A Cross Section of Intersection Theory

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In honour of the 90th birthday of Bartel Leendert van der Waerden

At the beginning of this century, the foundations of intersection theory were severely questioned. Van der Waerden took a constructive position in this debate. He gave rigorous proofs and justifications of some nineteenth-century methods, and found illustrative counterexamples to the sufficiency of some proposed definitions. In this paper we sketch three topics in intersection theory to which Van der Waerden contributed: Bézout's theorem, the principle of conservation of number, and the use of the length of a local ring as intersection multiplicity.

1. Bézout's Theorem

Consider in the plane the cubic C given by

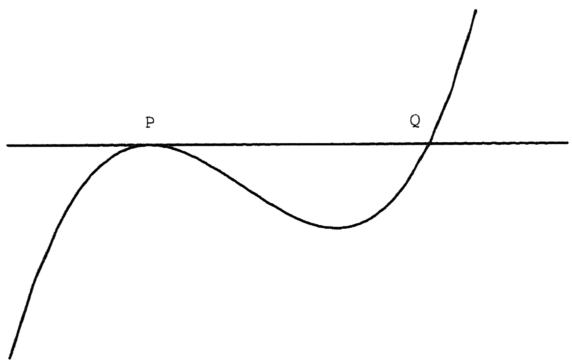

Figure 1. A transversal and a non-tranversal intersection

$$y - x^2(x-1) = 0$$

and the line L defined by $y = 0$. The intersection $C \cap L$ consists of the two points $P = (0,0)$ and $Q = (1,0)$, where both $y - x^2(x-1)$ and y are zero.

At P the line L is tangent to C and at Q the intersection is *transversal*: the tangents to C and to L do not coincide. Counting the transversal intersection with a *multiplicity* $i(Q; C \cdot L) = 1$ and the intersection where the tangents meet with a multiplicity $i(P; C \cdot L) = 2$, we obtain

$$\deg C \cdot \deg L = i(P; C \cdot L) + i(Q; C \cdot L).$$

This is an example of a general theorem in intersection theory named after Bézout. We will briefly describe the objects which appear in the theorem in its full generality. The ambient space in which things happen, is the *projective space* $\mathbf{P}^n$ of dimension n. One can think of $\mathbf{P}^n$ as the affine vector space $\mathbf{A}^n$ together with a natural notion of a hyperplane $\mathbf{P}^{n-1}$ at infinity. For instance, parallel lines in $\mathbf{A}^2$ intersect in $\mathbf{P}^2$ transversally at infinity. As a model for $\mathbf{P}^n$, one can take all lines through the origin in $\mathbf{A}^{n+1}$, and we denote by $(x_0 : \ldots : x_n)$ the point in $\mathbf{P}^n$ representing the line through $(x_0, \ldots, x_n) \in \mathbf{A}^{n+1} - \{0\}$. In $\mathbf{P}^n$ live *algebraic sets*. Such an algebraic set is defined as the set of points where some polynomials are zero. It can have several *irreducible components*, for instance $C \cap L$ consists of two components. Each component has a certain dimension. We will call the *dimension* of an algebraic set the maximum of the dimensions of its irreducible components. In algebraic geometry the notion of an algebraic set has been generalised to that of a *scheme* [4]. Let us consider two algebraic sets X and Y in $\mathbf{P}^n$, defined by sets of polynomials I and J respectively. The *intersection* $X \cap Y$, i.e. the set of common points, is defined as the set where all the polynomials of I and J are zero.

The theorem of Bézout relates the degree of the intersection $X \cap Y$ with that of X and Y under the condition that the intersection is proper (i.e. has minimal dimension). For the *degree*, we give here a geometric description. Let L be a linear subspace of $\mathbf{P}^n$ of dimension $n - \dim X$. Suppose that it intersects X properly and transversally, then the degree of X is the number of points in the intersection

$$\deg X = \# L \cap X.$$

These conditions on L are satisfied by almost all linear spaces of this dimension, and we will call such an L in *general position* with respect to X.

The general theorem of Bézout takes into account that each irreducible component Z of the intersection $X \cap Y$ has a certain *intersection multiplicity* $i(Z; X \cdot Y)$. The theorem says that

$$\deg X \cdot \deg Y = \sum i(Z; X \cdot Y) \deg Z,$$

where Z runs through the irreducible components of $X \cap Y$. Denoting the formal sum $\sum i(Z; X \cdot Y)[Z]$ by $X \cdot Y$, we can restate the theorem in a handsome formula

$$\deg X \cdot \deg Y = \deg X \cdot Y.$$

This formula can be further generalised to the intersection of several varieties $X_1, \ldots, X_r$. Although Bézout's name is attached to this general version, it is in fact due to VAN DER WAERDEN [12]. Bézout's original statement [1] only concerns the case where $r = n$ and each X_i is a hypersurface, i.e. an algebraic set defined by one polynomial and therefore of dimension $n - 1$.

In this article we touch upon some other topics in intersection theory where the work of Van der Waerden has been influential. Many nineteenth-century geometers (especially from Italy) worked with great virtuosity and geometric insight in algebraic geometric intersection theory. They used some basic principles that were plausible, but proved neither according to some of the standards of that time, nor according to modern standards. The debate about the validity became stronger at the beginning of the twentieth century. Also Hilbert was concerned, and he expressed this in one of his list of twenty three problems for the coming century. In his early work about the lack of rigour Van der Waerden put on firm grounds the notions of intersection multiplicity and of the principle of conservation of number. Having these tools, it is straightforward to prove Bézout's theorem in its full generality.

2. Principle of Conservation of Number

An interesting application of the theorem of Bézout is in the field of *enumerative geometry*: the study of varieties that satisfy some geometric conditions. Let us have a look at a typical but simple problem, and at the way it was solved in the nineteenth century: *How many conics (in the plane) are tangent to a given smooth cubic and pass through 4 given points (all in general position)?*

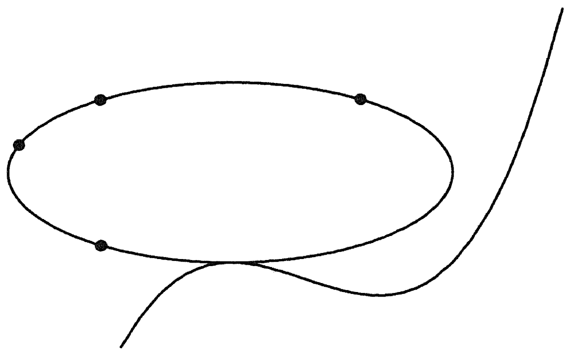

FIGURE 2. A conic passing through four points and tangent to a cubic

The idea is to look at the parameter space of all conics. This is a $\mathbf{P}^5$ since we have six monomials whose coefficients determine the conic and one degree of freedom in the homogeneity factor (a non-zero scalar multiple of a polynomial defines the same conic as the polynomial). The points representing conics that satisfy a condition like passing through a point or a condition like being tangent to a curve, form a hypersurface in $\mathbf{P}^5$. So by Bézout's theorem, the number of conics satisfying all conditions can be determined as the product of the degrees of the hypersurfaces.

Chasles [5] was the first to notice that such a hypersurface representing a

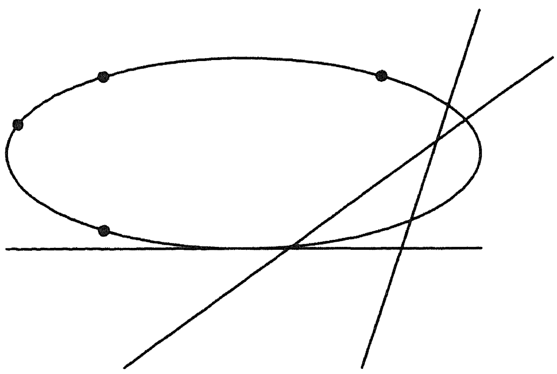

FIGURE 3. Similar situation according to the principle of conservation of number

condition on conics, can be written in terms of two "fundamental classes": μ of the conics passing through a point, and ν of the conics being tangent to a line. The first has degree 1 and the second has degree 2. Chasles knew that the condition of being tangent to a cubic can be written as $6\mu + 3\nu$, so the answer to the above question is

$$\mu \cdot \mu \cdot \mu \cdot \mu \cdot (6\mu + 3\nu) = 6\mu^5 + 3\mu^4\nu = 6 \cdot 1 + 3 \cdot 2 = 12.$$

How did Chasles determine the decomposition into fundamental classes of the condition of tangency to a cubic? For such a question, nineteenth century mathematicians often used the *principle of conservation of number*. This principle says that if such a condition is changed in a continuous way (e.g. by varying the cubic) and the number of solutions stays finite, then the number of solutions does not change. The intuition behind this principle was probably the same as we have for the fact that a continuous function with discrete values is constant on a connected component. In the case of the condition of tangency to a cubic, one can vary the cubic to obtain three lines with three points of intersection. Each line leads to a ν-term, each point to a double μ-term.

Schubert was able to extend this calculus of conditions to great height. His claim was that any condition of any codimension on a parameter space could be decomposed into fundamental classes. Therefore the problems of enumerative geometry reduce to finding the number of intersections of the fundamental classes with the family. This method was applied to all kinds of parameter spaces, not only $\mathbf{P}^n$.

One of the most famous numbers that Schubert computed is the number of twisted cubic space curves that are tangent to 12 given quadric surfaces (in general position):

$$5819539783680.$$

It took more than a century to verify this number [7]. Now history has shown that almost all numbers that Schubert computed are correct. However, at that time, many people were not convinced that the methods were sound.

Hilbert devoted one of his problems for the 20th century to this dispute. His 15th problem reads:

> "To establish rigorously and with exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.
>
> Although the algebra of today guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of final equations and the multiplicity of their solutions may be foreseen."

In 1975, when the status of all problems of Hilbert was reviewed, Kleiman was asked to report on the progress on the 15th problem. He wrote:

> "The foundations of the calculus were first secured for all applications by Van der Waerden [13]. [...] Van der Waerden saw that the (simplicial) topological intersection theory, developed by Lefschetz (1924, 1926) from some ideas of Poincaré and Kronecker, has all the necessary generality and rigor. In topological intersection theory, each algebraic set of the parameter variety is assigned a cohomology class. Continuously varying the set yields another set with the same cohomology class; in other words, the two sets are homologically equivalent. If two algebraic sets are in general position, then their intersection is assigned the (cup) product of their cohomology classes and their union is assigned the sum. Therefore, if several algebraic sets in general position intersect in a finite number of points, the number is conserved when the parameters of the sets are varied continuously because the number is equal to the degree of the product of the assigned cohomology classes and the classes are invariant; in other words, homological equivalence implies numerical equivalence. If the sets are defined by the conditions of an enumerative problem, it follows that the number of figures meeting the conditions is conserved when the parameters of the problem are varied continuously. [...] This is a rigorous justification of the principle of conservation of number within the context of an interpretation of the calculus of conditions by means of the calculus of algebraic cohomology classes."

Apart from justifying the principle of conservation of number and thus saving much of nineteenth-century work, the importance of this work of Van der Waerden lies also in the fact that it is an illustrative example of how topological methods can be applicable and useful in algebraic geometry.

3. Lengths and Multiplicities

Van der Waerden contributed also to another — related — problem in intersection theory. The way of defining the multiplicity $i(Z; X \cdot Y)$ of a component Z of a proper intersection $X \cap Y$ had been disputed since the end of the nineteenth century. Commonly, the length of the local ring $\mathcal{O}_{X \cap Y, Z}$ was taken as the intersection multiplicity (it works well in the case of plane curves). VAN DER WAERDEN [12] showed that this definition turns out to be wrong in the general case. Let us have a look why this goes wrong in an example in the projective 4 space over the complex numbers. The length of the local ring is here just the dimension of the local ring seen as a vector space over $\mathbf{C}$.

Take in $\mathbf{P}^4$ the algebraic set Y consisting of the union of two planes $L_1 : x = y = 0$ and $L_2 : z = w = 0$. The algebraic set Y is defined by the equations $xz = xw = yz = yw = 0$. We intersect Y with X defined by $x - z = y - w = 0$. These objects meet only at the origin. With help of the principle of conservation of number, it is easy to see that the multiplicity of the origin is 2 (just move X to get a transversal intersection with both L_1 and L_2). But for the length of the local ring, we get three, since

$$\begin{aligned} & \mathbf{C}(x,y,z,w)/(xz, xw, yz, yw, x-z, y-w) \\ \simeq\ & \mathbf{C}(x,y,z,w)/(z^2, zw, w^2, x-z, y-w) \\ \simeq\ & \mathbf{C}(z,w)/(z^2, zw, w^2) \\ \simeq\ & \mathbf{C} \oplus z\mathbf{C} \oplus w\mathbf{C}. \end{aligned}$$

In the dispute over intersection multiplicities, Severi defended the Italian school of geometers with the following dynamic definition: move the varieties X and Y, for instance with use of a general projective transformation, such that the intersection is proper and transversal, then $i(Z; X \cdot Y)$ is the number of components in which Z splits (see the summary in [10]). This definition had not been generally accepted because of its lack of rigour. But Van der Waerden [11] showed it is essentially correct, by developing a good notion of the degeneration of varieties, and hence a good notion of multiplicity. He published this in 1927 in the Mathematische Annalen. It was a key point in his work on Bézout's theorem (Mathematische Annalen, 1928 [12]) and on the justification of the principle of conservation of number (Mathematische Annalen, 1930 [13]).

In Van der Waerden's definition of multiplicity the group PGL_{n+1} of all projective transformations in $\mathbf{P}^n$ is used, which is a non-local object that prevents generalisations to the local situation. This was not a problem with the *diagonal construction*, that later, in a fundamental book on algebraic geometry [14], Weil introduced for the intersection of two varieties X and Y in the affine space $\mathbf{A}^n$. Consider two copies of $\mathbf{A}^n$, the first with variables $x_1, \ldots, x_n$ and the second with $y_1, \ldots, y_n$. In the Cartesian product $\mathbf{A}^n \times \mathbf{A}^n$ lies the product $X \times Y$ and also the *diagonal* Δ defined by $x_1 - y_1, \ldots, x_n - y_n$. It is easily seen that

$$X \cap Y = \Delta \cap (X \times Y),$$

and if $X \cap Y$ is proper, then $\Delta \cap (X \times Y)$ is proper too. Thus it suffices to define multiplicities for the right hand side. The advantage is that the space Δ is a complete intersection, i.e. defined by $\dim \mathbf{A}^n \times \mathbf{A}^n - \dim \Delta$ equations. This makes the situation easier to handle from a geometric point of view, since

Δ can be moved by varying the defining equations, but also from an algebraic point of view, because the ideal of Δ is generated by a regular sequence which becomes a system of parameters in the local ring $\mathcal{O}_{X\times Y,Z}$.

Meanwhile, CHEVALLEY [2] gave a definition of multiplicity in the setting of a complete local ring based on the algebraic side of the diagonal construction. SAMUEL [8] generalized this to the more general setting of any primary ideal in a noetherian local ring. Later SERRE [9] gave an explanation why the intuitive definition with the length does not work: also lengths of some modules $\mathrm{Tor}^i(\mathcal{O}_{X,Z}, \mathcal{O}_{Y,Z})$ interfere, which measure nilpotents in the local ring $\mathcal{O}_{X\cap Y,Z}$.

Nevertheless, the idea of the length of a local ring for a multiplicity was not completely absurd. But the obvious choice for the local ring was not the correct one. It took quite some time before a length of some other local ring associated in a geometric way to the intersection, appeared as an intersection multiplicity. We will try to show some of the concepts of an intersection theory that was developed by FULTON and MACPHERSON [3]. These concepts lead to a way of defining the intersection multiplicity as the length of a local ring.

For a variety V of dimension n the group Z_kV of *cycles* of dimension k is the free abelian group on subvarieties of dimension k in V. There is a so-called *rational equivalence* relation on Z_kV, which is roughly defined by stating that two varieties are rationally equivalent if they can be deformed into each other with a parameter running through a $\mathbf{P}^1$. By taking the rational equivalence classes, one obtains the *Chow group* A_kV. If V is non-singular and projective, then for any $[X] \in A_kV$ and $[Y] \in A_lV$, there is an intersection product $X \cdot Y \in A_{k+l-n}V$. This defines a ring structure on $A_*V = \bigoplus_k A_kV$.

Fulton and MacPherson define for any subvarieties S and S' of a variety V an intersection product $S \cdot S'$ if S is *regularly embedded* in V. (The algebraic set S of V of codimension d is called regularly embedded if it is locally cut out by d equations.) In the case of V non-singular, we can drop the condition on regularity, because the diagonal is then already regularly embedded in $V \times V$, so we can pass to the diagonal construction intersecting $S' = X \times Y$ with the diagonal Δ. This intersection product is a rational equivalence class living on the algebraic set $S \cap S'$ (which is in general smaller than V). An essential ingredient in their theory is the "normal cone."

The geometric idea is the following: since S is regularly embedded, there exists a vector bundle representing all normal directions to S: the normal bundle N_SV. Now we stretch V by "blowing up along S," and in this way we deform V to the normal bundle N_SV of S in V [3].

This process leads to a deformation of $S \hookrightarrow V$ into the embedding of the base in the normal bundle $S \hookrightarrow N_SV$. A subvariety S' of V deforms in this process to a *normal cone* $C_{S\cap S'}S$. The intersection product $S \cdot S'$ can be defined as $S \cdot C_{S\cap S'}S'$, where the second time S is the base (zero section) of the normal bundle. This last intersection is interesting: it turns out one can take here the length of the local ring of $C_{S\cap S'}S'$ along $S \cap S'$ to get the right notion of intersection multiplicity. Apparently, after the deformation to the normal bundle, the object representing normal directions has lost the information which is too particular to the local situation, and has become stable under movements of the algebraic sets.

We illustrate this with the previous example. Since the embedding of X in $\mathbf{P}^4$ is regular, we do not need to pass to the diagonal, but can work with the intersection of X and Y directly. The normal bundle of X in $\mathbf{P}^4$ has rank two, so the fibre over the origin is 2-dimensional. In the deformation, the two planes are both pushed to that fibre, so the normal cone consists of a plane counted twice. Hence the length of the local ring at the origin is two.

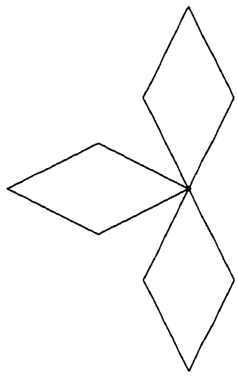

FIGURE 4. Before the deformation to the normal bundle

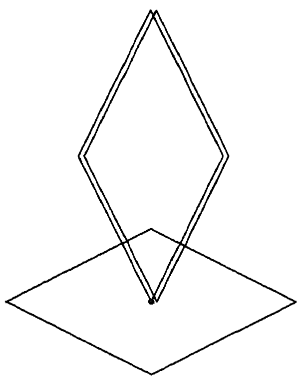

FIGURE 5. After the deformation to the normal bundle

So in a sense the circle is closed: the length of the local ring after the deformation to the normal bundle equals the intersection multiplicity.

Although the concepts and definitions are now generally accepted and no longer disputed thanks to Van der Waerden and others, intersection multiplicities are still intriguing. The computability is an issue: just write down an intersection which is not trivial, and try to compute the components and the multiplicities.

REFERENCES

1. E. BÉZOUT, 1979, *Théorie générale des équations algébriques*, Pierres, Paris.
2. C. CHEVALLEY, 1958, *Les classes d'équivalence rationelle, I, II*, Séminaire C. Chevalley, 2^e année, Anneaux de Chow et Applications, Secr. Math. Paris.
3. W. FULTON, 1984, Intersection Theory, *Ergebnisse der Math.* **3**, Bd. 2, Springer-Verlag, Berlin Heidelberg New York.
4. R. HARTSHORNE, 1977, Algebraic Geometry, *Graduate texts in math.* **52**, Springer-Verlag, Berlin Heidelberg New York.
5. S.L. KLEIMAN, 1980, Chasles's enumerative theory of conics: a historical introduction, in *Studies in algebraic geometry, Math. Assoc. Amer. Stud. Math.* **20**, 117-138.
6. S.L. KLEIMAN, 1976, Problem 15: Rigorous foundation of Schubert's calculus, *Proc. Sympos. Pure Math.* **28**, Amer. Math. Soc. 445-482.
7. S.L. KLEIMAN, S. STROMME, S. XAMBO, 1985, Schubert's number 5819539783680 of twisted cubics, *Proc. Conf. Rocca di Pappa.*
8. P. SAMUEL, 1951, La notion de multiplicité en algèbre et en géométrie algébrique, *J. Math. Pures Appl.* **30**, 159-274.
9. J.-P. SERRE, 1965, Algèbre Locale, Multiplicités, *Lect. Notes in Math.* **11**, Springer-Verlag, Berlin Heidelberg New York.
10. F. SEVERI, 1939, Über die Grundlagen der algebraischen Geometrie, *Abh. Math. Sem. Hamburg Univ.* **13**, 101-112.
11. B.L. VAN DER WAERDEN, 1927, Der Multiplizitätsbegriff in der algebraischen Geometrie, *Math. Ann.* **97**, 756-774.
12. B.L. VAN DER WAERDEN, 1928, Eine Veralgemeinerung des Bézoutschen Theorems, *Math. Ann.* **99**, 497-541.
13. B.L. VAN DER WAERDEN, 1930, Topologische Begründung des Kalküls der abzählenden Geometrie, *Math. Ann.* **102**, 337-362.
14. A. WEIL, 1946, *Foundations of Algebraic Geometry, Revised and enlarged edition,* Publ. **29**, 1962.