THE DOUBLE COVERING OF THE QUANTUM GROUP $SO_q(3)$ †

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A quantum analogue of the double covering of $SO(3)$ by $SU(2)$ is formulated and proved. Here the quantum group $SO_q(3)$ is defined by means of the R-matrix given by FRT for root systems of type B. An explicit basis for the deformed function algebra of $SO_q(3)$ is constructed as well as an algorithm to reduce any expression in the generators to a linear combination of basis elements.

1. The adjoint group of $SU_q(2)$

We shall make use of both the language of Hopf algebras and that of quantum groups. We view a Hopf algebra $A = \mathcal{O}(G)$ as the algebra of polynomial functions on an (algebraic) quantum group $G = \mathrm{Spec}(A)$. Hopf $*$-algebras then correspond to real forms of quantum groups. A morphism $\phi: G \to G'$ of quantum groups resp. real quantum groups is by definition a morphism $\phi: A' \to A$ of Hopf algebras resp. Hopf $*$-algebras. In order to be able to distinguish formally between these two kinds of morphisms, we shall usually write $\phi^\sharp$ for the Hopf algebra morphism dual to the quantum group morphism ϕ.

We recall the definition of the quantum group $SU_q(2)$. Let $q \in \mathbb{R}$, $q \neq 0$. The algebra $A_q = \mathcal{O}(SU_q(2))$ is the complex unital associative algebra generated by α, β, γ, δ subject to the following relations:

$$\begin{aligned} &\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\gamma = \gamma\beta, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \\ &\delta\alpha - q^{-1}\beta\gamma = 1, \quad \alpha\delta - q\beta\gamma = 1. \end{aligned} \tag{1.1}$$

By using the diamond lemma one can prove that a linear basis of A_q is formed by the elements $\alpha^k\beta^l\gamma^m$ $(k, l, m \geq 0)$ and $\delta^k\beta^l\gamma^m$ $(k \geq 1,\ l, m \geq 0)$. See [B], [Kl].

† This paper is in final form and no version of it will be submitted for publication elsewhere.

The comultiplication Δ and counit ε are defined by:

$$\Delta\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\otimes\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \quad \varepsilon\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{1.2}$$

this being shorthand notation for $\Delta(\alpha) = \alpha\otimes\alpha + \beta\otimes\gamma$ etc.

The antipode S and the Hopf algebra involution $*$ are defined by

$$S\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\delta & -q^{-1}\beta\\ -q\gamma & \alpha\end{pmatrix}, \quad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^* = \begin{pmatrix}\delta & -q\gamma\\ -q^{-1}\beta & \alpha\end{pmatrix}. \tag{1.3}$$

Let us recall that a mapping $*: A_q \to A_q$ is called a Hopf algebra involution if $(A_q, *)$ is a unital $*$-algebra such that Δ and ε are $*$-homomorphisms.

We recall that a *(matrix) corepresentation* of A_q is a matrix (t_{ij}) with coefficients in A_q such that

$$\Delta(t_{ij}) = \sum_k t_{ik}\otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij}. \tag{1.4}$$

A corepresentation (t_{ij}) is called *unitary* if $t_{ij}^* = S(t_{ji})$. Corepresentations of A_q are also called *representations* of the quantum group $SU_q(2)$. The finite-dimensional representation theory of $SU_q(2)$ is known to be exactly analogous to the classical theory. With respect to a suitable basis the corepresentation of A_q corresponding to the adjoint representation of $SU(2)$ is given by (see [Ko]):

$$\mathrm{Ad}_q = \begin{pmatrix}\alpha^2 & (1+q^{-2})^{\frac12}\alpha\beta & -\beta^2\\ (1+q^{-2})^{\frac12}\alpha\gamma & 1+(q+q^{-1})\beta\gamma & -(1+q^2)^{\frac12}\delta\beta\\ -\gamma^2 & -(1+q^2)^{\frac12}\delta\gamma & \delta^2\end{pmatrix}. \tag{1.5}$$

It is easily checked that the subalgebra B_q of A_q generated by the matrix coefficients t_{ij} of Ad_q is spanned by all $\alpha^k\beta^l\gamma^m$, $\delta^k\beta^l\gamma^m$ such that $k+l+m$ is even. B_q is obviously invariant under S and $*$. Moreover, $\Delta(B_q)\subset B_q\otimes B_q$, since B_q is generated by the matrix coefficients of a corepresentation. We conclude that B_q is a Hopf $*$-subalgebra of A_q. The quantum group corresponding to B_q is called the adjoint group of $SU_q(2)$ and denoted by $\mathrm{Ad}(SU_q(2))$.

We define an algebra anti-automorphism σ of A_q and an algebra isomorphism $\tau: A_q \to A_{q^{-1}}$ by putting

$$\sigma\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\delta & -\gamma\\ -\beta & \alpha\end{pmatrix}, \quad \tau\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}. \tag{1.6}$$

It is obvious that σ leaves B_q invariant and that τ maps B_q onto $B_{q^{-1}}$. On the generators t_{ij} the mappings σ and τ are given by

$$\sigma(T) = \begin{pmatrix}t_{33} & t_{32} & t_{31}\\ t_{23} & t_{22} & t_{21}\\ t_{13} & t_{12} & t_{11}\end{pmatrix}, \quad \tau(T) = \begin{pmatrix}t_{33} & t_{23} & t_{13}\\ t_{32} & t_{22} & t_{12}\\ t_{31} & t_{21} & t_{11}\end{pmatrix}, \tag{1.7}$$

T being shorthand notation for (t_{ij}). Note that $\sigma^2 = \tau^2 = \mathrm{id}$ and $\tau\sigma\tau\sigma = \mathrm{id}$.

Table 1

1. $t_{11}t_{12} = s^2 t_{12}t_{11}$
   $t_{32}t_{33} = s^2 t_{33}t_{32}$
   $t_{21}t_{11} = s^{-2} t_{11}t_{21}$
   $t_{33}t_{23} = s^{-2} t_{23}t_{33}$

2. $t_{12}t_{13} = s^2 t_{13}t_{12}$
   $t_{31}t_{32} = s^2 t_{32}t_{31}$
   $t_{23}t_{13} = s^{-2} t_{13}t_{23}$
   $t_{31}t_{21} = s^{-2} t_{21}t_{31}$

3. $t_{11}t_{13} = s^4 t_{13}t_{11}$
   $t_{31}t_{33} = s^4 t_{33}t_{31}$
   $t_{33}t_{13} = s^{-4} t_{13}t_{33}$
   $t_{31}t_{11} = s^{-4} t_{11}t_{31}$

4. $t_{22}t_{11} = t_{11}t_{22} + (s^{-2} - s^2)t_{12}t_{21}$
   $t_{33}t_{22} = t_{22}t_{33} + (s^{-2} - s^2)t_{23}t_{32}$

5. $t_{23}t_{12} = t_{12}t_{23} + (s^{-2} - s^2)t_{13}t_{22}$
   $t_{32}t_{21} = t_{21}t_{32} + (s^{-2} - s^2)t_{22}t_{31}$

6. $t_{21}t_{12} = t_{12}t_{21}$
   $t_{32}t_{23} = t_{23}t_{32}$

7. $t_{22}t_{13} = t_{13}t_{22}$
   $t_{31}t_{22} = t_{22}t_{31}$

8. $t_{23}t_{11} = s^{-2}t_{11}t_{23} + (s^{-2} - s^2)t_{13}t_{21}$
   $t_{33}t_{21} = s^{-2}t_{21}t_{33} + (s^{-2} - s^2)t_{23}t_{31}$
   $t_{32}t_{11} = s^{-2}t_{11}t_{32} + s^{-2}(s^{-2} - s^2)t_{12}t_{31}$
   $t_{33}t_{12} = s^{-2}t_{12}t_{33} + s^{-2}(s^{-2} - s^2)t_{13}t_{32}$

9. $t_{22}t_{12} = t_{12}t_{22} + s(s^2 - s^{-2})t_{13}t_{21}$
   $t_{32}t_{22} = t_{22}t_{32} + s(s^2 - s^{-2})t_{23}t_{31}$
   $t_{21}t_{22} = t_{22}t_{21} + s^{-1}(s^{-2} - s^2)t_{12}t_{31}$
   $t_{22}t_{23} = t_{23}t_{22} + s^{-1}(s^{-2} - s^2)t_{13}t_{32}$

10. $t_{21}t_{13} = s^2 t_{13}t_{21}$
    $t_{31}t_{23} = s^2 t_{23}t_{31}$
    $t_{31}t_{12} = s^{-2} t_{12}t_{31}$
    $t_{32}t_{13} = s^{-2} t_{13}t_{32}$

11. $t_{33}t_{11} = t_{11}t_{33} + (s - s^{-1})t_{23}t_{21} + (s - s^{-1})t_{12}t_{32}$

12. $t_{31}t_{13} = t_{13}t_{31}$

13. $t_{21}t_{23} = s^2 t_{23}t_{21} + s(s^{-2} - s^2)t_{13}t_{31}$
    $t_{32}t_{12} = s^{-2}t_{12}t_{32} + s^{-1}(s^2 - s^{-2})t_{13}t_{31}$

14. $t_{12}t_{12} = -s^2(s + s^{-1})t_{13}t_{11}$
    $t_{21}t_{21} = -s^{-2}(s + s^{-1})t_{11}t_{31}$
    $t_{32}t_{32} = -s^2(s + s^{-1})t_{33}t_{31}$
    $t_{23}t_{23} = -s^{-2}(s + s^{-1})t_{13}t_{33}$

15. $t_{11}t_{23} = -st_{12} + s^4 t_{13}t_{21}$
    $t_{21}t_{33} = -st_{32} + s^4 t_{23}t_{31}$
    $t_{11}t_{32} = -st_{21} + s^2 t_{12}t_{31}$
    $t_{12}t_{33} = -st_{23} + s^2 t_{13}t_{32}$

16. $t_{23}t_{22} = t_{23} - s^{-2}(s + s^{-1})t_{13}t_{32}$
    $t_{22}t_{21} = t_{21} - s^{-2}(s + s^{-1})t_{12}t_{31}$
    $t_{12}t_{22} = t_{12} - s^2(s + s^{-1})t_{13}t_{21}$
    $t_{22}t_{32} = t_{32} - s^2(s + s^{-1})t_{23}t_{31}$

17. $t_{21}t_{32} = t_{31} + s^2 t_{22}t_{31}$
    $t_{12}t_{23} = t_{13} + s^2 t_{13}t_{22}$

18. $t_{11}t_{22} = t_{11} + s^2 t_{12}t_{21}$
    $t_{22}t_{33} = t_{33} + s^2 t_{23}t_{32}$

19. $t_{12}t_{32} = s - st_{22} - s^2(s + s^{-1})t_{13}t_{31}$
    $t_{23}t_{21} = s^{-1} - s^{-1}t_{22} - s^{-2}(s + s^{-1})t_{13}t_{31}$

20. $t_{11}t_{33} = (1 - s^2) + s^2 t_{22} + s^4 t_{13}t_{31}$

21. $t_{22}t_{22} = 2t_{22} - 1 + (s + s^{-1})^2 t_{13}t_{31}$

Lemma 1.1 — *The $t_{ij} \in B_q$ satisfy all the relations listed in Table 1 with $s = q$.*

In order to minimize the amount of calculation we use the symmetry inherent in the relations of Table 1. In fact, in a given cluster i, one can obtain all the relations by repeatedly applying σ and τ to the first relation of that cluster (in some cases one has to work modulo linear combinations of relations of clusters $k < i$). It suffices therefore to check only the first relation of each cluster. This is straightforward using (1.5) and (1.1).

Theorem 1.2 — *The $t_{ij} \in B_q$ satisfy no other relations than those listed in Table 1 ($s = q$). A linear basis of the vector space B_q is formed by the elements*

$$\begin{array}{ll} t_{13}^m t_{22} t_{31}^n & (m, n \geq 0) \\ t_{13}^m t_{12}^i t_{11}^j t_{21}^k t_{31}^n & (m, j, n \geq 0,\ 0 \leq i, k \leq 1) \\ t_{13}^m t_{23}^i t_{33}^j t_{32}^k t_{31}^n & (m, j, n \geq 0,\ 0 \leq i, k \leq 1,\ i + j + k > 0) \end{array} \tag{1.8}$$

It is clear from (1.5) that the elements (1.8) span B_q. Their linear independence immediately follows from (1.5) and the fact that the elements $\alpha^k \beta^l \gamma^m$ $(k, l, m \geq 0)$ and $\delta^k \beta^l \gamma^m$ $(k > 0, l, m \geq 0)$ are linearly independent in A_q. Let now D be the abstract algebra generated by the t_{ij} subject to the relations in Table 1. It follows from [1.1]† that there is a unique algebra homomorphism $\phi\colon D \to B_q$ sending $t_{ij} \in D$ to $t_{ij} \in B_q$. This homomorphism is surjective, since the t_{ij} generate B_q. We prove that the elements (1.8) span D. To this end, we introduce a total ordering on the generators by putting $t_{13} < t_{12} < t_{11} < t_{23} < t_{22} < t_{21} < t_{33} < t_{32} < t_{31}$. We then order any two given monomials in the t_{ij} by length and, if they are of equal length, lexicographically with respect to the above ordering on the t_{ij}. Inspection of the relations in Table 1 shows that all of them express a monomial f in the t_{ij} as a linear combination of monomials strictly less than f. This implies that any monomial in the t_{ij} can be expressed as a linear combination of monomials $t_{i_1} t_{i_2} \ldots t_{i_n}$ such that for all $1 \leq j \leq n-1$ the monomial $t_{i_j} t_{i_{j+1}}$ does not occur on the left-hand side of any equation in Table 1. It can be easily read off from the relations in Table 1 that the monomials satisfying this last condition are precisely the ones listed in (1.8). This proves that the elements (1.8) span D. So ϕ maps a family of vectors that span D to a linearly independent family of vectors in B_q. This implies that ϕ is injective, which concludes the proof of the theorem.

The relations in Table 1 thus form a presentation of the algebra B_q. One easily derives from (1.5) and (1.3) that the involution $*$ is given on the generators t_{ij} by

$$T^* = \begin{pmatrix} t_{33} & q t_{32} & q^2 t_{31} \\ q^{-1} t_{23} & t_{22} & q t_{21} \\ q^{-2} t_{13} & q^{-1} t_{12} & t_{11} \end{pmatrix}. \tag{1.9}$$

Remark 1.3 — We use the terminology of [B]. The semigroup ordering on the monomials in the t_{ij} defined in the proof of [1.2] clearly satisfies the descending chain condition and is compatible with the reduction system specified by Table 1. It follows from [1.2] (without actually resolving a single ambiguity!) that all the ambiguities are resolvable. Therefore, the diamond lemma applies and we get an algorithm to reduce any monomial in the t_{ij} to

† Numbers between square brackets [] refer to lemmas, propositions, theorems etc.

its (unique) expression in terms of the basis elements (1.8). This algorithm can be easily implemented on a computer using a computer algebra package for symbolic manipulation. We found that the package Reduce (version 3.4) was best suited to our purposes. In this way, one can perform explicit computations in the algebra B_q that would have been tiresome to do by hand.

We shall now briefly deliberate upon notions such as quantum subgroup, kernel and short exact sequence (cf. [PW]).

Let A be a Hopf algebra. A subspace $\mathfrak{a}$ is called a two-sided coideal if $\Delta(\mathfrak{a}) \subset A\otimes\mathfrak{a}+\mathfrak{a}\otimes A$. A subspace $\mathfrak{a}$ is called a Hopf ideal if $\mathfrak{a}$ is a ideal and an S-invariant two-sided coideal. If $\mathfrak{a}$ is a Hopf ideal, then $A/\mathfrak{a}$ naturally inherits a Hopf algebra structure from A. A quantum group H is said to be a quantum subgroup of G if $\mathcal{O}(H)$ is the quotient of $\mathcal{O}(G)$ by a Hopf ideal $\mathfrak{a}$, which is then called the defining ideal of H. So the quantum subgroups of a given quantum group G are in 1-1 correspondence with the Hopf ideals in $\mathcal{O}(G)$.

Suppose $\phi: G \to G'$ is a morphism of quantum groups and let $\phi^\sharp: \mathcal{O}(G') \to \mathcal{O}(G)$ be the corresponding morphism of Hopf algebras. Define $\mathfrak{a}$ to be the ideal in $\mathcal{O}(G)$ generated by the image under $\phi^\sharp$ of $\ker(\varepsilon') \subset \mathcal{O}(G')$. It is trivial that $\mathfrak{a}$ is a Hopf ideal. The quantum subgroup of G corresponding to $\mathfrak{a}$ is called the kernel of the morphism ϕ and denoted $\ker(\phi)$. A sequence of morphisms of quantum groups

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\phi} G' \longrightarrow 1 \tag{1.10}$$

is called exact if $\phi^\sharp$ is injective and $i^\sharp$ surjective, and if $\ker(i^\sharp)$ is the defining ideal of $\ker(\phi)$.

We now apply the above terminology to the adjoint group of $SU_q(2)$. We have a morphism $\phi: SU_q(2) \to \mathrm{Ad}(SU_q(2))$ such that $\phi^\sharp$ is the canonical injection.

Proposition 1.4 — *The ideal in A_q generated by $\ker(\varepsilon_{|B_q})$ is equal to the ideal generated by β, γ, $\alpha^2 - 1$, $\alpha - \delta$.*

Let us write $\mathfrak{a}$ for the ideal generated by β, γ, $\alpha^2 - 1$, $\alpha - \delta$. It follows from (1.2) that $\ker(\varepsilon)$ is spanned by the elements $\alpha^k - 1$, $\delta^k - 1$, $\alpha^k\beta^l\gamma^m$, $\delta^k\beta^l\gamma^m$ ($k \geq 0$, $l+m > 0$) and generated as an ideal by β, γ, $\alpha - 1$, $\delta - 1$. So $\ker(\varepsilon_{|B_q})$ is generated as ideal in B_q by $\alpha^2 - 1$, $\delta^2 - 1$, $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\delta\beta$, $\delta\gamma$. This implies that $\mathfrak{a}$ contains $\ker(\varepsilon_{|B_q})$. On the other hand, multiplying $\alpha^2 - 1$ on the right by δ, we get $\alpha(\alpha\delta) - \delta = \alpha(1 + q\beta\gamma) - \delta = \alpha - \delta + q\alpha\beta\gamma$. Hence $\alpha - \delta$ lies in the ideal generated by $\ker(\varepsilon_{|B_q})$. Multiplying $\alpha\beta$ on the left by δ, we obtain $\delta\alpha\beta = \beta + q^{-1}\beta^2\gamma$. This implies that β lies in the ideal generated by $\ker(\varepsilon_{|B_q})$. One reasons similarly for γ. The assertion follows.

The quotient $A/\mathfrak{a}$ is generated by α subject to the relation $\alpha^2 = 1$. This implies that it is a two-dimensional Hopf $*$-algebra. In fact, $A/\mathfrak{a}$ is the Hopf $*$-algebra of functions on the finite group $\mathbb{Z}_2$. We now have proved:

Theorem 1.5 — *There is the following short exact sequence of quantum groups:*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU_q(2) \xrightarrow{\phi} \mathrm{Ad}(SU_q(2)) \longrightarrow 1.$$

This justifies our writing $\mathrm{Ad}(SU_q(2)) = SU_q(2)/\{1, -1\}$.

2. The quantum group $SO_q(3)$

We adopt the definition of $SO_q(3)$ given in [FRT] with the one exception that we add a determinant relation (see also [T2]). In fact, the quantum group defined in [FRT] is a quantization of $O_q(3)$ and not of $SO_q(3)$.

Let (t_{ij}) $(1 \leq i,j \leq 3)$ denote a family of formal indeterminates and let $\mathbb{C}\langle t_{ij}\rangle$ denote the free associative unital complex algebra generated by the t_{ij}. It will be convenient to arrange the indeterminates t_{ij} in matrix form $T = (t_{ij})$. We can then define 9×9 matrices T_1 resp. T_2 with coefficients in the algebra $\mathbb{C}\langle t_{ij}\rangle$ by putting $T_1 = T \otimes I$ resp. $T_2 = I \otimes T$. Here I denotes the 3×3 identity matrix.

Let e_1, e_2, e_3 be the canonical basis of $V = \mathbb{C}^3$. Then $V \otimes V$ has the basis $e_i \otimes e_j$ $(1 \leq i,j \leq 3)$. Let e_{ij} denote the linear endomorphism of V sending e_j to e_i and e_k $(k \neq j)$ to 0. Let $q > 0$, $q \neq 1$. We define a linear endomorphism R_q of $V \otimes V$ or, equivalently, a 9×9 matrix $(R^q_{ij,kl})$ with complex coefficients by putting:

$$\begin{aligned} R_q = {} & q\sum_{i\neq i'} e_{ii}\otimes e_{ii} + e_{22}\otimes e_{22} + \sum_{i\neq j,j'} e_{ji}\otimes e_{ij} + \\ & + q^{-1}\sum_{i\neq i'} e_{ii'}\otimes e_{i'i} + (q-q^{-1})\sum_{i>j} e_{jj}\otimes e_{ii} - (q-q^{-1})\sum_{i>j} q^{\rho_i-\rho_j} e_{i'j}\otimes e_{ij'}. \end{aligned}$$

Here $i' = 4 - i$ and the sequence (ρ_i) is defined as $(\rho_1, \rho_2, \rho_3) = (\frac{1}{2}, 0, -\frac{1}{2})$. Note that the matrix $(R^q_{ij,kl})$ is symmetric and hence diagonalizable. Straightforward computation shows that the eigenvalues of R_q are $q, -q^{-1}, q^{-2}$ and that a basis of eigenvectors for $V \otimes V$ is given by:

$$\begin{aligned} W_q: \quad & e_1\otimes e_1,\ e_3\otimes e_3 \\ & qe_1\otimes e_2 + e_2\otimes e_1,\ qe_2\otimes e_3 + e_3\otimes e_2 \\ & qe_1\otimes e_3 - (q^{\frac{1}{2}} + q^{-\frac{1}{2}})e_2\otimes e_2 + q^{-1}e_3\otimes e_1 \\[1em] W_{-q^{-1}}: \quad & e_1\otimes e_2 - qe_2\otimes e_1,\ e_2\otimes e_3 - qe_3\otimes e_2 \\ & e_1\otimes e_3 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})e_2\otimes e_2 - e_3\otimes e_1 \\[1em] W_{q^{-2}}: \quad & q^{-\frac{1}{2}}e_1\otimes e_3 + e_2\otimes e_2 + q^{\frac{1}{2}}e_3\otimes e_1 \end{aligned} \tag{2.1}$$

We consider the two-sided ideal I_q in $\mathbb{C}\langle t_{ij}\rangle$ generated by the relations (called *commutation relations*) coming from the matrix equation

$$R_q T_1 T_2 = T_1 T_2 R_q. \tag{2.2}$$

Equating the 9×9 coefficients on both sides yields the following explicit form for the defining relations of I_q:

$$\sum_{m,n} R^q_{ij,mn} t_{mk} t_{nl} = \sum_{m,n} t_{im} t_{jn} R^q_{mn,kl} \quad \text{for } 1 \leq i,j,k,l \leq 3.$$

Let V^* denote the linear dual of V with dual basis e_i^*. We define a linear mapping $\theta: V^* \otimes V^* \otimes V \otimes V \to \mathbb{C}\langle t_{ij}\rangle$ by putting

$$\theta(e_i^* \otimes e_j^* \otimes e_k \otimes e_l) = t_{ik}t_{jl}. \tag{2.3}$$

Let $R_q^*: V^* \otimes V^* \to V^* \otimes V^*$ denote the transpose of R_q. It is clear that it is diagonalizable with the same eigenvalues as R_q. Since the matrix $(R^q_{ij,kl})$ is symmetric, bases for the eigenspaces W_λ^* of R_q^* are obtained by replacing e_i by e_i^* in (2.1). We now have:

Proposition 2.1 — *The ideal I_q is generated by the subspace $\bigoplus_{\lambda\neq\mu} \theta(W_\lambda^* \otimes W_\mu)$ of $\mathbb{C}\langle t_{ij}\rangle$.*

Let us write $\xi = \theta \circ (\mathrm{id} \otimes R_q - R_q^* \otimes \mathrm{id})$. On the one hand, I_q is generated by the image of ξ since $\xi(e_i^* \otimes e_j^* \otimes e_k \otimes e_l) = \sum_{mn} R^q_{mn,kl} t_{im}t_{jn} - \sum_{mn} R^q_{ij,mn} t_{mk}t_{nl}$. On the other hand, $\xi(W_\lambda^* \otimes W_\mu) = (\mu - \lambda)\theta(W_\lambda^* \otimes W_\mu)$. We conclude that I_q is generated by $\bigoplus_{\lambda\neq\mu} \theta(W_\lambda^* \otimes W_\mu)$, since R_q^* is diagonalizable.

The quotient algebra $M_q = \mathbb{C}\langle t_{ij}\rangle / I_q$ is called the *algebra of polynomial functions on the complex orthogonal quantum matrix space of rank* 3. We indifferently write t_{ij} for the generators in $\mathbb{C}\langle t_{ij}\rangle$ or their canonical images in M_q.

There are unique algebra homomorphisms $\Delta : M_q \to M_q \otimes M_q$ and $\varepsilon : M_q \to \mathbb{C}$ such that

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij} \quad (i,j = 1,2,3).$$

With these mappings M_q becomes a bialgebra.

We now define the *quantized orthogonal exterior algebra* $\Lambda_q V$ to be the tensor algebra over V divided out by the ideal generated by the subspace $\overline{W}_q = W_q \oplus W_{q^{-2}} \subset V \otimes V$. It follows from (2.1) that a basis of $\overline{W}_q$ is formed by

$$\begin{aligned} &e_1 \otimes e_1,\ e_3 \otimes e_3,\ (q^{\frac{1}{2}} - q^{-\frac{1}{2}})e_1 \otimes e_3 - e_2 \otimes e_2 \\ &qe_1 \otimes e_2 + e_2 \otimes e_1,\ qe_2 \otimes e_3 + e_3 \otimes e_2,\ e_1 \otimes e_3 + e_3 \otimes e_1. \end{aligned} \tag{2.4}$$

So $\Lambda_q V$ is the algebra generated by e_1, e_2, e_3 subject to the relations

$$e_1^2 = 0,\ e_3^2 = 0,\ e_2e_1 = -qe_1e_2,\ e_3e_2 = -qe_2e_3,\ e_3e_1 = -e_1e_3,\ e_2^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})e_1e_3. \tag{2.5}$$

An application of the diamond lemma shows that $\Lambda_q V$ is an 8-dimensional vector space with basis $e_{i_1} \ldots e_{i_n}$ $(1 \leq i_1 < \ldots < i_n \leq 3)$. It can be shown that there is a unique algebra homomorphism $\delta: \Lambda_q V \to M_q \otimes \Lambda_q V$ such that

$$\delta(e_i) = \sum_j t_{ij} \otimes e_j \quad (i = 1,2,3). \tag{2.6}$$

The mapping δ defines a left coaction of M_q on $\Lambda_q V$, i.e. it satisfies the properties:

$$(\Delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta, \quad (\varepsilon \otimes \mathrm{id}) \circ \delta = \mathrm{id}. \tag{2.7}$$

It follows from (2.6) and (2.5) that there is a unique element $\det_q T \in M_q$ such that

$$\delta(e_1e_2e_3) = \det_q T \otimes e_1e_2e_3. \tag{2.8}$$

The equalities (2.7) then imply that

$$\Delta(\det_q T) = \det_q T \otimes \det_q T, \quad \varepsilon(\det_q T) = 1.$$

An explicit expression for $\det_q T$ is:

$$\begin{aligned}\det_q T = t_{11}t_{22}t_{33} - qt_{12}t_{21}t_{33} - qt_{11}t_{23}t_{32} + qt_{12}t_{23}t_{31} \\ + qt_{13}t_{21}t_{32} - q^2 t_{13}t_{22}t_{31} - q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})t_{12}t_{22}t_{32}.\end{aligned} \tag{2.9}$$

We introduce a 3×3 matrix C given by

$$\begin{pmatrix} 0 & 0 & q^{-\frac{1}{2}} \\ 0 & 1 & 0 \\ q^{\frac{1}{2}} & 0 & 0 \end{pmatrix}.$$

It satisfies $C^2 = I$. The so-called *orthogonality relations* are:

$$TC^tTC = C^tTCT = I. \tag{2.10}$$

It can be proved that the two-sided ideal J_q in M_q generated by the relations $\det_q T = 1$ and (2.10) is a biideal. We can now define the *algebra* $C_q = \mathcal{O}(SO_q(3))$ *of polynomial functions on the quantum group* $SO_q(3)$ as the quotient of M_q by the biideal J_q. C_q is a bialgebra by definition. It becomes a Hopf $*$-algebra by putting

$$S(T) = C^tTC = \begin{pmatrix} t_{33} & q^{-\frac{1}{2}}t_{23} & q^{-1}t_{13} \\ q^{\frac{1}{2}}t_{32} & t_{22} & q^{-\frac{1}{2}}t_{12} \\ qt_{31} & q^{\frac{1}{2}}t_{21} & t_{11} \end{pmatrix} \quad \text{and} \quad T^* = {}^tS(T). \tag{2.11}$$

Theorem 2.2 — *The algebra C_q is generated by the t_{ij} subject to the relations listed in Table 1 with $s = q^{\frac{1}{2}}$.*

The proof is completely elementary, although not entirely trivial, since one easily gets bogged down in a quagmire of relations. For this reason, we shall carefully indicate the line of reasoning to be followed, but not explicitly perform all the calculations. The proof consists of two parts.

We first prove that the linear span Z_q of the relations (2.2) in the free algebra $\mathbb{C}\langle t_{ij}\rangle$ is equal to the linear span of the relations (1) till (14) in Table 1 plus some extra relations (see below). To this end, we apply [2.1]. Let us identify V and V^* via the bases (e_i) and (e_i^*). Under this identification W_λ coincides with W_λ^*. It now follows from [2.1] that

$$Z_q = \theta(\overline{W}_q \otimes W_{-q^{-1}}) \oplus \theta(W_{-q^{-1}} \otimes \overline{W}_q) \oplus \theta(W_q \otimes W_{q^{-2}}) \oplus \theta(W_{q^{-2}} \otimes W_q).$$

The following remark may be of use. If $\omega: \mathbb{C}\langle t_{ij}\rangle \to \mathbb{C}\langle t_{ij}\rangle$ denotes the unique algebra homomorphism such that $\omega(t_{ij}) = t_{ji}$ then $\omega \circ \theta(W_\lambda \otimes W_\mu) = \theta(W_\mu \otimes W_\lambda)$. We use the bases of $\overline{W}_q$ resp. W_q, $W_{-q^{-1}}$, $W_{q^{-2}}$ given in (2.4) resp. (2.1). Let us call the basis vectors in (2.1) w_i $(1 \le i \le 9)$ and those in (2.4) $\bar{w}_j$ $(1 \le j \le 6)$, in the order in which they are introduced. Thus, $w_1 = e_1 \otimes e_1$, $w_2 = e_3 \otimes e_3$ etc. Straightforward calculation shows that the tensor products (in either order) of $\bar{w}_1$, $\bar{w}_2$ and w_6, w_7 (meaning $\bar{w}_1 \otimes w_6$, $w_6 \otimes \bar{w}_1$, $\bar{w}_1 \otimes w_7$,

$w_7 \otimes \bar{w}_1$, $\bar{w}_2 \otimes w_6$ etc.) yield precisely the relations (1) and (2) in Table 1. Similarly, from the tensor products (in either order) of $\bar{w}_4$, $\bar{w}_5$ and w_6, w_7 one derives precisely the relations (4), (5), (6) and (7). The relations coming from the tensor products (in either order) of $\bar{w}_1$, $\bar{w}_2$ and w_8 together with the relations coming from the tensor products of w_1, w_2 and w_9 are easily seen to be equivalent to relations (3) and (14). The tensor products $\bar{w}_4 \otimes w_8$, $w_6 \otimes \bar{w}_3$, $w_6 \otimes \bar{w}_6$ and $w_3 \otimes w_9$ lead to the following four equations:

$$\begin{aligned}
&qt_{11}t_{23} + q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})t_{12}t_{22} - qt_{13}t_{21} + t_{21}t_{13} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})t_{22}t_{12} - t_{23}t_{11} = 0 \\
&(q^{\frac{1}{2}} - q^{-\frac{1}{2}})t_{11}t_{23} - q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})t_{21}t_{13} - t_{12}t_{22} + qt_{22}t_{12} = 0 \\
&t_{11}t_{23} - qt_{21}t_{13} + t_{13}t_{21} - qt_{23}t_{11} \\
&q^{\frac{1}{2}}t_{11}t_{23} + qt_{12}t_{22} + q^{\frac{3}{2}}t_{13}t_{21} + q^{-\frac{1}{2}}t_{21}t_{13} + t_{22}t_{12} + q^{\frac{1}{2}}t_{23}t_{11} = 0.
\end{aligned} \tag{2.12}$$

We can eliminate $t_{23}t_{11}$ from the first and fourth equations of (2.12) by using the third equation. From the resulting two equations one can eliminate $t_{11}t_{23}$ by using the second equation of (2.12). One then obtains:

$$\begin{aligned}
&(q^{\frac{3}{2}} + q^{-\frac{1}{2}})t_{12}t_{22} - (q^{\frac{3}{2}} + q^{-\frac{1}{2}})t_{22}t_{12} - (q + q^{-1})t_{13}t_{21} + (q^2 + 1)t_{21}t_{13} = 0 \\
&(q^{\frac{3}{2}} + q^{-\frac{1}{2}})t_{12}t_{22} - (q^{\frac{3}{2}} + q^{-\frac{1}{2}})t_{22}t_{12} + (q + q^{-1})(q - 1)t_{13}t_{21} + (q^2 - q - q^{-1} + 1)t_{21}t_{13} = 0,
\end{aligned} \tag{2.13}$$

from which one deduces (10.a). Resubstituting (10.a) in the first equation of (2.13) resp. the third equation of (2.12) one obtains (9.a) resp. (8.a). Using (9.a) and (10.a) we can rewrite the second equation of (2.12) as

$$t_{11}t_{23} = -q^{\frac{1}{2}}t_{12}t_{22} - qt_{13}t_{21}, \tag{2.14}$$

and (8.a), (9.a), (10.a) and (2.14) are in fact equivalent to (2.12). In an exactly analogous way, one derives all the (other) relations of (8), (9) and (10) by using the tensor products $\bar{w}_4 \otimes w_8$, $\bar{w}_5 \otimes w_8$, $\bar{w}_6 \otimes w_6$, $\bar{w}_6 \otimes w_7$, $w_3 \otimes w_9$, $w_4 \otimes w_9$ and their images under the flip $v \otimes w \mapsto w \otimes v$. The analogues of (2.14) are:

$$\begin{aligned}
t_{21}t_{33} &= -q^{\frac{1}{2}}t_{22}t_{32} - qt_{23}t_{31} \\
t_{22}t_{21} &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}} - q^{-\frac{3}{2}})t_{12}t_{31} - q^{-\frac{1}{2}}t_{11}t_{32} \\
t_{23}t_{22} &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}} - q^{-\frac{3}{2}})t_{13}t_{32} - q^{-\frac{1}{2}}t_{12}t_{33}.
\end{aligned} \tag{2.15}$$

Finally, the tensor products $\bar{w}_3 \otimes w_8$, $\bar{w}_6 \otimes w_8$, $w_5 \otimes w_9$ and their images under the flip give rise to six equations which can be seen to be equivalent to the relations (11), (12), (13) plus the following relations:

$$\begin{aligned}
t_{23}t_{21} &= q^{-1}t_{12}t_{32} + q^{-\frac{1}{2}}(q - q^{-1})t_{13}t_{31} \\
t_{22}t_{22} &= t_{11}t_{33} + (q^{\frac{1}{2}} - 2q^{-\frac{1}{2}})t_{12}t_{32} + q^{-1}t_{13}t_{31}.
\end{aligned} \tag{2.16}$$

Summarizing, we have now proved that the relations (2.2) are equivalent to the relations (1) till (14) plus the relations in (2.14), (2.15) and (2.16).

In the second part of the proof we shall make use of the terminology laid down in [B]. As generators we take the t_{ij}. We order the monomials in the t_{ij} as in the proof of [1.2]. As reduction system we take the relations (1) till (14) plus the relations in (2.14), (2.15) and (2.16). Note that this reduction system is compatible with the ordering of the monomials in the t_{ij}. Given any relation σ in the t_{ij} we can first reduce it to an irreducible expression and then rewrite it in the form $f = \sum_i c_i f_i$ where the f_i are monomials strictly less than the monomial f. If σ is of degree at most two the result of these two operations on σ is uniquely determined. We call it the reduced form of σ. It clearly is compatible with the ordering of the monomials in the t_{ij}. If we apply this procedure to the orthogonality relations (2.10) we end up with a single relation:

$$t_{11}t_{33} = 1 - qt_{13}t_{31} - q^{\frac{1}{2}}t_{12}t_{32}. \tag{2.17}$$

We add (2.17) to our reduction system. Given any ambiguity, it either is resolvable or gives rise to a new relation which can be written in reduced form and added to the reduction system. Starting from the ambiguities $t_{32}t_{23}t_{11}$, $t_{33}t_{22}t_{12}$, $t_{33}t_{21}t_{11}$, $t_{32}t_{22}t_{11}$, $t_{33}t_{23}t_{11}$ and $t_{33}t_{22}t_{11}$ we get:

$$\begin{aligned}
t_{12}t_{23}t_{31} &= t_{13}t_{21}t_{32}\\
t_{12}t_{23}t_{32} &= (q+1)t_{13}t_{22}t_{32} + q(q^{\frac{1}{2}} + q^{-\frac{1}{2}})t_{13}t_{23}t_{31}\\
t_{11}t_{22}t_{32} &= 2q^2(q^{\frac{1}{2}} + q^{-\frac{1}{2}})t_{13}t_{21}t_{31} + q^{\frac{3}{2}}(q^{\frac{1}{2}} + 2q^{-\frac{1}{2}})t_{12}t_{22}t_{31} - q^{\frac{1}{2}}t_{21}\\
t_{12}t_{21}t_{32} &= q(q^{\frac{1}{2}} + q^{-\frac{1}{2}})t_{13}t_{21}t_{31} + (q+1)t_{12}t_{22}t_{31}\\
t_{12}t_{22}t_{33} &= 2q^2(q^{\frac{1}{2}} + q^{-\frac{1}{2}})t_{13}t_{23}t_{31} + q^{\frac{3}{2}}(q^{\frac{1}{2}} + 2q^{-\frac{1}{2}})t_{13}t_{22}t_{32} - q^{\frac{1}{2}}t_{23}\\
t_{11}t_{22}t_{33} &= (q^{-\frac{1}{2}} - q^{\frac{1}{2}} - q^{\frac{3}{2}})t_{12}t_{22}t_{32} - qt_{13}t_{22}t_{31} - 2q(q - q^{-1})t_{13}t_{21}t_{32} + t_{22}.
\end{aligned} \tag{2.18}$$

We add these relations to our reduction system. The reduced form of the relation $\det_q T = 1$ (see (2.9)) then becomes:

$$t_{12}t_{22}t_{32} = q^{\frac{1}{2}} - q^{\frac{1}{2}}t_{22} + q^2(q^{\frac{1}{2}} + q^{-\frac{1}{2}})t_{13}t_{22}t_{31} - 2q(q^{\frac{1}{2}} + q^{-\frac{1}{2}})t_{13}t_{21}t_{32}. \tag{2.19}$$

We add (2.19) to our reduction system too. We then get new ambiguities $t_{12}t_{12}t_{22}t_{32}$ and $t_{12}t_{22}t_{32}t_{32}$ from which one derives (16.c) and (16.d). We add these last two equations to our reduction system. (From now on, this will be done automatically every time we derive a new relation in Table 1.) The new (inclusion) ambiguities $t_{11}t_{22}t_{32}$ and $t_{12}t_{22}t_{33}$ then lead to (15.c) and (15.d). Since (2.19) is not irreducible anymore, we should rewrite it as:

$$t_{13}t_{21}t_{32} = (q+1)^{-1} - q^{-1}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{-1}t_{12}t_{32} + qt_{13}t_{22}t_{31} - (q+1)^{-1}t_{22}. \tag{2.20}$$

The ambiguities $t_{22}t_{32}t_{21}$ and $t_{21}t_{22}t_{32}$ lead to (17.a) and then the ambiguity $t_{13}t_{21}t_{32}$ leads to (19.a). Using the new reduction rules (19.a), (16.c/d) and (15.c/d), one sees that the relations (2.17), (2.14), (2.15) and (2.16) can be rewritten as (20), (15.a), (15.b), (16.b), (16.a), (19.b) and (21) respectively. The ambiguities $t_{23}t_{12}t_{22}$, $t_{11}t_{12}t_{32}$ and $t_{12}t_{32}t_{33}$ now lead to (17.b), (18.a) and (18.b) respectively. It is easy to see that the reduction rules (2.18) and (2.20) can now be discarded. This concludes the proof of the theorem.

Corollary 2.3 — *There is a unique algebra homomorphism $\chi^\sharp\colon C_{q^2} \to B_q$ such that*

$$\chi^\sharp(T) = \begin{pmatrix} \alpha^2 & (1+q^{-2})^{\frac{1}{2}}\alpha\beta & -\beta^2 \\ (1+q^{-2})^{\frac{1}{2}}\alpha\gamma & 1+(q+q^{-1})\beta\gamma & -(1+q^2)^{\frac{1}{2}}\delta\beta \\ -\gamma^2 & -(1+q^2)^{\frac{1}{2}}\delta\gamma & \delta^2 \end{pmatrix}.$$

The mapping $\chi^\sharp$ is an isomorphism of Hopf $*$-algebras.

It follows from [2.2] and [1.2] that $\chi^\sharp$ is well-defined and bijective. That $\chi^\sharp$ is a Hopf algebra morphism follows from the fact that Ad_q (see (1.5)) is a corepresentation. Finally, $\chi^\sharp$ respects the $*$-operations because of (1.9) and (2.11).

We identify $SO_{q^2}(3)$ and $\mathrm{Ad}(SU_q(2))$ via χ. We then have a morphism $\phi: SU_q(2) \to SO_{q^2}(3)$ (see above [1.4]) and [1.5] can be restated as

Corollary 2.4 — *There is the following exact sequence of quantum groups:*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU_q(2) \xrightarrow{\phi} SO_{q^2}(3) \longrightarrow 1.$$

During the winter school in Zdikov O. Ogievetsky pointed out to me that the quantum analogue of the classical isomorphism $A_1 \sim B_1$ had already been proved in [JO] in a more general context. In [T1] a result similar to [2.3] was announced, but to my knowledge a proof of this claim has never been published.

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