence $S_2(\sigma_{21})$ can be expressed as linear combinations of $I_{12}(\sigma)$ and $I_{11}(\sigma)$ at the σ-points of $S_2(\sigma_{21})$.

In total we need only $n_1 + n_2 + 2$ of those relations. So $Q_2(\cdot)$ and $Q_1(\cdot)$ are known, whenever $I_{12}(p)$ and $I_{11}(p)$ are known; see Remark 6.5 below for their uniqueness. To show that these functions are completely determined, note that a pair of meromorphic functions $\{M_2(p|\sigma_{21}), M_1(p|\sigma_{21})\}$, cf. (5.7), (5.8), is determined by $S_2(\sigma_{21})$ apart from a factor, because all residues $R_{2\sigma_{21}}(\cdot)$ of $M_2(p|\sigma_{21})$ and those of $M_1(p|\sigma_{21})$ at their poles are linear functions of $R_{2\sigma_{21}}(\sigma_{21}^{(1)})$ with $\sigma_{21}^{(1)} = \sigma_{21}$, cf. (4.24) and (4.27); actually they are all proportional to $R_{2\sigma_{21}}(\sigma_{21}^{(1)})$. By using the expression (4.24) for the residue $R_{2\sigma_{21}}(\sigma_{21}^{(1)})$ it is seen that $M_2(p|\sigma_{21})$ and $M_1(p|\sigma_{21})$ are completely determined, apart from a factor which is a linear function of $\Omega_1(P_{12}(\sigma_{21}^{(1)}))$, on the assumption that $A_2(\sigma_{21}^{(1)}) \neq 0$, cf. Remark 4.1 and Remark 6.5 below. If $A_2(\sigma_{21}^{(1)}) = 0$ then $S_2(\sigma_{21})$ does not generate a pole set. Similarly for the other pairs of meromorphic functions in (6.4), i.e. it remains to determine

$$\begin{aligned} &\Omega_1(p) \quad \text{for} \quad p = P_{12}(\sigma_{21}^{(1)}) \quad \text{and} \quad p = P_{12}(\sigma_{22}^{(1)}), \\ &\Omega_2(p) \quad \text{for} \quad p = P_{22}(\tau_{11}^{(1)}) \quad \text{and} \quad p = P_{22}(\tau_{12}^{(1)}). \end{aligned} \tag{6.28}$$

Hence by using (6.7) and (6.8), four linear equations are obtained for the unknowns

$$\Omega_1(P_{12}(\sigma_{21}^{(1)})), \qquad \Omega_1(P_{12}(\sigma_{22}^{(1)})), \qquad \Omega_2(P_{21}(\tau_{11}^{(1)})), \qquad \Omega_2(P_{22}(\tau_{12}^{(1)})), \tag{6.29}$$

since, as shown above, the coefficients in the polynomials $Q_2(p)$ and $Q_1(p)$ depend linearly on the unknowns in (6.29).

Remark 6.4. It has already been mentioned at the beginning of the present section that if $\omega_2(P_{12}(p),p)$ and $\omega_1(p, P_{22}(p))$ have fewer than four zeros in $|p| > 1$, then only the sequences generated by the existing zeros occur, as do the functions derived from these sequences. It is then readily seen that for the number of remaining unknowns we are left with a similar number of linear equations, whenever the corresponding terms in (6.1) are non-zero, cf. Remark 6.5 below.

For the ultimate determination of $\Omega(p_1,p_2)$, cf. (2.6) and (2.7), it remains to determine $\Omega(0,0)$ since $\Omega_1(p)$ and $\Omega_2(p)$ have been constructed above. By taking $p_2 = 1$ in (2.6), dividing the resulting expression by $p_1 - 1$, taking note that all coefficients in (2.6) are zero for $p_1 = p_2 = 1$, we obtain for $p_2 \to 1$ a linear relation for $\Phi(0,0)$ because the norming condition requires $\Phi(1,1) = 1$; $\Phi(0,0)$ so calculated is unique and positive (see the following remark).

Remark 6.5. Apart from Assumption 4.1, which has been introduced for technical reasons, cf. Remark 4.2, our analysis is essentially based on Assumption 2.1. The conditions (2.5) (i), (iii) guarantee that the z_n-process has a unique stationary distribution and so $\Phi(p_1,p_2)$ should be regular for $|p_1| < 1$, continuous for $|p_1| \leq 1$, for every fixed p_2 with $|p_2| \leq 1$; and, similarly, with p_1 and p_2 interchanged. It is seen that $\Omega_1(p)$ and $\Omega_2(p)$ as defined in (6.5) satisfy the conditions (2.8), independently of the values of the unknowns in (6.30). The relations (4.1) and (4.2), or equivalently (6.5), stem from the requirement that zeros of the kernel $Z(p_1,p_2), |p_1| \leq 1, |p_2| \leq 1$, should be zeros of the right-hand side of (2.6), because of the boundedness of $\Phi(p_1,p_2)$ in $|p_1| \leq 1, |p_2| \leq 1$. From these relations and the regularity properties of $\Omega_1(p)$ and $\Omega_2(p)$ a set of linear equations for the coefficients in the polynomials $Q_1(p)$ and $Q_2(p)$ and the unknowns in (6.30) has been obtained, the number of unknowns and that of the equations being equal, independently of the number of zeros of $\omega_2(P_{12}(p),p)$ and of $\omega_1(p,P_{22}(p))$ in $|p| > 1$, cf. Remark 6.4, but there is at least one such zero, because of (2.5) (iv), see Lemma B.1. Because there is a unique $\Phi(p_1,p_2)$ satisfying (2.6) and the mentioned regularity conditions, the set of linear equations just mentioned should have a unique solution, and the same holds for the determination of $\Phi(0,0)$, cf. Remark 6.4. This uniqueness of $\Phi(p_1,p_2)$ leads also to the conclusion that at least one of the inequalities in (6.1) should apply for the zeros of $\omega_2(P_{12}(p),p)$ and of $\omega_1(p,P_{22}(p))$ in $|p| > 1$. If it turns out that for such a zero the relevant inequality in (6.1) does not hold then this zero does not generate a pole set $S(\cdot)$, cf. (6.3), and the inherent functions $M_2(p|\cdot)$, $M_1(p|\cdot)$ are identically zero. Actually the conditions in (6.1) can be only verified if the relevant unknowns in (6.30) have been solved from the linear equations; on the other hand it is evident from the analysis above that only incidentally one or more of the conditions in (6.1) are not satisfied.

Appendix A. On the zeros of the kernel

For the analysis of the functional equation (2.12) we need several properties of the zero of the kernel (2.10). These properties are derived in this appendix.

From (2.4) and (2.10) we have

$$\begin{aligned} Z(p_1,p_2) &= \phi_3(p_1,p_2) - p_1p_2 \\ &= r_{-1,1}p_2^2 + r_{-1,0}p_2 + r_{-1,-1} + r_{0,-1}p_1 + r_{1,-1}p_1^2 - p_1p_2. \end{aligned} \tag{A.1}$$

Generally, a zero of $Z(p_1,p_2)$ is indicated by $(\hat{p}_1,\hat{p}_2)$. It is readily verified that (cf. also (2.5))

$$\begin{aligned} &\hat{p}_1 = 1 \Rightarrow \text{either } \hat{p}_2 = 1 \quad \text{or} \quad \hat{p}_2 = 1 + \frac{1-\nu_3}{r_{-1,1}} > 0, \\ &\hat{p}_1 = 1 \Rightarrow \text{either } \hat{p}_1 = 1 \quad \text{or} \quad \hat{p}_1 = 1 + \frac{1-\mu_3}{r_{1,-1}} > 0. \end{aligned} \tag{A.2}$$

Denote by $D_3(p_2)$ the discriminant of the right-hand side of (A.1), considered as a quadratic in p_1, i.e.

$$\begin{aligned} D_3(p_2) &= (1-4r_{1,-1}r_{-1,1})p_2^2 - 2p_2(r_{0,-1} + 2r_{1,-1}r_{-1,0}) - 4r_{1,-1}r_{-1,-1} + r_{0,-1}^2 \\ &= (1-4r_{1,-1}r_{-1,1})(p_2-\delta_{21})(p_2-\delta_{22}), \end{aligned} \tag{A.3}$$

with

(A.4) $\qquad \delta_{21}, \delta_{22}$ the two zeros of $D_3(p_2)$, $\qquad |\delta_{21}| \le |\delta_{22}|$.

Lemma A.1

(A.5) $$-1 < \delta_{21} < \delta_{22} < 1.$$

Proof. From (A.3) we have

(A.6) $$D_3\left(\frac{1}{q}\right) = \frac{1}{q^2}\{(1 - r_{0,-1}q)^2 - 4r_{1,-1}(r_{-1,1} + r_{-10}q + r_{-1,-1}q^2)\}.$$

Obviously, (A.2) implies that

(A.7) $$D_3(1) > 0.$$

It is seen that $q^2D_3(1/q)$ decreases on [0, 1], and

$$q^2D_3\left(-\frac{1}{q}\right) > q^2D_3\left(\frac{1}{q}\right) \quad \text{for} \quad q \in [0, 1],$$

so $D_3(p_2)$ has no zero for $p_2^{-1} = q \in [-1, 1]$. Because $D_3(r_{0,-1}) < 0$ it follows, cf. (A.7), that $D_3(1/q)$ has two real zeros and so (A.5) follows.

Lemma A.2. *The two zeros* $P_{11}(p_2)$, $P_{12}(p_2)$ *of* $Z(p_1, p_2)$ *may be defined so that*

(A.8) $$|P_{12}(p)| < |p| < |P_{11}(p)| \quad \textit{for} \quad |p| \ge 1, \quad p \ne 1,$$
$$P_{12}(1) = 1 < P_{11}(1) = 1 + \frac{1 - \mu_3}{r_{1,-1}},$$

and $P_{11}(p)$, $P_{12}(p)$ *are both regular functions of* p *for* $|p| \ge 1$ *and can be continued analytically from* $|p| = 1$ *into* $\{p : |p| < 1, p \notin (\delta_{21}, \delta_{22})\}$.

Proof. Put $p_1 = zp_2$ then $Z(p_1, p_2) = 0$ implies, cf. (A.1),

(A.9) $$z = \boldsymbol{E}\{z^{\xi_3}p_2^{\xi_3+\eta_3-2}\}.$$

From (2.4) it follows that $\xi_3 + \eta_3 \le 2$, and (2.5) (i), (ii), imply

$$\Pr\{\xi_3 + \eta_3 = 2\} = r_{-1,1} + r_{1,-1} < 1.$$

Hence for $|p_2| > 1$, $p_2 \ne 1$, $|z| = 1$,

(A.10) $$|\boldsymbol{E}\{z^{\xi_3}p^{\xi_3+\eta_3-2}\}| < 1.$$

Because both sides of (A.9) are regular functions of z for $|z| \le 1$, Rouché's theorem, cf. [16], p. 155, shows that (A.9) has a unique zero in $|z| < 1$ for $|p_2| \ge 1$, $p_2 \ne 1$. For $p_2 = 1$, (A.9) has one root in $|z| \le 1$, viz. $z = 1$, cf. (A.2).

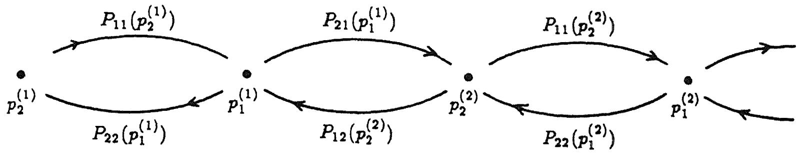

Figure 1

From (A.1) it is seen that (A.9) is a quadratic equation in z. Since for $|p_2| \geq 1$ it has exactly one root in $|z| \leq 1$, the other root is in $|z| > 1$. Take $P_{12}(p_2)$ for $|p_2| \geq 1$ as the zero of $Z(p_1, p_2)$ which corresponds to the zero in $|z| \leq 1$ and $P_{11}(p_2)$ as the zero corresponding to $|z| > 1$, then the relation (A.8) follows. Lemma A.1 implies that the branching points δ_{21} and δ_{22} of $P_{11}(p_2)$ and $P_{12}(p_2)$ are located in $(-1, 1)$, and so the zeros $P_{11}(p_2)$ and $P_{12}(p_2)$ are both regular in $|p_2| \geq 1$. Since δ_{21} and δ_{22} are their only branch points, (A.5) implies that they can be continued analytically into $\{p : |p| < 1, p \notin (\delta_{21}, \delta_{22})\}$.

Remark A.1. The analogous lemma for $Z(p_1, p_2)$ can be formulated for its zeros as a function of p_1 with $|p_1| > 1$. So for $|p_1| > 1$ we designate these zeros by $P_{21}(p_1)$ and $P_{22}(p_1)$ and such that

$$|P_{21}(p_1)| > |p_1| > |P_{22}(p_1)|, \qquad |p_1| > 1. \tag{A.11}$$

Remark A.2. Take $|p_2^{(1)}| > 1$ and put

$$p_1^{(1)} := P_{11}(p_2^{(1)}), \qquad p_2^{(2)} := P_{21}(p_1^{(1)}), \qquad p_1^{(2)} := P_{11}(p_2^{(2)}), \qquad p_2^{(3)} := P_{21}(p_1^{(2)}),$$

then from (A.8) and (A.11),

$$p_2^{(1)} = P_{22}(p_1^{(1)}), \qquad p_1^{(1)} = P_{12}(p_2^{(2)}).$$

These mappings are illustrated in Figure 1, for real $p_2^{(1)} > 1$. From Lemma A.2 and Remark A.1 it follows that

$$1 < |p_2^{(1)}| < |p_1^{(1)}| < |p_2^{(2)}| < |p_1^{(2)}| < \dots. \tag{A.12}$$

Lemma A.3. *The following limits exist:*

$$\begin{aligned}
\alpha_{12} &:= \lim_{p_2 \to \infty} P_{12}(p_2)/p_2 = \frac{1}{2r_{1,-1}}\{1 - (1 - 4r_{1,-1}r_{-1,1})^{\frac{1}{2}}\} > 0, \\
\alpha_{11} &:= \lim_{p_2 \to \infty} P_{11}(p_2)/p_2 = \frac{1}{2r_{1,-1}}\{1 + (1 - 4r_{1,-1}r_{-1,1})^{\frac{1}{2}}\} > 0, \\
\alpha_{21} &:= \lim_{p_1 \to \infty} P_{21}(p_1)/p_1 = \frac{1}{2r_{-1,1}}\{1 + (1 - 4r_{1,-1}r_{-1,1})^{\frac{1}{2}}\} > 0, \\
\alpha_{22} &:= \lim_{p_1 \to \infty} P_{22}(p_1)/p_1 = \frac{1}{2r_{-1,1}}\{1 - (1 - 4r_{1,-1}r_{-1,1})^{\frac{1}{2}}\} > 0.
\end{aligned} \tag{A.13}$$

Proof. From (A.1) we have

$$\hat{p}_1 = \frac{1}{2r_{1,-1}}\{\hat{p}_2 - r_{0,-1} \pm (1 - 4r_{-1,1}r_{1,-1})^{\frac{1}{2}}\sqrt{(\hat{p}_2 - \delta_{21})(\hat{p}_2 - \delta_{22})}\},$$

and from this relation together with (2.5) (ii), (A.8) and (A.11) the relations (A.13) follow.

Remark A.3. It is readily verified by using (2.5) (ii) that in the (p_1, p_2)-plane the curve $Z(p_1, p_2) = 0$, p_1 and p_2 both real, represents a hyperbola. Its centre is in the first quadrant and its asymptotic directions are given by $p_1 = \alpha_{11}p_2$ and $p_1 = \alpha_{12}p_2$, cf. (A.13). From Lemma A.1 it is seen that the two branches of the hyperbola are located inside the acute angles between the asymptotes; note $\alpha_{11} > 0$, $\alpha_{12} > 0$.

To $p_2^{(1)}$ in (A.12) corresponds a zero

(A.14) $$p_1^{(0)} := P_{12}(p_2^{(1)}), \qquad |p_1^{(0)}| < |p_2^{(1)}|,$$

as it follows from Lemma A.2. If $|p_1^{(0)}| > 1$ then we can again apply this lemma and define

$$p_2^{(0)} := P_{22}(p_1^{(0)}), \qquad |p_2^{(0)}| < |p_1^{(0)}|.$$

If $|p_2^{(0)}| > 1$ then again using the lemma we may define

$$p_2^{(-1)} := P_{12}(p_2^{(0)}),$$

and so on. So we may continue the sequence in (A.12) to the left, i.e. by elements which decrease in absolute value. From the location of the hyperbola just described and by using (A.5) it is seen that this completion of the sequence in (A.12) stops after a finite number of steps, because one of the $p_1^{(i)}$, $p_2^{(i)}$, $i = 0, -1, -2, \ldots$, will be less than or equal to 1 in absolute value.

It is finally noted that the iterated zeros of $Z(p_1, p_2)$ in Remark A.2 are all real if $p_2^{(1)}$ is real, and they all refer to the same branch of the hyperbola $Z(p_1, p_2) = 0$. Since one of these branches is located in the first quadrant and the other in the third quadrant, cf. (A.5), it is seen that in a sequence of iterated zeros these all have the same sign and do not have a finite point of accumulation.

Appendix B. On the zeros of $\omega_2(P_{12}(p), p)$

For the detailed analysis of the relations (4.1) and (4.2) we need information concerning the zeros of some of its coefficients. Put, cf. (2.9),

(B.1) $$f_2(p) := \omega_2(P_{12}(p), p) \equiv p - \phi_2(P_{12}(p), p),$$

hence for $p \neq 0$,

(B.2) $$p^{-3} f_2(p) = \frac{1}{p^2} - \boldsymbol{E}\left\{\left[\frac{P_{12}(p)}{p}\right]^{\xi_2} p^{\xi_2+\eta_2-3}\right\}.$$

For the investigation of the number of zeros of $f_2(p)$ in $|p| < 1$ and $|p| > 1$ we have to distinguish three cases.

(i) *The case* $w_{11} > 0$. It is seen from (2.2) and (2.4) that

(B.3) $$\xi_2 + \eta_2 - 3 \leq 0 \quad \text{and} \quad \Pr\{\xi_2 + \eta_2 = 3\} = w_{11} > 0.$$

Put, for $|t| < 1$,

(B.4) $$p^{-3} f_2(t,p) := \frac{1}{p^2} - t\boldsymbol{E}\left\{\left[\frac{P_{12}(p)}{p}\right]^{\xi_2} p^{\xi_2+\eta_2-3}\right\}.$$

Because $P_{12}(p)$ is a regular function of p for $|p| \geq 1$ and $|P_{12}(p)/p| \leq 1$ for $|p| \geq 1$, cf. Lemma A.2, it follows that for $|t| < 1$, $|p| \geq 1$, cf. (B.3),

$$\left| t\boldsymbol{E}\left\{\left[\frac{P_{12}(p)}{p}\right]^{\xi_2} p^{\xi_2+\eta_2-3}\right\}\right| < 1,$$

and so by applying Rouché's theorem, cf. [15], it is seen that for $|t| < 1, f_2(t,p)$ has two zeros in $|p| > 1$. These zeros are obviously continuous functions of t and both have a limit for $t \to 1$. Denote these limiting values by, say, σ_{21} and σ_{22}, then

(B.5) $$|\sigma_{21}| \geq 1, \qquad |\sigma_{22}| \geq 1.$$

From (B.3) and Lemma A.3 it is readily verified that

(B.6) $$\lim_{p\to\infty} \boldsymbol{E}\{[P_{12}(p)/p]^{\xi_2} p^{\xi_2+\eta_2-3}\} = \boldsymbol{E}\{\alpha_{12}^{\xi_2}(\xi_2 + \eta_2 = 3)\} > 0;$$

so that, since the expectation in (B.4) is equal to 1 for $p = 1$ it is easily seen that the zeros of (B.4) in $|p| > 1$ are both real for real t with $|t| < 1$. Hence, σ_{21} and σ_{22} are both real; one, say, σ_{21}, is negative, the other is positive, so that

(B.7) $$\sigma_{21} \leq -1, \qquad \sigma_{22} \geq 1.$$

Obviously, $p = 1$ is a zero of $f_2(p)$. To investigate whether $\sigma_{22} > 1$ or $\sigma_{22} = 1$ we consider the derivatives with respect to p^{-1} of both terms on the right-hand side of (B.2). So

(B.8) $$\frac{d}{du}\boldsymbol{E}\{[P_{12}(u^{-1})]^{\xi_2} u^{3-\eta_2}\}|_{u=1} = 3 - \nu_2 - \mu_2 \frac{d}{dp} P_{12}(p)|_{p=1}.$$

By the definition of $P_{12}(p)$ we have

$$P_{12}(p)p = \boldsymbol{E}\{[P_{12}(p)]^{\xi_3} p^{\eta_3}\},$$

and it follows that, cf. (2.5) (i),

$$\left.\frac{dP_{12}(p)}{dp}\right|_{p=1} = -\frac{1-\nu_3}{1-\mu_3}, \tag{B.9}$$

so by using (2.5) (iii),

$$\frac{d}{du}E\{[P_{12}(u^{-1})]^{\xi_2}u^{3-\eta_2}\}|_{u=1} = 2 - \left(\nu_2 - 1 - \mu_2\frac{1-\nu_3}{1-\mu_3}\right) > 2. \tag{B.10}$$

Hence by replacing p in (B.2) by u^{-1} it is easily seen that at $u = 1$ the slope of u^2 is less than that of $E\{[P_{12}(u^{-1})]^{\xi_2}u^{3-\eta_2}\}$, and hence, if $w_{11} > 0$, then $f_2(p)$ has three zeros in $|p| \geq 1$, viz. one at $p_2 = 1$ and two in $p_2 > 1$, i.e. $\sigma_{21} < -1$, $\sigma_{22} > 1$.

Elimination of p_1 from $\omega_1(p_1,p) = 0$ and $\omega_3(p_1,p) = 0$ leads to an algebraic equation of the sixth degree (note (2.5) (iii)) and so this equation has six zeros. It has been shown above that exactly three of these zeros are located in $|p| \geq 1$, one at $p = 1$, the other two in $|p| > 1$, and as it is readily seen the zeros in $|p| \geq 1$ all have multiplicity 1. Hence of the six zeros three are located in $|p| < 1$.

(ii) *The case* $w_{11} = 0$, $w_{10} + w_{01} > 0$. From (B.1) we have

$$p^{-2}f_2(p) = \frac{1}{p} - E\left\{\left[\frac{P_{12}(p)}{p}\right]^{\xi_2} p^{\xi_2+\eta_2-2}\right\}, \tag{B.11}$$

and, cf. (2.2) and (2.4),

$$w_{11} = 0, \quad w_{10} + w_{01} > 0 \Rightarrow \xi_2 + \eta_2 - 2 \leq 0, \quad \Pr\{\xi_2 + \eta_2 = 2\} = w_{10} + w_{01} > 0. \tag{B.12}$$

An analysis analogous with that of case (i) above shows that $f_2(p)$ has exactly two zeros in $|p| \geq 1$, one at $p = 1$, the other being positive; both have multiplicity 1. For the present case elimination of p_1 from $\omega_1(p_1,p) = 0$, $\omega_2(p_1,p) = 0$, yields an algebraic equation of the fourth degree, and so $f_2(p)$ has exactly two zeros in $|p| < 1$.

(iii) *The case* $w_{11} = w_{10} = w_{01} = 0$, $w_{1,-1} > 0$, $w_{0,-1} > 0$. From (B.1) we have

$$p^{-1}f_2(p) = 1 - E\left\{\left[\frac{P_{12}(p)}{p}\right]^{\xi_2} p^{\xi_2+\eta_2-1}\right\}, \tag{B.13}$$

and, cf. (2.2) and (2.4),

$$\begin{aligned} &w_{11} + w_{01} + w_{10} = 0, \quad w_{1,-1} > 0, \quad w_{0,-1} > 0 \Rightarrow \xi_2 + \eta_2 \leq 1, \\ &\Pr\{\xi_2 + \eta_2 = 1\} \equiv w_{1,-1} > 0. \end{aligned} \tag{B.14}$$

As above it is shown for the present case that $f_2(p)$ has exactly one zero in $|p| \geq 1$, viz. $p = 1$, with multiplicity 1. Because $f_2(p) = 0$ is now equivalent with an algebraic equation of the third degree, it has exactly two zeros in $|p| < 1$.

The analysis above leads to the following lemma, cf. also Adan [1], p. 48.

Lemma B.1. *The function* $w_2(P_{12}(p), p)$ *has a zero with multiplicity* 1 at $p = 1$ (note (2.6) (i)) *and*:

(i) *if* $w_{11} > 0$, *it has three zeros in* $|p| < 1$ *and two zeros in* $|p| \geq 1$, $p \neq 1$, *one negative, and the other positive;*

(ii) *if* $w_{11} = 0$, $w_{10} + w_{01} > 0$, *it has one zero in* $|p| \geq 1$, $p \neq 1$, *which is positive and has multiplicity* 1, *and two zeros in* $|p| < 1$;

(iii) *if* $w_{11} = w_{10} = w_{01} = 0$, $w_{1,-1} > 0$, $w_{0,-1} > 0$, *it has no zeros in* $|p| \geq 1$, $p \neq 1$, *and two zeros in* $|p| < 1$;

(iv) *if* p_2 *with* $|p_2| \geq 1$ *is a zero of* $\omega_2(P_{12}(p), p)$ *then* $\omega_2(P_{11}(p_2), p_2) \neq 0$.

Proof. The statements (i), (ii), (iii) have been proved above. The fourth statement follows from (2.4), (2.9) and Lemma A.2.

Remark B.1. For the function $\omega_1(p, P_{22}(p))$ a lemma analogous to Lemma B.1 may be proved, but its formulation and proof are similar, so they are omitted.

Appendix C. Asymptotics of $R_1(\tau^{(i)}), R_2(\sigma^{(i)})$ for $i \to \infty$

In this section we derive the asymptotic behaviour for $i \to \infty$ of the residues $R_1(\tau^{(i)})$ and $R_2(\sigma^{(i)})$, cf. (4.27).

We start with the asymptotic behaviour of $\sigma^{(i)}$ and $\tau^{(i)}$ for $i \to \infty$, cf. (4.5). From (2.5) (ii) it is seen that p_1 as a function of p_2, with $\phi_3(p_1, p_2) - p_1 p_2 = 0$, represents a hyperbola, cf. also Remark A.3, with asymptotic directions given by, cf. (A.13),

$$p_1 = \alpha_{11} p_2 \quad \text{and} \quad p_1 = \alpha_{12} p_2; \tag{C.1}$$

and so using (4.5), (A.8) and (A.11) it is readily seen that

$$\lim_{i \to \infty} |\sigma^{(i)}| = \infty, \qquad \lim_{i \to \infty} |\tau^{(i)}| = \infty. \tag{C.2}$$

Hence neither of the sequences $\{\sigma^{(i)}, i = 1, 2, \ldots\}$ and $\{\tau^{(i)}, i = 1, 2, \ldots\}$, has a finite accumulation point. From Lemma A.3 and (4.5) it follows that

$$\lim_{i \to \infty} \frac{\tau^{(i)}}{\sigma^{(i)}} = \alpha_{11} > 1, \qquad \lim_{i \to \infty} \frac{\tau^{(i)}}{\sigma^{(i+1)}} = \alpha_{12} < 1, \tag{C.3}$$

from which it follows readily that

$$\lim_{i \to \infty} \frac{\sigma^{(i+1)}}{\sigma^{(i)}} = \frac{\alpha_{11}}{\alpha_{12}} = \lim_{i \to \infty} \frac{\tau^{(i+1)}}{\tau^{(i)}} > 1. \tag{C.4}$$

Next we consider

$$\frac{\omega_2(P_{11}(\sigma^{(i)}),\sigma^{(i)})}{\omega_1(P_{11}(\sigma^{(i)}),\sigma^{(i)})}=\frac{\omega_2(\tau^{(i)},\sigma^{(i)})}{\omega_1(\tau^{(i)},\sigma^{(i)})}$$

$$(C.5)\qquad =\left.\frac{w_{11}p_2+w_{10}-\frac{1}{p_1}+w_{1,-1}\frac{1}{p_2}+w_{01}\frac{p_2}{p_1}+w_{0,-1}\frac{1}{p_2p_2}}{h_{11}p_1+h_{01}-\frac{1}{p_2}+h_{-1,1}\frac{1}{p_1}+h_{10}\frac{p_1}{p_2}+h_{-1,0}\frac{1}{p_1p_2}}\right|_{\substack{p_1=\tau^{(i)}\\ p_2=\sigma^{(i)}}}.$$

It follows from (2.5) (iv), (v), (C.2) and (C.5) that for $i\to\infty$,

$$\frac{\omega_2(\tau^{(i)},\sigma^{(i)})}{\omega_1(\tau^{(i)},\sigma^{(i)})}=\frac{1}{\alpha_{11}}\frac{w_{11}}{h_{11}}\left(1+O\left(\frac{1}{\tau^{(i)}}\right)\right)\qquad\text{for } w_{11}>0,h_{11}>0,$$

$$=\frac{1}{\tau^{(i)}}\frac{1}{h_{11}}\left\{w_{10}+w_{01}\frac{1}{\alpha_{11}}\right\}\left(1+O\left(\frac{1}{\tau^{(i)}}\right)\right)\qquad\text{for } w_{11}=0,h_{11}>0,$$

$$=\sigma^{(i)}w_{11}\{h_{01}+h_{10}\alpha_{11}\}^{-1}\left(1+O\left(\frac{1}{\sigma^{(i)}}\right)\right)\qquad\text{for } w_{11}>0,h_{11}=0,$$

$$=\frac{w_{10}+w_{01}\frac{1}{\alpha_{11}}}{h_{01}+h_{10}\alpha_{11}}\left(1+O\left(\frac{1}{\sigma^{(i)}}\right)\right)\qquad\text{for } w_{11}=0,h_{11}=0.$$

(C.6)

Similarly, we obtain for $i\to\infty$,

$$\frac{\omega_1(P_{12}(\sigma^{(i)}),\sigma^{(i)})}{\omega_2(P_{12}(\sigma^{(i)}),\sigma^{(i)})}=\frac{\omega_1(\tau^{(i-1)},\sigma^{(i)})}{\omega_2(\tau^{(i-1)},\sigma^{(i)})}$$

$$=\alpha_{12}\frac{h_{11}}{w_{11}}\left(1+O\left(\frac{1}{\sigma^{(i)}}\right)\right)\qquad\text{for } w_{11}>0,h_{11}>0,$$

$$=\tau^{(i-1)}h_{11}\left\{w_{10}+w_{01}\frac{1}{\alpha_{12}}\right\}^{-1}\left(1+O\left(\frac{1}{\sigma^{(i)}}\right)\right)\qquad\text{for } w_{11}=0,h_{11}>0,$$

$$=\frac{1}{\sigma^{(i)}}\frac{1}{w_{11}}\{h_{01}+h_{10}\alpha_{12}\}\left(1+O\left(\frac{1}{\sigma^{(i)}}\right)\right)\qquad\text{for } w_{11}>0,h_{11}=0,$$

$$(C.7)\qquad =\frac{h_{01}+h_{10}\alpha_{12}}{w_{10}+w_{01}\frac{1}{\alpha_{12}}}\left(1+O\left(\frac{1}{\sigma_i}\right)\right)\qquad\text{for } w_{11}=0,h_{11}=0.$$

For p_1, p_2 satisfying $Z(p_1,p_2)=0$, i.e. (cf. (2.10)),

$$r_{-1,1}p_2^2-p_1p_2+r_{1,-1}p_1^2+r_{-1,0}p_2+r_{0,-1}p_1+r_{-1,-1}=0,$$

we have

$$(C.8)\qquad \frac{dp_1}{dp_2}=-\frac{p_1}{p_2}\frac{1-2r_{-1,-1}p_2/p_1-r_{-1,0}/p_1}{1-2r_{1,-1}p_1/p_2-r_{0,-1}/p_2}.$$

With $p_1 = P_{11}(p_2)$, and $p_1 = P_{12}(p_2)$, $|p_2| > 1$, we obtain from Lemma A.3, after some algebra,

$$\text{(C.9)} \qquad \lim_{p\to\infty} \frac{dP_{11}(p)}{dp} = \alpha_{11}, \qquad \lim_{p\to\infty} \frac{dP_{12}(p)}{dp} = \alpha_{12}.$$

From (4.27) we have

$$\frac{R_2(\sigma^{(i+1)})}{R_2(\sigma^{(i)})} = \left\{ \frac{\omega_1(P_{12}(p),p)}{\omega_1(P_{12}(p),p)} \frac{\omega_2(P_{11}(p),p)}{\omega_1(P_{11}(p),p)} \left(\frac{d}{dp} P_{11}(p)\right) \left[\frac{d}{dp} P_{12}(p)\right]^{-1} \right\}_{p=\sigma^{(i)}}.$$

(C.10)

From (C.16), (C.7), (C.9) and (C.10) we find that

$$\text{(C.11)} \qquad \begin{aligned} \lim_{i\to\infty} \frac{R_2(\sigma^{(i+1)})}{R_2(\sigma^{(i)})} &= 1 && \text{for } w_{11} > 0, h_{11} > 0, \\ &= \frac{w_{01} + w_{10}\alpha_{11}}{w_{01} + w_{10}\alpha_{12}} \frac{\alpha_{12}}{\alpha_{11}} && \text{for } w_{11} = 0, h_{11} > 0, \\ &= \frac{h_{01} + h_{10}\alpha_{12}}{h_{01} + h_{10}\alpha_{11}} \frac{\alpha_{11}}{\alpha_{12}} && \text{for } w_{11} > 0, h_{11} = 0, \\ &= \frac{h_{01} + h_{10}\alpha_{12}}{h_{01} + h_{10}\alpha_{11}} \frac{w_{01} + w_{10}\alpha_{11}}{w_{01} + w_{10}\alpha_{12}} && \text{for } w_{11} = 0, h_{11} = 0; \end{aligned}$$

$$\text{(C.12)} \qquad \begin{aligned} \lim_{i\to\infty} \frac{R_1(\tau^{(i+1)})}{R_1(\tau^{(i)})} &= \frac{\alpha_{11}}{\alpha_{12}} \lim_{i\to\infty} \frac{\omega_2(P_{11}(\sigma^{(i+1)}),\sigma^{(i+1)})}{\omega_1(P_{11}(\sigma^{(i+1)}),\sigma^{(i+1)})} \frac{\omega_1(P_{12}(\sigma^{(i)}),\sigma^{(i)})}{\omega_2(P_{12}(\sigma^{(i)}),\sigma^{(i)})} \\ &= \lim_{i\to\infty} \frac{R_2(\sigma^{(i+1)})}{R_2(\sigma^{(i)})}, \end{aligned}$$

for all the four cases occurring in (C.11).

Remark C.1. Note that (2.5) (iv), (v) imply that the quotients in (C.11) are well defined.

For the definition of the meromorphic functions introduced in Section 5 we have to investigate the existence and determination of the smallest positive integers m_2 and m_1 for which the series

$$\text{(C.13)} \qquad \sum_{i=1}^{\infty} \frac{R_2(\sigma^{(i+1)})}{\{\sigma^{(i)}\}^{m_2+1}} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{R_1(\tau^{(i)})}{\{\tau^{(i)}\}^{m_1+1}},$$

converge absolutely, cf. (5.10).

Put, cf. (C.11) and (C.12),

$$\text{(C.14)} \qquad \rho := \lim_{i\to\infty} \frac{R_2(\sigma^{(i+1)})}{R_2(\sigma^{(i)})} = \lim_{i\to\infty} \frac{R_1(\tau^{(i+1)})}{R_1(\tau^{(i)})} > 0,$$

where the value of ρ varies with the four cases in (C.11). It follows for fixed but sufficiently large i and $k = 1, 2, \ldots$, cf. (C.4), that

$$\text{(C.15)} \qquad \frac{R_2(\sigma^{(i+k)})}{R_2(\sigma^{(i)})} \sim \rho^k, \qquad \frac{\sigma^{(i+k)}}{\sigma^{(i)}} \sim \left\{\frac{\alpha_{11}}{\alpha_{12}}\right\}^k.$$

Consequently, for every integer n,

$$\text{(C.16)} \qquad \frac{R_2(\sigma^{(i+k)})}{[\sigma^{(i+k)}]^{n+1}} \sim \frac{R_2(\sigma^{(i)})}{[\sigma^{(i)}]^{n+1}} \left\{\rho\left[\frac{\alpha_{12}}{\alpha_{11}}\right]^{n+1}\right\}^k, \qquad k = 1, 2, \ldots.$$

Because, cf. (2.5) (ii) and (A.13),

$$0 < \alpha_{12}/\alpha_{11} < 1,$$

we see that the first series in (C.13) converges absolutely for that value, say, m_2 of $n = 0, 1, 2, \ldots$, which satisfies

$$\text{(C.17)} \qquad \begin{aligned} &n = 0 && \text{if } \rho\frac{\alpha_{12}}{\alpha_{11}} < 1, \\ &\rho\left[\frac{\alpha_{12}}{\alpha_{11}}\right]^n \geq 1 > \rho\left[\frac{\alpha_{12}}{\alpha_{11}}\right]^{n+1} && \text{if } \rho\frac{\alpha_{12}}{\alpha_{11}} \geq 1, \end{aligned}$$

and (C.17) implies that m_2 exists.

Analogously, it follows from (C.4) and (C.14) that m_1 is that value of $n = 0$, 1, 2, ..., which satisfies (C.17) and hence it is seen that $m_2 = m_1$. Put

$$\text{(C.18)} \qquad m := m_1 = m_2.$$

Because $|\sigma_i| \to \infty$, cf. (C.2), it follows that for fixed p and sufficiently large i,

$$\text{(C.19)} \qquad 2|p| < |\sigma_i| \Rightarrow \frac{2}{3} < \left[\left|\frac{p}{\sigma_i}\right| - 1\right]^{-1} < 2.$$

Note that for $k = 1, 2, \ldots$,

$$\text{(C.20)} \qquad \frac{R_2(\sigma^{(i+k)})}{p - \sigma^{(i+k)}}\left[\frac{p}{\sigma^{(i+k)}}\right]^m = p^m\left[\frac{p}{\sigma^{(i+k)}} - 1\right]^{-1}\frac{R_2(\sigma^{(i+k)})}{\{\sigma^{(i+k)}\}^{m+1}},$$

and hence it follows from (C.16), (C.19), (C.20) and the definition of m_2 in (C.18), that the series

$$\text{(C.21)} \qquad \sum_{i=1}^{\infty}\frac{R_2(\sigma^{(i)})}{p - \sigma^{(i)}}\left[\frac{p}{\sigma^{(i)}}\right]^m \quad \text{and} \quad \sum_{i=1}^{\infty}\frac{R_1(\tau^{(i)})}{p - \tau^{(i)}}\left[\frac{p}{\tau_i}\right]^m$$

converge uniformly and absolutely in every circle $|p| < R$, whenever the terms with poles inside the circle with radius $R > 1$ are deleted, cf. [16] Chapter 7.4, [3], p. 219.

Remark C.2. It is readily seen that if in the series (C.21) the exponent m is replaced by $m + h$, with h any positive integer, then the series so obtained also converge uniformly and absolutely in the same sense as (C.21); see further Remark 6.3.

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