On Tutte's Characterization of Graphic Matroids—A Graphic Proof

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ABSTRACT

In this paper we present a relatively simple proof of Tutte's characterization of graphic matroids. The proof uses the notion of 'signed graph' and it is 'graphic' in the sense that it can be presented almost entirely by drawing (signed) graphs. © 1995 John Wiley & Sons, Inc.

1. INTRODUCTION: TUTTE'S CHARACTERIZATION OF GRAPHIC MATROIDS

The aim of this paper is to present a proof of the following result.

Theorem 1 (Tutte [8]). Let $\mathcal{M}$ be a binary matroid. Then $\mathcal{M}$ is graphic if and only if $\mathcal{M}$ has no F_7, F_7^*, $\mathcal{M}^*(K_5)$, or $\mathcal{M}^*(K_{3,3})$ minor.

We assume the reader to be familiar with matroid terminology. In fact we need the following notions (cf. Oxley [3], Truemper [7], or Welsh [11]): *binary matroid*; *representation (matrix)* (*over GF* (2)) of a binary matroid; *matroid-isomorphism* (denoted by $\sim$); *contraction, deletion* and *minor*; and *graphic matroid*, i.e., the *cycle matroid* (=*polygon matroid*) of a graph. Representation matrices over $GF(2)$) of the four matroids mentioned in Theorem 1 are

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} (F_7), \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} (F_7^*),$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \mathcal{M}^*(K_5)),$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} (\mathcal{M}^*(K_{3,3})).$$

Besides Tutte's original proof in [8] there exist proofs of Theorem 1 derived by: Ghouila-Houri [1]: Inukai and Weinberg [2], using Tutte's "wheels and whirls theorem" (Tutte [9]); Seymour [5], using results from [4]; Wagner [10] and Truemper [6]. Our proof of Theorem 1 is in Section 3. The notions we need for it are given in Section 2. To motive them and to guide the reader into the proof we have included the key ideas of our proof as observations in Section 2.

2. PRELIMINARIES-THE KEY IDEAS

Signed Graphs

A *signed graph* is a pair (G, Σ) where $G = (V(G), E(G))$ is an undirected graph, possibly with loops and parallel edges, and Σ is a subset $E(G)$. Edges in Σ are called *odd.* The other edges are called *even.* A circuit in G is called *odd (even)* in (G, Σ), if it contains an odd (even) number of odd edges.

The *extended even cycle matroid,* $S(G, \Sigma)$, of (G, Σ) is the binary matroid represented over $GF(2)$ by the matrix

$$\begin{bmatrix} 1 & \chi_\Sigma \\ 0 & M_G \end{bmatrix}.$$

Here χ_Σ denotes the characteristic vector of Σ as a subset of $E(G)$. M_G denotes the *node-edge incidence matrix* of G, i.e., the matrix in $GF(2)^{V(G)\times E(G)}$ defined by

$$(M_G)_{u,e} = \begin{cases} 1 & \text{if } e = uv, \text{ for some node } v \neq u\,; \\ 0 & \text{else.} \end{cases}$$

The element of $S(G, \Sigma)$ which is not in $E(G)$ will be denoted by σ. The circuits of $S(G, \Sigma)$ are: even circuits in (G, Σ), the union of two odd circuits in (G, Σ) when they have at most one node in common, and the union of an odd circuit in (G, Σ) with $\{\sigma\}$.

Re-signing (G, Σ) *on* $U \subseteq V(G)$, means replacing Σ by $\Sigma \Delta \delta(U)$ ($\Delta :=$ symmetric difference, $\delta(U) := \{uv \in E(G) \mid u \in U, v \notin U\}$). As re-signing leaves odd (even) circuits odd (even), it does not affect $S(G, \Sigma)$.

A signed graph that comes from (G, Σ) by a series of the following operations: re-signing, deletion of an edge, and contraction of an **even** edge, is called a *minor* of (G, Σ). Obviously, the extended even cycle matroid of a minor of (G, Σ) is a minor of $S(G, \Sigma)$. Moreover, contracting σ in $S(G, \Sigma)$ yields the cycle matroid, $\mathcal{M}(G)$, of G.

The following observation makes clear why we are interested in signed graphs.

Observation 1. Let $\mathcal{M}$ be a binary matroid, with representation M over $GF(2)$. If all its proper minors are graphic and $M \neq 0$, then $\mathcal{M}$ is the extended even cycle matroid of a signed graph.

Proof. We may assume that, possibly after applying row operations over $GF(2)$,

$$M = \begin{bmatrix} 1 & a^{\mathrm{T}} \\ 0 & N \end{bmatrix}.$$

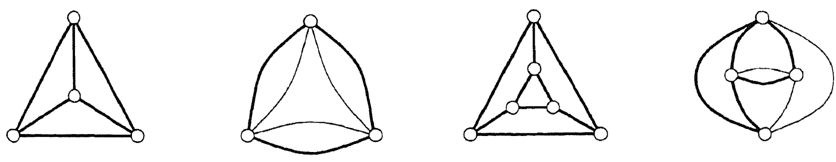
Matrix N represents a minor of $\mathcal{M}$. Hence N represents the cycle matroid of a graph, G say. If $\Sigma := \{e \in E(G) \mid a_e = 1\}$, then $\mathcal{M} \sim S(G, \Sigma)$. ∎

Examples—The Four Forbidden Minors

In view of Observation 1 one might expect that the four matroids in Theorem 1 are extended even cycle matroids of signed graphs. Indeed, the four signed graphs, depicted in Figure 1 satisfy

Observation 2. $S(\overline{K}_4) \sim F_7^*$, $S(\overline{K}_3^2) \sim F_7$, $S(\overline{P}_3) \sim \mathcal{M}^*(K_5)$, and $S(\overline{K}_4^2) \sim \mathcal{M}^*(K_{3,3})$.

Convention. In all figures bold lines denote odd edges and thin lines denote even edges. Dashed and dotted lines stand for pairwise openly disjoint paths. Dashed lines correspond to paths with at least one edge, dotted lines may have length zero. The word **odd** in a region denotes that the circuit bounding that region is odd.



$\overline{K}_4$ $\overline{K}_3^2$ $\overline{P}_3$ $\overline{K}_4^2$

FIGURE 1

Almost Bipartite Signed Graphs

A signed graph (G, Σ) is *bipartite* if $\Sigma = \delta(U)$ for some $U \subseteq V(G)$, in other words, if one can re-sign (G, Σ) to $(G, \emptyset)$. So a signed graph is bipartite if and only if it contains no odd circuits. A signed graph (G, Σ) is called *almost bipartite* if there exists a node $u \in V(G)$, called a *blocknode*, such that $u \in V(C)$ for each odd circuit in C in (G, Σ).

Observation 3. Let (G, Σ) be a signed graph. If (G, Σ) is almost bipartite, then $S(G, \Sigma)$ is graphic.

Proof. Let u be a blocknode in (G, Σ). Re-sign such that $\Sigma \subseteq \delta(u)$. We construct a new graph H as follows (see Fig. 2). We split node u into two new nodes u_1 and u_2. We split $\delta(u)$ into $\delta(u_1)$ and $\delta(u_2)$ such that $\delta(u_1) = \delta(u) \setminus \Sigma$ and $\delta(u_2) = \Sigma$. (One should understand this in such a way that an odd loop with endpoint u becomes an edge from u_1 to u_2.) Finally we add an edge e_σ from u_1 to u_2. It is not hard to see that $S(G, \Sigma) \sim \mathcal{M}(H)$ ∎

Switching—Essentially Almost Bipartite Signed Graphs

Let (G, Σ) be a signed graph. Moreover let G_1 and G_2 be two subgraphs of G, and let $\Sigma_i := \Sigma \cap E(G_i)$ for $i = 1, 2$. We say that $[(G_1, \Sigma_1), (G_2, \Sigma_2)]$ is a *k-partition* if the following holds:

$$E(G_1) \cup E(G_2) = E(G), \quad E(G_1) \cap E(G_2) = \emptyset,$$
$$V(G_1) \cup V(G_2) = V(G), \quad |V(G_1) \cap V(G_2)| = k.$$

We call the *k-partition strong* if both (G_1, Σ_1) and (G_2, Σ_2) are not bipartite, and $|E(G_1)|, |E(G_2)| \geq 3$.

If (G, Σ) has a 1-partition $[(G_1, \Sigma_1), (G_2, \Sigma_2)]$, and the signed graph (H, Θ) consists of two disjoint copies of (G_1, Σ_1) and (G_2, Σ_2), then we say that (H, Θ) comes from (G, Σ) by *breaking (at a cutnode)*. Conversely we say that (G, Σ) comes from (H, Θ) by *glueing (at a node)*.

Let (G, Σ) have a 2-partition $[(G_1, \Sigma_1), (G_2, \Sigma_2)]$. Let u and v be the two nodes that G_1 and G_2 have in common. Consider disjoint copies (H_1, Θ_1) and (H_2, Θ_2) of (G_1, Σ_1) and (G_2, Σ_2), respectively. For $i = 1, 2$, let u_i and v_i be the copies of u and v in H_i. Construct (H, Θ) from (H_1, Θ_1) and

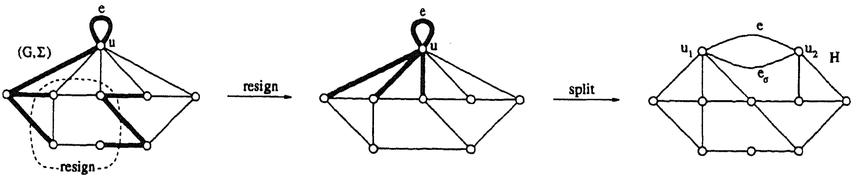


FIGURE 2

(H_2, Θ_2) by identifying u_1 with v_2 and u_2 with v_1. The operation carrying (G, Σ) into (H, Θ) is called *switching*. (See Fig. 3.)

We call a signed graph *essentially almost bipartite* if it can be turned into an almost bipartite graph by a series of breakings, glueings and switchings. It is obvious that these operations leave $S(G, \Sigma)$ invariant. Hence Observation 3 yields

Observation 4. Let (G, Σ) be a signed graph. If (G, Σ) is essentially almost bipartite, then $S(G, \Sigma)$ is graphic.

3. THE PROOF

Let $\mathcal{M}$ be a binary matroid with no F_7, F_7^*, $\mathcal{M}^*(K_5)$, or $\mathcal{M}^*(K_{3,3})$ minor. We have to prove that $\mathcal{M}$ is graphic. We may assume that all its proper minors are graphic. Hence, by Observation 1, $\mathcal{M}$ is the extended even cycle matroid of a signed graph, (G, Σ) say. The graphicness of $\mathcal{M}$ follows from Observations 2 and 4 and the result below (which is the special case of Tutte's Theorem 1 when restricted to extended even cycle matroids of signed graphs, see the remark toward the end of the paper).

Theorem 2. Let (G, Σ) be a signed graph with no $\overline{K}_4$, $\overline{K}_3^2$, $\overline{P}_3$, or $\overline{K}_4^2$ minor. Then (G, Σ) is essentially almost bipartite.

Proof. Let (G, Σ) be a signed graph with no $\overline{K}_4$, $\overline{K}_3^2$, $\overline{P}_3$, or $\overline{K}_4^2$ minor. Assume that all the proper minors of (G, Σ) are essentially almost bipartite. We consider three cases.

Case 1. (G, Σ) has a 0- or 1-partition.
We may assume that in fact (G, Σ) has a 0-partition. (If not, break.) As all components are proper minors of (G, Σ), we may assume that they are almost bipartite. (If not, switch.) Glueing the blocknodes of the components to one new node yields an almost bipartite graph. So (G, Σ) is essentially almost bipartite.

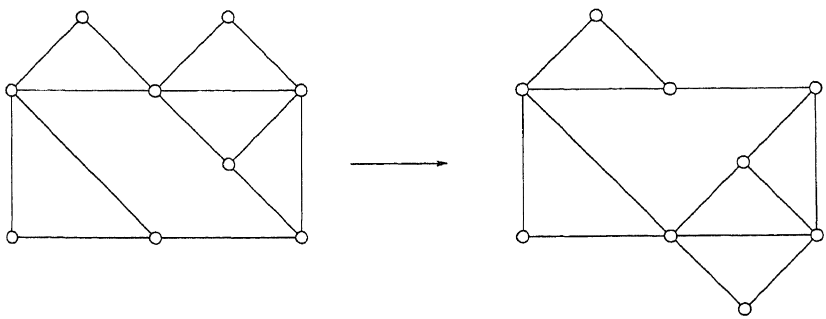


FIGURE 3

Case 2. (G, Σ) has no 0- or 1-partition, but has a strong 2-partition.

Let $[(G_1, \Sigma_1)], (G_2, \Sigma_2)]$ be a strong 2-partition of (G, Σ). Let u and v be the two nodes that G_1 and G_2 have in common. For $i = 1, 2$, let (G_i^+, Σ_i^+) denote the signed graph obtained from (G_i, Σ_i) by adding two edges, e_i and f_i say, from u to v, with e_i odd, and f_i even. (G_1^+, Σ_1^+) and (G_2^+, Σ_2^+) are minors of (G, Σ). Indeed, as the partition is strong, (G_2, Σ_2) contains an odd circuit, C say. As (G, Σ) has no 0- or 1-partition, by Menger's Theorem, in G_2 there exist two node disjoint paths P and Q from C to $\{u, v\}$ (see Fig. 4). From this it is easy to see that (G_1^+, Σ_1^+) is a minor of (G, Σ). By symmetry, so is (G_2^+, Σ_2^+).

It is not hard to see that every switching in (G_1^+, Σ_1^+) or in (G_2^+, Σ_2^+) can be carried out in (G, Σ). As (G_1^+, Σ_1^+) and (G_2^+, Σ_2^+) are proper minors of (G, Σ), they are essentially almost bipartite. Carry out all the switchings needed to make (G_1^+, Σ_1^+) and (G_2^+, Σ_2^+) almost bipartite in (G, Σ). Then all odd circuits in (G_1^+, Σ_1^+) contain u, or all these odd circuits contain v (since e_1 and f_1 form an odd circuit). The same is true for (G_2^+, Σ_2^+). So (G, Σ) is almost bipartite or can be made almost bipartite by just one switching.

Case 3. (G, Σ) has no 0-, 1-, or strong 2-partition.

An *odd-K_4* and an *odd-K_3^2* are as depicted in Figure 5.

Claim 1. (G, Σ) contains no odd-K_4 and no odd-K_3^2. Moreover (G, Σ) contains no pair of node-disjoint odd circuits.

Proof of claim 1. It is an easy exercise to show that if (G, Σ) would contain an odd-K_4 or an odd-K_3^2, then it has a $\overline{K}_4$ or a $\overline{K}_3^2$ minor. Suppose that C_1 and C_2 are two node-disjoint odd circuits. Assume that $|V(C_1)| + |V(C_2)|$ is as large as possible. First we show that C_1 or C_2 has two nodes only. Indeed, if this is not the case then (G, Σ) contains a signed subgraph as in Figure 6a. (This follows from Menger's Theorem and the fact that (G, Σ) has no 0-, 1-, or strong 2-partition.) As (G, Σ) contains no odd-K_4, each of the regions marked with "*" in Figure 6a is bounded by an even circuit. But this is impossible, as (G, Σ) has no $\overline{P}_3$ minor. So we may assume that

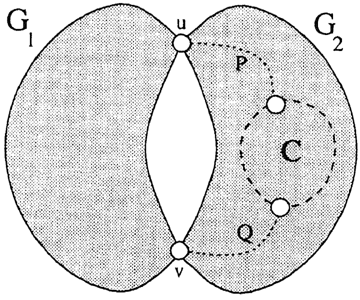


FIGURE 4

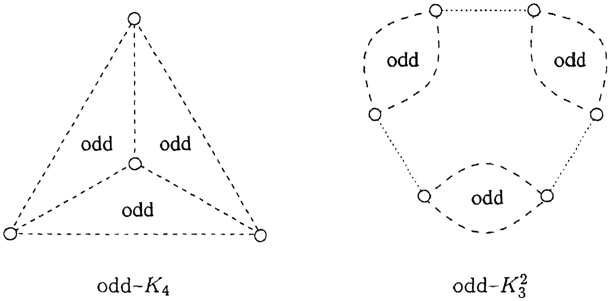


FIGURE 5

$V(C_1) := \{u_1, u_2\}$. Again by Menger's Theorem and the fact that (G, Σ) has no 0-, 1-, or strong 2-partition, it follows that there exist two paths P_1^1 and P_1^2 from u_1 to C_2 which have only u_1 in common and do not contain u_2. Similarly we have P_2^1 and P_2^2 from u_2 to C_2. P_1^i and P_2^j are node-disjoint, except possibly at their endpoints on C_2. (If not, we could find an odd circuit disjoint from C_2 and with more edges than C_1.) Hence (G, Σ) contains one of the signed graphs in Figure 6b and 6c. One easily sees now that (G, Σ) has $\overline{K}_4^2$ or $\hat{K}_4^2$ (see Fig. 7) as minor. As $\overline{K}_4$ is a minor of $\hat{K}_4^2$, this contradicts our assumptions on (G, Σ). ∎

Let C_1 and C_2 be two odd circuits such that their *intersection graph* P, i.e., $V(P) = V(C_1) \cap V(C_2)$, $E(P) = E(C_1) \cap E(C_2)$ is a path. It is easy to see that such two odd circuits exist. Assume they are chosen such that $|E(P)|$ is as small as possible. Let u be one of the endpoints of P. For $i = 1, 2$, let $e_i \in (E(C_i) \cap \delta(u)) \backslash E(P)$. (See Figure 8(a) and (b).) Let H denote the *union graph* of C_1 and C_2, i.e., $V(H) = V(C_1) \cup V(C_2)$, $E(H) = E(C_1) \cup E(C_2)$. We re-sign such that $\Sigma \cap E(H) = \{e_1, e_2\}$. An *st-arc* is a path from $s \in V(H) \backslash \{u\}$ to $t \in V(H) \backslash \{u\}$ which is internally node-disjoint from H. The arc is *odd (even)* if it contains an odd (even) number of odd edges.

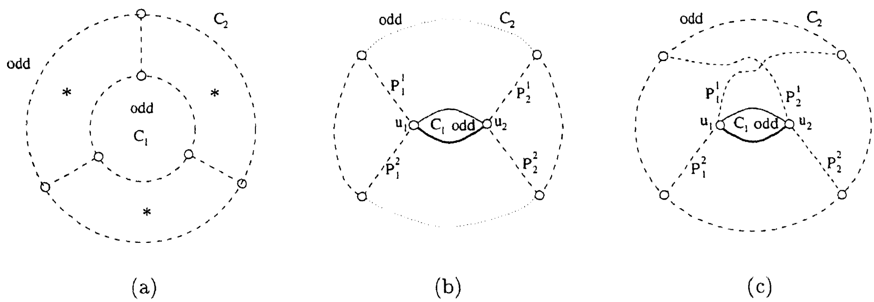
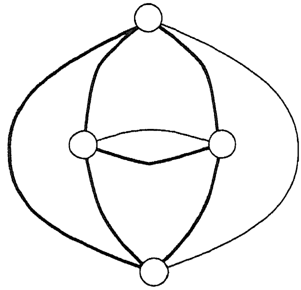


FIGURE 6



$\hat{K}_4^2$

FIGURE 7

Claim 2. If Q is an odd st-arc then $s \in V(C_1)\backslash V(C_2)$ and $t \in V(C_2)\backslash V(C_1)$ (or conversely). Moreover in that case $E(P) = \emptyset$.

Proof of claim 2. Let Q_H be the unique st-path with edges only in $E(H)\backslash\{e_1, e_2\}$. Then Q_H and Q form an odd circuit. Hence $\ell_i := |E(Q_H) \cap E(C_i)| \geq |E(P)|$ for $i = 1, 2$. If $\ell_1 = 0$ (or if $\ell_2 = 0$), then $|E(P)| = 0$, and one of the endpoints of Q is u, which contradicts the definition of an arc. So $\ell_1, \ell_2 > 0$. Hence $s \in V(C_1)\backslash V(C_2)$ and $t \in V(C_2)\backslash V(C_1)$ (or conversely). If $|E(P)| > 0$, then Q and H form an odd-K_4 (see Fig. 8(c)). ∎

Claim 3. If $E(P) = \emptyset$, then there exists no pair of internally node-disjoint paths of different parity from $V(C_1)\backslash V(C_2)$ to $V(C_2)\backslash V(C_1)$

Proof of claim 3. If such paths would exist, then $\overline{K}_3^2$ is a minor of (G, Σ) (see Fig. 8(d)). ∎

We now prove that u is a blocknode. Suppose this is not the case. Let C be an odd circuit not containing u. As it meets both C_1 and C_2 it is the disjoint union of paths on H (disjoint from u) and of arcs. As C is odd an odd

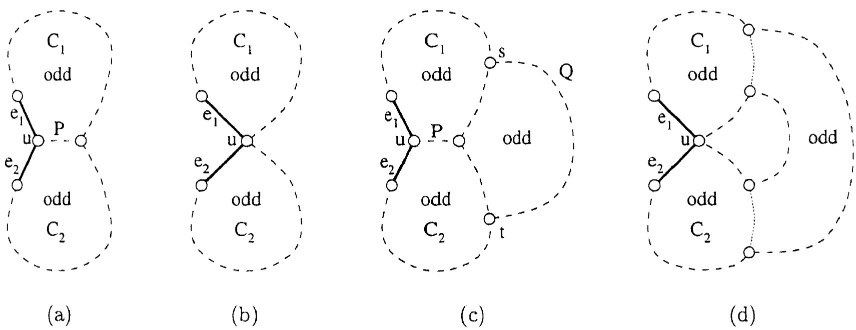


FIGURE 8

number of the arcs is odd. By Claims 2 and 3 it follows that $E(P) = \emptyset$ and that C contains an odd number of arcs from $V(C_1)\backslash V(C_2)$ to $V(C_2)\backslash V(C_1)$. This is absurd. ∎

Remark. That Theorem 2 is a special case of Theorem 1, is quite easy to see. First observe that the only signed graphs with an extended even cycle matroid among F_7, F_7^*, $\mathcal{M}^*(K_5)$ and $\mathcal{M}^*(K_{3,3})$ are $\overline{K}_4$, $\overline{K}_3^2$, $\overline{P}_3$, and $\overline{K}_4^2$. So if a signed graph (G, Σ) has none of these four signed graphs as a minor, then, by Theorem 1, $S(G, \Sigma)$ is graphic. Let H be a graph such that $S(G, \Sigma)$ is isomorphic to $\mathcal{M}(H)$. Let e_σ be the edge in H corresponding to the special element σ in $S(G, \Sigma)$. Let $\tilde{H}$ be obtained from H by contracting e_σ. Then G and $\tilde{H}$ have isomorphic cycle matroids. Hence, by Whitney's switching theorem (Whitney [12]), it follows that G can be turned into $\tilde{H}$ by breaking, glueing and switching. From this it easily follows that (G, Σ) is essentially almost bipartite. This shows how Theorem 2 follows from Theorem 1.

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