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A Short Proof of the Planarity Characterization of Colin de Verdière

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Colin de Verdière introduced an interesting new invariant $\mu(G)$ for graphs G, based on algebraic and analytic properties of matrices associated with G. He showed that the invariant is monotone under taking minors and moreover, that $\mu(G) \leqslant 3$ if only if G is planar. In this paper we give a short proof of Colin de Verdière's result that $\mu(G) \leqslant 3$ if G is planar.

1. INTRODUCTION

Let G be a connected undirected graph, which throughout this paper we assume without loss of generality to have vertex set $\{1, ..., n\}$. Then Colin de Verdière's invariant $\mu(G)$ [3] (English translation: [4]) is the largest corank of any symmetric $n \times n$ matrix $M = (m_{i,j})$

> with exactly one negative eigenvalue (of multiplicity 1) and with $m_{i,j} < 0$ if i and j are adjacent and $m_{i,j} = 0$ if i and j are not adjacent and $i \neq j$, (1)

so that M fulfils the "Strong Arnold Hypothesis". (The *corank* of a matrix is the dimension of its kernel.) For the "Strong Arnold Hypothesis" we refer to Colin de Verdière [3]; we do not need it in this paper.

In [3] it is proved that if G' is a minor of G, then $\mu(G') \leqslant \mu(G)$. (In proving this, the "Strong Arnold Hypothesis" is essential.) So for each fixed t, the class of graphs G satisfying $\mu(G) \leqslant t$ is closed under taking minors. Hence, by the theorem of Robertson and Seymour [6], there is a finite collection of "forbidden minors" for such a class of graphs.

Colin de Verdière [3] showed that the graphs G satisfying $\mu(G) \leqslant 1$ are exactly the paths, those satisfying $\mu(G) \leqslant 2$ are exactly the outerplanar graphs, and those satisfying $\mu(G) \leqslant 3$ are exactly the planar graphs. If $\mu(G) \leqslant 4$ then G is linklessly embeddable, since each graph G in the

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complete class of forbidden minors found by Robertson, Seymour, and Thomas [7] has $\mu(G)>4$ (cf. Bacher and Colin de Verdière [1]). In fact, Robertson, Seymour, and Thomas [8] conjecture that also the reverse implication holds.

Colin de Verdière's proof [3] of the fact that $\mu(G)\leqslant 3$ for planar graphs G uses notions of differential geometry and in particular Cheng's result [2] on the multiplicity of the second eigenvalue of a Laplacian on the 2-sphere. Bacher and Colin de Verdière [1] give a proof that uses the facts that under some conditions μ is invariant under ΔY- and $Y\Delta$-transformations and that a planar graph can be reduced to an edge by these transformations. We here give a direct combinatorial proof.

2. The Proof

We first give an auxiliary result. For any vector x, let supp(x) denote the support of x (i.e., the set $\{i \mid x_i \neq 0\}$). Moreover we denote $\text{supp}_+(x) := \{i \mid x_i > 0\}$ and $\text{supp}_-(x) := \{i \mid x_i < 0\}$. For any subset U of V let $\langle U \rangle$ denote the subgraph of G induced by U. If $x \in \mathbb{R}^n$ and $I \subseteq \{1, ..., n\}$, then x_I denotes the subvector of x induced by the indices in I. Similarly, if M is an $n \times n$ matrix and $I, J \subseteq \{1, ..., n\}$, then $M_{I \times J}$ denotes the submatrix of M induced by row indices in I and column indices in J. We say that a vector $x \in \ker(M)$ *has minimal support* if x is nonzero and if for each nonzero vector $y \in \ker(M)$ with $\text{supp}(y) \subseteq \text{supp}(x)$ one has $\text{supp}(y) = \text{supp}(x)$. Viewing the matrix M as a Laplacian the following proposition can be regarded as a Courant nodal theorem [5] for graphs.

Proposition 1. *Let G be a connected graph and let M satisfy* (1). *Let $x \in \ker(M)$ have minimal support. Then $\langle \text{supp}_+(x) \rangle$ and $\langle \text{supp}_-(x) \rangle$ are connected.*

Proof. Suppose for instance that $\langle \text{supp}_+(x) \rangle$ is disconnected. Let I and J be two components of $\langle \text{supp}_+(x) \rangle$. Let $K := \text{supp}_-(x)$. Since $m_{i,j} = 0$ if $i \in I$, $j \in J$, we have:

$$\begin{aligned} M_{I \times I} x_I + M_{I \times K} x_K &= 0, \\ M_{J \times J} x_J + M_{J \times K} x_K &= 0. \end{aligned} \tag{2}$$

Let z be an eigenvector of M with negative eigenvalue. By the Perron–Frobenius theorem we may assume $z > 0$. Let

$$\lambda := \frac{z_I^T x_I}{z_J^T x_J}. \tag{3}$$

Define $y \in \mathbb{R}^n$ by: $y_i := x_i$ if $i \in I$, $y_i := -\lambda x_i$ if $i \in J$, and $x_i := 0$ if $i \notin I \cup J$. By (3), $z^T y = z_I^T x_I - \lambda z_J^T x_J = 0$. Moreover, one has (since $m_{i,j} = 0$ if $i \in I$ and $j \in J$):

$$\begin{aligned} y^T M y &= y_I^T M_{I \times I} y_I + y_J^T M_{J \times J} y_J = x_I^T M_{I \times I} x_I + \lambda^2 x_J^T M_{J \times J} x_J \\ &= -x_I^T M_{I \times K} x_K - \lambda^2 x_J^T M_{J \times K} x_K \leqslant 0, \end{aligned} \tag{4}$$

(using (2)) since $M_{I \times K}$ and $M_{J \times K}$ are nonpositive, and since $x_I > 0$, $x_J > 0$ and $x_K < 0$.

Now $z^T y = 0$ and $y^T M y \leqslant 0$ imply that $My = 0$ (as M is symmetric and has exactly one negative eigenvalue, with eigenvector z). Therefore, $y \in \ker(M)$, contradicting the minimality of $\operatorname{supp}(x)$. ∎

From this we derive:

THEOREM 1. *If G is planar then $\mu(G) \leqslant 3$.*

Proof. Since $\mu(G)$ does not increase after deleting edges, we may assume that G is maximally planar. So G is 3-connected and contains a triangle which is a face. Let U be the set of vertices of this triangle. Assume that $\mu(G) > 3$. Let $M = (m_{i,j})$ be a matrix satisfying (1) with corank equal to $\mu(G)$. Since the corank of M is larger than 3, $\ker(M)$ contains a nonzero vector x with $x_i = 0$ for all $i \in U$. We may assume that x has minimal support.

Since G is 3-connected there exist 3 pairwise disjoint paths P_1, P_2, P_3, where each P_i starts in a vertex $v_i \notin \operatorname{supp}(x)$ adjacent to at least one vertex in $\operatorname{supp}(x)$, and ends in U. Now if M satisfies (1), then each vertex $v \notin \operatorname{supp}(x)$ adjacent to some vertex in $\operatorname{supp}_+(x)$ is also adjacent to some vertex in $\operatorname{supp}_-(x)$ and conversely. So each v_i is adjacent to at least one vertex in $\operatorname{supp}_+(x)$ and at least one vertex in $\operatorname{supp}_-(x)$.

By Proposition 1, $\operatorname{supp}_+(x)$ and $\operatorname{supp}_-(x)$ can be contracted to one vertex each. Delete all vertices of G not contained in $\operatorname{supp}(x)$ or in any P_i and contract each P_i to one vertex. Add a vertex inside the triangle and edges between this vertex and the vertices in U. The graph we obtain is still planar. But this graph contains $K_{3,3}$ as subgraph, hence we have a contradiction. ∎

The proof of this theorem is similar to the proof given in Cheng [2] for the maximal multiplicity of the second eigenvalue of a Laplacian on the 2-sphere. We have chosen a nonzero vector vanishing on the vertices of a triangle and then used a Courant nodal theorem for graphs to obtain a contradiction. Cheng showed that the multiplicity of the second eigenvalue of a Laplacian on the 2-sphere is at most 3, by choosing an eigenfunction whose value and both partial derivatives vanish at some given point of the

2-sphere. But the positive or negative support of this eigenfunction consists of more than one component, contradicting Courant nodal theorem.

Finally we mention the following corollary of Proposition 1.

COROLLARY 1. *Let G be a connected graph and let M satisfy* (1). *If* $\ker(M)$ *has dimension* 1 *and* $x \in \ker(M)$ *then* $\langle \text{supp}_+(x) \rangle$ *and* $\langle \text{supp}_-(x) \rangle$ *are connected.*

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