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006

# FINDING $k$ DISJOINT PATHS IN A DIRECTED PLANAR GRAPH*

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**Abstract.** It is shown that, for each fixed $k$, the problem of finding $k$ pairwise vertex-disjoint directed paths between given pairs of terminals in a directed planar graph is solvable in polynomial time.

**Key words.** disjoint, path, directed, planar, graph, polynomial-time, algorithm, free group, homologous, cohomologous

**AMS subject classifications.** 05C20, 05C85, 68R10

**1. Introduction and statement of result.** In this paper we show that the following problem, the *k disjoint paths problem for directed planar graphs*, is solvable in polynomial time, for any fixed $k$:

(1)
given : a directed planar graph $D = (V, A)$ and $k$ pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of $D$;
find : $k$ pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in $D$, where $P_i$ runs from $r_i$ to $s_i (i = 1, \ldots, k)$.

The problem is NP-complete if we do not fix $k$ (even in the undirected case; Lynch [2]). Moreover, it is NP-complete for $k = 2$ if we delete the planarity condition (Fortune, Hopcroft, and Wyllie [1]). This is in contrast to the undirected case (for those believing NP≠P), where Robertson and Seymour [4] showed that, for any fixed $k$, the $k$ disjoint paths problem is polynomial-time solvable for any graph (not necessarily planar).

In this paper we do not aim at obtaining the best possible running time bound, as we presume that there are much faster (but possibly more complicated) methods for (1) than the one we describe in this paper. In fact, recently Reed, Robertson, Schrijver, and Seymour [3] showed that for undirected planar graphs the $k$ disjoint paths problem can be solved in *linear* time, for any fixed $k$. This algorithm makes use of methods from Robertson and Seymour's theory of graph minors. A similar algorithm for directed planar graphs might exist but probably would require extending parts of graph minors theory to the directed case.

Our method is based on cohomology over free (nonabelian) groups. For the $k$ disjoint paths problem we use free groups with $k$ generators. It extends methods given in [5] for undirected graphs on surfaces based on homotopy. Cohomology is in a sense dual to homology and can be defined in any directed graph, even if it is not embedded on a surface. We apply cohomology to an *extension* of the planar graph dual of $D$—just applying homology to $D$ itself seems not powerful enough.

We remark that in our approach free groups and (co)homology are used mainly as a framework to formulate certain ideas smoothly; they give us a convenient tool for recording shifts of curves over the plane. No deep group theory or topology is used. We could avoid free groups and cohomology by adopting a more complex notation and terminology; that would, however, implicitly mimic free groups and cohomology. The present approach also readily allows application of the algorithm where the embedding of the graph in the plane is given combinatorially, that is, by a list of the cycles that bound the faces of the graph.

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## 2. The cohomology feasibility problem.

**2.1. Free groups.** The *free group* $G_k$, generated by the *generators* $g_1, g_2, \ldots, g_k$, consists of all words $b_1 b_2 \cdots b_t$ where $t \geq 0$ and $b_1, \ldots, b_t \in \{g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}$ such that $b_i b_{i+1} \neq g_j g_j^{-1}$ and $b_i b_{i+1} \neq g_j^{-1} g_j$ for $i = 1, \ldots, t-1$ and $j = 1, \ldots, k$. The product $x \cdot y$ of two such words is obtained from the concatenation $xy$ by deleting iteratively all occurrences of any $g_j g_j^{-1}$ and $g_j^{-1} g_j$. (So in our notation $x \cdot y \neq xy$ in general.) This defines a group, with unit element 1 equal to the empty word $\emptyset$. We call $g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}$ the *symbols*. The *size* $|x|$ of a word $x$ is the number of symbols occurring in it, counting multiplicities.

A word $y$ is called a *segment* of a word $w$ if $w = xyz$ for certain words $x, z$. It is called a *beginning segment* if $x = 1$ and an *end segment* if $z = 1$. A subset $\Gamma$ of a free group is called *hereditary* if for each word $y \in \Gamma$, each segment of $y$ belongs to $\Gamma$.

We define a partial order $\leq$ on $G_k$ by

$$x \leq y \Leftrightarrow x \text{ is a beginning segment of } y. \tag{2}$$

This gives a lattice if we extend $G_k$ with an element $\infty$ at infinity. We denote the meet and join by $\wedge$ and $\vee$. So $x \wedge y$ is equal to the longest common beginning segment of $x$ and $y$. Moreover, $x \vee y = \infty$ except if $x \leq y$ or $y \leq x$.

We make two easy observations.

LEMMA 2.1. *Let $\alpha$ be a symbol, and let $x, z \in G_k$. If $x \leq \alpha \cdot z$ and $z \leq \alpha^{-1} \cdot x$, then $x^{-1} \cdot \alpha \cdot z = 1$ or $x = z = 1$.*

*Proof.* Let $y := x^{-1} \cdot \alpha \cdot z$, and suppose that $y \neq 1$. Since $x \leq \alpha \cdot z$, it follows that $\alpha \cdot z = xy'$ for some $y'$, and hence $y = x^{-1} \cdot \alpha \cdot z = x^{-1} \cdot (xy') = y'$. Consequently $xy \in G_k$; and since $z \leq \alpha^{-1} \cdot x$, it follows similarly that $zy^{-1} \in G_k$, that is, $yz^{-1} \in G_k$. Since $y \neq 1$, this implies that $xyz^{-1} \in G_k$, and so $\alpha = x \cdot y \cdot z^{-1} = xyz^{-1}$. In particular, $1 = |\alpha| = |x| + |y| + |z| \geq |x| + 1 + |z|$. Therefore, $x = z = 1$. □

LEMMA 2.2. *Let $x, y \in G_k$. If $x \not\leq y$, then the last symbol of $x$ is equal to the last symbol of $y^{-1} \cdot x$.*

*Proof.* Let $z := x \wedge y$. Write $x = zx'$ and $y = zy'$, where $x' \neq 1$. Let $\alpha$ be the first symbol of $x'$. Since $z = x \wedge y$, we know $\alpha \not\leq y'$. Hence $(y')^{-1} \cdot x' = (y')^{-1} x'$ (i.e., no cancellation). Consequently,

$$y^{-1} \cdot x = ((y')^{-1} z^{-1}) \cdot (zx') = (y')^{-1} \cdot x' = (y')^{-1} x'. \tag{3}$$

Hence, as $x' \neq 1$, the last symbol of $x$ is equal to the last symbol of $y^{-1} \cdot x$. □

**2.2. The cohomology feasibility problem for free groups.** Let $D = (V, A)$ be a weakly connected directed graph, let $r \in V$, and let $(G, \cdot)$ be a group. (We allow directed graphs to have parallel arcs.) Two functions $\phi, \psi : A \longrightarrow G$ are called *$r$-cohomologous* if there exists a function $f : V \longrightarrow G$ such that

$$\begin{aligned} &\text{(i) } f(r) = 1; \\ &\text{(ii) } \psi(a) = f(u)^{-1} \cdot \phi(a) \cdot f(w) \text{ for each arc } a = (u, w). \end{aligned} \tag{4}$$

One easily checks that this gives an equivalence relation.

Consider the following *cohomology feasibility problem for free groups*:

(5)
given : a weakly connected directed graph $D = (V, A)$, a vertex $r$, a function $\phi : A \longrightarrow G_k$, and for each $a \in A$ a hereditary subset $\Gamma(a) of G_k$;
find : a function $\psi : A \longrightarrow G_k$ such that $\psi$ is $r$-cohomologous to $\phi$ and such that $\psi(a) \in \Gamma(a)$ for each arc $a$.

We show the following.

THEOREM 2.3. *The cohomology feasibility problem for free groups is solvable in time bounded by a polynomial in* $|A| + \sigma + k$.

Here $\sigma$ is the maximum size of the words $\phi(a)$ and the words in the $\Gamma(a)$. (In fact we can drop $k$ and assume that $G_k$ is the free group generated by the generators occurring in the $\phi(a)$ and the words in the $\Gamma(a)$.)

Note that, by the definition of $r$-cohomologous, equivalent to finding a $\psi$ as in (5) is finding a function $f : V \longrightarrow G_k$ satisfying

(6)
(i) $f(r) = 1$;
(ii) $f(u)^{-1} \cdot \phi(a) \cdot f(w) \in \Gamma(a)$ for each arc $a = (u, w)$.

We call such a function *f feasible.*

In solving the cohomology feasibility problem for free groups we may assume

(7)
(i) $\Gamma(a) \neq \emptyset$ for each arc $a$;
(ii) $|\phi(a)| \leq 1$ for each arc $a$;
(iii) with each arc $a = (u, w)$ also $a^{-1} = (w, u)$ is an arc, with $\phi(a^{-1}) = \phi(a)^{-1}$ and $\Gamma(a^{-1}) = \Gamma(a)^{-1}$.

Here $\Gamma(a)^{-1} := \{x^{-1} | x \in \Gamma(a)\}$. Condition (7)(ii) can be attained by replacing any arc $a = (u, w)$ such that $\phi(a) = \beta_1 \cdots \beta_t$ and $t \geq 2$ by a $u - w$ path $a_1 \cdots a_t$ with $\phi(a_i) := \beta_i$ $(i = 1, \ldots, t)$ and $\Gamma(a_1) := \Gamma(a)$ and $\Gamma(a_i) := \{1\}$ $(i = 2, \ldots, t)$. (Here and below we indicate a path $P$ by the string of arcs traversed by $P$ (in the order traversed by $P$). If $P$ traverses an arc $a$ in the backward direction, then we denote this in the string by $a^{-1}$. For instance, $P = a_1 a_2^{-1} a_3$ means that $P$ traverses first $a_1$ in the forward direction, next $a_2$ in the backward direction, and finally $a_3$ in the forward direction. The arcs need not be distinct.)

**2.3. Pre-feasible functions.** Let input $D = (V, A), r, \phi, \Gamma$ for the cohomology feasibility problem for free groups (5) be given, assuming (7). We call a function $f : V \longrightarrow G_k$ *pre-feasible* if

(8)
(i) $f(r) = 1$;
(ii) for each arc $a = (u, w)$ with $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$ one has $f(u) = f(w) = 1$.

Define a partial order $\leq$ on the set $G_k^V$ of all functions $f : V \longrightarrow G_k$ by

(9) $$f \leq g \Leftrightarrow f(v) \leq g(v) \quad \text{for each } v \in V. \quad \square$$

It is easy to see that $G_k^V$ forms a lattice if we add an element $\infty$ at infinity. Let $\wedge$ and $\vee$ denote the meet and join, respectively.

Pre-feasibility behaves nicely with respect to the lattice:

PROPOSITION 1. *If $f_1$ and $f_2$ are pre-feasible, then so is $f := f_1 \wedge f_2$.*

*Proof.* Clearly $f(r) = 1$. Let $a = (u, w)$ be an arc such that $y := f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$ while not $f(u) = f(w) = 1$. By Lemma 2.1 and by symmetry we may assume that $f(u) \not\leq \phi(a) \cdot f(w)$. Let $x$ and $x'$ be such that $f_1(u) = f(u)x$ and $f_2(u) = f(u)x'$, and let $z$ and $z'$ be such that $f_1(w) = f(w)z$ and $f_2(w) = f(w)z'$. Let $\alpha$ and $\beta$ be the first and last symbol, respectively, of $y$. Since $z_1 \wedge z_2 = 1$, we know $\beta^{-1} \not\leq z$ or $\beta^{-1} \not\leq z'$. Without loss of generality, $\beta^{-1} \not\leq z$.

Since $f(u) \not\leq \phi(a) \cdot f(w)$, by Lemma 2.2 the first symbol of $f(u)^{-1}$ is equal to $\alpha$. So $\alpha \not\leq x$, and hence

$$f_1(u)^{-1} \cdot \phi(a) \cdot f_1(w) = x^{-1} \cdot y \cdot z = x^{-1}yz. \tag{10}$$

So $y$ is a segment of $f_1(u)^{-1} \cdot \phi(a) \cdot f_1(w)$. By the heredity of $\Gamma(a)$ this implies that $f_1(u)^{-1} \cdot \phi(a) \cdot f_1(w) \notin \Gamma(a)$. So, as $f_1$ is pre-feasible, $f_1(u) = f_1(w) = 1$. Therefore $f(u) = f(w) = 1$. □

So for any function $f : V \longrightarrow G_k$ there exists a smallest pre-feasible function $\bar{f} \geq f$, provided there exists at least one pre-feasible function $g \geq f$. If no such $g$ exists, we set $\bar{f} := \infty$. We observe the following proposition.

PROPOSITION 2. *If $\bar{f}$ is finite, then*

(11)
(i) $f(r) = 1$ *and* $|f(v)| < 2\sigma|V|$ *for each vertex* $v$;

(ii) *for each arc* $a = (u, w)$ : *if* $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$, *then* $f(u) \leq \phi(a) \cdot f(w)$ *or* $f(w) \leq \phi(a^{-1}) \cdot f(u)$.

*Proof.* Clearly, $f(r) \leq \bar{f}(r) = 1$. Moreover, by induction on the minimum number $t$ of arcs in any $r - v$ path one shows $|\bar{f}(v)| \leq 2\sigma t$. Indeed, if $a = (u, v)$ is the last arc in the path, then $y := \bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(v)$ belongs to $\Gamma(a)$ or is equal to $\phi(a)$ and, hence, has size at most $\sigma$. So

$$|\bar{f}(v)| = |\bar{f}(u) \cdot \phi(a)^{-1} \cdot y| \leq |\bar{f}(u)| + |\phi(a)| + \sigma \leq 2\sigma(t-1) + 1 + \sigma \leq 2\sigma t. \tag{12}$$

This implies $|f(v)| \leq |\bar{f}(v)| < 2\sigma|V|$.

To see (ii), suppose $f(u) \not\leq \phi(a) \cdot f(w)$ and $f(w) \not\leq \phi(a^{-1}) \cdot f(u)$. The first implies (by Lemma 2.2) that the first symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the first symbol of $f(u)^{-1}$. The second implies (again by Lemma 2.2) that the last symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the last symbol of $f(w)$. Since $f \leq \bar{f}$, it follows that $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is a segment of $\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)$. So $\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w) \notin \Gamma(a)$. Hence, as $\bar{f}$ is pre-feasible, $\bar{f}(u) = \bar{f}(w) = 1$, and therefore $f(u) = 1$. This contradicts the fact that $f(u) \not\leq \phi(a) \cdot f(w)$. □

**2.4. A subroutine finding $\bar{f}$.** Let input $D = (V, A), r, \phi, \Gamma$ for the cohomology feasibility problem for free groups (5) be given, again assuming (7). We describe a polynomial-time subroutine that outputs $\bar{f}$ for any given $f : V \longrightarrow G_k$.

If $f$ is pre-feasible, output $\bar{f} := f$. If $f$ violates (11), output $\bar{f} := \infty$. Otherwise choose an arc $a = (u, w)$ satisfying $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$ and $f(w) \not\leq \phi(a^{-1}) \cdot f(u)$ (as $f$ is not pre-feasible and satisfies (11), such an arc exists by Lemma 2.1). Perform the following:

*Iteration*: Write $\phi(a) \cdot f(w) = xy$, with $y \in \Gamma(a)$ and $|y|$ as large as possible, reset $f(u) := x$, and start anew.

PROPOSITION 3. *In the iteration, resetting $f$ increases $|f(u)|$ and does not change $\bar{f}$.*

*Proof.* Consider the iteration. Denote by $f'$ the reset $f$. As (11)(ii) holds, $f(u) \leq \phi(a) \cdot f(w)$. Since $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$, $f(u)$ is a segment of $x$ with $f(u) \neq x$. So $|f'(u)| > |f(u)|$.

To see $\bar{f'} = \bar{f}$, we must show $f' \leq \bar{f}$, that is, $f'(u) \leq \bar{f}(u)$ if $\bar{f}$ is finite. Suppose that $\bar{f}$ is finite and that $f'(u) \not\leq \bar{f}(u)$. Let $\beta$ be the last symbol of $x = f'(u)$. As $x \not\leq \bar{f}(u)$ and as $\phi(a) \cdot f(w) = xy$, $\beta y$ is an end segment of $\bar{f}(u)^{-1} \cdot \phi(a) \cdot f(w)$.

Since $\phi(a) \cdot f(w) \not\leq \bar{f}(u)$ (as $x \leq \phi(a) \cdot f(w)$), by Lemma 2.2 the last symbol of $\bar{f}(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the last symbol of $\phi(a) \cdot f(w)$. Since $\phi(a) \cdot f(w) \not\leq f(u)$ (as $x \leq \phi(a) \cdot f(w)$ and $f(u) \leq \bar{f}(u)$), by Lemma 2.2 the last symbol of $\phi(a) \cdot f(w)$ is equal to the last symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$. Since $f(w) \not\leq \phi(a^{-1}) \cdot f(u)$, by Lemma 2.2 the last symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the last symbol of $f(w)$. Concluding, the last symbol of $\bar{f}(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the last symbol of $f(w)$. Hence $\bar{f}(u)^{-1} \cdot \phi(a) \cdot f(w)$ is a beginning segment of $\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)$. So $\beta y$ is a segment of $\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)$, and hence $\beta y$ belongs to $\Gamma(a)$. This contradicts the maximality of $y$. □

Since at each iteration $|f(u)|$ increases for some vertex $u$, after at most $2\sigma|V|^2$ iterations either we get a prefeasible function $f$ or (11) is violated. Thus the subroutine is polynomial time.

**2.5. Algorithm for the cohomology feasibility problem for free groups.** Let input $D = (V, A), r, \phi, \Gamma$ for the cohomology feasibility problem for free groups (5) be given. We find a feasible function $f$ as follows.

Again we may assume (7). For every $a = (u, w) \in A$ let $f_a$ be the function defined by $f_a(u) := \phi(a)$ and $f_a(v) := 1$ for each $v \neq u$. Let $E$ be the set of pairs $\{a, a'\}$ from $A$ for which $\bar{f}_a \vee \bar{f}_{a'}$ is finite and pre-feasible. Let $E'$ be the set of pairs $\{a, a^{-1}\}$ with $a \in A$ and $\phi(a) \notin \Gamma(a)$.

We search for a subset $X$ of $A$ such that each pair in $X$ belongs to $E$ and such that $X$ intersects each pair in $E'$. This is a special case of the two-satisfiability problem and, hence, can be solved in polynomial time.

PROPOSITION 4. *If $X$ exists, then the function $f := \bigvee_{a \in X} \bar{f}_a$ is feasible. If $X$ does not exist, then there is no feasible function.*

*Proof.* First assume that $X$ exists. Let $f$ be as given. Since $\bar{f}_a \vee \bar{f}_{a'}$ is finite and pre-feasible for each two $a, a'$ in $X$, $f$ is finite and $f(r) = 1$. Moreover, suppose $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$ for some arc $a = (u, w)$. Let $f(u) = \bar{f}_{a'}(u)$ and $f(w) = \bar{f}_{a''}(w)$ for $a', a'' \in X$. As $\bar{f}_{a'} \vee \bar{f}_{a''}$ is pre-feasible, $\bar{f}_{a'}(u) = \bar{f}_{a''}(w) = 1$. So $\phi(a) \notin \Gamma(a)$, and hence $a$ or $a^{-1}$ belongs to $X$. By symmetry we may assume $a \in X$. Then

$$\phi(a) = f_a(u) \leq \bar{f}_a(u) \leq f(u) = \bar{f}_{a'}(u) = 1, \tag{13}$$

a contradiction.

Assume conversely that there exists a feasible function $f$. Let $X$ be the set of arcs $a = (u, w)$ with the property that $\phi(a) \leq f(u)$. Then $X$ intersects each pair in $E'$. For let $a = (u, w)$ be an arc satisfying $a \notin X$ and $a^{-1} \notin X$, that is, $\phi(a) \not\leq f(u)$ and $\phi(a^{-1}) \not\leq f(w)$. Hence $f(u)^{-1} \cdot \phi(a) \cdot f(w) = f(u)^{-1}\phi(a)f(w)$. So $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ contains $\phi(a)$ as a segment (as $|\phi(a)| \leq 1$). So $\phi(a) \in \Gamma(a)$ and hence $\{a, a^{-1}\} \notin E'$.

Moreover, each pair in $X$ belongs to $E$. For let $\{a', a''\}$ be a pair in $X$. We show that $\{a', a''\} \in E$, that is, $f' := \bar{f}_{a'} \vee \bar{f}_{a''}$ is finite and pre-feasible. As $a' \in X$, we have $\phi(a') \leq f(u)$ and hence $f_{a'} \leq f$, implying $\bar{f}_{a'} \leq f$. Similarly, $\bar{f}_{a''} \leq f$. So $f'$ is finite and $f'(r) = 1$. Consider an arc $a = (u, w)$ with $y := f'(u)^{-1} \cdot \phi(a) \cdot f'(w) \notin \Gamma(a)$. We may assume $f'(u) = \bar{f}_{a'}(u)$ and $f'(w) = \bar{f}_{a''}(w)$ (since $\bar{f}_{a'}$ and $\bar{f}_{a''}$ are pre-feasible). To show $f'(u) = f'(w) = 1$, by Lemma 2.1 we may assume $f'(w) \not\leq \phi(a^{-1}) \cdot f'(u)$.

First assume $f'(u) \not\leq \phi(a) \cdot f'(w)$. Then by Lemma 2.2 the first and the last symbols of $y$ are equal to the first symbol of $f'(u)^{-1}$ and the last symbol of $f'(w)$, respectively. Since $f' \leq f$, this implies that $y$ is a segment of $f(u)^{-1} \cdot \phi(a) \cdot f(w) \in \Gamma(a)$. This contradicts the heredity of $\Gamma(a)$ as $y \notin \Gamma(a)$.

Second assume $f'(u) \leq \phi(a) \cdot f'(w)$. So $\phi(a) \cdot f'(w) = f'(u)y$ for some $y$. Since $\bar{f}_{a''}(u) \leq f'(u)$, it follows that $y$ is an end segment of

(14) $$\bar{f}_{a''}(u)^{-1}\cdot(f'(u)y)=\bar{f}_{a''}(u)^{-1}\cdot\phi(a)\cdot f'(w)=\bar{f}_{a''}(u)^{-1}\cdot\phi(a)\cdot\bar{f}_{a''}(w).$$

So $\bar{f}_{a''}(u)^{-1}\cdot\phi(a)\cdot\bar{f}_{a''}(w)\not\in\Gamma(a)$, since $y\not\in\Gamma(a)$. As $\bar{f}_{a''}$ is pre-feasible, this implies $\bar{f}_{a''}(u)=\bar{f}_{a''}(w)=1$; so $f'(w)=1$. Hence $f'(u)\leq\phi(a)$ and therefore, since $y\not\in\Gamma(a)$ and $|\phi(a)|\lneq 1$, $f'(u)=1$. □

Thus we have proved Theorem 2.3.

## 3. The $k$ disjoint paths problem for directed planar graphs.

**3.1. Directed planar graphs, $R$-homology, and flows.** Let input $D=(V,A)$, $r_1,s_1,\dots,r_k,s_k$ for problem (1) be given. In solving (1) we may assume that $D$ is weakly connected and that for each $i=1,\dots,k$, $r_i$ is incident with exactly one arc, which is leaving $r_i$, and $s_i$ is incident with exactly one arc, which is entering $s_i$. We fix an embedding of $D$. Let $\mathcal{F}$ denote the collection of faces of $D$ and let $R$ be the unbounded face of $D$.

Call two functions $\phi,\psi:A\longrightarrow G_k$ *$R$-homologous* if there exists a function $f:\mathcal{F}\longrightarrow G_k$ such that

(i) $f(R)=1$;

(15) (ii) $f(F)^{-1}\cdot\phi(a)\cdot f(F')=\psi(a)$ for each arc $a$, where $F$ and $F'$ are the faces at the left-hand side and right-hand side of $a$, respectively.

The relation to cohomologous is direct by duality. The *dual* graph $D^*=(\mathcal{F},A^*)$ of $D$ has as vertex set the collection $\mathcal{F}$ of faces of $D$, while for any arc $a$ of $D$ there is an arc $a^*$ of $D^*$ from the face of $D$ at the left-hand side of $a$ to the face at the right-hand side. Define for any function $\phi$ on $A$ the function $\phi^*$ on $A^*$ by $\phi^*(a^*):=\phi(a)$ for each $a\in A$. Then $\phi$ and $\psi$ are $R$-homologous (in $D$) if and only if $\phi^*$ and $\psi^*$ are $R$-cohomologous (in $D^*$).

For any solution $\Pi=(P_1,\dots,P_k)$ of (1) let $\psi_\Pi:A\longrightarrow G_k$ be defined by $\psi_\Pi(a):=g_i$ if path $P_i$ traverses $a$ ($i=1,\dots,k$), and $\psi_\Pi(a):=1$ if $a$ is not traversed by any of the $P_i$.

Call a function $\phi:A\longrightarrow G_k$ a *flow* if for each vertex $v\in V$ with $v\not\in\{r_1,s_1,\dots,r_k,s_k\}$ one has

(16) $$\phi(a_1)^{\varepsilon_1}\cdot\phi(a_2)^{\varepsilon_2}\cdot\dots\cdot\phi(a_n)^{\varepsilon_n}=1,$$

where $a_1,\dots,a_n$ are the arcs incident with $v$, in clockwise order, where $\varepsilon_i=+1$ if $a_i$ has its tail at $v$ and $\varepsilon_i=-1$ if $a_i$ has its head at $v$ (if $a_i$ happens to be a loop we take $\varepsilon_i=+1$ and $\varepsilon_i=-1$ at the corresponding positions in (16)), and if moreover for any arc $a$ incident with $r_i$ or $s_i$ one has $\phi(a)=g_i$ ($i=1,\dots,k$).

Clearly, if $\Pi$ is a solution of (1), then $\psi_\Pi$ is a flow. Moreover, if $\phi$ is a flow and $\phi'$ is $R$-homologous to $\phi$, then also $\phi'$ is a flow.

**3.2. Deriving a solution from a flow.** We first show the next proposition.

PROPOSITION 5. *There exists a polynomial-time algorithm that, for any flow $\phi$, either finds a solution $\Pi$ of* (1) *or concludes that there does not exist a solution $\Pi$ of* (1) *such that $\psi_\Pi$ is $R$-homologous to $\phi$.*

[Here *polynomial-time* means: polynomial-time in the size of $D$ and the maximum size of the $\phi(a)$. Note that it is not required that if we find a solution $\Pi$ of (1), then $\psi_\Pi$ is $R$-homologous to $\phi$.]

*Proof.* Let $\phi$ be a flow. Consider the dual graph $D^*=(\mathcal{F},A^*)$ of $D$. We construct the 'extended' dual graph $D^+=(\mathcal{F},A^+)$ by adding in each face of $D^*$ all chords. (So $D^+$ need not be planar.) More precisely, for any two vertices $F$, $F'$ of $D^*$ and any (undirected) $F-F'$

path $\pi$ on the boundary of any face of $D^*$, extend the graph with an arc, denoted by $a_\pi$, from $F$ to $F'$. Define $\phi^+ : A^+ \longrightarrow G_k$ by

(17)
(i) $\phi^+(a^*) := \phi(a)$ for each arc $a$ of $D$;
(ii) $\phi^+(a_\pi) := \phi(a_1)^{\varepsilon_1} \cdot \dots \cdot \phi(a_t)^{\varepsilon_t}$ for any path $\pi = (a_1^*)^{\varepsilon_1} \cdots (a_t^*)^{\varepsilon_t}$ with $\varepsilon_1, \dots, \varepsilon_t \in \{+1, -1\}$.

(As before, $(a_i^*)^{-1}$ means that $\pi$ traverses $a_i^*$ in the backward direction.) Moreover, let $\Gamma(a^*) := \{1, g_1, \dots, g_k\}$ and $\Gamma(a_\pi) := \{1, g_1, g_1^{-1}, \dots, g_k, g_k^{-1}\}$.

By Theorem 2.3 we can find in polynomial time a function $\psi$ that is $R$-cohomologous to $\phi^+$ in $D^+$, with $\psi(b) \in \Gamma(b)$ for each arc $b$ of $D^+$, provided that such a $\psi$ exists. If we find such a $\psi$, let $P_i$ be any directed $r_i - s_i$ path traversing only arcs $a$ satisfying $\psi(a^*) = g_i$ (for $i = 1, \dots, k$). (Such paths exist since $\phi$ is a flow.) Then $P_1, \dots, P_k$ form a solution to (1). Indeed, $P_1, \dots, P_k$ are vertex-disjoint, for suppose that there exist arcs $a$ and $b$ of $D$ that are both incident with a vertex $v$ and $\psi(a^*) = g_i^{\pm 1}$, $\psi(b^*) = g_j^{\pm 1}$, and $i \neq j$. Consider a shortest path $\pi$ along the face of $D^*$ corresponding to $v$ such that $\pi$ contains arcs $a^*$ and $b^*$. We may assume that we have chosen $a$ and $b$ such that $\pi$ is as short as possible. Then $|\psi(a_\pi)| \geq 2$, as $\psi(a_\pi)$ contains both $g_i^{\pm 1}$ and $g_j^{\pm 1}$ (neither of them can be cancelled, since $a$ and $b$ are chosen so that $\pi$ is shortest). This contradicts the fact that $\psi(a_\pi) \in \Gamma(a_\pi)$.

If we do not find such a function $\psi$, we may conclude that there does not exist a solution $\Pi$ of (1) such that $\psi_\Pi$ is $R$-homologous to $\phi$, since otherwise the cohomology feasibility problem has a solution, viz. $\psi := (\psi_\Pi)^+$. □

**3.3. Enumerating homology types.** In this section we show the following.

PROPOSITION 6. *For each fixed $k$, we can find in polynomial time flows $\phi_1, \dots, \phi_N$ with the property that for each solution $\Pi$ of (1), $\psi_\Pi$ is $R$-homologous to at least one of $\phi_1, \dots, \phi_N$.*

*Proof.* Consider systems $\Pi = (P_1, \dots, P_k)$ satisfying:

(18)
(i) $P_i$ is an undirected path from $r_i$ to $s_i$, not traversing the same edge more than once, and not having any self-crossing ($i = 1, \dots, k$);
(ii) $P_i$ and $P_j$ are edge-disjoint and do not have any crossing ($i, j = 1, \dots, k; i \neq j$).

(An *undirected* path is a path that may traverse arcs in the backward direction.)

For any such system $\Pi$, define $\psi_\Pi : A \longrightarrow G_k$ by $\psi_\Pi(a) := g_i$ if $P_i$ traverses $a$ in the forward direction, $\psi_\Pi(a) := g_i^{-1}$ if $P_i$ traverses $a$ in the backward direction ($i = 1, \dots, k$), and $\psi_\Pi(a) := 1$ if $a$ is not traversed by any $P_i$.

We will show that for each fixed $k$, we can find in polynomial time flows $\phi_1, \dots, \phi_N$ with the property that

(19) for each $\Pi$ satisfying (18), $\psi_\Pi$ is $R$-homologous to at least one of $\phi_1, \dots, \phi_N$.

This is stronger than what we need to show.

Consider a nonloop arc $a'$ not incident with any $r_i$ or $s_i$. Contract $a'$, yielding graph $D'$. Let $\phi'_1, \dots, \phi'_N$ be flows in $D'$ satisfying (19) with respect to $D'$. Then for each $j$ there is a unique flow $\phi_j$ in $D$ such that $\phi_j(a) = \phi'_j(a)$ for each arc $a \neq a'$. Moreover, if $\Pi$ satisfies (18) in $D$, then contracting $a'$ gives a system $\Pi'$ satisfying (18) in $D'$. Hence there exists a $j$ such that $\phi'_j$ is $R$-homologous to $\psi_{\Pi'}$ (in $D'$), implying that $\phi_j$ is $R$-homologous to $\psi_\Pi$ (in $D$).

Concluding, we can obtain from a system of flows satisfying (19) for $D'$ a system of flows satisfying (19) for $D$. Repeating this we obtain that we may assume that there is only one vertex $v$ in $V \setminus \{r_1, s_1, \ldots, r_k, s_k\}$ and that each arc not incident with $r_1, s_1, \ldots, r_k, s_k$ is a loop at $v$. We may assume that each loop is oriented clockwise (since presently we are interested in undirected paths). For each loop $l$ let $X_l$ be the set of vertices in $r_1, s_1, \ldots, r_k, s_k$ enclosed by $l$. Call loops $l, l'$ *parallel* if $X_l = X_{l'}$. Trivially, there are at most $2^{2k}$ parallel classes. (By Euler's theorem, there are at most $4k$ parallel classes, but we do not need this stronger bound, since $k$ is fixed.)

If $\Pi$ satisfies (18), then there is a system $\Pi'$ satisfying (18) such that $\psi_{\Pi'}$ is $R$-homologous to $\psi_\Pi$ and such that the paths in $\Pi$ do not contain any loop $l$ with $X_l = \emptyset$ and do not contain $l'l^{-1}$ or $l^{-1}l'$ for any two parallel loops $l, l'$. So we can restrict the systems $\Pi$ to systems having this additional property.

For any such system $\Pi$ and any two subsets $B, C \subseteq \{a, a^{-1} | a \in A\}$, let $x_\Pi(B, C)$ denote the number of occurrences of $bc$ in the paths in $\Pi$ such that $b \in B, c \in C$. Then $\Pi$ is up to $R$-homology fully determined by the system of numbers $x_\Pi(B, C)$, where $B$ and $C$ range over all sets

$$(20) \qquad \begin{array}{ll} L, L^{-1} & (L \text{ a parallel class of loops}), \\ \{(r_i, v)\}, \{v, s_i\} & (i = 1, \ldots, k), \end{array}$$

with $L^{-1} := \{l^{-1} | l \in L\}$. Since each such number $x_\Pi(B, C)$ is at most $|A|$ and since there are at most $2(k + 2^{2k})$ sets among (20), for fixed $k$ we can enumerate all possibilities in polynomial time. □

### 3.4. The disjoint paths problem.

THEOREM 3.1. *For each fixed k, the k disjoint paths problem for directed planar graphs* (1) *is solvable in polynomial time.*

*Proof.* By Proposition 3.3 we can find in polynomial time (fixing $k$) a list of flows $\phi_1, \ldots, \phi_N$ such that for each solution $\Pi$ of (1), $\psi_\Pi$ is $R$-homologous to at least one of the $\phi_j$.

Now for each $j = 1, \ldots, N$ we apply the algorithm of Proposition 3.2 to input $\phi_j$. If for some $j$ we find a solution $\Pi$ of problem (1), we are done. If for each of $j = 1, \ldots, N$ it concludes that there is no solution $\Pi$ of (1) such that $\psi_\Pi$ is $R$-homologous to $\phi_j$, we may conclude that (1) has no solution at all. □

Quite directly one can extend the method to the following problem:

(21)
given : a directed planar graph $D = (V, A)$, $k$ pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of $D$, and subsets $A_1, \ldots, A_k$ of $A$;
find : $k$ pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in $D$, where $P_i$ runs from $r_i$ to $s_i$ and uses only arcs in $A_i (i = 1, \ldots, k)$.

The polynomial-time solvability of this problem (for fixed $k$) follows by restricting in the proof of Proposition 3.2 each $\Gamma(a^*)$ to those $g_i$ for which $A_i$ contains $a$.

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