A Proof-theoretic Treatment of Assignments

WILFRIED MEYER-VIOL, *CWI, Kruislaan 413, 1098 SJ Amsterdam, and OTS Utrecht, Trans 10, 3512 JK Utrecht, The Netherlands.*
E-mail: wilfried@cwi.nl

Introduction

In a standard logical calculus the valuation of terms belongs firmly to the semantics of the language. Only the domain of these valuations is present in the syntax. In some cases, when we have names for all domain elements, the range of the valuations can also be identified as part of the language, but still the valuations themselves have no syntactic counterpart. On the other hand, for instance, in the LDS framework of [3], [4] assignments appear as statements within a proof-theoretic context. This occurs in the form of the Θ–function which instantiates labels, but also in the 'CHOOSE u=sue' construction which occurs in the handling of pronouns. In this paper we will undertake a proof-theoretic investigation of a language which allows assignments as formulas of the language. This can be seen as part of the ongoing program of internalizing semantics into proof theory. Once assignment statements are admitted to the language, they become amenable to proof-theoretic manipulation. In particular, rules can be formulated to *introduce*, *use*, and *eliminate* them.

In our calculus, assignments statements enter into a derivation as assumptions which assign *values* to 'expressive' variables.

The first paragraph will introduce the language of assignment statements and expressive variables. *Expressive variables* will be terms of the form $\nu x \varphi$, where ν is a *variable binding term operator* [1], x an individual variable and φ a formula in which x occurs free. Assignment statements will be formulas of the form $t := a$ where a, the *value* of the statement, is an individual constant standing for some domain element, and the term t, the *argument* of the statement, is some expressive variable. The arguments of assignment statements will incorporate the specific circumstances under which the statement has been introduced. Assignment statements always incorporate a *choice*: they assign a *value* to a (meta)variable. And this choice is made for a specific purpose. In this paper, assignment statements are introduced to handle the elimination of *existential quantifiers*. This reason for introducing an assignment is made explicit in the expressive (meta)variable. Once we have introduced the notion of a 'considered' choice to eliminate quantifiers, we may wonder whether we cannot describe a quantifier exhaustively in terms of assignment statements with the appropriate argument. That is, can a quantifier be proof-theoretically described purely in terms of the assignments that are used to eliminate and introduce it? It will turn out that this is possible for a whole range of quantifiers. In this paper we will describe some *indefinite* quantifiers with the classical existential quantifier as 'top–element'. The quantifiers in this family differ essentially in their attention to *dependencies*.

Dependencies between terms arise in sequences of choices, where the *value* of a choice is used in the statement of the *condition* for a subsequent choice. In our calculus, these dependencies are reflected in the *syntactic* structure of the terms involved: the conditions for previous choices are embedded in the condition for the present choice. In a standard logic these dependencies have no logical meaning: that is, in general we can find logically equivalent forms of a formula $R(t_1, \ldots, t_n)$ in which all possible variations of dependencies between $t_1, \ldots, t_n$ arise.

The second paragraph will introduce the basic proof theory for assignment statements. Here we will concentrate on the ubiquitous notion of an *arbitrary assignment*. This semantic notion will have the proof-theoretic form of a *dischargeable* assignment statement. The inference rules introduced for assignment statements will be such that dependencies between terms are respected.

The third paragraph then considers a treatment of the existential quantifier by means of assignment statements. We will first describe an interpretation of this quantifier which preserves dependencies. This logic will have all 'single quantifier' principles of the standard logic. Typically excluded are quantifier *permutations*. In order to get all standard principles for the existential quantifier we will extend the assignment logic with rules to give us *movement* of terms, creating fresh dependencies.

The natural interpretation of assignment statements is by value assignment functions of a model. It is not the object of this paper to discuss the *semantics* of assignment statements. However, the discussion about the proof theory of assignment statements will occasionally be interrupted to supply semantic interpretations informally. For more details on the semantic of this logic and for proofs of some of the statements in this paper we refer to [7].

1 The language of assignment statements

In this section we will introduce the first–order language we will be working with. It is a standard first–order language with some additional features. First of all, it contains a family of special terms, called ν–terms, which are associated with *formulas* of the language. Secondly, it contains a special binary predicate symbol ':=', the *assignment* predicate.

DEFINITION 1.1 (ALPHABET OF $\mathcal{L}$)
The alphabet of the language $\mathcal{L}$ consists of

1. a denumerably infinite set of *individual constants* $a, b, c, \ldots$ (also $a_1, a_2, \ldots$),
2. a denumerably infinite set of *individual variables* u, v, w, x, y, z (also x_1, $x_2, \ldots$) ,
3. a set of *predicate symbols* P, Q, $R \ldots$ (also P_1, $P_2 \ldots$),
4. the *logical symbols* $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$
5. *parentheses*) and (.

In addition to this classical alphabet, the language $\mathcal{L}$ also has

6. a set of *variable binding term operators* $\nu_1, \nu_2, \ldots$
7. A binary predicate ':='.

DEFINITION 1.2 ($\mathcal{L}$–TERMS)
1. Individual constants and variables are terms

2. If φ is an $\mathcal{L}$–formula, ν a term binding operator and x a variable occurring free in φ, then $\nu x\varphi$ is an ν– *term* of $\mathcal{L}$.
3. Only sequences defined by (1) and (2) are terms.

A variable binding term operator (vbto) ν takes a variable and a $\mathcal{L}$–formula in which the variable occurs free, to give an $\mathcal{L}$– term. So, there is no vacuous quantification by ν–binders. The family of ν–terms for all operators ν will be called *identified terms*, the formula φ in $\nu x\varphi$ will be called the *identifier* of the term. Whenever we fix a definite operator ν_i, then the terms constructed by this operator will be called ν_i–terms.

DEFINITION 1.3 ($\mathcal{L}$–FORMULAS)

1. If P is a k–place predicate symbol and $t_1, \ldots, t_k$ are terms, then $P(t_1, \ldots, t_k)$ is a formula
2. If t is a *closed* $\mathcal{N}$–term and t' an $\mathcal{L}$–term, then $t := t'$ is an $\mathcal{L}$–formula.
3. If φ and ψ are $\mathcal{L}$–formulas, then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$
4. If φ is an $\mathcal{L}$–formula and x an individual variable, then $\forall x\varphi$ and $\exists x\varphi$ are $\mathcal{L}$–formulas.
5. Only sequences defined by (1)–(5) are $\mathcal{L}$–formulas.

So the syntax of $\mathcal{L}$–formulas is the standard first–order syntax, with additional atomic formulas of the form $t := t'$, so–called *assignment statements*.

We assume familiarity with the notions of the *scope* of quantifiers and vbto's, the notion of *free* and *bound* variables, and the notion of a term t *being free for* a term t' in a formula φ.

Identified terms

An identified term consists of a specific vbto ν binding a variable x which occurs in φ, the *identifier* of the term. This identifier gives the possibility to relate terms and formulas of the language, We will exploit this relation proof-theoretically to make explicit the relationships between the derivational properties of the quantifier rules and the proper terms they introduce and eliminate. Because ν–terms may occur in formulas as well as other ν– terms, an obvious relation of *dependency* between terms can be defined.

DEFINITION 1.4 (SYNTACTIC DEPENDENCY)
Let $\mathcal{N}$ be the set of all ν–terms, $\nu_i x\varphi \in \mathcal{N}$ and t an $\mathcal{L}$–term.

1. $\nu_i x\varphi$ immediately depends on t, notation $\nu_i x\varphi \ll t$, if t occurs in φ not within the scope of a ν–symbol/
2. $\nu_i x\varphi$ depends on t, notation $\nu_i x\varphi \prec t$, if there is some finite chain of immediate dependency steps from $\nu x\varphi$ to t.
3. $\mathcal{N}_i(\varphi)$ is the set of all ν_i–terms occurring in φ.

The dependency relation between terms mentions only *occurrences* of terms in other terms. The notion of *subordination* extends this relation between terms to include quantificational dependencies.

DEFINITION 1.5 (SUBORDINATION)
Let $\nu_i x\varphi$, $\nu_j y\psi \in \mathcal{N}$ then $\nu_i x\varphi$ is subordinate to $\nu_j y\psi$, if $\nu_j y\psi \prec \nu_i x\varphi$ and some free occurrence of variable y in $\nu x\varphi$ is bound in $\nu_j y\psi$

So $\nu y Q(y,x)$ is subordinate to $\nu x R(x, \nu y Q(y,x))$ and $\nu y Q(y, \nu z T(z,x))$ subordinate to $\nu x R(x, \nu y Q(y, \nu z T(z,x)))$.

In general, the dependency relation between terms reflects, the *nesting* of the choice conditions they represent, while the subordination relation reflects *linkage* of these conditions by means of variable sharing. The ν–terms will function as expressive variables we introduce to eliminating quantifiers. The information such a variable should contain is reflected in the syntactic structure of the term. For instance, Hilberts *epsilon terms* $\epsilon x \varphi$ [5],[6] can be seen as a triple

$$\langle \epsilon, x, \varphi \rangle$$

of a *control parameter* 'ϵ', a variable 'x' and a formula 'φ' in which x occurs free: $\epsilon x \varphi$ then characterizes a variable x *existentially* bound in φ. Hilberts *tau terms* $\tau x \varphi$ have the same shape, but use a different control parameter: in this case the variable x is *universally* bound in φ. These well-known terms where introduced to analyze *quantificational* structure of first–order formulas, but the general shape of identified terms allows us to introduce a variety of control parameters

Assignment statements

The syntax of the assignment predicate := has been stated as follows

> If t is a *closed* $\mathcal{N}$–term and t' an $\mathcal{L}$–term, then $t := t'$ is an $\mathcal{L}$–formula.

Assignment statements can be complex or atomic. An *atomic* assignment statement is a formula of the form $t := a$, where a is an individual constant and t is a closed ν–terms such that there is no closed ν–term t' with $t \prec t'$. For example, $\epsilon x R(x,x) := a$, and $\epsilon x R(x, \epsilon y Q(x,y)) := a$ are atomic assignment statements, while $\epsilon x R(x, \epsilon y Q(y,y)) := a$, $\epsilon x R(x, \epsilon y Q(x,y, \epsilon x P(x))) := a$ and $\epsilon x R(x,x) := \epsilon y Q(y)$ are not. Complex assignment statement can be built from atomic ones, and complex statements analyzed into atomic ones, by means of *assignment rules*.

Assignment statements bring the assignments of values for expressive variables, ν–terms, into the proof-theoretic realm. Such statements represent *choice* actions: in a statement $\nu_i x \varphi := a$ a value is chosen for the term $\nu_i x \varphi$. The control parameter ν_i determines in what way the a value of the choice is related to the choice *condition* $\varphi(x)$.[1]

Assignments statements *internalize* semantic assignment functions. This internalization requires the introduction of *names* for the choice values. By these names an natural dependency relation between assignment statements arises: once a choice value has been given a name, subsequent choices may use this name in the formulation of the choice condition.

[1] The assignment statements resemble the Θ function of [3] which instantiates labels, but also the CHOOSE label of the same article, which relates pronouns to their antecedent: CHOOSE u = sue.

2 Basic logic of assignment statements

The logic of assignment statements is given by inference rules which relate assignments to ν–terms, and allow assignment statements to interact with $\mathcal{L}$–formulas, and in particular with other assignment statements. These rules extend some standard natural deduction calculus for classical or intuitionistic logic, for instance [8]. There are two rules for ν–terms in general. By IV, the value assigned to ν–terms is independent of

$$\frac{\nu x\varphi := t}{\nu y\varphi[y/x] := t}\ \text{(IV)}\qquad \frac{\nu x\varphi := t \qquad \forall x(\varphi(x) \leftrightarrow \psi(x))}{\nu x\psi := t}\ \text{(EQ)}$$

Where y is free for x in φ. (IV)

FIG. 1. Rules for ν–terms

the identity of the bound variable. By EQ, ν–terms with logically equivalent identifiers become indistinguishable. Specific classes $\mathcal{N}_i$ of ν_i–terms arise by the addition of rules for specific vbto's ν_i.

There are two basic proof rules for assignment statements.

The first rule, R1, deals with the *use* of assignments The second rule, R2, deals

$$\frac{\varphi \qquad t_1 := t_2}{\varphi[t_2/t_1]}\ \text{(R1)}\qquad \frac{\begin{matrix}\Sigma \qquad [t_1 := t_2] \\ \hline \vdots \\ \xi\end{matrix}}{\xi[t_1/t_2]}\ \text{(R2)}$$

R1: *Provided t_2 is free for t_1 in φ*

R2: *Provided t_2 does not occur in t_1, in Σ or in any term in Σ.*

FIG. 2. Basic Rules for Assignment Statements

with *discharge* of assignment statements. An assignment can be discharged if it is 'arbitrary': in order to discharge an assignment statement, no conclusion depending on an assignment should be based on some special property of the value of the discharged assignment statement. We can discharge an assignment statement if we can 'reverse' the corresponding substitution while maintaining derivationhood.

Notice that, in R2, no restrictions are imposed on occurrences of t_1 in Σ. This implies that special properties of this term may be used.

REMARK 2.1

Assignment statements will be interpreted by *value assignment functions* of a model. The kind of function used will depend on the kinds of the terms involved. The semantic interpretation of ν–terms $\nu x\varphi$ takes place by *choice functions* Φ assigning domain elements to all subsets of the domain. The IV and EQ rules our sound and complete with respect to this interpretation.

There are various choices for an interpretation of the *truth* of assignment statements on a model. Most straightforwardly, a statement $t_1 := t_2$ holds on a model $\mathcal{M}$ with respect to term valuation $V_{\mathcal{M},\Phi}$ if t_1 and t_2 are assigned the same value in $\mathcal{M}$ by $V_{\mathcal{M},\Phi}$. The soundness of R1 then follows from the standard *substitution lemma*. For R2, consider that under the conditions for discharge, a model $\mathcal{M}$ for the assumptions in Σ need not interpret the term t_2. Consequently, $\mathcal{M}$ can be *expanded* to a model interpreting t_2.

A second natural interpretation of the truth of $t_1 := t_2$ on a model is: $t_1 := t_2$ holds on model $\mathcal{M}$ with respect to term valuation $V_{\mathcal{M},\Phi}$ if t_1 and t_2 are assigned the same value in $\mathcal{M}$ by all *extensions* $V'_{\mathcal{M},\Phi}$ of $V_{\mathcal{M},\Phi}$ over the term t_2. In this case the interpretation of inference rules has to proceed by evaluating the conclusion by an extension of the valuation function of the premises (see remark 2.8).

Finally, a third interpretation goes as follows: $t_1 := t_2$ holds on a model $\mathcal{M}$ with respect to term valuation $V_{\mathcal{M},\Phi}$ if t_1 and t_2 are assigned the same value in $\mathcal{M}$ by a t_1–*variant* $V_{\mathcal{M},\Phi'}$ of $V_{\mathcal{M},\Phi}$. This interpretation will be required in paragraph 3.3.

EXAMPLE 2.2 (SOME BASIC DERIVATIONS)
The rule R1 gives us the standard well-definedness conditions for the internal assignments. Straightforwardly we have

$$\frac{\epsilon x\varphi(x,\epsilon y\psi) := a \qquad\qquad \epsilon y\psi := b}{\epsilon x\varphi(x,b) := a}$$

So we have $\{\epsilon x\varphi(x,\epsilon y\psi) := a, \epsilon y\psi := b\} \vdash_{R1} \epsilon x\varphi(x,b) := a$
In a classical context we can derive also $\{\epsilon x\varphi(x,b) := a, \epsilon y\psi := b\} \vdash_{R1} \epsilon x\varphi(x,\epsilon y\psi) := a$, by

$$\dfrac{\dfrac{\dfrac{[\neg\epsilon x\varphi(x,\epsilon y\psi) := a](1) \qquad \epsilon y\psi := b}{\neg\epsilon x\varphi(x,b) := a} \qquad \epsilon x\varphi(x,b) := a}{\bot}}{\epsilon x\varphi(x,\epsilon x\psi) := a\ (-1)}$$

The rule R2 gives us then $\{\epsilon y\psi := b\} \vdash_{R1-R2} \epsilon x\varphi(x,\epsilon\psi) := \epsilon x\varphi(x,b)$ and $\{\epsilon y\psi := b\} \vdash_{R1-R2} \epsilon x\varphi(x,b) := \epsilon x\varphi(x,\epsilon y\psi)$. The syntactic dependency relation on assignment statements is translated into a dependency relation on the ϵ–terms.

Proof-theoretic dependency

It can be clearly described under what circumstances constellations of assignment statements *can* and *cannot* be discharged. Let us assume that, in a derivation Δ of φ from Σ, proper terms, and the terms t_1, t_2, t_3 in Σ occur only in assignment statements, then there are three constellations of *assumptions* preventing discharge.

1. $\{t_1 := a, t_2 := a\}$. Neither of the assumptions can be discharged by R2 because the proper term occurs as value in the other assumption
2. $\{t_1(a) := a\}$. No discharge by R2 is possible, because a occurs in $t_1(a)$.

3. $\{t_1(a) := b, t_2(b) := c, t_3(b) := a\}$. No discharge by R2 is possible because each proper term occurs in a term of an assumption.

Assignments of the second kind will be called *reflexive* assignment statements. Sets of assignment statements of the third kind will be called *circular*. The conditions under which discharge is possible can be stated in terms of a syntactic dependency relation on assignment statements

DEFINITION 2.3 (PROOF-THEORETIC DEPENDENCY)
In a derivation Δ term assignment $t_1 := a$ *immediately* depends on assignment $t_2 := b$, notation $t_1 := a \ll t_2 := b$, if b occurs in φ.

The assignment statement $t_1 := a$ *depends* on $t_2 := b$, notation $t_1 := a < t_2 := b$, if there is a sequence of immediate dependency steps relating $t_1 := a$ to $t_2 := b$.

If $t_1 := a \ll t_2 := b$ then we will also say that the term a immediately depends on term b in Δ, notation $a \ll b$.

DEFINITION 2.4 (ASSIGNMENT STATEMENTS AND FUNCTIONS)
Let Δ be a derivation with discharged and undischarged assumptions in Σ. Then

1. $As_\Delta \subseteq \Sigma$ is the set of assignments statements in Δ.
   $\overline{As}_\Delta = \{t := t' \mid \Sigma \vdash t := t'\}$
2. $A_\Delta = \{\langle t, a\rangle \mid t := a \in As_\Delta\}$.
   $\overline{A}_\Delta = \{\langle t, a\rangle \mid t := a \in \overline{As}_\Delta\}$
3. For φ a formula occurrence in Δ, $As_\Delta(\varphi)$ is the set of assignment statements on which derivationally depends.

For arbitrary sets As of assignment statements, not related to a derivation, we will consider the same sets, where $\overline{As} = \{t := t' \mid As \vdash t := t'\}$ and A and $\overline{A}$ are defined accordingly. Now the above conditions give the following lemma for the use of proper terms.

LEMMA 2.5 (STRUCTURE OF ASSIGNMENTS)
Let Δ be a derivation with assumptions in Σ. Let proper terms occur in Σ only in As_Δ, then the elements of $As_\Delta(\varphi)$ can be discharged at formula occurrence φ if and only if

1. $A_\Delta(\varphi)$ is a many–to–one relation. i.e., if $\langle t_1, a\rangle, \langle t_2, a\rangle \in A_\Delta(\varphi)$, then $t_1 = t_2$.
2. The tuple $\langle A_\Delta, <\rangle$ is a *strict partial order*.

Because of the first condition, given a tuple $\langle t(a), a'\rangle \in A_\Delta(\varphi)$ we can always find at most a *unique* $\langle t', b\rangle \in A_\Delta(\varphi)$ such that $b = a$. I.e $t(a) := b$ has at most one $t' := a \in As_\Delta(\varphi)$. t has at most one $\ll$–successor for every term a occurring in t. Because of the uniqueness of $\ll$–successors, if all elements of a *dependency closed* A_Δ can be discharged, then A_Δ can divided in a family of *functional* dependency closed subsets: a family of internal assignments.

This structure of assignments is common to all frameworks where proper terms are used. for instance in the restrictions put on the use of proper terms in Existential Instantiation and Universal Generalisation Frameworks [9], and this structure also occurs as an ordering on *arbitrary objects* in the models of [2].

COROLLARY 2.6 (CONSERVATIVITY)
Let Σ be a set of $\mathcal{L}$–formulas, ψ an $\mathcal{L}$–formula, both ':='–free, and let $\vdash$ denote the derivability relation determined by some deduction system for classical logic, then

$$\Sigma \vdash_{R1-R2} \psi \Rightarrow \Sigma \vdash \psi$$

PROOF. Suppose $\Sigma \vdash_{R1-R2} \psi$ but $\Sigma \nvdash \psi$ for Σ and ψ satisfying the condition of the proposition. Then $\Sigma \cup \{\neg\psi\}$ is classically consistent, and so has a model $\mathcal{M}$. Because all assignment statements in the R1–R2 derivation Δ of ψ from Σ have been discharged, A_Δ can be strictly partially ordered by $<$, and none of the values assigned in As_Δ occur in Σ or ψ. Consequently, the model $\mathcal{M}$ for Σ can be supplied with a strict partially ordered set of *expansions* interpreting every new proper term a for $t := a \in As_\Delta$ by the value assigned to t. Because none of the proper terms statements in As_Δ occurs in ψ, none occur in $\neg\psi$. Consequently $\mathcal{M}$ can be expanded to a model for $\Sigma \cup As_\Delta \cup \{\neg\psi\}$. So $\Sigma \nvdash_{R1-R2} \psi$. ∎

The partially ordered set of *expansions* in the above proof represents a a partially ordered set of *internalizations* of assignments of the model. Each expansion interpretating $t := a$ can interpret t' for every assignment $t' := b$ such that $t' := b \ll t := a$. Because b does not occur in any of the assignments already internalized, we can expand the present model over b. If As_Δ has non–dischargeable assignment statements, matters are different. For instance, for a set of assignment statements like $\{t := a, t'(a) := a\}$ we can expand a model, interpreting t, over a. Now this model interprets the term $t(a)$, but $t(a) := a$ need not hold for arbitrary values of a. There is no guarantee that we can find an expansion internalizing the second assignment.

In a derivation of an ':='–free conclusion from ':='–premises, the assignment statements arise only as instruments that can be discarded upon reaching the conclusion. The assignment statements embody arbitrary assignment of names to choice values. A Derivation Δ with conclusion φ such that $As_\Delta(\varphi)$ is non–empty represents a *specific* choice of value: the conclusion φ follows only under the specific assignments present in $As_\Delta(\varphi)$. This is the case, for instance when we want to assign the same value to different identified terms, or need circular sets of assignment statements. The next paragraph will show reasons for wanting such (sets of) assignments.

Sets of assignment statements As determine 'internal' *assignment functions* $\overline{A}$. A derivation in which all assignment statements have been discharged represents a derivation where *arbitrary* values haven been assigned: no matter what value we choose in the assignments, the conclusion follows. For this arbitrariness to hold none of the assumptions of the derivation may contain *both* the argument and the value of an assignment statement.[2] If this is the case, then the *internal* assignments of the derivation can be interpreted by *arbitrary* external, i.e. semantic, assignment functions.

The behavior of these functions as *value assignments* is characterized by the following proposition:

[2]Violation of the restriction on the substituted term gives $R(x,t) \vdash R(x,x)$ by the derivation.

$$\cfrac{\cfrac{R(x,t) \qquad [x := t]}{R(t,t)}}{R(x,x)}$$

PROPOSITION 2.7
Let Δ be a derivation in classical first order logic extended by R1. Let As_Δ be a *dependency closed*[3] set of assignment statements and let A' be a *functional* dependency closed subset of A_Δ, such that $t_1, \in dom(\overline{A'})$. Then

$$\overline{A'}(t_1) = \overline{A'}(t_1[\overline{A'}(t_2)/t_2])$$

PROOF. Immediate from the derivations in example 2.2 and the fact that As_Δ is dependency closed. ∎

REMARK 2.8
In a derivation Δ every formula occurrence φ is accompanied by a set $As_\Delta(\varphi)$ of assignment statements on which it depends. Along a *path* $\varphi_1, \ldots, \varphi_n$ in Δ the sequence $As_\Delta(\varphi_1), \ldots, As_\Delta(\varphi_n)$ gives the development of an assignment function, and the model–valuation pair at a formula occurrence φ_i can be read off from the model $\mathcal{M}$ for the assumptions of Δ and the assignment $A_\Delta(\varphi_i)$.

Let all assumptions of Δ and its conclusion be without individual constants. If all assignment statements in Δ have been discharged, and φ_1 is an assumption of Δ, then $As_\Delta(\varphi_1)$ and $As_\Delta(\varphi_n)$ are empty. In this case the assignment statements occurring in the course of the derivation represent *arbitrary* assignments to the ν–terms, and we can conclude that the conclusion holds on a model for the assumptions for *every* assignment of values to the ν–terms.

3 The existential quantifier

In this paragraph we are going to put the assignment statements to use. As a case study, we will investigate the existential quantifier. First we define an elimination and introduction rule for the existential quantifier which require assignment statements involving Hilberts ϵ–terms. We will explore how far the basic assignment rules get us in defining the classical quantifier. It will turn out that the basic rules alone give us only a rudimentary quantifier which does not allow for quantifier interactions. We will suggest two rules to extend the basic framework which give us the interaction principles of the standard quantifier.

3.1 Elimination and introduction

Our interpretation of the existential quantifier is guided by an example from [2]. According to this author, the paradigmatic form of reasoning with existential information is exemplified by the following move in a mathematical argument

"There exists a bisector to the angle α. Call *it* B"

In our language we can give a straightforward rendering of this form of reasoning by a combined use of assignment statements and ϵ–terms. The formulation of this rule requires all components of the language.

[3] As_Δ is dependence closed if, whenever $t_1 := t_2 \in As_\Delta$ and $t_1 \prec t_3$, then there is a t_4 such that $t_3 := t_4 \in As_\Delta$

The elimination rule

$$\epsilon\text{E–Rule} \qquad \frac{\exists x\varphi \qquad\qquad \epsilon x\varphi := a}{\varphi[a/x]}$$

In the formulation of the ϵE–rule, $\exists x\varphi$, "There is a φ– er", is the major premise. The *assignment statement* $\epsilon x\varphi := a$, "Call it a", is the minor premise of the application. The major premise introduces a *generic proper term* in the proof discourse to which reference is possible in the subsequent proof (by '*it*', or '*that* φ–er').

In the language this object takes the form of an ϵ–term $\epsilon x\varphi$. The minor premise assigns a value a, the *proper term* to this object. In this rule, the formula $\varphi[a/x]$ is a *conclusion*[4]. Consequently, "There exists a bisector to the angle α. Call it a", has the conclusion: "So a is a bisector to angle α.

Semantically, the assignment $\epsilon x\varphi := a$ represents a 'friendly' choice of value to the variable x which has to satisfy φ. Given the premise $\exists x\varphi$, such a friendly choice can find such an element. This justifies the conclusion.

Proof-theoretically, notice that by the assignment $\epsilon x\varphi := a$, the proper term a proof-theoretically depends on all individual constants occurring in $\exists x\varphi$. By consequence, the conclusion $\varphi[a/x]$ of the ϵE–rule does not hold for arbitrary a satisfying φ, but only for those that depend on the parameters in φ: the existential quantifier ranges over *dependent* objects. The major premise of the ϵE–Rule, $\exists x\varphi$ tells us that there is an element dependent on the parameters in $\exists x\varphi$ satisfying φ. The minor premise gives a piece of dependency relation: a proper term a depending on the parameters in $\epsilon x\varphi$. The conclusion then gives us the fact that this dependent element a satisfies φ. By consequence, the corresponding rule *eliminating* the proper term (*introducing* the existential quantifier) should not allow us to conclude $\exists x\varphi$ from arbitrary $\varphi[a/x]$; we have to know that the term a in fact depends on the parameters in $\exists x\varphi$.

Putting the term defining rule, ϵE, and the second assignment rule, R2, together we get the following

$$\frac{\Sigma \qquad \dfrac{\exists x\varphi \qquad\qquad [\epsilon x\varphi := a](1)}{\varphi[a/x]}}{\dfrac{\begin{array}{c}\vdots \\ \xi\end{array}}{\xi[\epsilon x\varphi/a]\ (-1)}}$$

Because of the conditions on R2, the constant a does not occur in φ, in Σ or in any term in Σ. If a does not occur in ξ then we have the standard discharge conditions of the elimination rule for the existential quantifier.

Properties of ϵE The ϵE–rule combined with the standard logical rules give us the following interactions of ϵ–terms, existential formulas and their assignment statements:

1. $\{\exists x\varphi, \epsilon x\varphi := a\} \vdash_{\epsilon E} \varphi(a)$
   Given that there are φ-ers, then $\epsilon x\varphi$ is assigned a φ–er as value.
2. $\{\epsilon x\varphi := a, \neg\varphi(a)\} \vdash_{\epsilon E} \neg\exists x\varphi$
   If $\epsilon x\varphi$ is assigned a non–φ–er as value, then there are no φ–ers at all

[4] not an assumption as is the case in applications of the standard ($\exists$E) rule.

3. $\vdash_{\epsilon E} \exists x\varphi \to \forall y(\epsilon x\varphi := y \to \varphi(y))$
If there are φ–ers, then any value assigned to $\epsilon x\varphi$ will be a φ–er.

Combined use of ϵE and R1 gives us

4. $\{\exists x\varphi, \psi(\epsilon x\varphi), \epsilon x\varphi := a\} \vdash_{\epsilon E} \psi(a) \wedge \varphi(a)$.

Including also R2 we get

1. $\vdash_{\epsilon E} \exists x\varphi \to \varphi(\epsilon x\varphi)$

These give universal demands for all models with value assignments interpreting the ϵ–terms.

The introduction rule

Corresponding to the ϵE–rule which eliminates an existential quantifier, we have to formulate an introduction rule for this quantifier. The standard rule ($\exists$I),

$$\frac{\varphi[t/x]}{\exists x\varphi}$$

where t should be free for x in φ. This rule allows us to conclude $\exists x\varphi$ for *arbitrary* t such that $\varphi[t/x]$ holds. This reflects the interpretation of the existential quantifier in terms of non–emptiness of definable subsets. However, this rule is not symmetric to the elimination rule ϵE. The symmetric rule should only allow the conclusion $\varphi(t)$ for terms t dependent on the parameters of $\exists x\varphi$. We will consider the following introduction rule:

DEFINITION 3.1 (THE ϵI–RULE)

$$\frac{\varphi[a/x] \qquad\qquad \epsilon x\varphi := a}{\exists x\varphi}$$

From the fact that $\varphi[a/x]$ holds and the fact that a *is the value* of some assignment to $\epsilon x\varphi$, we can conclude to $\exists x\varphi$.

If the premise φ depends on assumptions that do not contain a, then the assignment can be discharged. If this is not the case, then $\exists x\varphi$ cannot be concluded solely on the basis of $\varphi[a/x]$. Notice that this rule need not remove all occurrences of the proper term from the major premise

$$\frac{R(x,a)[a/x] \qquad\qquad \epsilon x R(x,a) := a}{\exists x R(x,a)}$$

By the ϵI– rule we can abstract over some, but not all, occurrences of a given term in a formula. In this case we have to use a *reflexive* assignment statement. So this statement cannot have the status of an assumption. In derivations with the ϵE and ϵI–rules, the object is to have the right assignment statements present to introduce existential quantifiers. In the above example the statement $\epsilon x R(x,a) := a$ cannot be taken as an assumption if we want all assignment statements to be discharged at the conclusion. Extending the basic ϵE–ϵI calculus with rules to derive fresh assignment statements will achieve this.

The symmetry between the introduction and elimination rule has the curious consequence that the nature of the quantifier does not appear anymore from its introduction and elimination rule. The full load of determining the nature of the quantifier is is carried by the generic terms used in their application.

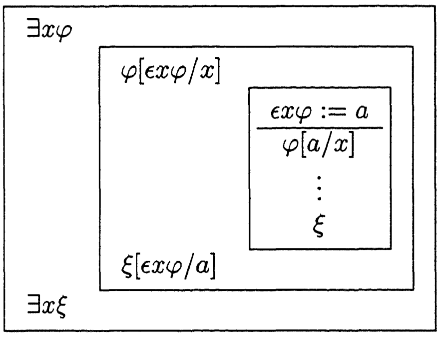
REMARK 3.2
For ϵ–terms the general choice functions evaluating ν–terms satisfy: for all $N \in \mathcal{P}(M)$:

$$\Phi(N) = \begin{cases} m \in N & \text{if } N \neq \emptyset \\ m \in M & \text{otherwise} \end{cases}$$

3.2 Pure term logic

Starting from the premises "There is a bisector to angle α." "Call *it* a." we have concluded "So a is a bisector to angle α". Now we use this instance to show "a has property ξ". We retract the specific instance chosen and conclude: "There exists a bisector to the angle α. *It* has property ξ".

Finally, to get to standard, anaphor-free logic, we have to rid ourselves of 'it'. For instance, by concluding "There exists a bisector to angle α. So there exists a ξ–er". In graphic form, we get the procedural nesting



The outer box contains the first–order language without ϵ–terms or individual constants. In this box the existential formulas consitute the *antecedents* for the ϵ–terms in the middle box. In this box we find the pure term fragment of the language $\mathcal{L}$. I.e. the $\mathcal{L}$ formulas without quantifier symbols (in formulas or ϵ–terms) and without individual constants. The terms in this box are assigned values in the deepest box. This box we leave by discharging the corresponding assignment statement. We leave the middle box by introducing the existential quantifier, i.e., setting up a fresh *antecedent*.[5]

In this calculus we can reduce all antecedents to logically equivalent anaphoric forms by the derivations

$$\frac{\dfrac{\exists x\varphi \qquad [\epsilon x\varphi := a](0)}{\varphi[a/x]}}{\varphi[\epsilon x\varphi/x]\ (-0)} \qquad\qquad \frac{\dfrac{\dfrac{\varphi[\epsilon x\varphi/x] \qquad [\epsilon x\varphi := a](0)}{\varphi[a/x]} \qquad [\epsilon x\varphi := a]}{\exists x\varphi}}{\exists x\varphi\ (-0)}$$

[5] A standard proof of conservativity of this calculus over first–order logic shows that everything that can be derived from the anaphoric framework based on non–anaphoric formulas can also be derived anaphor–free.

Consequently, $\vdash \exists x\varphi \leftrightarrow \varphi[\epsilon x\varphi/x]$. In fact, if we define a quantifier $\mathcal{E}$, by $\mathcal{E}x\varphi =_{df} \varphi[\epsilon x\varphi/x]$, then the assignment rules take over the role of the quantifier rules, given, of course, the specific interpretation of the ϵ–terms. The R1 rule then *eliminates* ϵ–terms and discharge is taken care of by R2. In pure assignment terms

$$\dfrac{\dfrac{\mathcal{E}x\varphi \qquad [\epsilon x\varphi := a](0)}{\varphi[a/x]}}{\mathcal{E}x\varphi\ (-0)}$$

This is a correct derivation if the term a does not occur in φ. Here the discharge of the assignment statement coincides with the introduction of the quantifier $\mathcal{E}x$.

By these rules, every ϵ–free first–order formula is logically equivalent to a pure, quantifier–free, term form.[6]

We will concentrate on this *defined* existential quantifier and consider the rules necessary to turn this operator into the quantifier of standard logic.

When we define the existential quantifier $\mathcal{E}$ as $\mathcal{E}x\varphi =_{df} \varphi[\epsilon x\varphi/x]$, we have to show that the corresponding identity $\nu y\mathcal{E}x\varphi =_{df} \nu y\varphi[\epsilon x\varphi/x]$ for ν–terms is well-defined. To show that this is the case we will derive $\nu y\mathcal{E}xR(x,y) := a \leftrightarrow \nu yR(\epsilon xR(x,y),y) := a$.

$$\dfrac{\dfrac{\dfrac{\mathcal{E}xR(x,y) \qquad [\epsilon xR(x,y) := b](1)}{R(b,y)}}{\mathcal{E}xR(x,y) \to R(b,y)}}{\mathcal{E}xR(x,y) \to R(\epsilon xR(x,y),y))\ (-1)}$$

and

$$\dfrac{\dfrac{\dfrac{\epsilon xR(x,y) := b \qquad \dfrac{R(\epsilon xR(x,y),y) \qquad [\epsilon xR(x,y) := b](1)}{R(b,y)}}{\mathcal{E}xR(x,y)}}{R(\epsilon xR(x,y),y) \to \mathcal{E}xR(x,y)}}{R(\epsilon xR(x,y),y) \to \mathcal{E}xR(x,y)\ (-1)}$$

The conclusion now follows with the EQ rule for ν–terms.

Introducing the existential quantifier by definition in this way, the full burden of determining the logical properties of the quantifier lies with the rules for ϵ–terms and assignment statements. Each family of indefinite terms is determined by its set of rules. For instance, the 'critical formula' schema

$$\varphi[t/x] \to \varphi[\epsilon x\varphi/x]$$

reduces the quantifier $\mathcal{E}$ to $\exists$ in one fell swoop [5]. For our purposes we will proceed more slowly

DEFINITION 3.3 (INDEFINITE QUANTIFIERS)
A variable binding term operator ν determines a family of *indefinite* quantifiers if it satisfies the schema

$$(\varphi(\nu x\varphi) \vee \psi(\nu x\psi)) \leftrightarrow (\varphi(\nu x(\varphi \vee \psi) \vee (\psi(\nu x(\varphi \vee \psi)).$$

We will reserve the symbol 'ϵ' for vbto's satisfying this schema.

[6] In this paper we will only deal with the existential quantifier. The universal quantifier will taken to be the standard one. However, it is easy to determine the pure term form of *dual* of the quantifier $\mathcal{E}$. For $\vdash \neg\mathcal{E}\neg x\varphi \leftrightarrow \varphi[\epsilon x\neg\varphi/x]$.

For ϵ an indefinite operator, the corresponding quantifier satisfies $\mathcal{E}x(\varphi \vee \psi) \leftrightarrow \mathcal{E}x\varphi \vee \mathcal{E}x\psi$. To get such an operator we can give the following rules for assignments to ϵ–terms This gives us the 'single quantifier' principles for the indefinite $\mathcal{E}x$.

$$\text{I1} \quad \frac{\varphi[a/x] \qquad\qquad \epsilon x\varphi := a}{\epsilon x(\varphi \vee \psi) := a} \qquad \text{I2} \quad \frac{\varphi[a/x] \qquad\qquad \epsilon x(\varphi \vee \psi) := a}{\epsilon x\varphi := a}$$

FIG. 3. Rules for Indefinite ν–Terms

EXAMPLE 3.4 (SOME TYPICAL PRINCIPLES)

1. $\mathcal{E}x(\varphi \wedge \psi) \rightarrow \mathcal{E}x\varphi \wedge \mathcal{E}x\varphi$

$$\begin{array}{c}
(\varphi \wedge \psi)[\epsilon x(\varphi \wedge \psi)/x] \\
\hline
(\varphi \wedge \psi)[\epsilon x(\varphi \wedge \psi)/x] \vee (\varphi \wedge \neg\psi)[\epsilon x(\varphi \wedge \neg\psi)/x] \\
\hline
(\varphi \wedge \psi)[\epsilon x((\varphi \wedge \psi) \vee (\varphi \wedge \neg\varphi))/x] \vee (\varphi \wedge \neg\psi)[\epsilon x((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi))/x] \\
\hline
(\varphi \wedge \psi)[\epsilon x\varphi/x] \vee (\varphi \wedge \neg\psi)[\epsilon x\varphi/x] \\
\hline
\varphi[\epsilon x\varphi/x]
\end{array}$$

2. $\mathcal{E}x(\mathcal{E}x\varphi \rightarrow \varphi)$.
   Notice that we have $\varphi[a/x], \epsilon x\varphi := a/\epsilon x(\psi \rightarrow \varphi) := a$ by I1 and the EQ rule. This gives

$$\dfrac{\dfrac{\dfrac{\overset{(0)(1)}{\epsilon x(\varphi[\epsilon x\varphi/x] \rightarrow \varphi) := a} \qquad \dfrac{[\varphi[\epsilon x\varphi/x](0) \qquad\qquad [\epsilon x\varphi := a](1)}{\varphi[a/x]}}{\varphi[\epsilon x(\varphi[\epsilon x\varphi/x] \rightarrow \varphi)/x]}}{\varphi[\epsilon x\varphi/x] \rightarrow \varphi[\epsilon x(\varphi[\epsilon x\varphi/x] \rightarrow \varphi)/x] \ (-0)}}{\varphi[\epsilon x\varphi/x] \rightarrow \varphi[\epsilon x(\varphi[\epsilon x\varphi/x] \rightarrow \varphi)/x] \ (-1)}$$

3. $\mathcal{E}x\varphi \wedge \psi[\epsilon x\varphi/x] \rightarrow \mathcal{E}x(\varphi \wedge \psi)$.
   This constitutes the extension of scope of an existential quantifier known from dynamic logics. R1 gives us already $\mathcal{E}x\varphi \wedge \psi[\epsilon x\varphi/x] \wedge \epsilon x\varphi := a \rightarrow \varphi[a/x] \wedge \psi[a/\epsilon x\varphi]$. Now the I2 rule supplies the relevant assignment $\epsilon x(\varphi \wedge \psi) := a$ (note that $\varphi \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)$).

REMARK 3.5

In semantic terms, the rules I1 and I2 mean that the value range of a subset N of the domain M of $\mathcal{M}$ under the set of choice functions of model $\mathcal{M}$ is the union of the value ranges of its subsets.

Limitations of the basic indefinite

The quantifier $\mathcal{E}x$ defined by the rules I1 and I2 gives us an indefinite quantifier which behaves like the existential modality of the *minimal* modal logic. But the rules I1 and I2 do not give us quantifier interaction principles of standard first–order logic. To see this we note the following. Every formula φ in the pure term language can be supplied with a structure $\langle \mathcal{E}(\varphi), \prec_\varphi \rangle$ of all ϵ–terms occurring in φ ordered by the *syntactic* dependency relation on ϵ–terms. In the course of a R1–R2 derivation with assumptions and conclusion in pure ϵ–terms, the dependency relation among the terms in a formula occurrence becomes a proof-theoretic one. That is, the dependencies can no longer be read from the syntactic structure of the terms, but derive from the proof-theoretic context; in particular, from the assignment statements that are assumptions to the formula occurrence. And the dependency structure in an R1–R2 derivation of a formula is 'preserved': if an atomic subformula φ of the conclusion can be traced to an atomic subformula ψ of an (undischarged) assumption, then the ϵ–terms on corresponding predicate locations in φ and ψ are dependency related in the same way to the remaining ϵ–terms in that subformula.[7] In particular we have the following lemma.

LEMMA 3.6
Let $R(t_1, \ldots, t_n)$ and $R(t'_1, \ldots, t'_n)$, be two quantifier free closed formulas, where every term t_i, t'_i, for $1 \leq i \leq n$, is quantifier free. Let $\vdash$ denote the derivability relation determined by the assignment rules R1,R2, the ν–term rules IV and EQ, and the ϵ–Rules ϵE, ϵI, I1 and I2, then, if $\vdash (R(t_1, \ldots, t_n) \leftrightarrow R(t'_1, \ldots, t'_n)$, then for all $1 \leq i, j \leq n$ $t_i \prec t_j$ iff $t'_i \prec t'_j$.

On the other hand, quantifier interaction principles generally change dependency structure, so these will not be derivable by the calculus of the above lemma. Consider, for instance, a derivation of $\mathcal{E}y\mathcal{E}xR(x,y)$ from $\mathcal{E}xR(x,x)$:

$$\dfrac{\dfrac{\dfrac{\mathcal{E}xR(x,x) \qquad [\epsilon xR(x,x) := a]\ (1)}{R(a,a)} \qquad \epsilon xR(x,a) := a(2)}{\mathcal{E}xR(x,a)} \qquad \epsilon y\mathcal{E}xR(x,y) := a(3)}{\mathcal{E}y\mathcal{E}xR(x,y)}$$

In pure term formulation, this is a derivation of $R(\epsilon xR(x, \overbrace{\epsilon y\mathcal{E}xR(x,y))}, \epsilon y\mathcal{E}xR(x,y))$ from $R(\epsilon xR(x,x), \epsilon xR(x,x))$. A dependency arises in the conclusion that is absent in the assumption. We note that none of the assignment statements (1),(2), or (3) can be discharged. Semantically, this entails that only specific models for $\mathcal{E}xR(x,x)$, with the right kind of choice functions, are models for $\mathcal{E}y\mathcal{E}xR(x,y)$.

In order to be able to derive the latter formula from the former we need rules to *derive* the reflexive assignment statement $\epsilon xR(x,a) := a$ from the premises $\mathcal{E}xR(x,x)$ and $\epsilon xR(x,x) := a$.

---

[7] To be more specific. Let Δ be a *normal* R1–R2 pure term derivation in which all assignment statements have been discharged and all minimal formula are atomic. Consider a *path* π in Δ of length n ending in the conclusion. Let χ_π be the *minimal* formula occurrence $R(t_1, \ldots, t_k)$ of π. We now identify with every node i of π a formula φ_i as follows. If the occurrence φ at node i of π is the major premise of an E–rule, then φ_i is the smallest subformula of φ such that χ_π is a subformula of φ_i. If φ is the conclusion of an introduction rule, then φ_i is the largest subformula of φ that is a subformula of χ_π.

Now consider the sequence $\varphi_i, \ldots \varphi_n$ of atomic formulas for path π. Let $\mathcal{T}(\varphi_i)$ be the set of individual constants and ϵ–terms occurring in φ_i. The elements of $\mathcal{T}(\varphi_i)$ are mapped to the terms in $\mathcal{T}(\varphi_{i+1})$ occurring at the same R^k location in φ_{i+1}. The composition of these mappings relates φ_1 to φ_n. If all assignments statements in Δ have been discharged, then identifying terms by their predicate location gives an isomorphism between the *syntactic* dependency structures of φ_1 and φ_n: terms occurring at the same locations in φ_1 and φ_n are in the same way syntactically related.

To show this, the only hitch we have to overcome is the fact that, by applications of the double negation elimination, only some occurrences of the argument of an assignment statement may be eliminated in favor of the value. I.e., $R(t,t), t := a / R(t,a)$ is allowed. In this situation we get splitting up of the term t in t and a (the cardinality of $\mathcal{T}(\varphi_{i+1})$ is larger than that of $\mathcal{T}(\varphi_i)$). However, because all assignment statements in Δ have been discharged, the different terms t and a must have been identified again (i.c. as t) at the conclusion.

3.3 Subordination

Whenever a set of assignment statements As has a circular subset C of cardinality n, we can derive from C n reflexive assignment statements of 'depth n'[8].

$$\{t_1(a) := b, t_2(b) := c, t_3(c) := a\} \vdash t_3(t_2(t_1(a))) := a$$

by the derivation

$$\dfrac{\dfrac{t_1(a) := b \qquad t_2(b) := c}{t_2(t_1(a)) := c} \qquad t_3(c) := a}{t_3(t_2(t_1(a))) := a}$$

Conversely, from a reflexive assignment statement of depth n we can create a circular set of n elements

$$\{t_1(t_2(t_3(a))) := a, t_3(a) := b, t_2(b) := c\} \vdash t_1(c) := a$$

by

$$\dfrac{\dfrac{t_1(t_2(t_3(a))) := a \qquad t_3(a) := b}{t_1(t_2(b)) := a} \qquad t_2(b) := c}{t_1(c) := a}$$

Notice that we have here circular *proof-theoretic* dependency, which cannot be reduced to pure *syntactic* dependency: the construction of ϵ–terms does not allow this. There is no way in which, for instance, $\epsilon x R(x,b) := a$ and $\epsilon y Q(a,y) := b$ can be reduced to an assignment statement with only ϵ–terms.

To get the full power of the classical existential quantifier we will need to have reflexive assignment statements available for ϵ–terms. Because these are non–dischargeable as *assumptions* of a derivation, rules are required which allow us to derive them from well–founded sets of assignment statements. The rules we will suggest consist of an introduction and elimination rule for *subordination*. These rules apply only for

$$\text{SI}\ \dfrac{\mathcal{E}x\varphi \qquad \epsilon x\varphi := a}{\epsilon x(\varphi[x/a]) := a} \qquad \text{SE}\ \dfrac{\mathcal{E}x(\varphi[x/a]) \qquad \epsilon x(\varphi[x/a]) := a}{\epsilon x\varphi := a}$$

FIG. 4. Subordination Rules

$\epsilon x\varphi \in \mathcal{E}$, i.e., if the variable x actually occurs free in φ.

With these rules, we can derive circular sets of assignments from well–founded ones. For instance, given $\mathcal{E}xR(x, \epsilon yQ(y,x))$ we have by SE and R1

$$\dfrac{\dfrac{\epsilon xR(x, \epsilon yQ(y,x)) := a}{\epsilon xR(x, \epsilon yQ(y,a)) := a} \qquad \epsilon yQ(y,a) := b}{\epsilon xR(x,b) := a}$$

[8] A circular assignment statement has depth n if the value of the statement occurs in the argument in the scope of n ν–symbols.

By eliminating subordination we create a circular set of assignment statements from $\{1, 2\}$, a well-founded set. Conversely circular sets allow us to introduce subordination. Given $\mathcal{E}xR(x, \epsilon yQ(y, a))$ we have by SI

$$\dfrac{\dfrac{\epsilon xR(x,b) := a \qquad\qquad \epsilon yQ(y,a) := b}{\epsilon xR(x, \epsilon yQ(y,a)) := a}}{\epsilon xR(x, \epsilon yQ(y,x)) := a}$$

So we see an intimate relationship between *subordination* and circular sets of assignment statements. A special case of the SE–rule is the following

$$\dfrac{\epsilon x\varphi(x,x) := a}{\epsilon x\varphi(x,a) := a}$$

REMARK 3.7

The soundness of these rules under the variant interpretation can be (informally) argued for as follows. Take assignment statement $\epsilon xR(x,x) := a$ and consider rule SE . A model $\mathcal{M}, \Phi$ for $\exists xR(x,x)$ where this assignment statement holds will have $I(a) = \Phi(\{b \mid \mathcal{M}, \Phi \models R(b,b)[g]\})$. But it need not be that $I(a) = V_{\mathcal{M},\Phi}(\epsilon xR(x,a))$. However, because $I(a) \in \{b \mid \mathcal{M}, \Phi \models R(b,a)[g]\}$, there will always be a $\epsilon xR(x,a)$–*variant* Φ' of Φ such that $I(a) = V_{\mathcal{M},\Phi'}(\epsilon xR(x,a))$. The existence of such a variant is guaranteed on any model for $\exists xR(x,x)$ and $\epsilon xR(x,x) := a$. The truth of $\exists xR(x,x,)$ is essential. If this formula where false on a model, then $V_{\mathcal{M},\Phi}$ would assign an arbitrary element to $\epsilon xR(x,x)$, for instance $I(a)$. But $\exists xR(x,a)$ might still be true on this model. In this case $\epsilon xR(x,a)$ cannot be assigned the value $I(a)$: there is no variant of Φ verifying $\epsilon xR(x,a) := a$.

Analogous argumentation shows the soundness of SI.

Weakening of the existential quantifier

Consider again weakening of existential quantifier. Now this is derivable by the assignment rules R1 and R2, and SI and SE.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{E}xR(x,x) \qquad [\epsilon xR(x,x) := a]\ (1)}{R(a,a)} \qquad \dfrac{(1)}{\epsilon xR(x,a) := a}}{\mathcal{E}xR(x,a)} \qquad \begin{array}{c}(1), \epsilon yR(y,y) := a \\ \vdots \\ \epsilon y\mathcal{E}xR(x,y) := a\end{array}}{\mathcal{E}y\mathcal{E}xR(x,y)}}{\mathcal{E}y\mathcal{E}xR(x,y)\ (-1)}$$

In this case we can derive the necessary assignment statements. We get $\epsilon xR(x,a) := a$ from $\epsilon xR(x,x) := a$ by SE straightforwardly. For the premise $\epsilon y\mathcal{E}xR(x,y) := a$ consider the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\epsilon yR(y,y) := a}{\epsilon yR(a,y) := a} \qquad \epsilon xR(x,a) := a}{\epsilon yR(\epsilon xR(x,a),y) := a}}{\epsilon yR(\epsilon xR(x,y),y) := a}}{\epsilon y\mathcal{E}xR(x,y) := a}$$

$\epsilon yR(y,y) := a$ follows from $\epsilon xR(x,x) := a$ by the IV rule for ν–terms.

Permutation of existential quantifiers

Both SI and SE rules are necessary to get the standard quantifier *permutations* and *weakenings*. We derive $\mathcal{E}x\mathcal{E}yR(x,y) \rightarrow \mathcal{E}y\mathcal{E}xR(x,y)$ as follows

$$\dfrac{\mathcal{E}x\mathcal{E}yR(x,y) \quad [\epsilon x\mathcal{E}yR(x,y) := a]\ (1)}{\mathcal{E}yR(a,y)}$$

$$\dfrac{\mathcal{E}yR(a,y) \quad [\epsilon yR(a,y) := b]\ (2)}{R(a,b)}$$

$$\dfrac{R(a,b) \quad \begin{matrix}(1,2)\\ \vdots \\ \epsilon xR(x,b) := a\ (3)\end{matrix}}{\mathcal{E}xR(x,b)}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{E}xR(x,b) \quad \begin{matrix}(1,2)\\ \vdots \\ \epsilon y\mathcal{E}xR(x,y) := b\ (4)\end{matrix}}{\mathcal{E}y\mathcal{E}xR(x,y)}}{\mathcal{E}y\mathcal{E}xR(x,y)\ (-2)}}{\mathcal{E}y\mathcal{E}xR(x,y)\ (-1)}$$

If we have to *assume* (3) and (4), then $\mathcal{E}y\mathcal{E}xR(x,y)$ follows only from $\{\mathcal{E}x\mathcal{E}yR(x,y),(1),(2),(3),(4)\}$, for none of these assumptions can be discharged at the conclusion.

The subordination rules, however, allow us to derive permutation of existential quantifiers, by deriving (3),(4) from (1),(2). This derivation uses the theorem

$$\vdash \forall x(\mathcal{E}yR(x,y) \leftrightarrow R(x,\epsilon yR(x,y))$$

Here it is

$$\dfrac{\dfrac{\dfrac{\epsilon x\mathcal{E}yR(x,y) := a\ (1)}{\epsilon xR(x,\epsilon yR(x,y)) := a}}{\epsilon xR(x,\epsilon yR(a,y)) := a} \quad \epsilon yR(a,y) := b\ (2)}{\epsilon xR(x,b) := a\ (3)}$$

$$\dfrac{\dfrac{\dfrac{\epsilon xR(x,b) := a\ (3) \quad \epsilon yR(a,y) := b\ (2)}{\epsilon yR(\epsilon xR(x,b),y) := b}}{\epsilon yR(\epsilon xR(x,y),y) := b}}{\epsilon y\mathcal{E}xR(x,y) := b\ (4)}$$

In fact, the subordination rules give us *all* standard permutations of quantifiers [7]. So to get the standard existential quantifier in pure term logic we have to supply the basic logic, which is characterized by dependency as a conversely well-founded strict partial order, with rules that derive circular dependencies. Derivations in which the rules SE and SI are used show the actual principles of *movement* which effect changes in the dependencies among terms.

Movement

A reversal of dependency involves more than is visible in the quantifier formulas $\mathcal{E}x\mathcal{E}yR(x,y)$ and $\mathcal{E}y\mathcal{E}xR(x,y)$. This becomes clear when the pure term equivalents are considered. We will follow the derivation of the permutation of existential quantifiers in pure term form. The assumption $\mathcal{E}x\mathcal{E}yR(x,y)$ then corresponds to

$$(i) \quad R(\underbrace{\epsilon xR(x,\epsilon yR(x,y))}_{a}, \underbrace{\epsilon yR(\underbrace{\epsilon xR(x,\epsilon yR(x,y))}_{a},y)}_{b})$$

and the conclusion $\mathcal{E}y\mathcal{E}xR(x,y)$ to

$$(iii)\quad R(\underbrace{\epsilon xR(x,\underbrace{\epsilon yR(\epsilon xR(x,y),y)}_{b})}_{a},\underbrace{\epsilon yR(\epsilon xR(x,y),y)}_{b}))$$

In standard logic these forms are logically equivalent, i.e., interderivable. Standard logic does not respect the dependencies within an atomic formula.[9] In pure term form an atomic formula φ determines a dependency structure $\langle\mathcal{E}(\varphi),\prec\rangle$. The basic rules of our calculus preserve this dependency: if $\vdash (R(t_1,\ldots,t_n)\leftrightarrow R(t'_1,\ldots,t'_n)$, then $t_i \prec t_j$ iff $t'_i \prec t'_j$. Logical equivalence in this system is sensitive to dependency. Classical logic abstracts over the dependencies. Atomic formulas with different dependencies among their terms may have the same (truth–functional) meaning. The derivation of one atomic formula from an equivalent carrying a different dependency structure shows the mechanisms by which the changes in this structure are achieved. What happens in the derivation of *(iii)* from *(i)* is that we reach an intermediate form in which the dependency relation between the locations in R has been reversed, but the original, unlifted 'antecedent', is still present, *embedded* in the new term. There is a *trace* of liftings, of movement.

The following derivation supplies, in pure term form, the reversal of the dependency between a and b. Given $\mathcal{E}xR(x,\epsilon yR(x,y))$ we have by SE and R2

$$\begin{array}{c}
[\epsilon xR(x,\epsilon yR(x,y)) := a](1) \\
\hline
\epsilon xR(x,\epsilon yR(a,y)) := a \\
\hline
\underbrace{\epsilon xR(x,\epsilon yR(\underbrace{\epsilon xR(x,\epsilon yR(x,y))}_{a},y))}_{b} := \underbrace{\epsilon xR(x,\epsilon yR(x,y))}_{a}\ (-1)
\end{array}$$

The term $\epsilon xR(x,\epsilon yR(x,y))$ can be lifted.[10] The lifted form replaces the old one in the derivation reversing the dependency relation with respect to $\epsilon yR(\epsilon xR(x,\epsilon yR(x,y)),y)$ $(=b)$.

$$(ii)\quad R(\underbrace{\epsilon xR(x,\underbrace{\epsilon yR(\overbrace{\epsilon xR(x,\epsilon yR(x,y))}^{a},y)}_{b})}_{a},\underbrace{\epsilon yR(\overbrace{\epsilon xR(x,\epsilon yR(x,y))i}^{a},y)}_{b})$$

(This corresponds to $\mathcal{E}xR(x,b)$ in the previous derivation). The dependency between a and b has reversed. Notice that the term a is the value of more than one constituent. By definition, this form is equivalent to $\mathcal{E}xR(x,\epsilon yR(\epsilon xR(x,\epsilon yR(x,y)),y))$. But this is not by definition the same as $\mathcal{E}y\mathcal{E}xR(x,y)$. To establish the latter we have to continue

[9] Notice that the *location* of the term in the formula $R(x,y)$ does not change. So if we assume that the atomic predicate assigns 'roles' to its argument places, then only the dependencies between terms change, not the roles assigned to them.

[10] Here it is clear that the assignment predicate (:=) can not be interpreted as identity (=), for, in the standard ϵ–calculus, , under the standard interpretation of ϵ–terms by *choice functions* Φ, the terms $\epsilon xR(x,\epsilon yR(\epsilon xR(x,\epsilon yR(x,y))$ and $\epsilon xR(x,\epsilon yR(x,y))$ need not receive the same value. However, there always is an $\epsilon xR(x,\epsilon yR(x,y))$–*variant* Φ' of Φ which assigns these terms the same value.

the derivation. The derivation of assignment statement (4) from (1),(2) gives us in pure term form

$$\epsilon y R(\epsilon x R(x, \overbrace{y}), y) := \underbrace{\epsilon y(\epsilon x R(x, \overbrace{\epsilon y R(x, y)}), y)}_{b}$$

This represents a *lowering* of terms [7]. Substituting the argument of this assignment for its value in *(ii)* gives the desired conclusion

$$(iii) \quad R(\underbrace{\epsilon x R(x, \underbrace{\epsilon y R(\epsilon x R(x, y), y)}_{b})}_{a}, \underbrace{\epsilon y R(\epsilon x R(x, y), y)}_{b})$$

Conclusion

In the logic of assignment statements we have developed a system in which dependencies between terms are taken seriously. These dependencies can occur in syntactic and in proof-theoretic form. The deductive machinery of the calculus supplies us with means to eliminate syntactic dependecies in favor of proof-theoretic ones and vice versa. In this calculus the basic rules are sensitive to the dependencies: logically equivalent instances of the same predicate must carry the same dependency structure. To make logical equivalence insensitive to dependency, rules for *movement* have to be added to the system. In this calculus these rules take the form of subordination rules.

In the set of term equivalents of first–order formulas, we can identify the *base generated* formulas as the ones that are derivable from first–order formulas by means of dependency preserving rules. The base generated formulas can be subjected to the subordination rules which effect movement of dependency. This identifies a set of *transformational variants* of the base generated formulas. Consequently, the logic of assignment statements suggests a proof-theoretic treatment of concepts central to linguistic theory.

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