Tuning of Gaussian Stochastic Control Systems

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***Abstract*—A closed-loop system consisting of a control system and an adaptive controller will be called tuning for a specified control objective if the real system and the ideal system defined below achieve the same value for the control objective. The real system is the system consisting of the unknown control system in closed loop with the adaptive controller in which the parameters of the adaptive controller have been determined by identification under feedback or in closed loop. The ideal system is the system consisting of the unknown control system in closed loop with a controller in which the controller has been synthesized with knowledge of the unknown control system and such that the closed-loop system satisfies the control objective. For which adaptive controllers does tuning hold? This question will be considered for both a Gaussian stochastic control system with full observations and with partial observations. The approach to the problem is based on stochastic realization theory for Gaussian systems. The stated question is answered positively for the control objectives of minimum variance control and pole placement. Necessary conditions for tuning are discussed.**

I. Introduction

The purpose of this paper is to explore the question: Which adaptive controllers for Gaussian stochastic control systems achieve tuning in the closed loop?

The problem of adaptive control is: given an unknown control system and control objectives, how can, we synthesize an adaptive controller such that the resulting closed-loop system satisfies the control objectives as much as possible? A major question is whether the adaptive controller will achieve the same value for the control objective as a controller synthesized with knowledge of the unknown control systems. Of the many adaptive control algorithms, attention is limited to those produced by the self-tuning synthesis procedure for this problem that was developed by Aström and Wittenmark [2].

A theoretical analysis of self-tuning controllers will have to establish that these controllers are indeed tuning. Although some progress has been made on this question [4], [16], [17], in general it is unsolved. A major difficulty in the analysis of this question is that identification takes place in the closed loop. A consequence of this limitation is that the control system cannot be determined uniquely in general. This nonuniqueness must be accounted for in an analysis of self-tuning controllers.

The problem of whether a controller designed by the self-tuning synthesis procedure is indeed tuning has been investigated for several classes of stochastic control systems. For Markov chains, the contributions by [6], [16] show that tuning in general does not hold but that under certain conditions tuning of specific control objectives does hold. In [24] an attempt is made to analyze tuning for Gaussian stochastic control systems. The approach is based on an analysis of the limit set of parameter estimation algorithms. Only a first-order system is treated. Related work may be found in [4], [12], [15], [17]–[20], [25]–[29], [31], [33].

In this paper a specific problem of tuning will be treated. The aim of this investigation is to explore the interaction of learning and control in adaptive control. Attention is restricted to the model class of Gaussian stochastic control systems, and it is assumed that the plant generating the observations may be represented by an element of the model class. Attention is focused on the consequences of identification of systems operating under feedback or in the closed loop.

Analysis of self-tuning control algorithms will have to treat two aspects: 1) the limitations of identification of a system operating under feedback or in closed loop; and 2) the convergence of parameter estimates. In the opinion of the author, both points need to be studied separately in adaptive control theory. This viewpoint is rather novel. Point 1) may be studied using results of realization theory. It is expected that this study will yield information on which controllers are tuning as defined in this paper. For earlier works along this line [see 27–29]. If a controller is tuning, then a study of point 2) may yield information on which parameter estimation algorithm in combination with the controller produces convergent parameter estimates and achieves tuning in practice. See [21, 23] for results on convergence of parameter estimates. In the past the technicalities of convergence analysis have obscured a study of point 1).

The formulation of the problem requires the introduction of several concepts. The ideal system is the system consisting of the unknown control system in the closed loop with a controller in which the controller has been synthesized with knowledge of the unknown control system and such that the closed-loop system satisfies the control objective. The real system is the system consisting of the unknown control system in closed loop with the adaptive controller in which the parameters of the adaptive controller have been determined by closed-loop identification. A closed-loop control system will be called tuning for a specified control objective if the identification condition defined in Section II implies that the ideal system and the real system achieve the same value for the control objective. In Section II these concepts are presented in more detail. This terminology applies to both deterministic and stochastic systems.

The problem considered in this paper is to establish whether or not a closed-loop control system is tuning as defined in this paper.

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The approach to the problem is stochastic realization theory. The weak stochastic realization problem is given an observed process to construct a state-space realization in the form of a stochastic system such that the output process of this system equals the given process in distribution. Stochastic realization theory has been developed mainly for Gaussian processes [14]. In this paper, characterizations of sets are based on the comparison of the ideal and the real system as stochastic realizations of their output processes.

The results obtained so far are as follows. For the full observations case, tuning holds for pole placement but not for linear quadratic control. For the partial observations case with minimum variance control, tuning holds not only for the variance but also for the family of finite-dimensional distributions of the output process. Tuning also holds for the pole placement control objective. The spectra of the ideal system and the real system will be different in this case. Tuning has also been analyzed for a controller consisting of a Kalman filter and a linear control law. The conclusions are as of yet unclear, but the indications are that tuning is not achieved in general.

Consideration has been given to the question: Which controllers achieve tuning for the spectrum of the output process? A minimum variance controller achieves this goal. Is this the only class? For a first-order system, it has been proven that tuning of the output process holds only for control laws based on either the minimum variance or the pole placement control objective. This question for higher order systems is under investigation. There are connections of this question with that of robustness of adaptive control algorithms; see [5].

A preliminary version of this paper without proofs appeared as [34]. Some of the results of this paper are close in spirit to those of [17]; however, the approach and the extent of this paper are different.

II. Problem Formulation

Let $(\Omega, F, \boldsymbol{P})$ be a probability space consisting of the set Ω, the σ-algebra F, and the probability measure $\boldsymbol{P}$. The time index set is $T = Z$, the set of the integers. Let R be the set of the real numbers and C be the set of the complex numbers. Consider the class of Gaussian stochastic control systems. The notation P will be used to denote a parameter set of this class. It is assumed that $P \subset R^r$ for some integer r. A Gaussian stochastic control system associated with parameter $p \in P$ is specified by the state-space representation

$$x(t+1) = A(p)x(t) + B(p)u(t) + M(p)v(t), \tag{1}$$

$$y(t) = C(p)x(t) + N(p)v(t) \tag{2}$$

where $v: \Omega \times T \to R^k$ is a Gaussian white noise process (a sequence of independent random variables) with for all $t \in Tv(t) \in G(0, V(p))$, $u: \Omega \times T \to R^m$ is the input process, $x: \Omega \times T \to R^n$ is the state process, $y: \Omega \times T \to R^p$ is the output process, and $A: P \to R^{n \times n}$ is a measurable map while the maps B, C, M, N are defined similarly. Denote this system by $\sigma(p)$ and the class of Gaussian stochastic control systems by $\{\sigma(p), p \in P\}$.

Assumption 2.1: There is a model class of control systems each system of which is indexed by a parameter vector $p \in P \subset R^r$. The notation $\{\sigma(p), p \in P\}$ is used for this class. The plant generating the data in the form of the outputs can be represented by an element in the model class. Suppose that the parameter corresponding to this system is denoted by $p_0 \in P$ and the system itself by $\sigma(p_0)$. Consider a control objective and a family of controllers $\Sigma_c = \{\sigma_c(p), p \in P\}$ such that for each $p \in P$ the closed-loop system $\sigma_{cl}(p, p)$, consisting of the control system $\sigma(p)$ and the controller $\sigma_c(p)$ (see Fig. 3) achieves the control objectives as well as possible.

The setting of the adaptive control problem is that of a technical system which is uncertain. Suppose that a control objective is given. The adaptive control problem is then to construct an adaptive controller such that the closed-loop control system satisfies the control objective.

The self-tuning synthesis procedure for adaptive control has been introduced in [2]. One proceeds as follows. For every parameter value $p \in P$ construct a controller $\sigma_c(p)$ that in the closed loop with the plant $\sigma(p)$ satisfies the control objectives as well as possible. The adaptive controller at time t proceeds:

1) Estimate the value p_0 of the unknown parameter. Denote the parameter estimate by $\hat{p}(t)$
2) Apply the input value produced by the controller indexed by $\hat{p}(t)$.

The performance of such controllers has been investigated. The following questions on the performance of self tuning controllers are of interest:

1) *Transient behavior.* Does $as - \lim_{t\to\infty} \hat{p}(t)$ exist? The expression $as - \lim$ denotes the almost sure limit of the process $\{\hat{p}(t), t \in T\}$.
2) *Control.* If the parameter estimates converge does the resulting closed-loop system then satisfy the control objective?

The first question concerns the convergence of the parameter estimates. As such it deals with the transient behavior of the algorithm. The limit, if it exists, may be different from the value of the parameter of the system generating the data. It is not assumed that the limit of the parameter estimates equals the value of the parameter of the system that generates the data. That the first question can always be solved is shown by the Bayesian embedding technique in [17] and by the extended least-squares method in [23].

The second question concerns the controller that one will use if the parameter estimates have converged. This question involves the identification in closed loop. It will be made clear that the answer to the second question is negative for some classes of controllers. The analysis of this paper concentrates exclusively on the second question.

The problem formulation requires an introduction and the statement of several definitions. The basic objective of study is the closed-loop system consisting of a control system in combination with a controller. Both the class of control systems and the class of controllers are indexed by a parameter vector in the parameter set. By Assumption 2.1 the plant that generates the data is an element of the model class and its parameter vector is given by $p_0 \in P$.

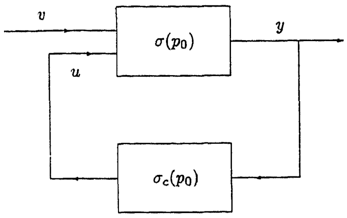

Fig. 1. The ideal system $\sigma_{cl}(p_0, p_0)$.

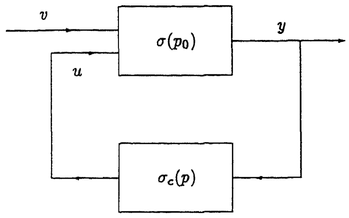

Fig. 2. The real system $\sigma_{cl}(p_0, p)$.

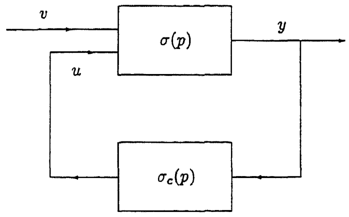

Fig. 3. The imaginary system $\sigma_{cl}(p, p)$.

According to the self tuning synthesis procedure, a controller is selected on the basis of identification. Suppose that the identification procedure produces the parameter vector $p \in P$. The controller used is then $\sigma_c(p)$. It is useful for the subsequent discussion to introduce the following concepts.

Definition 2.2: Consider the notation of Assumption 2.1 and Figs. 1, 2, and 3. Let $p \in P$ be an arbitrary element.

1) The ideal system is the closed-loop system $\sigma_{cl}(p_0, p_0)$ consisting of the unknown control system $\sigma(p_0)$ and the controller $\sigma_c(p_0)$ corresponding to it according to the control objective.
2) The real system corresponding to $p \in P$ is the closed-loop system $\sigma_{cl}(p_0, p)$ consisting of the unknown control system $\sigma(p_0)$ and the controller $\sigma_c(p)$.
3) The imaginary system corresponding to $p \in P$ is the closed-loop system $\sigma_{cl}(p, p)$ consisting of the control system $\sigma(p)$ and the controller $\sigma_c(p)$.

The interpretation of the ideal, the real, and the imaginary system is as follows. If the unknown system is in the model class and represented by $\sigma(p_0)$, then the controller that achieves the control objectives as best as possible is by definition $\sigma_c(p_0)$. Because the parameter value p_0 is unknown, however, the controller $\sigma_c(p_0)$ cannot be constructed in general. Therefore the associated closed-loop system is called the ideal system. The closed-loop system that represents what will be used if the parameter adjustment has been completed is called the real system. It consists of the unknown control system $\sigma(p_0)$ and the controller $\sigma_c(p)$, and it will be denoted by $\sigma_{cl}(p_0, p)$. This closed-loop system corresponds to reality, hence the name real system. If a parameter estimate has been obtained say the parameter p, then the control engineer will think that the unknown control system is represented by $\sigma(p)$. Hence he will think that the closed-loop system consists of $\sigma_{cl}(p, p)$. Since the closed-loop system consisting of $\sigma(p)$ and $\sigma_c(p)$ is conceived only in the mind of the control engineer, it is called the imaginary system. The abbreviation *cl* stands for closed loop.

Because of the imposed restriction to the study of control aspects, there is no parameter estimation algorithm and no sequence of parameter estimates. Yet, one needs a condition that corresponds to the consistency of a sequence of parameter estimates. For this purpose the identification condition is introduced that is stated below. This condition is phrased in terms of stochastic realization theory. Consider two closed-loop stochastic systems as defined above. Such Gaussian stochastic systems are called output equivalent if the family of finite-dimensional distributions of the output processes of these systems are the same. If these output processes are stationary, Gaussian, and have zero mean value function, then the stochastic systems are equivalent iff the covariance functions are the same.

Definition 2.3: For the self-tuning control setup considered, the identification condition for the real and the imaginary system is said to hold for $p \in P$ if one of the following equivalent conditions hold:

1) The family of finite-dimensional distributions of the output processes of the real system $\sigma_{cl}(p_0, p)$ equals that of the imaginary system $\sigma_{cl}(p, p)$.
2) The systems $\sigma_{cl}(p_0, p)$ and $\sigma_{cl}(p, p)$ are equivalent weak stochastic realizations.
3) The covariance functions of the output process y of the systems $\sigma_{cl}(p_0, p)$ and $\sigma_{cl}(p, p)$ are equal.

In this case one also says that the associated systems are output equivalent. Let

$$P_{id}(p_0, \Sigma_c) = \{p \in P | \sigma_{cl}(p_0, p), \sigma_{cl}(p, p) \text{ are output equivalent}\}. \tag{3}$$

For the interpretation of the identification condition the reader should recall that by assumption the plant generating the data is represented by the control system $\sigma(p_0)$ of the model class. Thus the real system $\sigma_{cl}(p_0, p)$ corresponds to what is the plant in closed loop with the adaptive controller $\sigma_c(p)$, and the output process of this system corresponds to the observed data. The imaginary system is a mathematical model for the closed-loop system in which the control system and the plant are indexed by the same parameter vector. The identification condition implies that an imaginary system $\sigma_{cl}(p, p)$ or a $p \in P$ has been selected such that its output process is equivalent to the observed data. This condition thus formulates in the framework of this paper what an identification procedure is supposed to achieve. The convergence results of [17], [23] imply that asymptotically the identification Condition 2.3 holds.

Definition 2.4: The closed-loop control system of Assumption 2.1 will be called tuning if the identification condition for the real and the imaginary system implies that the ideal and the real system achieve the same value for the control objective. Thus tuning holds if for $p \in P$ for which the real system $\sigma_{cl}(p_0, p)$ and the imaginary system $\sigma_{cl}(p, p)$ are output equivalent, one may conclude that the ideal system $\sigma_{cl}(p_0, p_0)$ and the real system $\sigma_{cl}(p_0, p)$ achieve the same value for the control objective. Let

$$P_{fb}(p_0, \Sigma_c) = \left\{ p \in P \;\middle|\; \begin{array}{l} \sigma_{cl}(p_0, p_0), \sigma_{cl}(p_0, p) \text{ achieve the same} \\ \text{value for the control objective} \end{array} \right\}. \tag{4}$$

Then the closed-loop system is tuning iff

$$P_{id}(p_0, \Sigma_c) \subset P_{fb}(p_0, \Sigma_c). \tag{5}$$

Examples of control objectives for which tuning will be investigated are minimum variance control and pole placement.

The definition of tuning may be interpreted as follows. Consider an adaptive controller based on the self-tuning synthesis procedure that is put in closed loop with the unknown control system. In practice the parameter value is then adjusted until the output process of the real system and that of the imaginary system are rather close. In the context of this paper, this condition is translated to the condition that the output processes of the real and the imaginary system are the same as stationary Gaussian processes. If these output processes are equivalent then the identification condition is said to hold. The set $P_{id}(p_0, \Sigma_c)$ characterizes the set of parameter values to which a parameter estimation algorithm may converge. If the closed-loop system is tuning, then the identification condition implies that the closed-loop system will be such that the ideal and the real system achieve the same value for the control objective. Hence the ultimate aim of adaptive control is achieved.

In the literature tuning is sometimes defined as convergence of the parameter estimates in combination with the property that the resulting closed-loop system achieves the prescribed control objective. As argued in Section I, it is useful for the development of adaptive control theory to separate the convergence issue from the control issue. The author proposes to use the term tuning for the property of the closed-loop system as specified in Definition 2.4 Another concept of tuning has been presented in [26].

Problem 2.5: Given the class of Gaussian stochastic control systems and a control objective, determine whether the closed-loop system associated with the control problem for this system is tuning. Three subproblems may be deduced from this problem:

1) Characterize the subclass of the class of Gaussian stochastic control systems such that the associated real system and imaginary system are output equivalent, or determine the set $P_{id}(p_0, \Sigma_c)$; see Definition 2.3.
2) Characterize the subclass of the class of Gaussian stochastic control systems such that the ideal system and the real system achieve the same value for the control objective, or determine the set $P_{fb}(p_0, \Sigma_c)$; see Definition 2.4.
3) Determine whether or not tuning holds, or whether

$$P_{id}(p_0, \Sigma_c) \subset P_{fb}(p_0, \Sigma_c). \tag{6}$$

Tuning is important not only for adaptive control but for the installation of control systems in practice. Note that in any installation of a controller tuning takes place in the closed loop as indicated above.

It is conjectured that tuning holds as defined in Definition 2.4 iff the value of the control objective can be determined from the output of the closed-loop system. If this conjecture is true, then it has important consequences for synthesis of adaptive control systems and for control systems in general.

There are several issues that complicate solution of the above problem:

1) *Identification in closed loop.* When identification takes place in the closed loop then there may exist more than one parameter value that produces the same output process of the closed-loop system.
2) *Nonuniqueness of the controller.* For a given control objective and a stochastic control system, the associated controller may not be unique. In the case that the control objective is pole placement the controller is nonunique in general.
3) *Cancellation of dynamics in closed loop.* A minimal realization of the closed-loop system may have a state-space dimension that is less than that of the plant and the controller combined.

These issues will be treated in the subsequent sections.

The problem of tuning has been isolated first for Markov chains although it was not called by that name. In [25] a condition of identifiability was imposed which ensures that the only element in the set $P_{id}(p_0, \Sigma_c)$ is p_0. Because always $p_0 \in P_{fb}(p_0, \Sigma_c)$, tuning takes place. In [6] it was pointed out that in general

$$\{p_0\} \neq P_{id}(p_0, \Sigma_c) \tag{7}$$

and that in general tuning does not hold. In a series of papers Kumar *et al.* [18], [19] explored self tuning and showed that tuning, defined in a way that is different from that of Definition 2.4, may take place for specific control objectives under certain conditions.

III. Complete Observations Case

In this section the problem formulated in Section II is solved for a Gaussian stochastic control system in which the state is observed. The cases of adaptive control of a finite state Markov chain and that of a controlled diffusion process are not treated. For convergence results and tuning for an adaptively controlled finite-state Markov chain see [7]. For convergence results and tuning for a partially observed Markov chain see [1], and for a countable-state Markov chain [33, Theorem 7.1]. For convergence results of an observed controlled diffusion process, see [8], [9], [10].

Definition 3.1: Consider the class of Gaussian stochastic control systems with complete observations. Let $\sigma(p)$ denote a system in the model class represented by the equation

$$x(t+1) = A(p)x(t) + B(p)u(t) + M(p)v(t),\quad v(t) \in G(0, V(p)). \tag{8}$$

Denote the parameter set by $P \subset R^r$. Assume that the unknown control system generating the data is in the model class and represented by $p_0 \in P$. Depending on the control objective, attention is limited to a subset $P_1 \subset P$ of the parameter set P. On P_1 the feedback law is said to be well defined and given by $F: P_1 \to R^{m\times n}$, $u(t) = F(p)x(t)$.

The adaptive control problem for the above defined class of systems has been considered in [12], [13].

Next the real and the imaginary systems are described. The real system $\sigma_{cl}(p_0, p)$ is specified by the equation

$$x(t+1) = [A(p_0) + B(p_0)F(p)]x(t) + M(p_0)v(t),\quad v(t) \in G(0, V(p_0)). \tag{9}$$

The imaginary system $\sigma_{cl}(p, p)$ is specified by

$$x(t+1) = [A(p) + B(p)F(p)]x(t) + M(p)v(t),\quad v(t) \in G(0, V(p)). \tag{10}$$

A stochastic system of the form (9) or (10) will be called stochastically stable if the state process is a stationary square-integrable process with zero mean value function and finite variance. This elementary definition will suffice for our purpose because attention is restricted to stationary processes. Let $C^- = \{c \in C \| c| < 1\}$. The set of eigenvalues of a matrix A is denoted by $sp(A)$.

Definition 3.2: Define for $p \in P$ the conditions:

1) The feedback law $F: P \to R^{m\times n}$ is well defined, or $p \in P_1$.
2) The real and the imaginary system are stochastically stable.
3) The real and the imaginary system are output equivalent (see Definition 2.3) where the output equals the state in the case of Definition 3.1.

Let

$$P_{id}(p_0, F) = \{p \in P |\, \text{conditions 1, 2, and 3 above hold}\} \tag{11}$$

Theorem 3.3: Consider the model class of Definition 3.1. The set $P_{id}(p_0, F)$ is characterized by

$$P_{id}(p_0, F) = \{p \in P | p \in P_1, \text{ and (13), (14), (15) hold}\} \tag{12}$$

1) $sp([A(p) + B(p)F(p)]) \subset C^-$ and $sp([A(p_0) + B(p_0)F(p)]) \subset C^-$; (13)
2) $A(p_0) + B(p_0)F(p)|im(Q(p_0, p)) = A(p) + B(p)F(p)|im(Q(p, p))$ (14) where the notation $|im(Q(p, p))$ stands for the restriction to the image of this matrix and $Q(p_0, p) = Q(p, p)$ as established in the proof;
3) $M(p_0)V(p_0)M(p_0)^T = M(p)V(p)M(p)^T$; (15) where the definitions of $Q(p_0, p)$ and $Q(p, p)$ are given in (66), respectively (68).

The proof may be found in the appendix.

Next the question of tuning is briefly discussed. Since the conditions of Theorem 3.3 are similar to those obtained for a deterministic system, we limit attention to mentioning the result on tuning. Consider as control objective $F(p_0) = F(p)$. For other control objectives corresponding results may be obtained.

Definition 3.4: Let

$$P_{fb}(p_0, F) = \{p \in P | F(p) \text{ well defined},\ \sigma_{cl}(p_0, p) \text{ stoc. stable}, F(p) = F(p_0)\}. \tag{16}$$

Theorem 3.5 ([27]–[29]): Consider the model class of Definition 3.1 with the additional condition that the input space has dimension one. Assume that for all $p \in P$ the pair $(A(p) + B(p)F(p), (M(p)V(p)^{1/2})$ is reachable.

a) If a pole assignment control law that is stabilizing is used, then tuning holds, or $P_{id} \subset P_{fb}$.
b) If a LQ control law is used then the intersection
$$P_{id}(p_0, F) \cap P_{fb}(p_0, F) \tag{17}$$
   - either is a singleton and $P_{id}(p_0, F)$ contains an open and dense subset that is C^ω diffeomorphic to an open and dense subset of an n-dimensional manifold;
   - or is contained in the boundary of a subset of $P_{id}(p_0, F)$ while this subset is C^ω diffeomorphic to an open and dense subset of an n-dimensional manifold.

   In either case (17) is a negligible subset of $P_{id}(p_0, F)$. If the identification procedure produces a parameter vector in $p \in P$ in the set $P_{id}(p_0, F)$ then tuning holds only if $p \in P_{id}(p_0, F) \cap P_{fb}(p_0, F)$. Because the intersection is either a singleton in the set P_{id} which is of dimension n or because the intersection is contained in a set of dimension strictly smaller than that of $P_{id}(p_0, F)$, tuning holds only in exceptional circumstances.

The proof of this result is easily deduced from Theorem 3.3, in particular from condition (14), and from the indicated references.

In the paper [12] a particularly structured Gaussian stochastic control system in continuous time is considered. The model class consists of systems described by the equation

$$dx(t) = \left[A_0 + \sum_{i=1}^{r} \alpha_i A_i\right] x(t)dt + Bu(t)dt + Mdv(t) \tag{18}$$

in which $v: \Omega \times T \to R^k$ is a standard Brownian motion, $u: \Omega \times T \to$ is the input process, and $x: \Omega \times T \to R^n$ is the state process. Of the parameters of the model class, the values of $A_0, A_1, \cdots, A_r, B, M$, are assumed known, while those of $\alpha_1, \cdots, \alpha_r$ are assumed unknown. It is further assumed that the matrices in the set $\{A_1, \cdots, A_r\}$ are linearly independent. Assume also that the system generating the data is in the model set and denote the corresponding parameter by $p_0 = (\alpha_{01}, \cdots, \alpha_{0r}) \in P$. As before the set of admissible control laws is specified by $F: P \to R^{m\times n}$, $u(t) = F(p)x(t)$. Assume that the real and the imaginary system are stochastically stable or that for all $p \in P$ the matrices

$$A_0 + \sum_{i=1}^{r} \alpha_{0i} A_i + BF(p), \qquad A_0 + \sum_{i=1}^{r} \alpha_i A_i + BF(p)$$

are stable. Assume finally that for all $p \in P$

$$\left(\left[A_0 + \sum_{i=1}^{r} \alpha_i A_i + BF(p) \right], \qquad M(p)V(p)^{1/2} \right) \tag{19}$$

is a reachable pair. Theorem 3.3 has been derived for a discrete-time stochastic system. The following proposition is based on a continuous-time version of Theorem 3.3. The extension of Theorem 3.3 to a continuous-time system as presented in (18) is straightforward.

Proposition 3.6: Consider the model class described by (18) and the assumptions stated between (18) and this proposition. Then

$$P_{id}(p_0, F) = \{p_0 \in R^r\}$$

where the left-hand side is as defined in (11).

Proof: Note that $p_0 \in P_{id}(p_0, F)$ because of the assumptions. Let $p \in P_{id}(p_0, F)$. Let $Q(p, p)$ and $Q(p_0, p)$ be the variance of the state processes of respectively the imaginary and the real system. These variables are the solutions of continuous-time Lyapunov equations that differs from the discrete-time Lyapunov equation given in (66), (68). Condition (19) implies that $im(Q(p, p)) = R^n$. By the proof of Theorem 3.3 $Q(p, p) = Q(p_0, p)$. Then

$$\begin{aligned} & A_0 + \sum_{i=1}^{r} \alpha_{0i} A_i + BF(p) \\ & = A_0 + \sum_{i=1}^{r} \alpha_i A_i + BF(p), \text{ by (14)} \\ & \Leftrightarrow \sum_{i=1}^{r} (\alpha_{0i} - \alpha_i) A_i = 0 \\ & \Leftrightarrow \alpha_{0i} - \alpha_i = 0, \quad \text{for } i = 1, \cdots, r, \Leftrightarrow p_0 = p \end{aligned}$$

where the linear independence of $\{A_1, \cdots, A_r\}$ has been used. □

Thus in Proposition 3.6 $P_{id}(p_0, F) = \{p_0 \in P\}$. Hence tuning for any control objective will always take place. The conclusion of Proposition 3.6 is that the tuning result for this model class is due to the imposed identifiability condition, not due to an adaptive control technique.

IV. Partial Observations Case

In this section the problem formulated in Section II is considered for a Gaussian stochastic control system with partial observations. A first-order system is treated in Appendix A. In this section the polynomial description of Gaussian stochastic control systems is used.

Definition 4.1: Consider the model class of Gaussian stochastic control systems in ARMAX representation

$$a(q^{-1})y(t) = q^{-d}b(q^{-1})u(t) + c(q^{-1})w(t) \tag{20}$$

in which w is Gaussian white noise with $w(t) \in G(0, 1)$, $q^{-1}y(t) = y(t-1)$

$$a(q^{-1}) = \overline{a}_0 + \overline{a}_1 q^{-1} + \cdots + \overline{a}_n q^{-n}, \tag{21}$$

$$b(q^{-1}) = \overline{b}_0 + \overline{b}_1 q^{-1} + \cdots + \overline{b}_{n-d} q^{-(n-d)}, \tag{22}$$

$$c(q^{-1}) = \overline{c}_0 + \overline{c}_1 q^{-1} + \cdots + \overline{c}_n q^{-n}. \tag{23}$$

Assume that $b_0 \neq 0$ and that $d = 1$. Let

$$P = R^n(q^{-1}) \times R^{n-d}(q^{-1}) / \{\overline{b}_0 = 0\} \times R^n(q^{-1}) \times R_+ \tag{24}$$

denote the set of polynomial triples (a, b, c) parameterizing this model class. Suppose that the unknown system is in the model class and represented by $p_0 = (a_0, b_0, c_0) \in P$.

For controllers attention is restricted to those described in polynomial form by

$$f(q^{-1})u(t) = g(q^{-1})y(t) \tag{25}$$

where

$$f(q^{-1}) = f_0 + f_1 q^{-1} + \cdots + f_m q^{-m}, \tag{26}$$

$$g(q^{-1}) = g_0 + g_1 q^{-1} + \cdots + g_k q^{-k}. \tag{27}$$

A polynomial is called stable if all its zeroes are strictly inside C^-.

If the control objective is minimum variance control then f and g are determined by the polynomial equation

$$c = af - q^{-d}bg. \tag{28}$$

For given polynomials a, b, c this equation has a solution in the set of polynomials f, g. In general the solution is not unique. From the available literature it is not clear whether the polynomial f is stable, even if the system is minimum phase or if c is stable. For minimum variance control see [32].

Let the control objective be pole placement. Assume given a polynomial h at which the poles of the spectrum are to be placed. Moreover, it is assumed that the zeroes of the polynomial are strictly inside C^-. Then the polynomials f, g of the controller are solutions of the polynomial equation

$$h = af - q^{-d}bg. \tag{29}$$

Given a, b, all solutions of this equation are given by, see [11],

$$f = hu + q^{-d}bm, \qquad g = hu + am$$

where m is an arbitrary polynomial and, if a, b are coprime, u, v are the unique polynomials such that

$$au - q^{-d}bv = 1.$$

Among all solutions (f, g) there is a solution in which f is of minimal degree. The polynomial equation with this condition has an unique solution [22, 3.1].

The real system $\sigma_{cl}(p_0, p)$ is represented in polynomial form by

$$[a_0(q^{-1})f(q^{-1}) - q^{-d}b_0(q^{-1})g(q^{-1})]y(t) = c_0(q^{-1})f(q^{-1})w(t). \quad (30)$$

Similarly, the imaginary system $\sigma_{cl}(p, p)$ is represented by

$$[a(q^{-1})f(q^{-1}) - q^{-d}b(q^{-1})g(q^{-1})]y(t) = c(q^{-1})f(q^{-1})w(t). \quad (31)$$

These systems are called stochastically stable if the output process y is a stationary process with zero mean value function and finite variance.

The following result is partly also stated in [2]. It is proven there by a method different from that of the stochastic realization approach of this paper. It is partly also stated in [17] again proven by a different method.

Theorem 4.2: Consider the model class introduced above. Assume that the unknown system is an element of the model class and represented by $p_0 = (a_0, b_0, c_0) \in P$. Assume further that b_0 and c_0 are stable polynomials and that the pair (a_0, b_0) is coprime. Let the control objective be minimum variance control. Define for $p \in P$ the conditions:

1) The real system and the imaginary system are stochastically stable.
2) The real and the imaginary system are minimum phase.
3) The real system and the imaginary system are output equivalent.

Let

$$P_{id}(p_0, \Sigma_c) = \{p \in P | \text{ conditions 1, 2, and 3 above hold}\}, \quad (32)$$

$$P_{fb}(p_0, \Sigma_c) = \left\{ p \in P \,\middle|\, \begin{array}{l}\text{the real system and the ideal}\\ \text{system have the same variance}\end{array} \right\}. \quad (33)$$

a) Then

$$P_{id}(p_0, \Sigma_c) = \left\{ p = (a, b, c) \in P \,\middle|\, \begin{array}{l} f,\ a_0 f - q^{-d} b_0 g \\ \text{are stable polynomials,} \\ \text{and } c_0 = a_0 f - q^{-d} b_0 g \end{array} \right\}, \quad (34)$$

where f, g are solutions of (28) corresponding to $p = (a, b, c) \in P$.

b) Tuning holds

$$P_{id}(p_0, \Sigma_c) \subset P_{fb}(p_0, \Sigma_c). \quad (35)$$

c) Also the output spectra of the real, the imaginary, and the ideal system are all identical and equal to $f_0 f_0^*$.

The proof may be found in the appendix. Note that use of the self-tuning minimum variance controller not only provides asymptotically a controller that achieves the same variance for the real system as would be obtained with the ideal system, but also achieves the same family of finite-dimensional distributions of the output process. The assumption on stochastic stability for the real and the imaginary system is necessary in the stochastic realization framework. The minimum-phase assumption is an identifiability condition.

The result below is partly also stated and proven in [17].

Theorem 4.3: Consider the model class and the assumptions of Theorem 4.2 except that the control objective is pole placement. The poles are to be placed at the zeroes of the stable polynomial h. Assume that for all $p \in P$ there are no cancellations in closed loop in the real and the imaginary system. Hence, $a_0 f - q^{-d} b_0 g$ and $c_0 f$ are coprime, and $af - q^{-d}bg$ and cf are coprime. Let $P_{id}(p_0, \sigma_c)$ be as defined in (32), and let

$$P_{fb}(p_0, \Sigma_c) = \left\{ p \in P \,\middle|\, \begin{array}{l}\text{the real and the ideal system have}\\ \text{both } h \text{ as the polynomial}\\ \text{of their poles}\end{array} \right\}.$$

Then

a) $P_{id}(p_0, \Sigma_c) = \{p \in P | p = (a, b, c)$ conditions 1, 2, and 3 defined below hold$\}$. (36)

Let f, g be associated with $p = (a, b, c)$ and h according to (29) and let f_0, g_0 be similarly be associated with $p_0 = (a_0, b_0, c_0)$ and h. Define for $p \in P$ the conditions:

1) c, f are stable polynomials.
2) $a_0 f - q^{-d} b_0 g$ is a stable polynomial.
3) $cf = c_0 f_0 - q^{-d} b_0 m$, and $cg = c_0 g_0 - q^{-d} a_0 m$ for some polynomial m.

b) $P_{id}(p_0, \Sigma_c) \subset P_{fb}(p_0, \Sigma_c)$ hence tuning holds.

c) A minimum-phase stable spectral factor of the real system is cf/h while one of the ideal system is $c_0 f_0/h$. Thus, although the poles of these spectral factors are the same, their zeroes may be different.

The proof may be found in the appendix.

V. Necessity Conditions for Tuning

In this section necessity conditions for tuning are investigated. For related work, see [30].

Necessity in the Case $n = 1$

Consider the model class of Appendix A, that of the first-order case. The Gaussian stochastic control system is thus represented by

$$x(t+1) = ax(t) + bu(t) + (a + c)w(t),$$

$$y(t) = x(t) + w(t).$$

Let $p = (a, b, c, v) \in P = R^3 \times R_+$, $P_1 = R \times R/\{0\} \times R \times R_+$. Define an output feedback control by $P_2 \subset P_1$, $f: P_2 \rightarrow R, u(t) = f(p)y(t)$. Assume that the system generating the data is in the model class and denote the corresponding parameter by $p_0 = (a_0, b_0, c_0, v_0)$. Assume in addition that $p_0 \in P_2$.

Theorem 5.1: Consider the above defined model class and control law. Assume that $p_0 \in P_2 \subset P_1$ is such that $b_0 \neq 0$, $|c_0| < 1$, $v_0 > 0$, and $|a_0 + b_0 f(p_0)| < 1$. Let $P_{id}(p_0, f)$ be as defined in (58). Then

$$P_{id}(p_0, f) = \{p = (a, b, c, v) \in P_1 | \text{ either case 1 or case 2 defined below holds}\}. \quad (37)$$

Case 1) $p \in P_1$ is such that:

1) $b \neq 0$ and $|c| < 1$.
2) $p \in P_2$.
3) $v = v_0$.
4) $f(p) = (-[a + c]/b) = (-[a_0 + c_0]/b_0)$ (38)

Case 2) $p \in P_1$ is such that:

1) $b \neq 0$, $|c| < 1$, and $|a + bf(p)| < 1$.
2) $p \in P_2$.
3) $cv = c_0 v_0$ and $(c_0^2 + 1)v_0 = (c^2 + 1)v$.
4) $a + bf(p) = a_0 + b_0 f(p)$.

Theorem 5.2: Consider the notation and the assumptions of Theorem 5.1. Let

$$P_{fb}(p_0, f) = \{p \in P_1 | f(p) \text{ is well defined and } f(p) = f(p_0)\} \quad (39)$$

If tuning holds as defined in Definition 2.4 or if $P_{id}(p_0, F) \subset P_{fb}(p_0, F)$ then the control law has either one of the following forms:

1) Minimum variance control

$$f(p) = \frac{-[a + c]}{b}. \quad (40)$$

2) Pole placement, or for all $p \in P_{id}(p_0, f)$

$$f(p) = \frac{e - a}{b} \quad (41)$$

for some $e \in R$ with $|e| < 1$.

The proof may be found in the appendix.

Necessity in the Case $n > 1$

Consider the model class of Definition 4.1 with ARMAX representation

$$a(q^{-1})y(t) = q^{-d}b(q^{-1})u(t) + c(q^{-1})w(t).$$

Consider again a controller of the form

$$f(q^{-1})u(t) = g(q^{-1})y(t).$$

The ideal, the real, and the imaginary system are then represented, respectively, by

$$[a_0 f_0 - q^{-d} b_0 g_0]y(t) = c_0 f_0 w(t),$$

$$[a_0 f - q^{-d} b_0 g]y(t) = c_0 f w(t),$$

$$[af - q^{-d} bg]y(t) = cfw(t).$$

The necessity question requires the specification of a control objective. Instead of a control objective, tuning of the spectrum of the output process is considered. Thus let

$$P_{fb}(p_0, \sigma_c) = \{p \in P | \text{ the ideal and the real system are output equivalent}\}.$$

The set $P_{id}(p_0, \sigma_c)$ is as defined in (32). The necessity question is then: What does the condition of tuning

$$P_{id}(p_0, \sigma_c) \subset P_{fb}(p_0, \sigma_c)$$

imply about the controller? No progress on this question can be made without assumptions on the cancellations of dynamics.

If in the imaginary system cancellation of dynamics takes place so that $c = af - q^{-d}bg$, then one recovers the minimum variance case. If no cancellation takes place, one can say more.

Proposition 5.3: Consider the single-input–single-output Gaussian stochastic control system in ARMAX representation and a linear time-invariant controller with the conventions above. Assume that:

1) The ideal and the real system are stochastically stable.
2) The ideal and the real system are minimum phase.
3) There is no cancellation of dynamics in either the ideal or the real system.

Then the ideal and the real system are output equivalent iff

$$\frac{g_0}{f_0} = \frac{g}{f}. \quad (42)$$

Proof: By the assumptions 1) and 2) above, the formulas

$$\frac{c_0 f_0}{a_0 f_0 - q^{-d} b_0 g_0}, \qquad \frac{c_0 f}{a_0 f - q^{-d} b_0 g}$$

are minimum phase stable spectral factors of, respectively, the ideal and the real system. The ideal and the real system are then output equivalent iff

$$q^{-d} b_0 c_0 [f g_0 - f_0 g] = 0$$

iff the condition mentioned in the theorem holds. □

The conclusion from Proposition 5.3 is that the ideal and the real system are output equivalent iff the impulse response functions of $\sigma_c(p_0)$ and $\sigma_c(p)$ are the same. This is a rather stringent requirement. Note that under the stability and minimum phase assumptions $p \in P_{id}(p_0, \sigma_c)$ iff

$$\frac{c_0 f}{a_0 f - q^{-d} b_0 g} = \frac{cf}{af - q^{-d} bg}. \quad (43)$$

Thus if tuning holds, then (43) implies (42). This condition restricts the map $(a, b, c) \mapsto (f, g)$ that constructs the controller. The consequences of this remain to be worked out.

VI. Conclusion

A concept of tuning of a stochastic control system is stated. Tuning holds if identification of the closed-loop system implies that the control objective for the closed-loop system achieves the same value as when the parameter values were known.

For a single-input–single-output Gaussian stochastic control system, it is proven that tuning holds for the control objectives of minimum variance control and of pole placement. A necessity condition for tuning of a first-order Gaussian stochastic control system is presented.

The results of the paper show the limitations of the synthesis procedure of self-tuning regulation. Tuning is achieved only for some control objectives. An alternative synthesis procedure is to apply an excitation signal to the unknown stochastic control system. This synthesis procedure is well known but not deeply investigated. More research is needed on this procedure, in particular on the interaction of identification and control.

Appendix A
First-Order Case

This appendix contains a result on tuning for a first-order Gaussian stochastic control system with partial observations. The characterizations are more explicit than those in Section IV. This appendix may also be read as an introduction to that section.

Consider the model class of a first-order Gaussian stochastic control system in ARMAX representation

$$y(t+1) = ay(t) + bu(t) + w(t+1) + cw(t) \tag{44}$$

in which w is a Gaussian white noise process with $w(t) \in G(0, v)$. This representation may be converted to a state-space representation by the transformation

$$x(t) = y(t) - w(t). \tag{45}$$

The resulting Gaussian stochastic control system in state-space representation is then

$$x(t+1) = ax(t) + bu(t) + (a+c)w(t), \tag{46}$$

$$y(t) = x(t) + w(t). \tag{47}$$

Let $P = R^3 \times R_+$ and denote $p = (a, b, c, v) \in P$. This stochastic system is defined to be stochastically stable if the output process is a stationary square integrable process with zero mean value function and a finite variance.

Consider the control objective of minimum variance control, that is of minimization of $E[y(t)^2]$. It is understood that the closed-loop system must be such that the output process is stochastically stable. The control law is called admissible if the input $u(t)$ depends only on the past of the output process y, thus on $y(t), y(t-1), \cdots$. If $b \neq 0$ and if the parameter $c \in R$ satisfies the condition that $|c| < 1$, hence the stochastic control system is strictly positive real, then the minimum variance control law is given by (see [3, 12.2])

$$u(t) = -\frac{(a+c)}{b} y(t). \tag{48}$$

The resulting closed-loop stochastic control system is then

$$x(t+1) = -cx(t) \tag{49}$$

$$y(t) = x(t) + w(t). \tag{50}$$

Because $|c| < 1$ the output process of this stochastic control system is equivalent as stationary process to that of the system

$$y(t) = w(t). \tag{51}$$

Next the real system and the imaginary system are displayed. Consider the real system. It consists of the unknown Gaussian stochastic control system

$$x(t+1) = a_0 x(t) + b_0 u(t) + (a_0 + c_0) w(t), \tag{52}$$

$$y(t) = x(t) + w(t) \tag{53}$$

with $p_0 = (a_0, b_0, c_0, v_0) \in P$. Let for $p = (a, b, c, v) \in P$, with $b \neq 0$

$$f(p) = \frac{-(a+c)}{b}. \tag{54}$$

Then the control law is $u(t) = f(p)y(t)$. The real system $\sigma_{cl}(p_0, p)$ is then represented by

$$x(t+1) = [a_0 + b_0 f(p)]x(t) + [a_0 + c_0 + b_0 f(p)]w(t), \tag{55}$$
$$y(t) = x(t) + w(t). \tag{56}$$

The imaginary system is given in (51).

Theorem A.1: Consider the class of Gaussian stochastic control systems with representation (46), (47) and the control objective of minimum variance control. Let $P = R^3 \times R_+$ with $p = (a, b, c, v) \in P$ and let

$$P_1 = R \times (R/\{0\}) \times R \times R_+,\ f: P_1 \to R,$$
$$f(p) = -\frac{(a+c)}{b}. \tag{57}$$

Assume that the unknown system is in the model class and is represented by $p_0 = (a_0, b_0, c_0, v_0) \in P$ with $b_0 \neq 0$, $|c_0| < 1$, and $v_0 > 0$. Define for $p \in P$ the conditions:

1) The feedback law $f(p)$ is well defined.
2) The real system $\sigma_{cl}(p_0, p)$ and the imaginary system $\sigma_{cl}(p, p)$ are stochastically stable and minimum phase.
3) The real and the imaginary system, $\sigma_{cl}(p_0, p)$ and $\sigma_{cl}(p, p)$, are output equivalent.

Define finally

$$P_{id}(p_0, f) = \{p \in P \mid \text{conditions 1, 2, and 3 above hold}\}. \tag{58}$$

a) Then

$$P_{id}(p_0, f) = \left\{ p = (a, b, c, v) \in P \;\middle|\; \begin{matrix} b \neq 0,\ |c| < 1,\ v > 0, \\ \frac{a+c}{b} = \frac{a_0 + c_0}{b_0},\ v = v_0 \end{matrix} \right\}. \tag{59}$$

b) The variance of the ideal respectively the real system is

$$v_0,\ v + \frac{[a_0 + c_0 + b_0 f(p)]^2 v}{1 - (a_0 + b_0 f(p))^2}. \tag{60}$$

The proof of this result is not presented here. Theorem A.1 is a special case of Theorem 4.2. The proof is similar to that of Theorem 5.1. Note that the expression for $P_{id}(p_0, f)$ is such that there may be other parameter values than p_0 that can occur as limits of parameter estimation algorithms.

Theorem A.2: Consider the notation and the assumptions of Theorem A.1. Let

$$P_{fb}(p_0, f) = \left\{ p \in P \,\middle|\, \begin{array}{l} f(p) \text{ is well defined, real and} \\ \text{ideal system have identical variance} \end{array} \right\}. \quad (61)$$

a) Then

$$P_{fb}(p_0, f) = \{p = (a, b, c, v) \in P | b \neq 0, \text{ and (63) holds}\}, \quad (62)$$

$$v_0 = v + \frac{[a_0 + c_0 + b_o f(p)]^2 v}{1 - (a_0 + b_0 f(p))^2}. \quad (63)$$

b) Moreover, tuning holds

$$P_{id}(p_0, f) \subset P_{fb}(p_0, f). \quad (64)$$

The proof is not presented here. It follows directly from Theorem A.1.

The paper [24] discusses a related problem. The approach presented here differs from that of the quoted paper by the control system considered and by the approach to the problem. The starting problem of the quoted paper is the limit set for the ordinary differential equation approach to convergence analysis of recursive algorithms. The approach of this paper is based on stochastic realization theory. The results of the quoted paper are termed there as of a preliminary nature.

The author has formulated and proven a theorem like such as Theorem A.1 for the case of the pole assignment control objective. He has also obtained a characterization for the case of a controller consisting of a Kalman filter in combination with a linear control law. The latter case yields a complicated characterization of $P_{id}(p_0, f)$ which does not provide much information.

Appendix B
Proofs

Proof of Theorem 3.3: Let $p \in P_{id}(p_0, F)$; that the real and the imaginary system are stochastically stable is equivalent with condition (13). The proof of the sufficiency of this statement follows from a standard result for stochastic system theory and that of necessity from consideration of the mean value function. The covariance function of the state process of the real system is given by

$$W_x(t, p_0, p) = \begin{cases} Q(p_0, p), & \text{if } t = 0, \\ [A(p_0) + B(p_0)F(p)]^t Q(p_0, p), & \text{if } t > 0, \\ Q(p_0, p)[A(p_0) + B(p_0)F(p)]^{-t}, & \text{if } t < 0, \end{cases} \quad (65)$$

$$Q(p_0, p) = [A(p_0) + B(p_0)F(p)]Q(p_0, p) \cdot [A(p_0) + B(p_0)F(p)]^T + M(p_0)V(p_0)M(p_0)^T. \quad (66)$$

Similarly for the imaginary system

$$W_x(t, p, p) = \begin{cases} Q(p, p) & \text{if } t = 0, \\ [A(p) + B(p)F(p)]^{t-1} Q(p, p), & \text{if } t > 0, \end{cases} \quad (67)$$

$$Q(p, p) = [A(p) + B(p)F(p)]Q(p, p) \cdot [A(p) + B(p)F(p)]^T + M(p)V(p)M(p)^T. \quad (68)$$

Then $p \in P_{id}(p_0, F)$ and condition 3) of Definition 3.2 imply that the covariance functions (65), (67) are equal. Hence $Q(p, p) = Q(p_0, p)$. Because the state process is stationary and Gaussian, the support of the state process of the real respectively the imaginary system is determined by $im(Q(p_0, p))$ and $im(Q(p, p))$. Because $Q(p, p) = Q(p_0, p)$, $im(Q(p_0, p)) = im(Q(p, p))$. From the equality of the covariance functions follows that (14) holds. Subtracting (66) from (68) yields condition (15).

Conversely, let p be an element of the right-hand side of (12). From (13), (14), (15), of Theorem 3.3, and the uniqueness of a solution of the Lyapunov equation (66), it follows that $Q(p, p) = Q(p_0, p)$. It follows then from (14), (65), and (67) that the covariance functions of the real and the imaginary system are the same. Hence $p \in P_{id}(p_0, F)$. □

Proof of Theorem 4.2: Let $p \in P_{id}(p_0, F)$. Let

$$k_1 = c_0 f / [a_0 f - q^{-d} b_0 g] \quad \text{and} \quad k_2 = cf/[af - q^{-d} bg].$$

Because the control objective is minimum variance control, f, g are determined by a, b, c via the polynomial equation

$$c = af - q^{-d} bg.$$

Hence, $k_2 = cf/c = f$. If $k_1(q^{-1})$ is a polynomial then denote $k_1^*(q^{-1}) = k_1(q)$. The spectrum of the output process of the real system is $k_1 k_1^*$ and that of the imaginary system is $k_2 k_2^*$. By condition 3) the real and the imaginary system are output equivalent, hence have the same spectrum for the output process. By condition 2) the real and the imaginary system are minimum phase, hence the minimum-phase stable spectral factors are equal $k_1 = k_2$ or

$$c_0 = a_0 f - q^{-d} b_0 g. \quad (69)$$

Hence $a_0 f - q^{-d} b_0 g$ is a stable polynomial. Given a_0, b_0, c_0, with according to the assumption a_0, b_0 coprime, the equation for f_0, g_0, with the restriction that $f_0 \in R$ and g_0 a polynomial of degree $n - d$

$$c_0 = a_0 f_0 - q^{-d} b_0 g_0 \quad (70)$$

has an unique solution. Because by (69), the pair f, g is also a solution of (70), there results

$$f = f_0, \qquad g = g_0.$$

Thus the controller $\sigma_c(p)$ with $fu(t) = gy(t)$ and the controller $\sigma_c(p_0)$ with $f_0 u(t) = g_0 y(t)$ have identical impulse response functions.

The ideal system is given by

$$[a_0 f_0 - q^{-d} b_0 g_0] y(t) = c_0 f_0 w(t)$$

so the minimum-phase stable spectral factor is $c_0 f_0 / c_0 = f_0$. The minimum-phase stable spectral factor of the real system is $c_0 f / c_0 = f = f_0$. Thus the ideal and the real system are output equivalent, $p \in P_{fb}(p_0, \sigma_c)$, and $P_{id}(p_0, \sigma_c) \subset P_{fb}(p_0, \sigma_c)$. A straightforward verification establishes that the right-hand side of (34) is contained in $P_{id}(p_0, \sigma_c)$. □

Proof of Theorem 4.3: Let $p \in P_{id}(p_0, \Sigma_c)$. The real system is stable iff the polynomial

$$h_0 = a_0 f - q^{-d} b_0 g \tag{71}$$

is stable. The imaginary system is stable iff the polynomial

$$h = af - q^{-d} bg \tag{72}$$

is stable. Because by assumption there are no pole-zero cancellations in the closed loop, the imaginary system is minimum-phase iff the polynomial cf is stable. The spectra of the real and the imaginary system are, respectively

$$\left(\frac{c_0 f}{a_0 f - q^{-d} b_0 g}\right)\left(\frac{c_0 f}{a_0 f - q^{-d} b_0 g}\right)^*, \quad \left(\frac{cf}{af - q^{-d} bg}\right)\left(\frac{cf}{af - q^{-d} bg}\right)^*.$$

Now c_0 is a stable polynomial by assumption, c and f are stable because of the minimum phase condition, h is stable by assumption, and $a_0 f - q^{-d} b_0 g$ is stable because the real system is. Thus the minimum-phase stable spectral factors of the real and the imaginary system, which are equal by $p \in P_{id}$, are

$$\frac{c_0 f}{a_0 f - q^{-d} b_0 g} = \frac{cf}{h}.$$

Because $h = a_0 f_0 - q^{-d} b_0 g_0$ this is equivalent with

$$c_0[a_0 f_0 - q^{-d} b_0 g_0] = c[a_0 f - q^{-d} b_0 g] \Leftrightarrow a_0[c_0 - cf] - q^{-d} b_0 [c_0 g_0 - cg] = 0.$$

The assumption that a_0, b_0 are coprime implies that

$$c_0 f_0 - cf = q^{-d} b_0 m \quad \text{and} \quad c_0 g_0 - cg = a_0 m$$

for some polynomial m. Thus

$$cf = c_0 f_0 - q^{-d} b_0 m \quad \text{and} \quad cg = c_0 g_0 - a_0 m.$$

Note that the poles of the real system are from (30) given by

$$\begin{aligned} a_0 f - q^{-d} b_0 g &= c^{-1}[a_0 c_0 f_0 - a_0 q^{-d} b_0 m - q^{-d} b_0 c_0 g_0 + q^{-d} b_0 a_0 m] \\ &= c^{-1} c_0 [a_0 f_0 - q^{-d} b_0 g_0] = c^{-1} c_0 h \end{aligned} \tag{73}$$

and the spectral factor is

$$\frac{c_0 f}{c^{-1} c_0 h} = \frac{cf}{h}. \tag{74}$$

A spectral factor of the ideal system is

$$\frac{c_0 f_0}{a_0 f_0 - q^{-d} b_0 g_0} = \frac{c_0 f_0}{h}. \tag{75}$$

Thus both the real and the ideal system have the same closed-loop poles and hence $p \in P_{fb}(p_0, \sigma_c)$.

Conversely, let p be an element of the right-hand side of (36). Because of condition 2), the real system is stable. From the pole placement objective follows that the imaginary system is stable. Because of condition 1) the imaginary system is minimum phase. That the real system and the imaginary system are output equivalent follows from the equality of the spectral factors via

$$\begin{aligned} \frac{c_0 f}{a_0 f - q^{-d} b_0 g} &= \frac{c_0 f}{c^{-1} c_0 h} \text{ by condition 3 and (73)} \\ &= \frac{cf}{h} = \frac{cf}{af - q^{-d} bg} \end{aligned}$$

because of the pole placement objective. Thus $p \in P_{id}(p_0, \Sigma_c)$. □

Proof of Theorem 5.1: The real system is represented by the equations

$$x(t+1) = (a_0 + b_0 f(p))x(t) + (a_0 + c_0 + b_0 f(p))w(t), \tag{76}$$

$$y(t) = x(t) + w(t) \tag{77}$$

with $w(t) \in G(0, v_0)$. The covariance function of the real system's output process is given by

$$w_y(t, p_0, p, f) = \begin{cases} q_0 + v_0, & \text{if } t = 0, \\ (a_0 + b_0 f(p))^{t-1} g_0, & \text{if } t > 0 \end{cases} \tag{78}$$

where

$$q_0 = \frac{[a_0 + c_0 + b_0 f(p)]^2 v_0}{1 - (a_0 + b_0 f(p))^2}, \tag{79}$$

$$g_0 = [a_0 + b_0 f(p)] q_0 + [a_0 + c_0 + b_0 f(p)] v_0. \tag{80}$$

The corresponding equations for the imaginary system are

$$x(t+1) = (a + bf(p))x(t) + (a + c + bf(p))w(t), \tag{81}$$

$$y(t) = x(t) + w(t) \tag{82}$$

with $w(t) \in G(0, v)$. The covariance function of the imaginary system's output process is given by

$$w_y(t, p, p, f) = \begin{cases} q + v, & \text{if } t = 0, \\ (a + bf(p))^{t-1} g, & \text{if } t > 0 \end{cases} \tag{83}$$

where

$$q = \frac{[a + c + bf(p)]^2 v}{1 - (a + bf(p))^2}, \tag{84}$$

$$g = [a + bf(p)] q + [a + c + bf(p)] v. \tag{85}$$

$\Rightarrow$ Let $p \in P_{id}(p_0, f)$. The real and the imaginary system are stable iff

$$|a_0 + b_0 f(p)| < 1 \text{ and } |a + bf(p)| < 1. \tag{86}$$

These systems are minimum phase iff $|c_0| < 1$ and $|c| < 1$. By assumption $|c_0| < 1$. These systems are output equivalent iff

$$\begin{cases} q_0 + v_0 = q + v, \\ [a_0 + b_0 f(p)]^{t-1} g_0 = [a + bf(p)]^{t-1} g, & \forall t \geq 1 \end{cases}$$

iff

$$\begin{cases} q_0 + v_0 = q + v, \\ g_0 = g, \\ [a_0 + b_0 f(p)] = [a + bf(p)], & \text{if } g_0 = g \neq 0. \end{cases} \tag{87}$$

A calculation using (84) and (85) shows that

$$g = \frac{v[a + c + bf(p)][c(a + bf(p)) + 1]}{1 - (a + bf(p))^2}. \tag{88}$$

One distinguishes the cases $g = g_0 = 0$ and $g = g_0 \neq 0$.

Suppose that $g = g_0 = 0$. Now $g = 0$ implies by (86) and (88) that either $v = 0$, or $a + c + bf(p) = 0$, or $c[a + bf(p)] + 1 = 0$. If $v = 0$ then $q = 0$ by (84). Hence $q_0+v_0 = q+v = 0$. Because $q_0 \geq 0$ and by assumption $v > 0$, this is a contradiction. Thus $v > 0$. If $c[a+bf(p)]+1 = 0$ then

$$a + bf(p) = \frac{-1}{c}.$$

The stability condition (86) implies then that $|1/c| < 1$ while the minimum phase assumption requires that $|c| < 1$. Therefore this case is also excluded. If $a+c+bf(p) = 0$ then

$$f(p) = \frac{-(a+c)}{b}. \tag{89}$$

Similarly $g_0 = 0$ leads to

$$f(p) = \frac{-(a_0+c_0)}{b_0}. \tag{90}$$

In this case $a + c + bf(p) = 0$ implies by (84) that $q = 0$, and similarly by (90) that $q_0 = 0$, hence by (87) that $v = v_0$. This proves case 1).

Suppose that $g = g_0 \neq 0$. The conditions obtained so far are $p \in P_2$, $b \neq 0$, $q_0 + v_0 = q + v$, $g = g_0 \neq 0$, $a_0 + b_0 f(p) = a + bf(p)$, $|c| < 1$, $a + bf(p)| < 1$. Note that

$$q = [a+bf(p)]q + [a+c+bf(p)]v = [a+bf(p)][q+v] + cv. \tag{91}$$

Hence $g = g_0$ implies that $cv = c_0 v_0$. From this, $q_0 + v_0 = q + v$, (79), and (84) follows with a calculation that

$$v_0(c_0^2 + 1) = v(c^2 + 1). \tag{92}$$

$\Leftarrow$ Conversely, let p be an element of the right-hand side of (37). Suppose that Case 1) holds. Then $p \in P_2$, and $f(p)$ is well defined. Then $a + bf(p) = -c$ and $|c| < 1$ imply that the imaginary system is stable. Similarly, $f(p) = -[a_0 + c_0]/b_0$ and $|c_0| < 1$ imply that the real system is stable. From (88) follows (38) that $g = g_0 = 0$. Similarly it follows from (79) and (84) that $q = 0 = q_0$. The condition $v = v_0$ and (87) then imply that the real and the imaginary system are output equivalent.

Suppose that Case 2) holds. Then $b \neq 0$ implies that $p \in P_2$, hence f is well defined. Conditions 1) and 4) imply that the real and the imaginary system are stable. The assumption $|c_0| < 1$ and the condition $|c| < 1$ imply that these systems are minimum phase. Note that

$$\begin{aligned} q_0 + v_0 \doteq q + v &\Leftarrow v_0(c_0 + a_0 + b_0 f(p))^2 \\ &\quad + v_0[1 - (a_0 + b_0 f(p))^2] \\ &= v(c + a + bf(p))^2 + v[1 - (a + bf(p))^2] \\ &\quad \text{by (79), (84), and Condition 4} \\ &\Leftarrow 2c_0 v_0(a_0 + b_0 f(p)) + v_0 c_0^2 + v_0 \\ &= 2cv(a + bf(p)) + vc^2 + v \\ &\Leftrightarrow v_0(c_0^2 + 1) = v(c^2 + 1) \\ &\quad \text{by Conditions 3 and 4.} \end{aligned} \tag{93}$$

This relation, (80), (85), and conditions 3) and 4) imply that $g = g_0$. From (87) then follows that the real and the imaginary system are output equivalent. □

Proof of 5.2: Let $p \in P_{id}(p_0, f)$. According to Theorem 5.1 one of the following cases holds.

Case 1: Minimum variance control

$$f(p) = \frac{-(a+c)}{b}. \tag{94}$$

Case 2: By condition 4) $a + bf(p) = a_0 + b_0 f(p)$. Because $p \in P_{id}(p_0, f) \subset P_{fb}(p_0, f)$, $f(p) = f(p_0)$. Thus

$$a + bf(p) = a_0 + b_0 f(p_0) =: e, \ |e| < 1$$

for all $p \in P_{id}$ in which e does not depend on p. Hence

$$f(p) = \frac{e - a}{b}$$

for all $p \in P_{id}$ and f is effectively pole assignment. □

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REFERENCES

[1] A. Arapostathis and S. I. Marcus, "Analysis of an identification algorithm arising in adaptive estimation of Markov chains," *Math. Contr. Sign. Syst.*, vol. 3, pp. 1–29, 1990.

[2] K. J. Aström and B. Wittenmark, "On self tuning regulators," *Automatica*, vol. 9, pp. 185–199, 1973.

[3] ——, *Computer Controlled Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1984.

[4] A. Becker, P. R. Kumar, and C. Z. Wei, "Adaptive control with the stochastic approximation algorithm: geometry and convergence," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 330–338, 1985.

[5] R. R. Bitmead, M. Gevers, and V. Wertz, "Adaptation and robustness in predictive control," in *Proc. 28th Conf. Decis. Contr.*, New York, 1989, pp. 2114–2119.

[6] V. Borkar and P. Varaiya, "Adaptive control of Markov chains I: Finite parameter set," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 953–957, 1979.

[7] V. S. Borkar, "The Kumar-Becker-Lin scheme revisited," *J. Optim. Th. Appl.*, vol. 66, pp. 289–309, 1990.

[8] ——, "Self-tuning control of diffusions without the identifiability condition," *J. Optim. Th. Appl.*, vol. 68, pp. 117–138, 1991.

[9] V. S. Borkar and M. K. Ghosh, "Ergodic control of multidimensional diffusions II: Adaptive control," *Appl. Math. Optim.*, vol. 21, pp. 191–220, 1990.

[10] ——, "Ergodic and adaptive control of nearest neighbor motions," *Math. Contr. Sign. Syst.*, vol. 4, pp. 81–98, 1991.

[11] F. M. Callier and C. A. Desoer, *Multivariable Feedback Systems.* New York: Springer-Verlag, 1982.

[12] T. E. Duncan and B. Prasik-Duncan, "Adaptive control of continuous-time linear stochastic systems," *Math. Contr. Sign. Syst.*, vol. 3, pp. 45–60, 1989.

[13] J. B. Farison, R. E. Graham, and R. C. Shelton Jr., "Identification and control of linear discrete systems," *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 438–442, 1967.

[14] P. Faurre, "Stochastic realization algorithms," in *System Identification—Advances and Case Studies*, R. K. Mehra and D. G. Lainiotis, Eds. New York: Academic, 1976, pp. 1–25.

[15] L. Guo and H.-F. Chen, "The Aström-Wittenmark self-tuning regulator revisited and ELS-based adaptive trackers," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 802–812, 1991.

[16] P. R. Kumar, "A survey of some results in stochastic adaptive control," *SIAM J. Control Optim.*, vol. 23, pp. 329–380, 1985.

[17] ——, "Convergence of adaptive control schemes using least-squares parameter estimates," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 416–424, 1990.

[18] P. R. Kumar and A. Becker, "A new family of optimal adaptive controllers for Markov chains," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 137–146, 1982.

[19] P. R. Kumar and W. Lin, "Optimal adaptive controllers for unknown Markov chains," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 765–774, 1982.
[20] P. R. Kumar and L. Praly, "Self-tuning trackers," *SIAM J. Contr. Optim.*, vol. 25, pp. 1053–1071, 1987.
[21] P. R. Kumar and P. Varaiya, *Stochastic Systems: Estimation, Identification, and Adaptive Control.* Englewood Cliffs, NJ: Prentice-Hall, 1986.
[22] V. Kučera, *Discrete Linear Control—The Polynomial Equation Approach.* Prague: Czechoslovak Academy of Sciences, 1979.
[23] T. L. Lai and C. Z. Wei, "Extended least squares and their application to adaptive control and prediction in linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 898–906, 1986.
[24] W. Lin, P. R. Kumar, and T. I. Seidman, "Will the self-tuning approach work for general cost criteria?" *Syst. Contr. Lett.*, vol. 6, pp. 77–85, 1985.
[25] P. Mandl, "Estimation and control in Markov chains," *Adv. Appl. Probab.*, vol. 6, pp. 40–60, 1974.
[26] A. S. Morse, "Towards a unified theory of parameter adaptive control—tunability," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1002–1012, 1990.
[27] J. W. Polderman, "A note on the structure of two subsets of the parameter space in adaptive control problems," *Syst. Contr. Lett.*, vol. 7, pp. 25–34, 1986.
[28] ——, "On the necessity of identifying the true parameter in adaptive lq control," *Syst. Contr. Lett.*, vol. 8, pp. 87–91, 1986.
[29] ——, "Adaptive control and identification: Conflict or conflux?" Ph.D. dissertation, Univ. Groningen, Groningen, 1987.
[30] J. W. Polderman and C. Praagman, "The closed-loop identification problem in indirect adaptive control," in *Proc. Conf. Decis. Contr.*, New York, 1989, pp. 2120–2124.
[31] L. Praly, S.-F. Lin, and P. R. Kumar, "A robust adaptive minimum variance controller," *SIAM J. Contr. Optim.*, vol. 27, pp. 235–266, 1989.
[32] U. Shaked and P. R. Kumar, "Minimum variance control of discrete time multivariable systems," *SIAM J. Contr. Optim.*, vol. 24, pp. 396–411, 1986.
[33] A. Shwartz and A. M. Makowski, "Comparing policies in Markov decision processes: Mandl's lemma revisited," *Math. Oper. Res.*, vol. 15, pp. 155–174, 1990.
[34] J. H. van Schuppen, "Asymptotic self-tuning for Gaussian stochastic control systems," in *Proc. 1st European Contr. Conf.*, Paris, 1991, pp. 258–263.

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