Notre Dame Journal of Formal Logic
Volume 36, Number 1, Winter 1995

Partial Up and Down Logic

JAN O. M. JASPARS

Abstract This paper presents logics for reasoning about extension and reduction of partial information states. This enterprise amounts to nonpersistent variations of certain constructive logics, in particular the so-called logic of constructible falsity of Nelson. We provide simple semantics, sequential calculi, completeness and decidability proofs.

1 Introduction The most simple logical means for knowledge representation is the semantic concept of partial truth-assignment. Propositions with a definite truth-value reflect the knowledge of a chosen agent. Propositions which are mapped to 1 are the things that the agent knows to be true, while propositions which have value 0 cover the information that the agent knows to be false. Propositions whose truth-values are left underspecified denote the agent's ignorance.

In this paper we develop dynamic extensions over these simple static representations, that is formalisms which provide logical means for reasoning about changing partial information states. We will follow van Benthem and de Rijke's style of dynamic modal logic (see van Benthem [4] and de Rijke [20]), where such formalisms are defined on the basis of total information states. We will focus on two kinds of changes: *enrichment* and *reduction*. These kinds of manipulations of states can easily be defined using a structural extension order $\leq$ which evolves naturally from the definition of partiality. Given the *static* meaning $[\![\varphi]\!]$ of a proposition φ, i.e., the partial states which support this proposition, the *dynamic* meaning $[\![\varphi]\!]_{dy}$ is induced by the extension order:

$$\{\langle s, t\rangle \mid s \leq t \;\&\; t \in [\![\varphi]\!]\}.$$

It represents a relational description of what happens to a state s when it is extended with the information φ. In an analogous way we specify the *negative dynamic* meaning $[\![\varphi]\!]^{-}_{dy}$ of φ, that is, the ways a situation s can shrink when the information φ has been removed from it:

$$\{\langle s, t\rangle \mid t \leq s \;\&\; t \notin [\![\varphi]\!]\}.$$

These two dynamic denotations are the basic relations for dynamic modal reasoning over extension and reduction. Such explicit dynamics will be accommodated by operators $[\varphi]_u$ and $[\varphi]_d$ for making universal statements over extensions and reductions, respectively. Their dual existential counterparts will be called $\langle\varphi\rangle_u$ and $\langle\varphi\rangle_d$. A proposition of the form $[\varphi]_u\,\psi$ says that extending the current state with the information that φ *necessarily* leads to a state which supports ψ, while $\langle\varphi\rangle_d\,\psi$ means that it is *possible* to retract φ from the current state in such a way that ψ holds afterwards.

(Note that these extension and reduction relations are only a small fragment of the relational wealth which has been employed in [4]. Van Benthem uses further relational constructions to interpret more complex dynamic operations, which facilitates definition of minimal variations of the extension and reduction relations. A negative side effect of the richness of van Benthem's system is its undecidability, see [20] and de Rijke [21].)

2 Dynamic, constructive, and nonmonotonic logic The above-mentioned simple dynamic setting originates from Kripke's semantic analysis of intuitionistic logic [14]. Intuitionistic logic can be seen as a dynamic logic of possessing *mathematical proofs*, and because this kind of information is taken to be *persistent*, that is proofs cannot be forgotten or retracted, only the extension relation is used for interpreting intensional connectives like implication and negation. In a dynamic modal setting intuitionistic implication $\varphi \rightarrow \psi$ can be described as $[\varphi]_u\,\psi$, while intuitionistic negation of φ boils down to $[\varphi]_u\,\bot$, where $\bot$ is the absurd or unprovable proposition.

The latter interpretation of negative information has led to discussion among constructivists, and also inspired different constructivistic axiomatizations of mathematical reasoning. One of these alternatives has been proposed in Nelson [17] (for a thorough essay on different treatments of negative information in constructive logic see Wansing [31]). Nelson's logic of *constructible falsity* treats negative information in the same fashion as positive information by taking *refutation* as a second mathematical construction. Proofs determine constructible truth, while refutations register constructible falsity. This logic reinstalls classical laws like the double negation and de Morgan equivalences in constructive logic, without accepting the principle of the excluded middle. Nelson's logic is of particular importance here, because it completely describes the persistent 'upward' part of the logics of this paper. Technically speaking, the logics we consider naturally arise from extending the expressivity of Nelson's logic over its Kripke semantics, which is principally the dynamics over partial states which has been described above. Kripke semantics for Nelson's logic can be found in Thomason [26]. (Nelson's logic has also been propagated outside the field of mathematical logic, a paper which demonstrates its use in default logic and logic programming is Pearce [18].)

In Gabbay [8] a nonpersistent extension of intuitionistic logic has been introduced by means of adding existential expressivity over the extension relation. The reason is to capture the *consistency*-operator M of the original default logic of Reiter [19] in an explicit fashion. The statement $M\varphi$ means that the current state can be extended with the information φ. It can be defined in the dynamic modal setting by $\langle\top\rangle_u\,\varphi$, where $\top$ is the trivial proposition which is always true (proved). In Turner [28] this idea has been incorporated in the setting of partial logic. The kind of Kripke models for Nelson's logic and the up-and-down logics of this paper are also used there.

These nonpersistent variations can be seen as subsystems of the 'upward' parts of the up-and-down logics of this paper. We will stick to classical definitions of semantic consequence and validity, and subsequently our systems will behave perfectly monotonic, transitive, commutative, etc. More unorthodox nonmonotonic entailment relations can be defined within the language of our up-and-down logics. For example, an obvious nonmonotonic candidate is the following: ψ follows from the assumption sequence $\varphi_1, \ldots, \varphi_n$ if extending an arbitrary state consecutively with φ_1 through φ_n always leads to a state which verifies ψ. In other words, $[\varphi_1]_u \ldots [\varphi_n]_u \psi$ holds always. Nonmonotonicity immediately pops up, because $\langle\top\rangle_u \varphi$ follows from itself, while it does not follow from the extended sequence $\langle\top\rangle_u \varphi$, $[\varphi]_u \bot$. Commutativity also fails in an obvious way: $\bot$ follows from $[\varphi]_u \bot$, $\langle\top\rangle_u \varphi$, while it does not follow necessarily from $\langle\top\rangle_u \varphi$, $[\varphi]_u \bot$.

In Section 3 we give a brief presentation of the semantics of partial logic and corresponding sequential axiomatizations. In Section 4 we follow the same procedure for their dynamic modal extensions. Finally, in Section 5 we prove completeness and decidability for the sequential systems of the first two sections.

3 Partial Logic In this section we shortly present a simple setting of partial propositional logics. As partial logics are most often inspired by semantic motivations, we wish to start with some of their basic modeltheoretic concepts.

3.1 Partial valuations

Definition 3.1 A *partial valuation* V is a partial function which assigns truth-values to a given set of propositional variables $I\!P$. In order to distinguish partial functions from total functions we replace the normal functional arrow $\longrightarrow$ by $\rightsquigarrow$. In short, $V : I\!P \rightsquigarrow \{0, 1\}$. The collection of all partial valuations is denoted by $\mathfrak{P}$. The *domain* of $V \in \mathfrak{P}$, $\mathfrak{Dom}(V)$, is the set of all propositional variables which obtain a truth-value by V:

$$\mathfrak{Dom}(V) := \{p \in I\!P \mid V(p) = 1 \text{ or } V(p) = 0\}.$$

Here partial valuations forbid the possibility for a proposition to be true and false at the same time. A technical removal of this 'excluded fourth value' boils down to redefining partial valuations V as relations between propositional variables $I\!P$ and truth-values: $V \subseteq I\!P \times \{0, 1\}$. Such liberalism has been defended for epistemic purposes by Belnap in his [2]. In Jaspars [11] the reader finds some arguments against this position. A technical advantage of going four-valued is that the classical symmetry between negative and positive information in partial logic gets restored, see for example Wagner [30].

If $\mathfrak{Dom}(V) = I\!P$ then V is said to be *total*. V' is said to be an *extension* of V whenever V' and V agree on all the propositional variables in the domain of V. We write $V \sqsubseteq V'$ if this relation holds:

$$V \sqsubseteq V' \overset{\text{def}}{\Longleftrightarrow} \forall p \in \mathfrak{Dom}(V) : V(p) = V'(p).$$

This last relation is of particular interest. $V \sqsubseteq V'$ says that V' contains at least as much information as V. Given this information order we are able to develop the kind of dynamics which has been mentioned in the Introduction.

3.2 Languages with static denotation There are many different partial logics. Loss of two-valuedness creates a lot of freedom, and subsequently leads to dispute and confusion. Even the basic choices of the interpretation of ordinary static connectives have led to divergent opinions. Many conflicting choices, however, are due merely to the underlying motivations of different applications of partial logic. This flexibility has led to many different partial logics.

The basic static language $\mathcal{L}$ which we will use is defined below. The reason why we have chosen $\mathcal{L}$ as our basic static partial equipment will be motivated on semantical grounds later on in this subsection.

Definition 3.2 Let $\mathbb{P}$ be a nonempty enumerable set of *propositional variables* or *atoms*. The language $\mathcal{L}$ is the smallest superset of $\mathbb{P}$ such that

$$\varphi, \psi \in \mathcal{L} \Rightarrow (\neg\varphi), (\varphi \wedge \psi) \in \mathcal{L} \text{ and } \bot \in \mathcal{L}.$$

These connectives are called *negation*, *conjunction*, and *falsum* respectively.

We will avoid superfluous use of parentheses, and take binary connectives to dominate over unary connectives. For example $\neg\varphi \wedge \psi$ means $((\neg\varphi) \wedge \psi)$ and not $(\neg(\varphi \wedge \psi))$. Furthermore, we will also use convenient abbreviations, like $\top := \neg\bot$ (*verum*), $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$ (*disjunction*). The letters p, q, r, possibly with additional sub- or superscripts, are used as atoms. Greek lower case letters are used to denote arbitrary formulas, while Greek capitals denote sets of formulas. Throughout the text we will also use sets of formulas in the scope of connectives and operators. Such expressions should be read in the most straightforward distributive manner. For example, $\neg\Gamma = \{\neg\varphi \mid \varphi \in \Gamma\}$ and $\varphi \wedge \Gamma = \{\varphi \wedge \psi \mid \psi \in \Gamma\}$.

For a given $V \in \mathfrak{P}$ the members of $\mathcal{L}$ obtain truth-values according the following inductive scheme:

Table 1

$V \models p \Leftrightarrow V(p) = 1 \quad (p \in \mathbb{P})$	$V \dashv p \Leftrightarrow V(p) = 0 \quad (p \in \mathbb{P})$
$V \not\models \bot$	$V \dashv \bot$
$V \models \neg\varphi \Leftrightarrow V \dashv \varphi$	$V \dashv \neg\varphi \Leftrightarrow V \models \varphi$
$V \models \varphi \wedge \psi \Leftrightarrow V \models \varphi \;\&\; V \models \psi$	$V \dashv \varphi \wedge \psi \Leftrightarrow V \dashv \varphi$ or $V \dashv \psi$

Clearly, there are other interpretations of negation and conjunction which are feasible as well. The choices which have been made in Table 1 are called *strong* or *exclusive* negation for $\neg$ and *strong Kleene* conjunction for $\wedge$. The *weak Kleene* conjunction Δ gives the same results whenever both conjuncts have a determined truth-value, and is undefined whenever one of the conjuncts is undefined. This entails the same truth conditions, but strengthens the falsity of conjunctions. This weak Kleene conjunction can be defined in terms of $\mathcal{L}$:

$$\varphi \vartriangle \psi := \neg(\neg(\varphi \wedge \psi) \wedge \neg(\varphi \wedge \neg\varphi) \wedge \neg(\psi \wedge \neg\psi)).$$

The language $\mathcal{L}$ has no complete expressive power over partial valuations. This means that there are other truth-value functional connectives which cannot be expressed in terms of $\mathcal{L}$ in the way the weak Kleene conjunction above has been defined. A simple example is weak negation $\sim$, which expresses that its argument is not true. Even when this connective is added to the language some expressive power is still lacking. Complete expressivity is reached when the 0-ary connective $\circledast$ has been added as well, which is the proposition which is always undefined. The following table adds the truth-values for these additional connectives.

Table 2

$V \models {\sim}\varphi \Leftrightarrow V \not\models \varphi$	$V \dashv {\sim}\varphi \Leftrightarrow V \models \varphi$
$V \not\models \circledast$	$V \not\dashv \circledast$

A proof of this full expressivity of $\mathcal{L}^{\circledast,\sim}$ can be found in Langholm [15]. In van Benthem [3] the reader finds a functional completeness proof for $\mathcal{L}^{\sim}$ with respect to the class of *closed* and *persistence preserving* connectives. Closedness refers to truth-value determination for the connected proposition whenever its connected parts have all determined truth-values. Persistence preservation of a connective means that persistence of its parts is preserved. A functional completeness proof for $\mathcal{L}^{\circledast}$ with respect to persistence preservation is due to Blamey [5]. In Thijsse [24] the reader finds an extensive survey on definability in partial logic with additional results for other languages.

The connectives in Table 2 have been distinguished from those in Table 1 on purpose. Their separation embodies the difference between partial and three-valued logics. In our view, three-valued logics are logics with three, equally qualified truth-values, while partial logic treats undefinedness as pure non-truth-valuedness. This distinction of determinate truth-values and undefinedness entails two crucial constraints for 'real' partial logics. First, whenever all the parts of some proposition have obtained a truth-value, then the proposition ought to get a truth-value as well, and second, if a proposition contains undefined parts then it may only get a truth-value whenever at least one part has a truth-value. Adherence to these dogmas of partiality leads to abandonment of connectives like $\sim$, by the latter constraint, and $\circledast$, by the former requirement. Technically, these two claims boil down to closed persistence preservation. By van Benthem's functional completeness result for $\mathcal{L}$ (see [3]), the partiality constraints precisely give us our linguistic means for partial propositional logic. We will not commit ourselves strictly to these principles of partiality, but instead, keep 'nonpartial' connectives separated.

Definition 3.3 The static $\mathfrak{P}$-denotation $[\![\varphi]\!]_{\mathfrak{P}}$ of a proposition $\varphi \in \mathcal{L}$ is given by the set of partial valuations which support φ, i.e., $\{V \in \mathfrak{P} \mid V \models \varphi\}$. We say that a set of formulas $\Delta \subseteq \mathcal{L}$ is a $\mathfrak{P}$-valid consequence of $\Gamma \subseteq \mathcal{L}$ whenever all $V \in \mathfrak{P}$ which verify all members of Γ verify at least one of the formulas in Δ.[1] We write:

$$\Gamma \models_{\mathfrak{P}} \Delta \stackrel{\text{def}}{\Longleftrightarrow} \left[\bigcap_{\varphi \in \Gamma} [\![\varphi]\!]_{\mathfrak{P}}\right] \subseteq \left[\bigcup_{\psi \in \Delta} [\![\psi]\!]_{\mathfrak{P}}\right].$$

When an argument in the consequence relation is left blank, then this argument is taken to be the empty set.

Below we will use analogous definitions for other classes of models and languages. A simple replacement of $\mathfrak{P}$ and $\mathcal{L}$ is enough to get the right definitions on the right place.

Observation 3.4 Significant classical validities which are $\mathfrak{P}$-invalid are *contraposition* and the *principle of the excluded middle*:

$$\Gamma \models_{\mathfrak{P}} \Delta \not\Rightarrow \neg\Delta \models_{\mathfrak{P}} \neg\Gamma \qquad \neg\Gamma \models_{\mathfrak{P}} \neg\Delta \not\Rightarrow \Gamma \models_{\mathfrak{P}} \Delta$$
$$\not\models_{\mathfrak{P}} \neg\varphi, \varphi.$$

The contraposition of the excluded middle, the *ex falso* principle, is a $\mathfrak{P}$-validity: $\neg\varphi, \varphi \models_{\mathfrak{P}}$, which also immediately provides a counterexample for contraposition. The structural reason behind this phenomenon is the following nonduality: $[\![\varphi]\!]_{\mathfrak{P}} \cap [\![\neg\varphi]\!]_{\mathfrak{P}} = \varnothing$, while $[\![\varphi]\!]_{\mathfrak{P}} \cup [\![\neg\varphi]\!]_{\mathfrak{P}} \neq \mathfrak{P}$ in general. Many other classical principles are inherited by partial logic, e.g., de Morgan principles, double negation, and the distribution principle for conjunction and disjunction.

3.3 Sequential axiomatizations of partial logics In this subsection we give a short presentation of a Gentzen-style sequential axiomatization of $\mathfrak{P}$-validity. There are two main reasons to choose this style of deduction. First of all, sequential systems turn out to be very practical when it comes to metatheory of partial logics, and secondly, they show the logical difference with classical systems very clearly.

Definition 3.5 In general, we define our sequential format as follows:

$$\frac{\Gamma_1 \vdash \Delta_1 \ \ldots \ \Gamma_n \vdash \Delta_n}{\Gamma_{n+1} \vdash \Delta_{n+1}}. \tag{1}$$

Γ_i and Δ_i are sets of formulas for all $i \in \{1, \ldots, n+1\}$. The symbol $\vdash$ denotes the derivation relation between these sets of formulas. $\Gamma \vdash \Delta$ is called a *sequent*, Γ is the *assumption set* of this sequent and Δ its *conclusion set*. The fraction notation in (1) must be interpreted as a conditional. The sequents $\Gamma_i \vdash \Delta_i$ with $i \leq n$ are the conditions of the rule in (1), and $\Gamma_{n+1} \vdash \Delta_{n+1}$ is the consequence of this rule. If $n = 0$ then the set of conditions is empty. In this case the rule is said to be *axiomatic*. Because the arguments of the derivation relation are sets, the notations Γ, φ and Γ, Γ' refer to $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \Gamma'$, respectively. Again, empty arguments of sequents refer to the empty set.

A *sequential system* S is a set of such sequential rules. If $\mathcal{L}_S$ is the underlying language, and $\Gamma, \Delta \subseteq \mathcal{L}_S$, then we say that $\Gamma \vdash_S \Delta$ is an S-*sequent*, or Δ is S-*derivable* from Γ, whenever $\Gamma \vdash \Delta$ can be established after a finite number of applications of the rules in S. We write $\Gamma \equiv_S \Delta$ if $\Gamma \vdash_S \Delta$ and $\Delta \vdash_S \Gamma$.

The arguments of sequents have been chosen to be sets on purpose. It reduces the amount of structural rules. The following table presents the structural rules which are left.

Table 3 STRUCTURAL RULES

$\Gamma \vdash \Delta$ if $\Gamma \cap \Delta \neq \varnothing$	START
$\dfrac{\Gamma \vdash \Delta \quad \Gamma \subseteq \Gamma'}{\Gamma' \vdash \Delta}$	L-MON
$\dfrac{\Gamma \vdash \varphi, \Delta \quad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$	CUT
$\dfrac{\Gamma \vdash \Delta \quad \Delta \subseteq \Delta'}{\Gamma \vdash \Delta'}$	R-MON

The left- and right-hand introduction of connectives are defined in two manners. It may be introduced straight away, the TRUE-introductions, and under the scope of a single negation, the FALSE-rules. This entails four possible introduction rules for every connective. The table below presents the TRUE- and FALSE-rules separately.[2]

Table 4 TRUE

$\Gamma, \bot \vdash \Delta$	L-TRUE $\bot$
$\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}$	L-TRUE $\neg$
$\dfrac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}$	L-TRUE $\wedge$
$\dfrac{\Gamma \vdash \varphi, \Delta \quad \Gamma' \vdash \psi, \Delta'}{\Gamma, \Gamma' \vdash \varphi \wedge \psi, \Delta, \Delta'}$	R-TRUE $\wedge$

FALSE

$\dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma, \neg\neg\varphi \vdash \Delta}$	L-FALSE $\neg$
$\dfrac{\Gamma, \neg\varphi \vdash \Delta \quad \Gamma', \neg\psi \vdash \Delta'}{\Gamma, \Gamma', \neg(\varphi \wedge \psi) \vdash \Delta, \Delta'}$	L-FALSE $\wedge$
$\Gamma \vdash \neg\bot, \Delta$	R-FALSE $\bot$
$\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \neg\neg\varphi, \Delta}$	R-FALSE $\neg$
$\dfrac{\Gamma \vdash \neg\varphi, \neg\psi, \Delta}{\Gamma \vdash \neg(\varphi \wedge \psi), \Delta}$	R-FALSE $\wedge$

The set of rules in Tables 3 and 4 is the system **P**. The only difference with classical propositional logic is the absence of:

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta} \text{ R-TRUE}\neg.$$

This rule, in combination with L-TRUE ¬, establishes contraposition for classical propositional logic. This also means that all FALSE-rules are superfluous in classical logic. They are merely meant as local repairs of the absence of contraposition in partial logics.

Observation 3.6 If $\Gamma \vdash_P \Delta$ then there exists finite $\Gamma', \Delta' \subseteq \mathcal{L}$ such that $\Gamma' \vdash_P \Delta'$. This can be proved easily by an induction on the length of **P**-derivations and the finite nature of **P**-derivability. All considered systems in this paper share this finiteness property. We will make use of it without explicit reference.

The following table presents rules for axiomatization of $\mathfrak{P}$-validity over the corresponding $\mathcal{L}$-extensions.

Table 5 RULES FOR ⊛ AND ∼

$\Gamma, \circledast \vdash \Delta$	L-TRUE ⊛
$\Gamma, \neg\circledast \vdash \Delta$	L-FALSE ⊛
$\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \sim\varphi \vdash \Delta}$	L-TRUE ∼
$\dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma, \neg\sim\varphi \vdash \Delta}$	L-FALSE ∼
$\dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \sim\varphi, \Delta}$	R-TRUE ∼
$\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \neg\sim\varphi, \Delta}$	R-FALSE ∼

The systems which contain the ⊛-rules and/or the ∼-rules for the languages $\mathcal{L}^{\circledast}$, $\mathcal{L}^{\sim}$ and $\mathcal{L}^{\circledast,\sim}$ are called $\mathbf{P}^{\circledast}$, $\mathbf{P}^{\sim}$ and $\mathbf{P}^{\circledast,\sim}$, respectively. The same policy will be maintained for the system **ud** in the next section.

Theorem 3.7 *The system* **P** *is sound and complete for* $\mathfrak{P}$*-validity over the language* $\mathcal{L}$*. For all* $\Gamma, \Delta \subseteq \mathcal{L}$*:* $\Gamma \vdash_P \Delta \iff \Gamma \models_{\mathfrak{P}} \Delta$*. The same results hold for the extended static derivation systems with weak negation and/or* ⊛.

Soundness results are omitted here. They can all be proved by a straightforward induction on the length of derivations. The completeness results are postponed to Section 5 where appropriate metatheoretical equipment will be introduced.

4 Dynamic extensions of partial logic The extension relation over partial valuations has been given in Definition 3.1. If $V \sqsubseteq V'$ then V' assigns the same truth-values as V does to all the atoms which appear in the domain of V, but it may have a larger domain than V. Interpreting partial valuations as information states, the extension relation says that V' contains at least as much 'hard' or factual information as V.

4.1 Information models In this section we will develop dynamic modal logics over the extension relation $\sqsubseteq$. For this purpose we extend the basic language(s) of the previous section with up- and down-operators: $[\varphi]_u$, $\langle\varphi\rangle_u$, $[\varphi]_d$, $\langle\varphi\rangle_d$. If $\mathcal{L}'$ is some language for partial logic which is closed under the connectives that it employs, then $\mathcal{L}'_{ud}$ will be used to denote the indicated dynamic extension, i.e., the smallest superset of $\mathcal{L}'$ which is closed under the $\mathcal{L}'$-connectives and the above-mentioned dynamic operators.

The interpretation of the up- and down-operators is analogous to the standard necessity and possibility operators in ordinary modal logic over the relations $[\![\varphi]\!]_{dy}$ and $[\![\varphi]\!]^{-}_{dy}$ which we have briefly introduced in the preamble of this paper. Possible world models which establish a complete interpretation of this modal framework are so-called information models.

Definition 4.1 An *information* model is a triple $M = \langle W, \leq, V\rangle$, such that W is a nonempty set of worlds, or *information states*, $\leq$ is a preorder over W, which is called the *information relation* of M, and V is a *monotonic global valuation function*, i.e., $V : W \longrightarrow \mathfrak{P}$ is such that for all $w, v \in W$ if $w \leq v$ then also $V(w) \sqsubseteq V(v)$. The class of all information models is denoted by $\mathfrak{N}$.

The up-down extension $\mathcal{L}_{ud}$ of $\mathcal{L}$ obtains an obvious truth-conditional semantics by combining the static semantics of $\mathcal{L}$ with an interpretation of the up- and down-operators over the information relation.

Table 6 Let $M = \langle W, \leq, V\rangle \in \mathfrak{N}$ and $w \in W$:

$M, w \models p \Leftrightarrow V(w)(p) = 1$	$M, w =\!\mid p \Leftrightarrow V(w)(p) = 0$

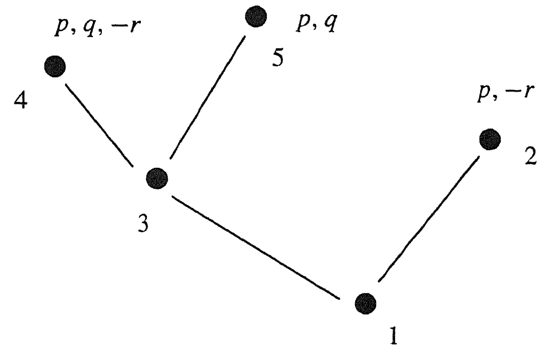
The $\mathcal{L}$-connectives obtain truth-values according to the decomposition as in Table 1. The additional connectives for the static extensions in the preceding section follow the same decomposition as in Table 2.

$M, w \models [\varphi]_u\psi \Leftrightarrow \forall v \geq w$	:	$M, v \models \varphi \Rightarrow M, v \models \psi$
$M, w =\!\mid [\varphi]_u\psi \Leftrightarrow \exists v \geq w$	:	$M, v \models \varphi \;\&\; M, v =\!\mid \psi$

$M, w \models [\varphi]_d\psi \Leftrightarrow \forall v \leq w$	:	$M, v \not\models \varphi \Rightarrow M, v \models \psi$
$M, w =\!\mid [\varphi]_d\psi \Leftrightarrow \exists v \leq w$	:	$M, v \not\models \varphi \;\&\; M, v =\!\mid \psi$

Here is a simple information model M. The proposition letters are the atoms which are locally verified. The minus symbol refers to local falsification.

Figure 1



Definition 4.2 The following sets stipulate different *interpretation sets* for a given proposition φ.

$$[\![\varphi]\!]_{\mathfrak{N}} = \{\langle M, w\rangle \mid M, w \models \varphi\}$$
$$[\![\varphi]\!]^{M}_{\mathfrak{N}} = \{w \text{ in } M \mid M, w \models \varphi\}$$
$$[\![\varphi]\!]^{\langle M,w\rangle}_{\mathfrak{N}} = \{u \text{ in } M \mid w \leq u \;\&\; M, u \models \varphi\}$$
$$[\![\varphi]\!]^{\langle M,w\rangle,-}_{\mathfrak{N}} = \{u \text{ in } M \mid u \leq w \;\&\; M, u \not\models \varphi\}$$

The first set represents the *global static meaning* of φ, while the second represents the *local*—with respect to $M \in \mathfrak{N}$—*static meaning* of φ. The two last sets denote context sensitive interpretations of φ. The first of them is the *contextual*—with respect to the information state w in M—*meaning* of φ, that is, the extensions of w which verify φ. The second set is the *negative contextual meaning* of φ with respect to w in M. These contextual interpretations entail the local dynamic relational interpretations by abstracting over the contextual information states:

$$[\![\varphi]\!]^{M,(-)}_{\mathfrak{N},dy} = \{\langle w, u\rangle \mid u \in [\![\varphi]\!]^{\langle M,w\rangle,(-)}_{\mathfrak{N}}\}.$$

We define $\langle\varphi\rangle_u$ and $\langle\varphi\rangle_d$ by means of the strong negation: $\neg[\varphi]_u\neg$ and $\neg[\varphi]_d\neg$, respectively. This yields an ordinary polymodal $\Box\Diamond$-format over the local dynamic relations above.

Every state of information has its factual static information specified by means of a local partial valuation, and the information relation specifies a structural extension relation between the states. This information relation is a subrelation of the extension relation over the local partial valuations, and *not* identical to it. Information states also contain information in the way they *can* be extended. Additional dynamic information constrains the set of possible local partial valuations as extensions. The example model in Figure 1 illustrates clearly the context sensitivity of dynamic interpretation. For example, $M, 3 \models [p]_u q$ while $M, 1 \not\models [p]_u q$, still, their local valuations are the same (empty). Speaking in dynamic terms, p has the same meaning as q in 3. This is certainly not the case in context 1.

An important aspect of formulas is their preservation behavior with respect to the information order. Formulas that are *persistent* are the ones which are maintained in upward direction of the information relation. *Antipersistent* information is information which will never be lost when going downwards. Examples of persistent formulas are provided by the complete static language $\mathcal{L}$, and formulas of the form $[\varphi]_u \psi$ and $\langle\varphi\rangle_d \psi$. Examples of antipersistent formulas are formulas of the form $[\varphi]_d \psi$ and $\langle\varphi\rangle_u \psi$.

Definition 4.3 A formula φ is *persistent* if for all $M \in \mathfrak{N}$ with information relation $\leq$ and w, v in M: $M, w \models \varphi \;\&\; w \leq v \implies M, v \models \varphi$. A formula φ is *antipersistent* if for all $M \in \mathfrak{N}$ with information relation $\leq$ and w, v in M: $M, w \models \varphi \;\&\; v \leq w \implies M, v \models \varphi$.

4.2 Application of information models Information models have been employed in different fields of pure and applied logic. With respect to the former category these models closely resemble the kind of Kripke structures which are used as models for Heyting's intuitionistic logic, see Kripke [14] and Fitting [7]. They differ from the information models of the previous subsection only in the global valuation function. In this case the valuation function is taken to be a map from the states to subsets of atoms which is monotonic over the information order. Falsity does not have an intuitionistic status. Nelson [17] extended intuitionistic logic with a constructive notion of falsity. Information models provide a precise semantics for this logic of *constructible falsity*, see Gurevich [9]. In fact, this logic is a subsystem of the up and down formalism of the previous section. The language consists of $\mathcal{L}$ with an additional implication $\rightarrow$. The truth of $\varphi \rightarrow \psi$ coincides with $[\varphi]_u \varphi$ as in intuitionistic logic, while its falsity has an extensional denotation: $\varphi \wedge \neg\psi$.

In the field of nonmonotonic logic information models have been used by Turner [28]. Turner defines an ordinary $\Box\Diamond$ modal logic over the information relation on the basis of an extension of $\mathcal{L}$ with these standard modal operators. $\Box\varphi$ is the same as $[\top]_u \varphi$ and $\Diamond\varphi$ is dually defined: $\neg\Box\neg\varphi$.

A slight variation of information models has been employed by Veltman [29] as so-called data semantics for model theoretic analysis of natural language conditionals. The models which are used there are the same as the information models above with an additional *refinability constraint*. This constraint says that every information state can be extended with the truth of a proposition φ or its falsity. For a model $M = \langle W, \leq, V\rangle$:

$$\forall s \in W\, \forall\varphi\, \exists t \in W : s \leq t \text{ and } (M, t \models \varphi \text{ or } M, t \dashv \varphi).$$

Veltman's conditionals $\varphi \rightsquigarrow \psi$ obtain the same meaning of $[\varphi]_u \psi$ both for truth and falsity.

4.3 Axiomatizations for partial up and down logics The following Tables 7 and 8 present a sequential axiomatization of the partial up and down logics which have been defined in the previous subsection. The system, which is obtained by putting **P** and the rules of the two next tables together, is called **ud**. To begin with we need to register many so-called *persistence rules* and some variations.

Table 7 PERSISTENCE RULES

$\dfrac{\Gamma \vdash p, \Delta \quad p \in I\!P}{\Gamma \vdash [\varphi]_u\, p, \Delta}$	PERS $I\!P$
$\dfrac{\Gamma \vdash \neg p, \Delta \quad p \in I\!P}{\Gamma \vdash [\varphi]_u \neg p, \Delta}$	PERS $\neg I\!P$
$\dfrac{\Gamma \vdash [\psi]_u\, \chi, \Delta}{\Gamma \vdash [\varphi]_u [\psi]_u\, \chi, \Delta}$	PERS UP
$\dfrac{\Gamma \vdash \langle\psi\rangle_d\, \chi, \Delta}{\Gamma \vdash [\varphi]_u \langle\psi\rangle_d\, \chi, \Delta}$	PERS DOWN
$\dfrac{\Gamma, \langle\psi\rangle_u\, \chi \vdash \Delta}{\Gamma, \langle\varphi\rangle_u \langle\psi\rangle_u\, \chi \vdash \Delta}$	C-PERS UP
$\dfrac{\Gamma, [\psi]_d\, \chi \vdash \Delta}{\Gamma, \langle\varphi\rangle_u [\psi]_d\, \chi \vdash \Delta}$	C-PERS DOWN
$\dfrac{\Gamma \vdash \langle\psi\rangle_u\, \chi, \Delta}{\Gamma \vdash [\varphi]_d \langle\psi\rangle_u\, \chi, \Delta}$	A-PERS UP
$\dfrac{\Gamma \vdash [\psi]_d\, \chi, \Delta}{\Gamma \vdash [\varphi]_d [\psi]_d\, \chi, \Delta}$	A-PERS DOWN
$\dfrac{\Gamma, [\psi]_u\, \chi \vdash \Delta}{\Gamma, \langle\varphi\rangle_d [\psi]_u\, \chi \vdash \Delta}$	C-A-PERS UP
$\dfrac{\Gamma, \langle\psi\rangle_d\, \chi \vdash \Delta}{\Gamma, \langle\varphi\rangle_d \langle\psi\rangle_d\, \chi \vdash \Delta}$	C-A-PERS DOWN

The first two rules record the persistence of literals. This means that literals are preserved when we extend information states. This captures the monotonicity of the global valuation functions over information models. The second pair of rules takes care of persistence for formulas of the form $[\varphi]_u\, \psi$ and $\langle\varphi\rangle_d\, \psi$. The third pair of rules are contrapositional formulations of these persistence rules. They need to be installed, because **ud** lacks contraposition just like **P**. The two last pairs arrange the antipersistence for formulas of the form $\langle\varphi\rangle_u\, \psi$ and $[\varphi]_d\, \psi$ in the same manner.

The following table presents the introduction rules for the dynamic modal operators:

Table 8 UP AND DOWN RULES

$\dfrac{\Gamma \vdash \varphi, \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', [\varphi]_u\, \psi \vdash \Delta, \Delta'}$	L-TRUE UP
$\dfrac{\Gamma, \varphi, \neg\psi \vdash \neg\Delta}{[\varphi]_u \Gamma, \neg[\varphi]_u\, \psi \vdash \neg[\varphi]_u \Delta}$	L-FALSE UP

$$\frac{\Gamma, \varphi \vdash \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', [\varphi]_d \psi \vdash \Delta, \Delta'} \quad \text{L-TRUE DOWN}$$

$$\frac{\Gamma, \neg\psi \vdash \varphi, \neg\Delta}{[\varphi]_d \Gamma, \neg[\varphi]_d \psi \vdash \neg[\varphi]_d \Delta} \quad \text{L-FALSE DOWN}$$

$$\frac{\Gamma, \varphi \vdash \psi, \neg\Delta}{[\varphi]_u \Gamma \vdash [\varphi]_u \psi, \neg[\varphi]_u \Delta} \quad \text{R-TRUE UP}$$

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma' \vdash \neg\psi, \Delta'}{\Gamma, \Gamma' \vdash \neg[\varphi]_u \psi, \Delta, \Delta'} \quad \text{R-FALSE UP}$$

$$\frac{\Gamma \vdash \varphi, \psi, \neg\Delta}{[\varphi]_d \Gamma \vdash [\varphi]_d \psi, \neg[\varphi]_d \Delta} \quad \text{R-TRUE DOWN}$$

$$\frac{\Gamma, \varphi \vdash \Delta \quad \Gamma' \vdash \neg\psi, \Delta'}{\Gamma, \Gamma' \vdash \neg[\varphi]_d \psi, \Delta, \Delta'} \quad \text{R-FALSE DOWN}$$

These rules look pretty entangled, but removing the Γ and Δs make them look far more familiar. If we take $\Gamma = \Delta = \varnothing$ in the TRUE UP-rules, modus ponens and a weak version of the deduction rule (implication introduction) appear. Removing the Γ and Δs from the other rules give different permutational completions of these well-known rules:

Example 1

MODI PONENTES (M.P.)

$[\varphi]_u \psi, \varphi \vdash_{ud} \psi$	$[\varphi]_d \psi \vdash_{ud} \varphi, \psi$
$\varphi, \psi \vdash_{ud} \langle\varphi\rangle_u \psi$	$\psi \vdash_{ud} \langle\varphi\rangle_d \psi, \varphi$

DEDUCTION RULES

$\varphi \vdash_{ud} \psi \Rightarrow \vdash_{ud} [\varphi]_u \psi$	$\vdash_{ud} \varphi, \psi \Rightarrow \vdash_{ud} [\varphi]_d \psi$
$\varphi, \psi \vdash_{ud} \Rightarrow \langle\varphi\rangle_u \psi \vdash_{ud}$	$\psi \vdash_{ud} \varphi \Rightarrow \langle\varphi\rangle_d \psi \vdash_{ud}$

The deduction rules are only valid with an empty assumption set. In general we do not have $\Gamma, \varphi \vdash_{ud} \psi \Rightarrow \Gamma \vdash_{ud} [\varphi]_u \psi$. This only holds when all members of Γ are all *persistent* in a deductive way, i.e., in terms of **ud**. If Δ is also **ud**-*antipersistent*, we even have: $\Gamma, \varphi \vdash_{ud} \psi, \Delta \Rightarrow \Gamma \vdash_{ud} [\varphi]_u \psi, \Delta$.

Definition 4.4 Let $\Gamma \subseteq \mathcal{L}_{ud}$. The **ud**-persistent part $\mathbf{p}_{ud}\Gamma$ of Γ is the set $\{\varphi \in \Gamma \mid \varphi \vdash_{ud} [\top]_u \varphi\}$; the **ud**-antipersistent part $\mathbf{ap}_{ud}$ of Γ is $\{\varphi \in \Gamma \mid \varphi \vdash_{ud} [\bot]_d \varphi\}$. In other words, for **ud**-persistent formulas we can derive by means of the **ud**-rules that they are preserved in upward direction. For **ud**-antipersistent we can derive that they are preserved downwards.

Example 2 STRENGTHENED DEDUCTION RULES

For all $\Gamma \subseteq \mathbf{p}_{ud}\mathcal{L}_{ud}$, $\Delta \subseteq \mathbf{ap}_{ud}\mathcal{L}_{ud}$:	
$\Gamma, \varphi \vdash_{ud} \psi, \Delta \Rightarrow \Gamma \vdash_{ud} [\varphi]_u \psi, \Delta$	$\Gamma, \varphi, \psi \vdash_{ud} \Delta \Rightarrow \Gamma, \langle\varphi\rangle_u \psi \vdash_{ud} \Delta$
$\Delta \vdash_{ud} \varphi, \psi, \Gamma \Rightarrow \Delta \vdash_{ud} [\varphi]_d \psi, \Gamma$	$\Delta, \psi \vdash_{ud} \varphi, \Gamma \Rightarrow \Delta, \langle\varphi\rangle_d \psi \vdash_{ud} \Gamma$

Note that Δ and Γ have mutually exchanged their sequential position in the last two rules. For getting a complete deduction rule for the down-operators an antipersistent assumption set and a persistent conclusion set are required. Some other important classes of **ud**-sequents are given in the following example.

Example 3

SIMPLIFICATION OF $\langle\varphi\rangle_u$ AND $[\varphi]_d$

$$\langle\varphi\rangle_u \psi \equiv_{ud} \langle\psi\rangle_u \varphi \equiv_{ud} \langle\top\rangle_u (\varphi \wedge \psi)$$
$$[\varphi]_d \psi \equiv_{ud} [\psi]_d \varphi \equiv_{ud} [\bot]_d (\varphi \vee \psi)$$

DUALITY PRINCIPLES

$\langle\top\rangle_u [\bot]_d \varphi \vdash_{ud} \varphi$	$\varphi \vdash_{ud} [\top]_u \langle\bot\rangle_d \varphi$
$\langle\bot\rangle_d [\top]_u \varphi \vdash_{ud} \varphi$	$\varphi \vdash_{ud} [\bot]_d \langle\top\rangle_u \varphi$

MODALITY REDUCTIONS (M.R.)

$$[\top]_u [\varphi]_u \psi \equiv_{ud} \langle\bot\rangle_d [\varphi]_u \psi \equiv_{ud} [\varphi]_u \psi$$
$$[\bot]_d \langle\varphi\rangle_u \psi \equiv_{ud} \langle\top\rangle_u \langle\varphi\rangle_u \psi \equiv_{ud} \langle\varphi\rangle_u \psi$$
$$[\bot]_d [\varphi]_d \psi \equiv_{ud} \langle\top\rangle_u [\varphi]_d \psi \equiv_{ud} [\varphi]_d \psi$$
$$[\top]_u \langle\varphi\rangle_d \psi \equiv_{ud} \langle\bot\rangle_d \langle\varphi\rangle_d \psi \equiv_{ud} \langle\varphi\rangle_d \psi$$

The duality principles illustrate the converse interpretation of the up- and down-operators, which are known from temporal logic. Briefly, the modality reductions rephrase the persistence and antipersistence.

Theorem 4.5 *The system* **ud** *is sound and complete for $\mathfrak{N}$-validity over the language $\mathcal{L}_{ud}$: for all $\Gamma, \Delta \subseteq \mathcal{L}_{ud}$: $\Gamma \vdash_{ud} \Delta \iff \Gamma \models_{\mathfrak{N}} \Delta$. These results also hold for the extended up and down systems* $\mathbf{ud}^{\circledast}$, $\mathbf{ud}^{\sim}$ *and* $\mathbf{ud}^{\circledast,\sim}$.

Proof: Soundness of the **ud**-system is omitted. The completeness is postponed to the next section.

5 Completeness and decidability In this section the completeness proof for **ud** is presented. We follow the Henkin procedure on the basis of so-called *saturated sets.* This concept is a generalization of maximally consistent sets which are used for this purpose in standard modal logic, see Hughes and Cresswell [10]. A maximally consistent set is a consistent set which cannot be extended without losing its consistency. A decidability proof of **ud** can be obtained by means of a fairly simple filtration technique.

5.1 Saturated sets

Definition 5.1 Let S be a certain sequential derivation system, and let $\mathcal{L}_S$ be its language. S is *consistent* iff $\varnothing \not\vdash_S \varnothing$. A set of formulas $\Gamma \subseteq \mathcal{L}_S$ is said to be S-*consistent*, whenever $\Gamma \not\vdash_S \varnothing$. A set of formulas $\Gamma \subseteq \mathcal{L}_S$ is said to be S-*saturated* whenever for all $\Delta \subseteq \mathcal{L}_S$:

$$\Gamma \vdash_S \Delta \Rightarrow \Delta \cap \Gamma \neq \varnothing.$$

The collection of all S-saturated sets will be denoted by $\mathfrak{Sat}_S$ in the sequel of the text. $\Lambda \subseteq \mathcal{L}_S$ is an S-*saturator* of a set $\Gamma \subseteq \mathcal{L}_S$ whenever for all $\Delta \subseteq \mathcal{L}_S$:

$$\Gamma \vdash_S \Delta \Rightarrow \Delta \cap \Lambda \neq \varnothing.$$

We will call Γ an S-*saturant* of Λ. We abbreviate this relation between Γ and Λ by $\Gamma \trianglelefteq_S \Lambda$.

The following proposition shows that if negation may be shifted according to L- and R-TRUE $\neg$ saturation and maximal consistency most often coincide.

Proposition 5.2 *For every system* S *which contains the* START, *the* L-MON *rule and the* L- *and* R-TRUE $\neg$ *all* S-*saturated sets are maximally* S-*consistent.*

Proof: Let S be a system which contains the above-mentioned rules. Both (1) $\varphi, \neg\varphi \vdash_S$, and (2) $\vdash_S \varphi, \neg\varphi$. Let $\Gamma, \Delta \in \mathfrak{Sat}_S$ with $\Gamma \subsetneq \Delta$, which says that there exists $\varphi \in \mathcal{L}_S$ such that (3) $\varphi \notin \Gamma$ and (4) $\varphi \in \Delta$. From $\Gamma \in \mathfrak{Sat}_S$, (2) and (3), we have $\neg\varphi \in \Gamma$, and so, $\neg\varphi \in \Delta$. This conclusion, in combination with (4) and (1), yields $\Delta \vdash_S \varnothing$, which contradicts $\Delta \in \mathfrak{Sat}_S$.

This proposition proves that for classical propositional logic the two notions are equal. In partial logic they are obviously different. Maximal consistency implies saturation, but not the other way around.

The notion of saturated sets has been introduced in the field of intuitionistic logic by Aczel [1] and Thomason [25].[3] In these papers saturated sets are defined by three independent properties which we obtain by substitution of 0, 1 and 2 for the cardinality of Δ in the definition of saturation above. Such definitions work perfectly when the underlying language contains a disjunction which captures the multiplicity of the right-hand arguments of the sequents.

Observation 5.3 Let S be a sequential derivation system with language $\mathcal{L}_S$ which contains a disjunction $\vee$ such that for all $\Gamma, \Delta \subseteq \mathcal{L}_S$ and $\varphi, \psi \in \mathcal{L}_S$: $\Gamma \vdash_S \varphi, \psi, \Delta \Longleftrightarrow \Gamma \vdash_S \varphi \vee \psi, \Delta$. A set of formulas is S-saturated iff

$\Gamma \nvdash \varnothing$,

$\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma$,

$\Gamma \vdash \varphi \vee \psi \Rightarrow \varphi \in \Gamma$ or $\psi \in \Gamma$.

The first two properties immediately follow from the definition of saturation. The first has been defined as consistency. Sets which obey the second property are called *theories*. The last property is often called saturation, but we have chosen this name for the sequential definition, which captures all the three properties and which also applies to longer conclusion arguments of sequents. This is very useful when we deal with a disjunction-free language.

The definition of a saturator is particularly important for proving completeness for partial intensional logics like **ud**. We will prove that for every system which contains the structural rules of **P** the relation $\Gamma \trianglelefteq_S \Lambda$ is the same as the existence of an **S**-saturated set between Γ and Λ. The relevance of this result is that saturators entail an upper bound for searching saturated sets, which is often required in proving completeness in the Henkin tradition for partial intensional logics. Usually one looks for 'states' which contain certain information but which may not be too specified. Many completeness results for partial modal logics can easily be obtained by proving saturation relations of this kind, see Jaspars [13].

Lemma 5.4 *Let* **S** *be a sequential derivation system which contains the* CUT *rule. If* $\Gamma \trianglelefteq_S \Lambda$ *and* $\Gamma \vdash_S \Delta$ *for a finite set* $\Delta \subseteq \mathcal{L}_S$*, then there exists* $\delta \in \Delta$ *such that* $\Gamma \cup \{\delta\} \trianglelefteq_S \Lambda$.

Proof: Let $\Gamma \trianglelefteq_S \Lambda$ and $\Gamma \vdash_S \Delta$ with Δ finite, and suppose that $\Gamma \cup \{\delta\} \ntrianglelefteq_S \Lambda$ for all $\delta \in \Delta$. This means that for all $\delta \in \Delta$ there exists $\Sigma_\delta \subseteq \mathcal{L}_S$ such that

$$\Gamma, \delta \vdash_S \Sigma_\delta \text{ and } \Sigma_\delta \cap \Lambda = \varnothing.$$

Let $\Sigma := \bigcup_{\delta \in \Delta} \Sigma_\delta$. R-MON yields $\Gamma, \delta \vdash_S \Sigma$ for all $\delta \in \Delta$. Applying CUT to this last **S**-sequent and the assumption $\Gamma \vdash_S \Delta$ yields $\Gamma \vdash_S \Delta - \delta, \Sigma$. Repetition of CUT-application for all δs completely eliminates Δ from the last **S**-sequent. In short, $\Gamma \vdash_S \Sigma$. Because $\Gamma \trianglelefteq_S \Lambda$ we conclude $\Sigma \cap \Lambda \neq \varnothing$. This contradicts that $\Sigma_\delta \cap \Lambda = \varnothing$ for all $\delta \in \Delta$.

This lemma shows that saturants can be extended in such a way that they remain saturants of the same saturator. In fact, a saturant can always be saturated in this way. The following lemma which formulates this result is called the *Bounded Saturation Lemma.*

Lemma 5.5 *Suppose* **S** *is a sequential derivation system containing the structural rules* START, L-MON, R-MON *and* CUT. *If* $\Lambda \subseteq \mathcal{L}_S$ *is an* **S**-*saturator of* $\Gamma \subseteq \mathcal{L}_S$*, then* Λ *contains an* **S**-*saturated set* Γ^* *such that* $\Gamma \subseteq \Gamma^*$.

Proof: Let $\Gamma \trianglelefteq_S \Lambda$ and let $\{\varphi_i\}_{i \in \mathbb{N}}$ be an enumeration of Λ. We define the following sequence of subsets of $\mathcal{L}_S$

$\Gamma_0 := \Gamma$

$$\Gamma_{n+1} := \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \trianglelefteq_S \Lambda \\ \Gamma_n & \text{otherwise.} \end{cases}$$

Furthermore we take $\Gamma^* \subseteq \mathcal{L}_S$ to be the limit of this sequence:

$$\Gamma^* := \bigcup_{n \in \mathbb{N}} \Gamma_n.$$

$\Gamma \subseteq \Gamma^* \subseteq \Lambda$ is immediately clear from the definition of Γ^* above. Another direct consequence of the construction above is $\Gamma_n \trianglelefteq_S \Lambda$ for all $n \in \mathbb{N}$. What is left to show is $\Gamma^* \in \mathfrak{Sat}_S$.

Suppose $\Gamma^* \vdash_S \Delta$. We need to prove $\Gamma^* \cap \Delta \neq \varnothing$. The assumption set can be reduced to a finite sequence $\gamma_1, \ldots, \gamma_m$ in Γ^* such that $\gamma_1, \ldots, \gamma_m \vdash_S \Delta$ (see Observation 3.6). Because every member of Γ^* is a member of some Γ_i, this means that there exists Γ_k such that $\{\gamma_1, \ldots, \gamma_m\} \subseteq \Gamma_k$. This implies $\Gamma_k \vdash_S \Delta$ by L-MON. Since $\Gamma_k \trianglelefteq_S \Lambda$, we also have $\Delta \cap \Lambda \neq \varnothing$. Because $\Delta \subseteq \mathcal{L}_S$ has been picked arbitrarily as an **S**-conclusion set of Γ^* we have $\Gamma^* \trianglelefteq_S \Lambda$. This conclusion, combined with Lemma 5.4, guarantees the existence of a formula $\delta \in \Delta$ such that

$$\Gamma^* \cup \{\delta\} \trianglelefteq_S \Lambda.$$

This result also ensures that $\Gamma_n \cup \{\delta\} \trianglelefteq_S \Lambda$ for all $n \in \mathbb{N}$. Obviously, $\delta \in \Lambda$, which means that there exists $l \in \mathbb{N}$ such that $\varphi_l = \delta$. Because $\Gamma_l \cup \{\varphi_l\} \trianglelefteq_S \Lambda$, we know that $\delta \in \Gamma_{l+1}$ by the inductive definition of the sequence $\{\Gamma_n\}_{n \in \mathbb{N}}$. We conclude $\delta \in \Gamma^*$, and so $\Gamma^* \cap \Delta \neq \varnothing$. This establishes the desired result: $\Gamma^* \in \mathfrak{Sat}_S$.

Observation 5.6 In fact this lemma is equivalent (given the **P**-structural rules) with the so-called Saturation Lemma or generalized Lindenbaum Lemma. This result says that if $\Gamma \nvdash_S \Delta$ then there exists a $\Sigma \in \mathfrak{Sat}_S$ such that $\Gamma \subseteq \Sigma$ and $\Delta \cap \Sigma = \varnothing$.[4] Note that whenever **S** contains the rule L-MON then $\Gamma \trianglelefteq_S \Lambda \iff \Gamma \nvdash_S \mathcal{L}_S \setminus \Lambda$. So, if **S** contains the structural rules of **P** and **ud**, then the Bounded Saturation Lemma is the same as the Saturation Lemma by means of this equivalence.

The equivalence of the normal Saturation Lemma with the bounded version may give the impression that Lemma 5.5 is superfluous here. Technically speaking it is, but its upper bound formulation has made completeness proofs for partial modal logics far more transparent.[5] As said earlier, due to the bounded formulation, many completeness proofs of partial modal systems come down to the establishment of one or more saturation equations.

Moreover, the proof of Lemma 5.5 is a generalization of the standard proof of Lindenbaum's Lemma, which says that every consistent set has a maximally consistent extension. This result would immediately follow when $\Lambda = \mathcal{L}_S$ is chosen in the proof of Lemma 5.5. Many proofs of the ordinary Saturation Lemma have a somewhat deviant nature (e.g., Troelstra and van Dalen [27]).

Note that the proof of Lemma 5.5 and the formulation are linguistically independent. Due to our sequential setting and the general definition of saturation, it can be used for many logics with poor expressivity, and does not rely on the presence of certain connectives like the disjunction.

6 The completeness of partial logics The completeness proofs of **P** and its extensions is fairly easy. Take $\mathfrak{Sat}_P$, and associate to every $\Sigma \in \mathfrak{Sat}_P$ a partial valuation function V_Σ which is defined by its content:

$$V_\Sigma(p) = \begin{cases} 1 & \text{iff } p \in \Sigma \\ 0 & \text{iff } \neg p \in \Sigma. \end{cases}$$

This definition together with the individual derivation rules ensure that $V_\Sigma \models \varphi$ iff $\varphi \in \Sigma$ for all $\Sigma \in \mathfrak{Sat}_P$ and $\varphi \in \mathcal{L}$ (1). This can be proved by a straightforward induction, and can be extended for the extended systems in the same fashion. If $\Gamma \not\vdash_P \Delta$ then there exists $\Theta \in \mathfrak{Sat}_P$ such that $\Gamma \subseteq \Theta$ and $\Delta \cap \Theta = \varnothing$. According to (1) above, this means that $V_\Theta \models \varphi$ and $V_\Theta \not\models \psi$ for all $\varphi \in \Gamma$ and $\psi \in \Delta$, and therefore, $\Gamma \not\models_{\mathfrak{P}} \Delta$.

***7 The completeness of* ud** The canonical model for the system **ud**, which we need to run the Henkin procedure, is given by the following definition:

Definition 7.1 The **ud**-canonical model is the triple $M_{ud} = \langle \mathfrak{Sat}_{ud}, \ll_{ud}, V_{ud} \rangle$ where for all $\Gamma, \Delta \in \mathfrak{Sat}_{ud}$ and $p \in I\!P$:

$\Gamma \ll_{ud} \Delta \iff \mathbf{p}_{ud}\Gamma \subseteq \Delta \;\&\; \mathbf{ap}_{ud}\Delta \subseteq \Gamma$, and

$$V_{ud}(\Gamma)(p) = \begin{cases} 1 & \text{iff } p \in \Gamma \\ 0 & \text{iff } \neg p \in \Gamma. \end{cases}$$

Recall that $\mathbf{p}_{ud}\Gamma = \{\varphi \in \Gamma \mid \varphi \vdash_{ud} [\top]_u \varphi\}$ and $\mathbf{ap}_{ud}\Delta = \{\varphi \in \Delta \mid \varphi \vdash_{ud} [\bot]_d \varphi\}$ (see Definition 4.4).

Observation 7.2 We leave it to the reader to show that $M_{ud} \in \mathfrak{M}$, i.e., V_{ud} is monotonic over $\ll_{ud}$ and $\ll_{ud}$ is a preorder.

We give the so-called *Truth Lemma* of **ud** first. This lemma almost establishes the desired result.

Lemma 7.3 *$M_{ud}, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma$ and $M_{ud}, \Gamma \dashv \varphi \Leftrightarrow \neg\varphi \in \Gamma$ for all $\Gamma \in \mathfrak{Sat}_{ud}$, $\varphi \in \mathcal{L}_{ud}$.*

Proof: By induction on the construction of $\mathcal{L}_{ud}$-formulas. We skip the basic step and the proofs of the static connectives. For the dynamic modal operators there are four cases which are nearly immediately obtainable from the definition of $\ll_{ud}$. These four "easy" cases are:

(*i*) $[\varphi]_u \psi \in \Gamma \Rightarrow M_{ud}, \Gamma \models [\varphi]_u \psi$,

(*ii*) $M_{ud}, \Gamma \dashv [\varphi]_u \psi \Rightarrow \neg[\varphi]_u \psi \in \Gamma$,

(*iii*) $[\varphi]_d \psi \in \Gamma \Rightarrow M_{ud}, \Gamma \models [\varphi]_d \psi$,

(*iv*) $M_{ud}, \Gamma \dashv [\varphi]_d \psi \Rightarrow \neg[\varphi]_d \psi \in \Gamma$.

We will demonstrate the first and the last step. The two others are left to the reader.

$[\varphi]_u \psi \in \Gamma \implies$ ($[\varphi]_u \psi \vdash_{ud} [\top]_u [\varphi]_u \psi$, Example 3: M.R.)
$\forall \Delta \gg_{ud} \Gamma : [\varphi]_u \psi \in \Delta \implies$ ($\varphi, [\varphi]_u \psi \vdash_{ud} \psi$, Example 1: M.P.)
$\forall \Delta \gg_{ud} \Gamma : \varphi \in \Delta \Rightarrow \psi \in \Delta \implies$ (induction hypothesis)
$\forall \Delta \gg_{ud} \Gamma : M_{ud}, \Delta \models \varphi \Rightarrow M_{ud}, \Delta \models \psi \implies M_{ud}, \Gamma \models [\varphi]_u \psi$.

$\neg[\varphi]_d \psi \notin \Gamma \implies$ ($\neg[\varphi]_d \psi \vdash_{ud} [\top]_u \neg[\varphi]_d \psi$, Example 3: M.R.)
$\forall \Delta \ll_{ud} \Gamma : \neg[\varphi]_d \psi \notin \Delta \implies$ ($\varphi \vdash_{ud} \neg[\varphi]_d \psi, \neg\psi$, Example 1: M.P.)
$\forall \Delta \ll_{ud} \Gamma : \varphi \notin \Delta \Rightarrow \neg\psi \notin \Delta \implies$ (induction hypothesis)
$\forall \Delta \ll_{ud} \Gamma : M_{ud} \not\models \varphi \Rightarrow M_{ud} \not\Dashv \psi \implies M_{ud}, \Gamma \not\Dashv [\varphi]_d \psi$.

The completing converse results of these four "easy" cases are consequences of the following sequential statements, in combination with the Bounded Saturation Lemma (Lemma 5.5). In these saturation equations $\Gamma \cup \{\varphi\}$ and $\Gamma \setminus \{\varphi\}$ are abbreviated by $\Gamma + \varphi$ and $\Gamma - \varphi$ respectively. Furthermore, the non-**ud**-persistent part, $\mathcal{L}_{ud} \setminus \mathbf{p}_{ud}\mathcal{L}_{ud}$ and the non-**ud**-antipersistent part, $\mathcal{L}_{ud} \setminus \mathbf{ap}_{ud}\mathcal{L}_{ud}$ of $\mathcal{L}_{ud}$ are abbreviated by NP and NAP, respectively.

(v)	$[\varphi]_u \psi \notin \Gamma$	$\Rightarrow$	$\mathbf{p}_{ud}\Gamma + \varphi \trianglelefteq_{ud} \Gamma \cup \text{NAP} - \psi$
(vi)	$\neg[\varphi]_u \psi \in \Gamma$	$\Rightarrow$	$\mathbf{p}_{ud}\Gamma + \varphi + \neg\psi \trianglelefteq_{ud} \Gamma \cup \text{NAP}$
(vii)	$[\varphi]_d \psi \notin \Gamma$	$\Rightarrow$	$\mathbf{ap}_{ud}\Gamma \trianglelefteq_{ud} \Gamma \cup \text{NP} - \varphi - \psi$
($viii$)	$\neg[\varphi]_d \psi \in \Gamma$	$\Rightarrow$	$\mathbf{ap}_{ud}\Gamma + \neg\psi \trianglelefteq_{ud} \Gamma \cup \text{NP} - \varphi$

These saturation relations may seem complicated statements. The following simple derivations explain why they lead to immediate success. For the sake of brevity we only prove that the claims (v) and ($viii$) give us the desired results: (v) $\Rightarrow M_{ud}, \Gamma \not\models [\varphi]_u \psi$ and ($viii$) $\Rightarrow M_{ud}, \Gamma \Dashv [\varphi]_d \psi$.

$$\begin{aligned} (v) &\implies \exists \Delta \in \mathfrak{Sat}_{ud} : \mathbf{p}_{ud}\Gamma \subseteq \Delta \subseteq \Gamma \cup \text{NAP} \;\&\; \varphi \in \Delta \;\&\; \psi \notin \Delta \\ &\implies \Gamma \ll_{ud} \Delta \;\&\; M_{ud}, \Delta \models \varphi \;\&\; M_{ud}, \Delta \not\models \psi \\ &\implies M_{ud}, \Gamma \not\models [\varphi]_u \psi. \end{aligned}$$

The first step consists of the application of the Bounded Saturation Lemma to (v). $\Gamma \ll_{ud} \Delta$ follows from the consequence and the simple observation that $\mathbf{ap}_{ud}(\Gamma \cup \text{NAP}) = \mathbf{ap}_{ud}\Gamma \subseteq \Gamma$, and therefore $\mathbf{ap}_{ud}\Delta \subseteq \Gamma$. The last step is due to application of the induction hypothesis.

$$\begin{aligned} (viii) &\implies \exists \Delta \in \mathfrak{Sat}_{ud} : \mathbf{ap}_{ud}\Gamma \subseteq \Delta \subseteq \Gamma \cup \text{NP} \;\&\; \varphi \notin \Delta \;\&\; \neg\psi \in \Delta \\ &\implies \Delta \ll_{ud} \Gamma \;\&\; M_{ud}, \Delta \not\models \varphi \;\&\; M_{ud}, \Delta \Dashv \psi \\ &\implies M_{ud}, \Gamma \Dashv [\varphi]_d \psi. \end{aligned}$$

The first step is an application of the Bounded Saturation Lemma again. The result implies $\Delta \ll_{ud} \Gamma$ because $\mathbf{p}_{ud}(\Gamma \cup \text{NP}) = \mathbf{p}_{ud}\Gamma \subseteq \Gamma$, and so $\mathbf{p}_{ud}\Delta \subseteq \Gamma$. Again, the last step follows from the induction hypothesis.

The proofs of (vi) $\Rightarrow M_{ud}, \Gamma \Dashv [\varphi]_u \psi$ and (vii) $\Rightarrow M_{ud}, \Gamma \not\models [\varphi]_d \psi$ are left to the reader. What is left to show is the validity of the claims (v) – ($viii$). We only prove the first and the last claim. The other two can be reproduced through mere analogy.

Claim (v): Suppose $[\varphi]_u \psi \notin \Gamma$.

Let $\Sigma \subseteq \mathcal{L}_{ud}$ such that $\mathbf{p}_{ud}\Gamma, \varphi \vdash_{ud} \Sigma$. We need to prove that:

(a) $\Sigma \cap (\Gamma \cup \text{NAP} - \psi) \neq \varnothing$.

If $\Sigma \cap (\text{NAP} - \psi) \neq \varnothing$, then we are done. So, suppose $\Sigma \cap (\text{NAP} - \psi) = \varnothing$, which is the same as $\Sigma \subseteq \mathbf{ap}_{ud}\mathcal{L}_{ud} + \psi$. In other words, *all* non-ψ-elements of Σ are **ud**-antipersistent, i.e., $\mathbf{ap}_{ud}(\Sigma - \psi) = \Sigma - \psi$. This yields the following minimal derivation:

(1) $\mathbf{p}_{ud}\Gamma, \varphi \vdash_{ud} \Sigma - \psi, \psi$ R-MON
(2) $\Gamma \vdash_{ud} \Sigma - \psi, [\varphi]_u \psi$ Example 2, $\mathbf{p}_{ud}\Gamma \subseteq \Gamma$ & L-MON.

Because $\Gamma \in \mathfrak{Sat}_{ud}$, the last **ud**-sequent above, and the assumption $[\varphi]_u \psi \notin \Gamma$ entail $(\Sigma - \psi) \cap \Gamma \neq \varnothing$, and therefore also $\Sigma \cap (\Gamma \cup \text{NAP} - \psi) \neq \varnothing$ (a).

Claim (viii): Suppose $\neg[\varphi]_d \psi \in \Gamma$.

Let $\Sigma \subseteq \mathcal{L}_{ud}$ with $\mathbf{ap}_{ud}\Gamma + \neg\psi \vdash_{ud} \Sigma$. We need to prove that:

(b) $\Sigma \cap (\Gamma \cup \text{NP} - \varphi) \neq \varnothing$.

If $\Sigma \cap (\text{NP} - \varphi) \neq \varnothing$, then we immediately have our desired result. So, let $\Sigma \subseteq \mathbf{p}_{ud}\mathcal{L}_{ud} + \varphi$. This means that $\mathbf{p}_{ud}(\Sigma - \varphi) = \Sigma - \varphi$. The following derivation settles this complementary case:

(1) $\mathbf{ap}_{ud}\Gamma, \neg\psi \vdash_{ud} \Sigma - \varphi, \varphi$ R-MON
(2) $\Gamma, \neg[\varphi]_d \psi \vdash_{ud} \Sigma - \varphi$ Example 2, $\mathbf{ap}_{ud}\Gamma \subseteq \Gamma$, L-MON & $\mathbf{p}_{ud}(\Sigma - \varphi) = \Sigma - \varphi$
(3) $\Gamma \vdash_{ud} \Sigma - \varphi$ $\neg[\varphi]_d \psi \in \Gamma$.

Because $\Gamma \in \mathfrak{Sat}_{ud}$, we conclude $\Sigma \cap (\Gamma - \varphi) \neq \varnothing$, which also establishes (b).

These derivations settle (*v*) and (*viii*).

With this result we have almost completed the completeness proof for **ud**. Suppose that $\Gamma \nvdash_{ud} \Delta$. According to the Saturation Lemma 5.6, there exists $\Sigma \in \mathfrak{Sat}_{ud}$ such that $\Gamma \subseteq \Sigma$ and $\Delta \cap \Sigma = \varnothing$. According to the Truth Lemma above, this yields $M_{ud}, \Sigma \models \varphi$ and $M_{ud}, \Sigma \not\models \psi$ for all $\varphi \in \Gamma$ and $\psi \in \Delta$. Because $M_{ud} \in \mathfrak{N}$, this shows that $\Gamma \not\models_{\mathfrak{N}} \Delta$.

Completeness for the systems $\mathbf{ud}^{\circledast}$, $\mathbf{ud}^{\sim}$ and $\mathbf{ud}^{\circledast,\sim}$ can be proved in precisely the same manner. The induction steps for the additional connectives in the corresponding Truth Lemmas are straightforward.

7.1 Decidability Decidability for finite **ud**-sequents can be established by a finite variation of the equipment of the previous sections.

Definition 7.4 Let $\Phi \subseteq \mathcal{L}_S$. An **S**-$\Phi$-*saturated* set is a set $\Gamma \subseteq \Phi$ such that for all $\Delta \subseteq \Phi$: $\Gamma \vdash_S \Delta \implies \Gamma \cap \Delta \neq \varnothing$. The collection of **S**-Φ-saturated sets is abbreviated by $\mathfrak{Sat}_S^\Phi$. Λ is called a **S**-Φ-*saturator* of $\Gamma \subseteq \Phi$ iff $\Gamma \vdash \Delta \implies \Lambda \cap \Delta \neq \varnothing$ for all $\Delta \subseteq \Phi$. This relation is abbreviated by $\Gamma \trianglelefteq_S^\Phi \Lambda$.

Lemma 7.5 *Let* $\Phi, \Lambda \subseteq \mathcal{L}_S$, *and* $\Gamma \subseteq \Phi$. *If* $\Gamma \trianglelefteq_S^\Phi \Lambda$ *then there exists* $\Gamma^* \in \mathfrak{Sat}_S^\Phi$ *such that* $\Gamma \subseteq \Gamma^* \subseteq \Lambda$.

Proof: This proof runs completely in the same fashion as that of Lemma 5.5. An appropriate reformulation of Lemma 5.4 is needed. Furthermore, the sequence φ_i in the proof of Lemma 5.5 should be taken from $\Lambda \cap \Phi$ (note that $\Gamma \trianglelefteq_S^\Phi \Lambda \Rightarrow \Gamma \trianglelefteq_S^\Phi \Lambda \cap \Phi$).

In order to prove the decidabity of **ud** we construct a finite counter model for a given finite non-**ud**-sequent: $\Pi \not\vdash_{ud} \Xi$. Let Σ be the set of subformulas of $\Pi \cup \Xi$ and their negations. Clearly, Σ is a finite set. Next consider the model $M^{\Sigma}_{ud} = \langle \mathfrak{Sat}^{\Sigma}_{ud} \ll^{\Sigma}_{ud}, V^{\Sigma}_{ud} \rangle$ with $\ll^{\Sigma}_{ud}$ and V^{Σ}_{ud} defined in the same way as $\ll_{ud}$ and V_{ud} but then restricted to $\mathfrak{Sat}^{\Sigma}_{ud}$. This construction yields a restricted version of the Truth Lemma for **ud**with respect to M^{Σ}_{ud}:

$$M, \Gamma \models \varphi \iff \varphi \in \Gamma \;\&\; M, \Gamma \dashv \varphi \iff \neg\varphi \in \Gamma.$$

for all Π- and Ξ-subformulas φ and $\Gamma \in \mathfrak{Sat}^{\Sigma}_{ud}$. This result can be proved just like Lemma 7.3. Because M^{Σ}_{ud} is finite and of fixed size, this immediately establishes the desired decidability results.

Theorem 7.6 **ud** *is decidable for finite sequents.*

This technique also applies to the systems $\mathbf{ud}^{\circledast}$, $\mathbf{ud}^{\sim}$ and $\mathbf{ud}^{\circledast,\sim}$. No further filtration techniques have to be used there.

The given filtration technique yields exponential time upper bounds for deciding $\mathfrak{N}$-validity for finite subsets of $\mathcal{L}_{ud}$. However, by making use of established complexity results and known embedding results, a much more refined result can be given. Statman [23] shows that validity for intuitionistic propositional logic is *PSPACE*-complete. This result immediately settles *PSPACE*-hardness for **ud**-validity, because intuitionistic propositional logic is a fragment of **ud**. Furthermore, by the polynomial time translation of **ud** into temporal **S4** given in [13], and the *PSPACE*-completeness result for this logic of Spaan [22], we obtain *PSPACE*-completeness for **ud**.

*8 **Conclusions and reflections*** Information models have been employed as Kripke structures to define dynamic modal logics for reasoning about extension and reduction of partial states. The bounded version of the Saturation Lemma has been particularly helpful in establishing a completeness and decidability result for the underlying calculus **ud**.

Of course, our main technical concern has been to guide the congregation of partial and dynamic modal logic. With respect to the dynamic modal logics of van Benthem and de Rijke, the relational part of our formalism is restricted. The inevitable consequence of this poverty is that minimal extensions and reductions do not appear in our formalism. Such minimal dynamic denotations can semantically be specified in the following manner:

$$\begin{aligned} [\![\varphi]\!]^{M,*}_{\mathfrak{N},dy} &= \{\langle s,t\rangle \in [\![\varphi]\!]^{M}_{\mathfrak{N},dy} \mid s \leq u \;\&\; u \in [\![\varphi]\!]^{M}_{\mathfrak{N}} \;\&\; u \leq t \Longrightarrow t \leq u\} \\ [\![\varphi]\!]^{M,-,*}_{\mathfrak{N},dy} &= \{\langle s,t\rangle \in [\![\varphi]\!]^{M,-}_{\mathfrak{N},dy} \mid u \leq s \;\&\; u \not\in [\![\varphi]\!]^{M}_{\mathfrak{N}} \;\&\; t \leq u \Longrightarrow u \leq t\}. \end{aligned}$$

A future research challenge is to develop adequate sequential calculi for an extension of the up and down calculus of this paper with additional modal operators over the relations above. Keeping the undecidability of van Benthem and de Rijke's formalism in mind, one should be aware of the possible technical dangers of such an enterprise.

Acknowledgment Much of this paper has been taken from my doctoral dissertation [13], which also describes different modal extensions of the system **ud** for reasoning about the 'up and down' dynamics for multiple agents. I would like to express particular thanks to Johan van Benthem, whose invaluable supervision of my dissertation helped me with the technical presentation of the main arguments of the completeness proofs. For the same reasons I would like to thank Elias Thijsse, the equivalence in Observation 5.6 was pointed out by him. I would also like to express my gratitude to Maarten de Rijke, Jan van Eijck and the anonymous referee for helpful comments. Bill Rounds told me about the complexity result of Statman [23]. My work was sponsored by CEC-project LRE-62-051 (FraCaS), funded by the European Community.

NOTES

1. There is some freedom here. The so-called *double barreled* consequence definition has also been used, e.g., Muskens [16]. This refers to a stricter notion of validity: "all models of Γ verify at least one of Δ *and* all models which falsify all formulas in Δ falsify at least one element of Γ." This notion of validity is propagated mainly because it structurally behaves better than our single barreled definition. The underlying reason is that it restores contraposition. In [24] the reader finds a classification of different sorts of definitions of valid consequence for partial logics.

2. In Fenstad [6] a slightly more elegant way of dealing with these four different places of introduction has been proposed. The authors introduce *quadrants* which are four-placed variants of sequents. There are two additional stacks, RIGHT and LEFT, for keeping false formulas separate. This presents a structurally elegant fashion of deduction. Because its style is somewhat unusual and the notation unpractical, we kept to an ordinary sequential style.

3. Intuitionistic logic only has a restricted version of R-TRUE $\neg$. It may be applied only with an empty conclusion set: $\Gamma, \varphi \vdash \varnothing \Longrightarrow \Gamma \vdash \neg\varphi$. This restricted version keeps saturation and maximal consistency apart as well.

4. Most often this result is formulated for singleton Δs, see Aczel [1]. The sequential variant can be found in [24].

5. Finding completeness proofs for partial modal logic with incomplete static expressivity has turned out to be pretty troublesome, see [24]. Normal form techniques also used long proofs, see Jaspars [12].

REFERENCES

[1] Aczel, P., "Saturated intuitionistic theories," pp. 1–13 in *Contributions to Mathematical Logic*, edited by H. Schmidt, K. Schütte, and H. Thiele, North-Holland, Amsterdam, 1968.

[2] Belnap, N., "A useful four-valued logic," pp. 8–37 in *Modern Uses of Multiple Valued Logic*, edited by G. Epstein and M. Dunn, Reidel, Dordrecht, 1977.

[3] van Benthem, J. F. A. K., *A Manual of Intensional Logic*, CSLI, Stanford, 1984.

[4] van Benthem, J. F. A. K., "Logic and the flow of information," pp. 693–724 in *Proceedings of the 9th International Congress of Logic, Methodology and Philosophy of Science (Uppsala, Sweden)*, edited by D. Prawitz, B. Skyrms and D. Westerstahl, 1991.

[5] Blamey, S., " Partial logic," pp. 1–70 in *Handbook of Philosophical Logic, Vol III: Alternatives to Classical Logic*, edited by D. M. Gabbay and F. Guenthner, Reidel, Dordrecht,1986.

[6] Fenstad, J. E., T. Langholm and E. Vespren, "Representations and interpretations," pp. 31–95 in *Computational Linguistics and Formal Semantics*, edited by M. Rosner and R. Johnson, Cambridge University Press, Cambridge, 1992.

[7] Fitting, M. C., *Intuitionistic Logic Model Theory and Forcing*, North-Holland, Amsterdam, 1969.

[8] Gabbay, D. M., "Intuitionistic basis for non-monotonic logic," pp. 260–273 in *Proceedings of the 6th Conference on Automated Deduction* edited by D. W. Loveland, Springer, Heidelberg, 1982.

[9] Gurevich, Y., "Intuitionistic logic with strong negation," *Studia Logica*, vol. 36 (1977), pp. 49–59.

[10] Hughes, G. E., and M. J. Cresswell, *A Companion to Modal Logic*, Methuen, New York, 1984.

[11] Jaspars, J. O. M., "Logical omniscience and inconsistent belief," pp. 129–146 in *Diamonds and Defaults*, edited by M. de Rijke, Kluwer, Dordrecht, 1993.

[12] Jaspars, J. O. M., "Normal forms in partial modal logic," pp. 37–50 in *Algebraic Methods in Logic and in Computer Science*, edited by C. Rauszer, Polish Academy of Sciences, Warszawa, 1993.

[13] Jaspars, J. O. M., *Calculi for Constructive Communication: A Study of the Dynamics of Partial States*, PhD Thesis, University of Tilburg, 1994.

[14] Kripke, S. A., "Semantical Analysis of Intuitionistic Logic I," pp. 92–130 in *Formal Systems and Recursive Functions*, edited by J. Crossley and M. Dummett, North-Holland, Amsterdam, 1965.

[15] Langholm, T., *Partiality, Truth and Persistence*, CSLI, Stanford, 1988.

[16] Muskens, R. A., *Meaning and Partiality*, PhD Thesis, University of Amsterdam, 1989.

[17] Nelson, D., "Constructible falsity," *Journal of Symbolic Logic*, vol. 14 (1949), pp. 16–26.

[18] Pearce, D., "Default logic and constructive logic," pp. 309–313 in *Proceedings of the 10th European Conference on Artificial Intelligence (ECAI92)*, edited by B. Neumann, Wiley, Chicester, 1992.

[19] Reiter, R., "A logic for default reasoning," *Artificial Intelligence*, vol. 13 (1980), pp. 81–132.

[20] de Rijke, M., "A system of dynamic modal logic," *Journal of Philosophical Logic*, forthcoming.

[21] de Rijke, M., *Extending Modal Logic*, PhD Thesis, University of Amsterdam, 1993.

[22] Spaan, E., "The complexity of propositional tense logics," pp. 287–307 in *Diamonds and Defaults*, edited by M. de Rijke, Kluwer Academic Publishers, Dordrecht, 1993.

[23] Statman, R., "Intuitionistic propositional logic is polynomial-space complete," *Theoretical Computer Science*, vol. 9 (1979), pp. 67–72.

[24] Thijsse, E. G. C., *Partial Logic and Knowledge Representation*, PhD Thesis, University of Tilburg, 1992.

[25] Thomason, R. H., "On the strong semantical completeness of the intuitionistic predicate logic," *Journal of Symbolic Logic*, vol. 33 (1968), pp. 1–7.

[26] Thomason, R. H., "A semantical study of constructible falsity," *Zeitschrift für Mathematischen Logik und Grundlagen der Mathematik*, vol. 15 (1969), pp. 247–257.

[27] Troelstra, A. S., and D. van Dalen, *Constructivism in Mathematics vol. I*, North Holland, Amsterdam, 1990.

[28] Turner, R., *Logics for Artificial Intelligence*, Ellis Horwood, Chicester, 1984.

[29] Veltman, F., *Logics for Conditionals*, PhD Thesis, University of Amsterdam, 1985.

[30] Wagner, G., *Vivid Logic*, Springer, Heidelberg, 1994.

[31] Wansing, H., *The Logic of Information Structures*, Springer, Heidelberg, 1993.

Centrum voor Wiskunde en Informatica
PO Box 94079, NL–1090 GB Amsterdam
The Netherlands
email: jaspars@cwi.nl