BOLYAI SOCIETY
MATHEMATICAL STUDIES

Combinatorics,
Paul Erdős is Eighty (Volume 1)
Keszthely (Hungary), 1993, pp. 379–402.

Applying Free Partially Commutative Groups to Paths and Circuits in Directed Graphs on Surfaces

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A *free partially commutative group* is a group generated by finitely many generators $g_1, \ldots, g_k$ and relations $g_i g_j = g_j g_i$ for some of the pairs g_i, g_j. It turns out that these groups are useful in studying disjoint paths and curves problems in graphs embedded on surfaces. The relation is through cohomology. In this paper we give a survey of the method.

1. Introduction

In [16] we showed that the following problem, the k *disjoint paths problem for directed planar graphs,* is solvable in polynomial time, for any fixed k:

given: a planar directed graph $D = (V, E)$ and k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D; (1)
find: k pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i $(i = 1, \ldots, k)$.

The problem is NP-complete if we do not fix k (even in the undirected case; Lynch [6]). Moreover, it is NP-complete for $k = 2$ if we delete the planarity condition (Fortune, Hopcroft, and Wyllie [5]). This is in contrast to the undirected case (for those believing NP$\neq$P), where Robertson and Seymour [13] showed that, for any fixed k, the k disjoint paths problem is polynomial-time solvable for any graph (not necessarily planar).

The proof of the polynomial-time solvability of problem (1) is based on considering cohomology over free groups. In this paper we survey a more general approach, viz. cohomology over free partially commutative groups. We discuss applications to disjoint paths and curves problems in planar graphs, and in graphs embedded on more general surfaces. For example, free partially commutative groups can be used in showing that the following problem can be solved in polynomial time for each fixed k and each fixed surface S:

given: a directed graph $D = (V, A)$ embedded in S, k pairs $(r_1, s_1),\ldots,(r_k, s_k)$ of vertices of D, subsets $A_1, \ldots, A_k$ of A, and a set E of pairs $\{i, j\}$ from $\{1, \ldots, k\}$; (2)

find: k pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i and uses only arcs in A_i $(i = 1, \ldots, k)$, and where P_i and P_j are vertex-disjoint if $\{i, j\} \notin E$.

Also similar problems for disjoint trees can be solved in polynomial time.

2. Free partially commutative groups

The method uses the framework of combinatorial group theory, viz. groups defined by generators and relations. For background literature on combinatorial group theory see [8] and [7].

Let $g_1, \ldots, g_k$ be 'generators'. Call the elements $g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}$ *symbols*. Define $(g_i^{-1})^{-1} := g_i$. A *word* (of *size* t) is a sequence $a_1 \ldots a_t$ where each a_j is a symbol. The empty word (of size 0) is denoted by $\emptyset$. Define $(a_1 \ldots a_t)^{-1} := a_t^{-1} \ldots a_1^{-1}$.

Let E be a set of unordered pairs $\{i, j\}$ from $\{1, \ldots, k\}$ with $i \neq j$. Then the group $G = G_E$ is the group generated by the generators $g_1, \ldots, g_k$ with relations

$$g_i g_j = g_j g_i \text{ for each pair } \{i, j\} \in E. \tag{3}$$

Such a group is called a *free partially commutative group* or a *graph group*. (These groups are studied *inter alia* in [1], [4], [17], [22]. However, in this paper we do not use the results of these papers.)

Note that if $E = \emptyset$ the group G_E is the *free group* generated by $g_1, \ldots, g_k$. If E consists of *all* pairs, G_E is isomorphic to $\mathbb{Z}^k$. We say that symbols a and b are *independent* if $a \in \{g_i, g_i^{-1}\}$ and $b \in \{g_j, g_j^{-1}\}$ for some $\{i,j\} \in E$ with $i \neq j$. So if a and b are independent then $ab = ba$ and $b \neq a^{\pm 1}$. (It follows from (6) below that also the converse implication holds.)

By definition, the elements in G come from all words by identifying any two words w and w' if w' arises from w by iteratively:

(i) replacing $x\alpha\alpha^{-1}y$ by xy or vice versa, where α is a symbol; (4)

(ii) replacing $x\alpha\beta y$ by $x\beta\alpha y$ where α and β are independent.

By *commuting* we mean applying (ii) iteratively.

Now for any word $w = \alpha_1 \dots \alpha_t$ one has:

$w = 1$ if and only if there exists a perfect matching M on $\{1, \dots, t\}$ such that (5)

(i) if $\{i,j\} \in M$ then $\alpha_j = \alpha_i^{-1}$;

(ii) if two pairs $\{i,j\}, \{i',j'\}$ in M cross then α_i and $\alpha_{i'}$ are independent.

(A *perfect matching* is a partition into pairs. Pairs $\{i,j\}$ and $\{i',j'\}$ are said to *cross* if $i < i' < j < j'$ or $i' < i < j' < j$ (assuming without loss of generality $i < j, i' < j'$).)

We say that two words x and y are *independent* if any symbol in x and any symbol in y are independent. (In particular, $\beta \neq \alpha^{\pm 1}$ for any symbols α in x and β in y.)

(5) directly implies that testing $w = 1$ is easy: just replace (iteratively) any segment $\alpha y \alpha^{-1}$ by y where α is a symbol and y is a word independent of α. The final word is empty if and only if $w = 1$. This gives a test for equivalence of words w and x: just test if $wx^{-1} = 1$. So the 'word problem' for free partially commutative groups is easy. (In fact it can be solved in linear time — see Wrathall [22].)

We call a word w *reduced* if it is not equal (as a word) to $x\alpha y\alpha^{-1}z$ for some symbol α independent of each symbol in y. Note that reducedness is invariant under commuting. (5) also implies the stronger statement:

Let w and x be reduced words with $w = x$. Then word x can be obtained from w by a series of commutings. (6)

It follows that we can define the *size* $|x|$ of an element x in G as the size of any reduced word $w = x$. Trivially, $|x^{-1}| = |x|$ and $|xy| \leq |x| + |y|$. Hence the function $\mathrm{dist}(x, y) := |x^{-1}y|$ is a distance function. That is, $\mathrm{dist}(x, x) = 0$, $\mathrm{dist}(x, y) = \mathrm{dist}(y, x)$ and $\mathrm{dist}(x, z) \leq \mathrm{dist}(x, y) + \mathrm{dist}(y, z)$ for all x, y, z. Note that $\mathrm{dist}(zx, zy) = \mathrm{dist}(x, y)$ for all x, y, z.

3. THE PARTIAL ORDER $\leq$

Let x and y be two elements of G_E. We write $x \leq y$ if there are reduced words $x' = x$ and $y' = y$ such that x' is a beginning segment of y'. So $x \leq y$ if and only if $|y| = |x| + |x^{-1}y|$.

It is not difficult to see that

$$\leq \text{ is a partial order on } G. \tag{7}$$

In fact, the partial order $\leq$ yields a lattice if we add to G an element ∞ at infinity. So $x \vee y$ is finite if and only if $x \leq w$ and $y \leq w$ for some w.

There is an easy algorithm to calculate $x \wedge y$ from x and y. Just iteratively select a symbol α such that $\alpha \leq x$ and $\alpha \leq y$ and replace x and y by $\alpha^{-1}x$ and $\alpha^{-1}y$. Doing this until there is no such symbol any more, the symbols selected form $x \wedge y$.

Next $x \vee y$ can be found as follows. Let $x' := (x \wedge y)^{-1}x$ and $y' := (x \wedge y)^{-1}y$. Then $x \vee y$ is finite if and only if x' and y' are independent. Moreover, $x \vee y = (x \wedge y)x'y'$.

In fact,

For each $x \in G$, the set $\{y \in G | y \leq x\}$, partially ordered by $\leq$, forms a distributive lattice. (8)

(The whole lattice on $G \cup \{\infty\}$ is generally not distributive: if a and b are distinct generators then $a \wedge (b \vee b^{-1}) = a \wedge \infty = a$ while $(a \wedge b) \vee (a \wedge b^{-1}) = 1 \vee 1 = 1$.)

An element x is called *join-irreducible* if $x \neq 1$ and $x = x' \vee x''$ implies $x' = x$ or $x'' = x$. Let $x_1, \ldots, x_t$ be all join-irreducible elements $\leq x$. Then $x = x_1 \vee \cdots \vee x_t$ and $t = |x|$.

We also note the following:

Let $x_1, \ldots, x_t \in G$ be such that $x_i \vee x_j$ is finite for all i, j. Then $x_1 \vee \ldots \vee x_t$ is finite. (9)

The partial order $\leq$ is clearly not invariant under mappings $x \mapsto zx$ for $z \in G$. The following formula expresses how $\wedge$ behaves under such an operation.

$$\text{For all } x, y, z \in G \text{ one has } z^{-1}x \wedge z^{-1}y = z^{-1}((x \wedge y) \vee (x \wedge z) \vee (y \wedge z)). \tag{10}$$

It follows from (10) that $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is the unique element w that is on the three shortest paths (with respect to the distance function dist) from x to y, x to z, and y to z. So $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is the 'median' in the sense of Sholander [19], [20], [21] (cf. Birkhoff [2]).

4. Convex sets and ideals

We call a subset H of G *left-convex* if H is nonempty and if $x, z \in H$ and $\text{dist}(x, y) + \text{dist}(y, z) = \text{dist}(x, z)$ then $y \in H$. Since the distance function is invariant under functions $x \mapsto yx$, if H is left-convex also yH is left-convex for any $y \in G$. It is not difficult to see that a nonempty subset H of G is left-convex if and only if

(i) if $x \leq y \leq z$ and $x, z \in H$ then $y \in H$; (11)

(ii) if $x, y \in H$ then $x \wedge y \in H$ and, if $x \vee y$ is finite, $x \vee y \in H$.

In particular, each left-convex set has a unique minimum element. We call a subset H of G *right-convex* if H^{-1} is left-convex, and *convex* if H is both left- and right-convex. (As usual, $H^{-1} := \{x^{-1} | x \in H\}$.)

A subset H of G is called a *left-ideal* if

(i) $1 \in H$; (12)

(ii) if $x \in H$ and $y \leq x$ then $y \in H$;

(iii) if $x, y \in H$ and $x \vee y$ is finite then $x \vee y \in H$.

It is not difficult to see that H is a left-ideal if and only if H is left-convex and $1 \in H$.

We call H a *right-ideal* if H^{-1} is a left-ideal, and an *ideal* if H is both a left- and a right-ideal.

Clearly, the intersection of any number of left-convex sets is again left-convex. Moreover, left-convex sets satisfy the following 'Helly-type' property:

Let $H_1, \ldots, H_t$ be left-convex sets with $H_i \cap H_j \neq \emptyset$ for all $i, j = 1, \ldots, t$. Then $H_1 \cap \cdots \cap H_t \neq \emptyset$. (13)

(13) implies the following. As usual, define $HH' := \{xx' | x \in H, x' \in H'\}$.

Let H be left-convex and let H' and H'' be right-convex, with $H' \cap H'' \neq \emptyset$. Then $HH' \cap HH'' = H(H' \cap H'')$. (14)

Indeed $HH' \cap HH'' \supseteq H(H' \cap H'')$ is easy. To see $\subseteq$, let $x \in HH' \cap HH''$. Then $x^{-1}H \cap (H')^{-1}$, $x^{-1}H \cap (H'')^{-1}$ and $(H')^{-1} \cap (H'')^{-1}$ are nonempty, and hence by (13), $x^{-1}H \cap (H')^{-1} \cap (H'')^{-1} \neq \emptyset$. So $x \in H(H' \cap H'')$.

For any nonempty left-convex set H and $x \in G$, there is a unique element y in H 'closest' to x, that is, one minimizing $\mathrm{dist}(x, y)$. To see this we may assume H is a left-ideal. Then by (12) y is the largest element in H satisfying $y \leq x$.

We denote this element by $\mathrm{cl}_H(x)$. Then

Let H be left-convex and let H' be a right-ideal. Let $x \in G$ and $y := \mathrm{cl}_H(x)$. Then $x \in HH'$ if and only if $y^{-1}x \in H'$. (15)

(15) implies that if H is a left-ideal and H' is a right-ideal, and if we can test in polynomial time whether any given word x belongs to H and to H', then we can also test in polynomial time if any given word y belongs to HH'. We first find $z = \mathrm{cl}_H(x)$. This can be done as follows: if we have a reduced word $x'x'' = x$ with $x' \in H$, find a symbol α such that $\alpha \leq x''$ and $x'\alpha \in H$; reset $x' := x'\alpha$ and $x'' := \alpha^{-1}x''$, and iterate. If no such α exists we set $z := x'$. Now by (15) $x \in HH'$ if and only if $y^{-1}x \in H'$.

Similarly, if H is left-convex and H' is right-convex and if we can test in polynomial time if any word belongs to H and to H' and moreover we know at least one word w in H and at least one word w' in H', then we can test if any given word x belongs to HH': we just test if $w^{-1}xw'^{-1}$ belongs to $(w^{-1}H)(H'w'^{-1})$. Note that $w^{-1}H$ is a left-ideal and that $H'w'^{-1}$ is a right-ideal.

From (15) one may also derive:

Let H be left-convex and let H' be an ideal. Then HH' is left-convex. (16)

For any $x \in G$ define

$$H_x^{\uparrow} := \{y \in G | y \geq x\} \text{ and } H_x^{\downarrow} := \{y \in G | y \leq x\}. \tag{17}$$

It is not difficult to see that $H_x^{\uparrow}$ and $H_x^{\downarrow}$ are left-convex. One has for any ideal H and any $x, y, z \in G$:

$x^{-1}yz$ belongs to H if and only if: (18)
(i) $\exists u \leq x \; \exists w \leq z : u^{-1}yw \in H$;
(ii) $\exists u \leq x \; \exists w \geq z : u^{-1}yw \in H$;
(iii) $\exists u \geq x \; \exists w \leq z : u^{-1}yw \in H$;
(iv) $\exists u \geq x \; \exists w \geq z : u^{-1}yw \in H$.

Proof. Necessity being trivial we show sufficiency. Assertion (i) means $y \in H_x^{\downarrow} H H_z^{\downarrow -1}$, and assertion (ii) means $y \in H_x^{\downarrow} H H_z^{\uparrow -1}$. Hence by Proposition (14), $y \in H_x^{\downarrow} H (H_z^{\downarrow -1} \cap H_z^{\uparrow -1}) = H_x^{\downarrow} H z^{-1}$. Similarly, $y \in H_x^{\uparrow} H z^{-1}$. Again by Proposition (14), $y \in (H_x^{\downarrow} \cap H_x^{\uparrow}) H z^{-1} = xHz^{-1}$. Therefore $x^{-1}yz \in H$. ■

Let H be an ideal and let x, a be such that $x^{-1}ax \in H$. Then there exists a y such that $y^{-1}ay \in H$ and such that $y \leq a^t$ for some t. (19)

This implies:

Let H be an ideal and let x, a be such that $x^{-1}a^s x \in H$ for some s. Then there exists a y such that $y^{-1}a^s y \in H$ and such that $y \leq a^{|a|}$. (20)

5. The cohomology feasibility problem

Let $D = (V, A)$ be a directed graph and let G be a group. Two functions $\phi, \psi : A \longrightarrow G$ are called *cohomologous* if there exists a function $f : V \longrightarrow G$ such that $\psi(a) = f(u)^{-1}\phi(a)f(w)$ for each arc $a = (u, w)$. One directly checks that this gives an equivalence relation.

Consider the following *cohomology feasibility problem:*

given: a directed graph $D = (V, A)$, a group G, a function $\phi : A \longrightarrow G$, and for each $a \in A$, a subset $H(a)$ of G; (21)

find: a function $\psi : A \longrightarrow G$ such that ψ is cohomologous to ϕ and such that $\psi(a) \in H(a)$ for each $a \in A$.

This problem can be solved in polynomial time in case G is an free partially commutative group and each $H(a)$ is an ideal.

In the algorithm it is not required that the $H(a)$ are given explicitly. It suffices that an algorithm is given that tests for any a and any word x whether or not x belongs to $H(a)$. (So $H(a)$ might be infinite.) The running time of the algorithm for the cohomology feasibility problem is bounded by a polynomial in $n := |V|$, $\sigma := \max\{|\phi(a)||a \in A\}$, and τ, where τ is the maximum time needed to test membership of x in $H(a)$ for any given arc a and any given word x of length bounded by a polynomial in n and σ. (The number k of generators can be bounded by $n\sigma$, since we may assume that all generators occur among the $\phi(a)$.)

Note that, by the definition of cohomologous, equivalent to finding a ψ as in (21), is finding a function $f : V \longrightarrow G$ satisfying:

$$f(u)^{-1}\phi(a)f(w) \in H(a) \text{ for each arc } a = (u, w). \tag{22}$$

We call such a function f *feasible.*

Note that if f is feasible and P is an $s-t$ path, then $f(s)^{-1}\phi(P)f(t) \in H(P)$. Here we use the following notation and terminology.

A *path* is defined as a 'word' $a_1 \ldots a_m$ where a_i or a_i^{-1} is an arc of D such that the head of a_i is equal to the tail of a_{i+1} ($i = 1, \ldots, m-1$). It is called an $s-t$ path if a_1 has tail s and a_m has head t. If $s = t$ we call it a *cycle*. Moreover, if $P = a_1 \ldots a_m$ then $P^{-1} := a_m^{-1} \ldots a_1^{-1}$.

If $P = a_1 \ldots a_m$ then:

$$\begin{aligned} \phi(P) &:= \phi(a_1) \ldots \phi(a_m), \\ H(P) &:= H(a_1) \ldots H(a_m), \end{aligned} \tag{23}$$

where $\phi(a^{-1}) := \phi(a)^{-1}$ and $H(a^{-1}) = H(a)^{-1}$.

This gives an obvious necessary (but generally not sufficient) condition for problem (21) having a solution:

for each cycle P there exists an $x \in G$ such that $x^{-1}\phi(P)x$ belongs to $H(P)$. (24)

6. Pre-feasible functions

In order to describe the algorithm solving the cohomology feasibility problem, we need the concept of 'pre-feasible' function. Let $D = (V, A)$ be a directed graph, let G be a free partially commutative group, let $\phi : A \longrightarrow G$ and for each $a \in A$, let $H(a)$ be an ideal of G.

We call a function $f : V \longrightarrow G$ *pre-feasible* if for each arc $a = (u, w)$ of D there exist $x \geq f(u)$ and $z \leq f(w)$ such that $x^{-1}\phi(a)z \in H(a)$. Clearly, each feasible function is pre-feasible. There is a trivial pre-feasible function f, defined by $f(v) := 1$ for each $v \in V$.

The collection of pre-feasible functions is closed under certain operations on the set G^V of all functions $f : V \longrightarrow G$. This set can be partially ordered by: $f \leq g$ if and only if $f(v) \leq g(v)$ for each $v \in V$. Then G^V forms a lattice if we add an element ∞ at infinity. Let $\wedge$ and $\vee$ denote meet and join.

> Let f_1 and f_2 be pre-feasible functions. Then $f_1 \wedge f_2$ and, if $f_1 \vee f_2 < \infty$, $f_1 \vee f_2$ are pre-feasible again. (25)

It follows that for each function $f : V \longrightarrow G$ there is a unique smallest pre-feasible function $\bar{f} \geq f$, provided that there exists at least one pre-feasible function $g \geq f$. If no such g exists we set $\bar{f} := \infty$. Note that $\overline{f \vee g} = \bar{f} \vee \bar{g}$ for any two functions f, g with $f \vee g$ finite.

7. A subroutine finding $\bar{f}$

Let input $D = (V, A), \phi, H$ for the cohomology feasibility problem be given. We describe a polynomial-time subroutine that outputs $\bar{f}$ for any given function f, under the assumption that (24) holds.

For any arc $a = (u, w)$ and any $x \in G$ let $\beta_a(x)$ be the smallest element z in G such that there exists an $x' \geq x$ with $(x')^{-1}\phi(a)z \in H(a)$. This is unique, as z is the minimum element in the left-convex set $\phi(a)^{-1}H_x^{\uparrow}H(a)$. Note that for any $f : V \longrightarrow G$ one has:

> f is pre-feasible if and only if $\beta_a(f(u)) \leq f(w)$ for each arc $a = (u, w)$. (26)

For any given x we can determine $\beta_a(x)$ in polynomial time if we can test in polynomial time if any given word belongs to $H(a)$ (as $\beta_a(x)$ is the minimal element of $\phi(a)^{-1}H_x^{\uparrow}H(a)$).

Subroutine to find $\bar{f}$: If f is pre-feasible, output $\bar{f} := f$. Otherwise, choose an arc $a = (u, w)$ such that $\beta_a(f(u)) \not\leq f(w)$. If $f(w) \vee \beta_a(f(u)) = \infty$, output $\bar{f} := \infty$. Otherwise reset $f(w) := f(w) \vee \beta_a(f(u))$, and start anew.

Proposition 1. *The output in the subroutine is correct.*

Proof. Clearly, if $f(w) \vee \beta_a(f(u)) = \infty$ then $\bar{f} = \infty$. If $f(w) \vee \beta_a(f(u)) < \infty$, let f' denote the reset function. Then $f \leq f'$. Moreover, if $\bar{f}$ is finite, then $f \leq f' \leq \bar{f}$, since $f'(w) = f(w) \vee \beta_a(f(u)) \leq \bar{f}(w) \vee \beta_a(\bar{f}(u)) = \bar{f}(w)$, since $\bar{f}$ is pre-feasible. ∎

8. Running time of the subroutine

After at most $2^{15}n^9k^9\sigma^8 + 2^2n^2k^2\rho$ iterations the subroutine gives an output, where:

$$\begin{aligned} n &:= |V|, \\ \sigma &:= \max\{|\phi(a)| \mid a \in A\}, \\ \rho &:= \max\{|f(v)| \mid v \in V\}. \end{aligned} \tag{27}$$

We give a proof of this in case G is a free group. To this end, define for each path P in D and each $x \in G$, $\beta_P(x)$ inductively by: $\beta_\emptyset(x) := x$ and $\beta_{Pa}(x) := \beta_a(\beta_P(x))$. Then:

Proposition 2. *For each path P in D, each $x \in G$ and each $y \in H(P)$ we have $\beta_P(x) \leq \phi(P)^{-1}xy$.*

Proof. If $P = \emptyset$, the assertion is trivial. If $P = a$, then for $z := \phi(a)^{-1}xy$ one has $x^{-1}\phi(a)z = y \in H(a)$, and hence $\beta_a(x) \leq z = \phi(a)^{-1}xy$.

Consider next a path Pa and let $y = y'y'' \in H(Pa)$, with $y' \in H(P)$ and $y'' \in H(a)$. Then by induction,

$$\begin{aligned} \beta_{Pa}(x) &= \beta_a(\beta_P(x)) \leq \beta_a(\phi(P)^{-1}xy') \\ &\leq \phi(a)^{-1}(\phi(P)^{-1}xy')y'' = \phi(Pa)^{-1}xy. \qquad \blacksquare \end{aligned} \tag{28}$$

We introduce the following further structure. Let f_t denote the function f as it is after t iterations. At each iteration t of the subroutine we make a path P_t as follows. If at iteration t we choose arc $a = (u, w)$ and reset $f_t(w) := \beta_a(f_{t-1}(u))$, then P_t is equal to the path $P_i a$ if $f_{t-1}(u) \neq f(u)$ and if the value of $f_{t-1}(u)$ was obtained at iteration i; and $P_t := a$ if $f_{t-1}(u) = f(u)$.

Note that $P_1, \ldots, P_t$ satisfy (if G is the free group):

for each vertex v with $f_t(v) \neq f(v)$ there is a vertex r and an $r - v$ path P_i such that $\beta_{P_i}(f(r)) = f_t(v)$. (29)

Proposition 3. *For free groups, the subroutine takes at most* $t := 2^9 n^7 k^6 \sigma^5 + 2^2 n^2 k \rho$ *iterations.*

Proof. Suppose we have performed t iterations. We first show:

At least one of the paths $P_1, \ldots, P_t$ contains $T := 2^8 n^5 k^5 \sigma^4$ arcs. (30)

Since there are n vertices and $2k$ symbols, there exist vertices r, v and a symbol α such that there are at least $t/2n^2k = 2^8 n^5 k^5 \sigma^5 + \rho$ indices i such that P_i runs from r to v and such that $\beta_{P_i}(f(r))$ has maximum symbol α. (We say that an element x of the free group has *maximum symbol* α if α is the last symbol in the reduced word representing x.)

Let J be the collection of such indices i, and let X denote the collection $\{\beta_{P_i}(f(r)) | i \in J\}$. Since each $x \in X$ satisfies $x \leq f_t(v)$, the elements in X form a chain (i.e., are totally ordered by $\leq$). Let w be the maximum element in X, and let j be the maximum element in J. Then

$$|w| \geq |X| \geq 2^8 n^5 k^5 \sigma^5 + \rho. \tag{31}$$

Let m be the number of arcs in P_j. Then by Proposition 2, $w = \beta_{P_j}(f(r)) \leq \phi(P_j)^{-1} f(r)$; so $|w| \leq |\phi(P_j)| + |f(r)| \leq m\sigma + \rho$. Hence with (31), $m \geq 2^8 n^5 k^5 \sigma^4$. Since each beginning segment of a path in Π_t again belongs to Π_t we have (30).

Let P have beginning segments $Q_0, Q_1, \ldots, Q_T$, and let $v_0, v_1, \ldots, v_T$ be the end vertices of $Q_0, Q_1, \ldots, Q_T$ respectively (with $v_0 = r$). Let $y_i := \beta_{Q_i}(f(r))$ for $i = 0, \ldots, T$.

For each vertex v of D and each symbol α let $I_{v,\alpha}$ denote the set of indices $i \in \{0, \ldots, T\}$ such that $v_i = v$ and such that y_i has maximum symbol α. Then there exists a vertex w and a symbol β such that $|I_{w,\beta}| \geq T/2nk$. Let L be the largest index in $I_{w,\beta}$. Since for all $i, i' \in I_{w,\beta}$ and $i < i'$ we have $y_i < y_{i'}$ (since y_i has maximum symbol β and satisfies $y_i \leq f_t(w)$ for each $i \in I_{w,\beta}$ and since $y_{i'} \not\leq y_i$), we know

$$|y_L| \geq |I_{w,\beta}| \geq T/2nk = 2^7 n^5 k^5 \sigma^4. \tag{32}$$

Let $M := 2^3n^2k^2\sigma$, $N := 2nk\sigma$ and $p := M\sigma$. Since $N = M/2nk$, there exists a vertex u and a symbol α such that $I_{u,\alpha}$ contains at least $N+1$ indices i satisfying $L - M \le i \le L$. Choose $N+1$ such indices $i_0 < i_1 < \cdots < i_N$. Define

$$x_0 := y_{i_0}, x_1 := y_{i_1}, \ldots, x_N := y_{i_N}. \tag{33}$$

For $j = 1,\ldots,N$ let C_j be the $u-u$ path $v_{i_0}, v_{i_0+1}, \ldots, v_{i_j-1}, v_{i_j}$. We show that C_N violates (24). Note that

$$|\phi(C_j)| \le (i_j - i_0)\sigma \le M\sigma = p \tag{34}$$

for each $j = 1,\ldots,N$.

Since $x_j \le f_t(u)$, we know that $x_0 < x_1 < \cdots < x_N$. Moreover,

$$x_0 < x_j \le \beta_{C_j}(x_0) \le \phi(C_j)^{-1}x_0 \tag{35}$$

for each $j = 1,\ldots,N$ (by Proposition 2).

By Proposition 2, $y_L \le \beta_Q(x_0) \le \phi(Q)^{-1}x_0$, where Q denotes the path v_{i_0}, v_{i_0+1}, ..., v_{L-1}, v_L. Hence

$$|x_0| \ge |y_L| - |\phi(Q)| \ge 2p^2 - p > (p+1)p \ge (p+1)|\phi(C_j)|. \tag{36}$$

Let $y_j := x_0^{-1}\phi(C_j)^{-1}x_0$. Then $x_0 < x_j \le \phi(C_j)^{-1}x_0$ and $|x_0| > (p+1)|\phi(C_j)|$ imply that $x_0 = r_jy_j^p$ for some $r_j \le x_0$ with maximum symbol α.

Let y_j have m_j symbols α and let x_0 have m symbols α. Write $x_0 = z_mz_{m-1}\ldots z_1$, where each z_i has maximum symbol α. (Such a decomposition is unique.) Since $x_0 = r_jy_j^p$ for each j, we know that $m \ge pm_j$ and that $z_i = z_{i'}$ if $i \equiv i' \pmod{m_j}$ and $i, i' \le pm_j$. Hence, since $p \ge \max\{m_1,\ldots,m_N\}$, for $m := \gcd\{m_1,\ldots,m_N\}$, $z_i = z_{i'}$ if $i \equiv i'$ $\pmod{m}$ and $i, i' \le pm$.

Let $a := z_mz_{m-1}\ldots z_1$ and $n_j := m_j/m$ for $m = 1,\ldots,N$. Then $y_j = a^{n_j}$ for each $j = 1,\ldots,N$.

Moreover,

$$n_1 < n_2 < \cdots < n_N. \tag{37}$$

For suppose that $n_{j+1} \le n_j$ for some $j = 1,\ldots,N-1$. Let C be the closed path satisfying $C_{j+1} = C_jC$. Then

$$\begin{aligned} x_{j+1} &\le \beta_C(x_j) \le \phi(C)^{-1}x_j = \phi(C_{j+1})^{-1}\phi(C_j)x_j \\ &= (\phi(C_{j+1})^{-1}x_0)(x_0^{-1}\phi(C_j)x_0)(x_0^{-1}x_j) = x_0y_{j+1}y_jx_0^{-1}x_j \\ &= x_0a^{n_{j+1}}a^{-n_j}(x_0^{-1}x_j). \end{aligned} \tag*{((38)}$$

This implies $|x_{j+1}| \le |x_0a^{n_{j+1}-n_j}| + |x_0^{-1}x_j| \le |x_0| + |x_0^{-1}x_j| = |x_j|$. (The inequality $|x_0a^{n_{j+1}-n_j}| \le |x_0|$ follows from the fact that $x_0 = fa^{n_j-n_{j+1}}$ for some f, since $0 \le m_j - m_{j+1} \le m_j \le M\sigma$.) This contradicts the fact that $x_{j+1} > x_j$, thus showing (37).

Now $|a| \le M\sigma/N = 2nk\sigma$, since $|a^{n_N}| = |y_N| \le M\sigma$ and $n_N \ge N$ (by (37)). By (24), there exists an $x \in G$ such that $x^{-1}\phi(C_N)x \in H(C_N)$. Hence there exists a $y \in G$ such that $y^{-1}a^{n_N}y \in H(C_N)^{-1}$. It is not difficult to see that we may assume that $y \le a$. Hence $a^{n_N-1} \in H(C_N)^{-1}$. Now by Proposition 2, $x_N \le \beta_{C_N}(x_0) \le \phi(C_N)^{-1}x_0a^{1-n_N}$, and hence $x_0^{-1}x_N \le x_0^{-1}\phi(C_N)^{-1}x_0a^{1-n_N} = a$. Therefore $|x_0^{-1}x_N| \le |a| \le 2nk\sigma$. Since $x_0 < x_1 < \cdots < x_N$ we know $|x_0^{-1}x_N| \ge N$. Hence $2nk\sigma \ge N = 4nk\sigma$, a contradiction. ∎

9. A POLYNOMIAL-TIME ALGORITHM FOR THE COHOMOLOGY FEASIBILITY PROBLEM FOR FREE PARTIALLY COMMUTATIVE GROUPS

We now describe the algorithm for the cohomology feasibility problem for free partially commutative groups. Let $D = (V, A)$ be a directed graph, let G be a free partially commutative group, let $\phi : A \longrightarrow G$ and let $H(a)$ be an ideal of G, for each $a \in A$. We assume that with each arc $a = (u, w)$ also $a^{-1} = (w, u)$ is an arc, with $\phi(a^{-1}) = \phi(a)^{-1}$ and $H(a^{-1}) = H(a)^{-1}$.

Let $\mathcal{U}$ be the collection of all functions $f : V \longrightarrow G$ such that for each arc $a = (u, w)$ there exist $x \ge f(u)$ and $z \ge f(w)$ satisfying $x^{-1}\phi(a)z \in H(a)$. For any given function f one can check in polynomial time whether f belongs to $\mathcal{U}$. Trivially, if $f \in \mathcal{U}$ and $g \le f$ then $g \in \mathcal{U}$. Moreover:

Proposition 4. *Let $f_1, \ldots, f_t$ be functions such that $f_i \vee f_j \in \mathcal{U}$ for all i, j. Then $f := f_1 \vee \cdots \vee f_t \in \mathcal{U}$.*

Let X be the set of pairs (u, x) where $u \in V$ and where x is join-irreducible such that there exists an arc $a = (u, w)$ with $x \le \phi(a)$. So X has size polynomially bounded by n and σ. For any $(u, x) \in X$, let $f_{u,x}$ be the function defined by

$$\begin{aligned} f_{u,x}(u) &:= x, \\ f_{u,x}(v) &:= 1 \text{ for all } v \ne u. \end{aligned} \tag{39}$$

Let E be the set of pairs $\{(u, x), (w, z)\}$ from X such that there exists an arc $a = (u, w)$ such that

$$\text{for all } x', z' \in G, \text{ if } (x')^{-1}\phi(a)z' \in H(a) \text{ then } x \le x' \text{ or } z \le z'. \tag{40}$$

Note that this holds if and only if $\phi(a) \notin \Delta_x H(a)\Delta_z^{-1}$, where for any join-irreducible element y, Δ_y is the left-ideal $\{y'|y' \not\geq y\}$. So (40) can be tested in polynomial time by (15) and (16).

Let E' be the collection of all pairs $\{(v,x),(v',x')\}$ from X such that $\bar{f}_{v,x} \vee \bar{f}_{v',x'}$ does not belong to $\mathcal{U}$ (possibly $(v,x) = (v',x')$).

Choose a subset Y of X such that $e \cap Y \neq \emptyset$ for each $e \in E$ and such that $e \not\subseteq Y$ for each pair $e \in E'$. This is a special case of the 2-satisfiability problem, and hence can be solved in polynomial time.

Proposition 5. *If no such Y exists, there is no feasible function.*

Proof. Suppose f is a feasible function. Then $Y := \{(v,x) \in X | v \in V, x \leq f(v)\}$ would have the required properties. ■

If we find Y, define f by:

$$f(v) := \bigvee \{\bar{f}_{v,x} | (v,x) \in Y\}. \tag{41}$$

Proposition 6. *f is a feasible function.*

Proof. Since $\bar{f}_{v,x} \vee \bar{f}_{v',x'} < \infty$ for each pair $\{(v,x),(v',x')\} \subseteq Y$, we know $f < \infty$. Moreover, f is the join of a finite number of pre-feasible functions, and hence f is pre-feasible. So by (18) it suffices to show that for each arc $a = (u,w)$:

(42)
(i) there exist $x \geq f(u)$ and $z \geq f(w)$ such that $x^{-1}\phi(a)z \in H(a)$;
(ii) there exist $x \leq f(u)$ and $z \leq f(w)$ such that $x^{-1}\phi(a)z \in H(a)$.

To see (42)(i), note that it is equivalent to: $f \in \mathcal{U}$. As $\bar{f}_{v,x} \vee \bar{f}_{v',x'} \in \mathcal{U}$ for all $(v,x),(v',x') \in Y$, Proposition 4. gives $f \in \mathcal{U}$.

To see (42)(ii), note that it is equivalent to:

$$\phi(a) \in H^{\downarrow}_{f(u)} H(a) (H^{\downarrow}_{f(w)})^{-1}. \tag{43}$$

Suppose (43) does not hold. Let b be the largest element in $H^{\downarrow}_{f(u)}H(a)$ satisfying $b \leq \phi(a)$. So by (15), $b^{-1}\phi(a) \notin (H^{\downarrow}_{f(w)})^{-1}$; that is, $\phi(a^{-1})b \not\leq f(w)$. Hence there exists a join-irreducible element $z \leq \phi(a^{-1})b$ such that $z \not\leq f(w)$. So $\phi(a^{-1})b \notin \Delta_z$ and hence by (15), $\phi(a) \notin H^{\downarrow}_{f(u)}H(a)\Delta_z^{-1}$. Note that since $b \leq \phi(a)$ and $z \leq \phi(a^{-1})b$ we have $z \leq \phi(a^{-1})$ and hence $(w,z) \in X$.

Let c be the largest element in $\Delta_z H^{-1}$ such that $c \leq \phi(a^{-1})$. By (15), $\phi(a)c \notin H^{\downarrow}_{f(u)}$; that is $\phi(a)c \not\leq f(u)$. Hence there exists a join-irreducible element x of $\phi(a)c$ such that $x \not\leq f(u)$. Again, since $c \leq \phi(a^{-1})$ and $x \leq \phi(a)c$ we have $x \leq \phi(a)$ and hence $(u,x) \in X$. As $\phi(a)c^{-1} \notin \Delta_x$, by (15) we know $\phi(a) \notin \Delta_x H \Delta_z^{-1}$. So $\{(u,x),(w,z)\} \in E$ and hence Y contains at least one of $(u,x),(w,z)$. So $x \leq f(u)$ or $z \leq f(w)$, a contradiction. ■

Thus we have proved:

Theorem 1. *The cohomology feasibility problem for free partially commutative groups is solvable in polynomial time.* ■

10. THE 2-SATISFIABILITY PROBLEM

In the algorithm we use a polynomial-time algorithm for the 2-satisfiability problem. Conversely, the 2-satisfiability problem can be seen as a special case of the cohomology feasibility problem for free groups. To see this, first note that any instance of the 2-satisfiability problem can be described as one of solving a system of inequalities in $\{0,1\}$ variables $x_1,\ldots,x_n$ of the form:

$$\begin{aligned} &x_i + x_j \geq 1 \text{ for each } \{i,j\} \in E, \\ &x_i + x_j \leq 1 \text{ for each } \{i,j\} \in E', \end{aligned} \tag{44}$$

where E and E' are given collections of pairs and singletons from $\{1,\ldots,n\}$. (So we allow $i = j$ in (44), yielding $2x_i \geq 1$ or $2x_i \leq 1$.)

Let G be the free group generated by the elements g and h. Make a directed graph with vertices $v_1,\ldots,v_n$ and with arcs:

(i) $a = (v_i,v_j)$, with $\phi(a) := ghg^{-1}$, for each $\{i,j\} \in E$; (45)

(ii) $a = (v_i,v_j)$, with $\phi(a) := h$, for each $\{i,j\} \in E'$.

Moreover, set $H(a) := \{w \in G|\ |w| \leq 2\}$ for each arc a.

Now the cohomology feasibility problem in this case is equivalent to solving (44) in $\{0,1\}$ variables. Indeed, if $x_1,\ldots,x_n$ is a solution of (44) then define $p(v_i) := g$ if $x_i = 1$ and $p(v_i) := 1$ if $x_i = 0$. Then p is a feasible function. Conversely, if p is a feasible function, define $x_i := 1$ if $p(v_i) \neq 1$ and the first symbol of $p(v_i)$ is equal to g, and $x_i := 0$ otherwise. Then $x_1,\ldots,x_n$ is a solution of (44).

11. A GOOD CHARACTERIZATION

One may derive from the algorithm a 'good' characterization of the feasibility of the cohomology feasibility problem for free partially commutative groups, i.e., one showing that the problem belongs to NP∩co-NP.

Theorem 2. *Let be given a directed graph $D = (V, A)$, a free partially commutative group G, a function $\phi : A \longrightarrow G$, and for each arc a, an ideal $H(a)$ of G. Then there exists a function $\psi : A \longrightarrow G$ such that ψ is cohomologous to ϕ and such that $\psi(a) \in H(a)$ for each arc a, if and only if*

for each vertex u and each two $u-u$ paths P, Q there exists an $x \in G$ such that $x^{-1}\phi(P)x \in H(P)$ and $x^{-1}\phi(Q)x \in H(Q)$. (46)

Remark 1. Condition (46) cannot be relaxed to requiring that for each cycle P there exists an $x \in G$ such that $x^{-1}\phi(P)x$ belongs to $H(P)$. To see this, let G be the free group generated by g and h. Let D be the directed graph with one vertex v and two loops, a and b, attached at v. Define $\phi(a) := h, H(a) := \{1, h, g, g^{-1}, g^{-1}h, hg\}$ and $\phi(b) := ghg^{-1}, H(b) := \{1, h, g, g^{-1}, hg^{-1}, gh\}$. If $x^{-1}\phi(a)x \in H(a)$ then the first symbol of x is not equal to g. If $x^{-1}\phi(b)x^{-1} \in H(b)$ then the first symbol of x is equal to g. So there is no x such that both hold.

On the other hand, for each path P there is an x such that $x^{-1}\phi(P)x \in H(P)$. Indeed, for each $k \in \mathcal{Z}$, $\phi(ab^k) \in H(ab^k)$ and $\phi(b^k a) \in H(b^k a)$. It follows that if P starts or ends with a or a^{-1}, then $\phi(P) \in H(P)$. Moreover, for each $k \in \mathcal{Z}$, $g^{-1}\phi(a^k b)g \in H(a^k b)$ and $g^{-1}\phi(ba^k)g \in H(ba^k)$. So if P starts and ends with b or b^{-1} then $g^{-1}\phi(P)g \in H(P)$. ∎

The fact that Theorem 2 is a good characterization relies on the facts that if the cohomology feasibility problem for free partially commutative groups has a solution, it has one of small size, and that if paths P, Q violating (46) would exist, there are such paths of polynomial length. (Both facts follow from the polynomial-time solvability of the subroutine.) We can check in polynomial time whether or not for given $u-u$ paths P and Q there exists an $x \in G$ such that $x\phi(P)x^{-1}$ belongs to $H(P)$ and $x\phi(Q)x^{-1}$ belongs to $\phi(Q)$. (By the fact that $H(P)$ and $H(Q)$ are ideals we have to consider for x only beginning segments of $\phi(P), \phi(P)^{-1}, \phi(Q), \phi(Q)^{-1}$. The number of such candidates for x is polynomially bounded.)

12. Directed graphs on surfaces and homologous functions

We now show how the results of the previous sections can be applied to disjoint paths and circuit problems in graphs embedded on surfaces. We first show how the polynomial-time solvability of the cohomology feasibility problem implies that, for each fixed k, the *k disjoint paths problem for directed planar graphs:*

given: a planar directed graph $D = (V, E)$ and k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D; (47)

find: k pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i $(i = 1, \ldots, k)$

is solvable in polynomial time.

Theorem 3. *For each fixed k, the k disjoint paths problem for directed planar graphs (47) is solvable in polynomial time.*

We sketch the proof. As group G we take the free group with k generators $g_1, \ldots, g_k$.

Let input $D = (V, A), r_1, s_1, \ldots, r_k, s_k \in V$ for (47) be given. We may assume that D is weakly connected, and that $r_1, s_1, \ldots, r_k, s_k$ are distinct, each being incident with exactly one arc. Fix an embedding of D in the plane, and let $\mathcal{F}$ denote the collection of faces of D.

Call two function $\phi, \psi : A \longrightarrow G$ *homologous* if there exists a function $f : \mathcal{F} \longrightarrow G$ such that

$$f(F)^{-1}\phi(a)f(F') = \psi(a) \tag{48}$$

for each arc a, where F and F' are the faces at the left-hand side and at the right-hand side of a, respectively.

For any solution $\Pi = (P_1, \ldots, P_k)$ of (47) let $\phi_\Pi : A \longrightarrow G$ be defined by:

$\phi_\Pi(a) := g_i$ if path P_i traverses a $(i = 1, \ldots, k)$, and (49)
$\phi_\Pi(a) := 1$ if a is not traversed by any of the P_i.

Now one can show:

For each fixed k we can find in polynomial time functions $\phi_1, \ldots, \phi_N : A \longrightarrow G$ with the property that for each solution Π of (47), ϕ_Π is homologous to at least one of $\phi_1, \ldots, \phi_N$. (50)

[This is the only reason why we can prove the polynomial-time solvability of problem (47) for fixed k only.]

It follows that it suffices to describe a polynomial-time method for the following problem:

$$\begin{aligned} &\text{given: a function } \phi : A \longrightarrow G; \\ &\text{find: a solution } \Pi \text{ of (47) such that } \phi_\Pi \text{ is homologous} \\ &\quad\text{to } \phi. \end{aligned} \tag{51}$$

Indeed, we can apply such an algorithm to each ϕ_j in (50). If we do not find Π for any ϕ_j, then (47) has no solution.

In order to solve (51) with the cohomology feasibility algorithm, we consider the *dual* graph $D^* = (\mathcal{F}, A^*)$ of D, having as vertex set the collection $\mathcal{F}$ of faces of D, while for any arc a of D there is an arc of D^*, denoted by a^*, from the face of D at the left-hand side of a to the face at the right-hand side of a. Define for any function ϕ on A the function ϕ^* on A^* by

$$\phi^*(a^*) := \phi(a) \tag{52}$$

for each arc a of D. Then any two functions ϕ and ψ are homologous (in D), if and only if ϕ^* and ψ^* are cohomologous (in D^*).

We extend the dual graph to the 'extended' dual graph $D^+ = (\mathcal{F}, A^+)$ by adding in each face of D^* all chords. (So D^+ need not be planar.) To be more precise, for any two vertices F, F' of D^* and any (undirected) $F - F'$ path π on the boundary of any face of D^*, extend D^* with an arc, called a_π, from F to F'. For any $\phi : A \longrightarrow G$ define $\phi^+ : A^+ \longrightarrow G$ by:

$$\begin{aligned} &\phi^+(a^*) := \phi^*(a^*) \text{ for each arc } a \text{ of } D; \\ &\phi^+(a_\pi) := \phi^*(\pi) \text{ for any path } \pi \text{ as above.} \end{aligned} \tag{53}$$

Moreover, let

$$\begin{aligned} &H(a^*) := \{1, g_1, \ldots, g_k\} \text{ and} \\ &H(a_\pi) := \{1, g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}. \end{aligned} \tag{54}$$

So each of these sets is an ideal in G.

Now let input ϕ of problem (51) be given. As the cohomology feasibility problem is solvable in polynomial time in this case, we can find in polynomial time a function ψ that is cohomologous to ϕ^+ in D^+, with $\psi(b) \in H(b)$ for each arc b of D^+, provided that such a ψ exists. If we find one, let P_i be any directed $r_i - s_i$ path traversing only arcs a satisfying $\psi(a^*) = g_i$ $(i = 1, \ldots, k)$. If such paths exist, they form a solution of (47).

If we do not find such a function ψ and such paths we may conclude that problem (51) has no solution. For suppose that ϕ_Π is homotopic to ϕ for some $\Pi = (P_1, \ldots, P_k)$. Then there exists a ψ as above, viz. $\psi := (\phi_\Pi)^+$. Moreover, for any ψ' cohomologous to $(\phi_\Pi)^+$ there exists for each $i = 1, \ldots, k$ a directed $r_i - s_i$ path P_i' traversing only arcs a such that g_i occurs in $\psi'(a^*)$. So we would find a solution, contradicting our assumption.

13. Further applications to disjoint paths

By extending (50) we obtain the following generalization. For any directed graph $G = (V, A)$ embedded on a compact surface, and any subset X of V, let $\tau(X)$ denote the minimum number t for which there exist faces $F_1, \ldots, F_t$ of D such that $X \subseteq \text{bd}(F_1 \cup \cdots \cup F_t)$. Then we have:

Theorem 4. *For each fixed t, the disjoint paths problem for directed planar graphs is solvable in polynomial time for inputs satisfying $\tau(\{r_1, s_1, \ldots, r_k, s_k\}) \leq t$.*

This can be shown as (50) can be extended to fixing $\tau(\{r_1, s_1, \ldots, r_k, s_k\})$ instead of fixing k. Theorem 4 extends a theorem of Robertson and Seymour [11] for undirected planar graphs with $t = 2$. (Recently, Ripphausen, Wagner, and Weihe [10] gave a *linear-time* algorithm if $t = 2$.)

One may also derive that the following problem:

given: a directed planar graph $D = (V, A)$, k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D, and subsets $A_1, \ldots, A_k$ of A; (55)

find: k pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i and uses only arcs in A_i $(i = 1, \ldots, k)$,

is solvable in polynomial time, for fixed k. This follows by restricting in (54) the $H(a^*)$ to those g_i for which A_i contains a.

More generally, consider the problem:

given: a directed planar graph $D = (V, A)$, k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D, subsets $A_1, \ldots, A_k$ of A, and a set E consisting of some pairs $\{i, j\}$ from $1, \ldots, k$; (56)

find: k pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i and uses only arcs in A_i $(i = 1, \ldots, k)$ and where P_i and P_j are disjoint if $\{i, j\} \notin E$.

Theorem 5. *For each fixed t, problem (56) is solvable in polynomial time, for inputs satisfying* $\tau(\{r_1, s_1, \ldots, r_k, s_k\}) \leq t$.

This follows as above from the polynomial-time solvability of the cohomology feasibility problem applied to the free partially commutative group with generators $g_1, \ldots, g_k$ and with relations $g_i g_j = g_j g_i$ for each $\{i, j\} \in E$. This result was shown for $t = 1$ by Ding, Schrijver, and Seymour [3].

A special case of this is the following *disjoint trees problem:*

given: a directed planar graph $D = (V, A)$ and k pairs $(r_1, S_1), \ldots, (r_k, S_k)$ with $r_1, \ldots, r_k \in V$ and $S_1, \ldots, S_k \subseteq V$; (57)

find: k pairwise vertex-disjoint rooted trees $T_1, \ldots, T_k$ in D, where T_i has root r_i and covers S_i $(i = 1, \ldots, k)$.

Theorem 6. *For each fixed t, problem (57) is solvable in polynomial time for inputs satisfying* $\tau(\{r_1, \ldots, r_k\} \cup S_1 \cup \cdots \cup S_k) \leq t$.

This follows from Theorem 5 by taking all pairs (r_i, s') with $i \in \{1, \ldots, k\}$ and $s' \in S_i$, and defining E so that directed paths will be disjoint if they correspond to distinct r_i and r_j. This was shown for undirected graphs in [15].

These polynomial-time solvability results for directed planar graphs, in fact also hold for directed graphs embedded on some compact surface S, as long as we keep this surface fixed. We mention the following:

given: a directed graph $D = (V, A)$ embedded on a compact surface S and k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D; (58)

find: k pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i $(i = 1, \ldots, k)$.

Theorem 7. *For each fixed compact surface S and each fixed t, problem (58) is solvable in polynomial time.*

One of the most general results in this direction concerns the following problem:

given: a directed graph $D = (V, A)$ embedded on a compact surface S, k pairs $(r_1, s_1), \dots, (r_k, s_k)$ of vertices of D, subsets $A_1, \dots, A_k$ of A, and a set E of pairs $\{i, j\}$ from $\{1, \dots, k\}$; (59)

find: k pairwise vertex-disjoint directed paths $P_1, \dots, P_k$ in D, where P_i runs from r_i to s_i and uses only arcs in A_i $(i = 1, \dots, k)$, and where P_i and P_j are vertex-disjoint if $\{i, j\} \notin E$.

Theorem 8. *For each fixed compact surface S and each fixed t, problem (59) is solvable in polynomial time for inputs satisfying $\tau(\{r_1, s_1, \dots, r_k, s_l\}) \leq t$.*

14. Disjoint closed curves in graphs on a compact surface

Let S be a compact surface. A *closed curve* on S is a continuous function $C : S^1 \longrightarrow S$, where S^1 is the unit circle in the complex plane. Two closed curves C and C' are called *freely homotopic,* in notation $C \sim C'$, if there exists a continuous function $\Phi : S^1 \times [0, 1] \longrightarrow S$ such that $\Phi(z, 0) = C(z)$ and $\Phi(z, 1) = C'(z)$ for each $z \in S^1$.

For any pair of closed curves C, D on S, let $\mathrm{cr}(C, D)$ denote the number of crossings of C and D, counting multiplicities. Moreover, $\mathrm{mincr}(C, D)$ denotes the minimum of $\mathrm{cr}(C', D')$ where C' and D' range over all closed curves freely homotopic to C and D, respectively. That is,

$$\mathrm{mincr}(C, D) := \min\{\mathrm{cr}(C', D') | C' \sim C, D' \sim D\}. \tag{60}$$

Let $G = (V, E)$ be an undirected graph embedded on S. (We identify G with its embedding on S.) For any closed curve D on S, $\mathrm{cr}(G, D)$ denotes the number of intersections of G and D (counting multiplicities):

$$\mathrm{cr}(G, D) := |\{z \in S^1 | D(z) \in G\}|. \tag{61}$$

The following was shown in [14] (motivated by [12]):

Theorem 9. *Let $G = (V, E)$ be an undirected graph embedded on a compact surface S and let $C_1, \ldots, C_k$ be pairwise disjoint simple closed curves on S, each non-nullhomotopic. Then there exist pairwise disjoint simple closed curves $C'_1, \ldots, C'_k$ in G such that $C'_i \sim C_i$ for $i = 1, \ldots, k$, if and only if for each closed curve D on S:*

$$\mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D), \tag{62}$$

with strict inequality if D is doubly odd.

Here we call a closed curve D on S *doubly odd* (with respect to G and $C_1, \ldots, C_k$) if D is the concatenation $D_1 \cdot D_2$ of two closed curves D_1 and D_2 such that $D_1(1) = D_2(1) \notin G$ and such that

$$\mathrm{cr}(G, D_j) \not\equiv \sum_{i=1}^{k} \mathrm{cr}(C_i, D_j) \pmod 2, \tag{63}$$

for $j = 1, 2$.

It is easy to see that the condition in the theorem is necessary — the essence is sufficiency. The theorem can be derived from Theorem 2 by considering simple closed curves $C_{k+1}, \ldots, C_t$ such that each component of $S \setminus (C_1 \cup \cdots \cup C_t)$ is an open disk and such that $C_1, \ldots, C_t$ have a minimum number of mutual crossings. Let $E := \{\{i, j\} | C_i \text{ and } C_j \text{ cross}\}$. We can find with a method similar to that described in Section 12 simple closed curves $C'_1, \ldots, C'_t$ in G such that $C'_i \sim C_i$ $(i = 1, \ldots, t)$ and such that C'_i and C'_j are disjoint if $\{i, j\} \notin E$. Then $C'_1, \ldots, C'_k$ are the closed curves as required.

The theorem can be extended to directed circuits in directed graphs embedded on a compact orientable surface, although the condition becomes more difficult to formulate (for the torus, see Seymour [18]). In any case, the method yields a polynomial-time algorithm finding the directed circuits.

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