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Bernoulli Polynomials Old and New: Generalizations and Asymptotics

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We consider two aspects of generalized Bernoulli polynomials $B_n^\mu(z)$. One is connected with defining functions instead of polynomials by making the degree n of the polynomial a complex variable. In the second problem we are concerned with the asymptotic behaviour of $B_n^\mu(z)$ when the degree n tends to infinity.

1. Introduction

At present Bernoulli numbers are introduced through generating functions, as we shall do below, but historically they arose in connection with the sums of the $p-$th power of the first $n-1$ integers $1+2^p+\cdots(n-1)^p$. The Greeks, Hindus and Arabs all had rules amounting to

$$\begin{aligned}
\sum_{i=1}^{n-1} i &= \tfrac{1}{2}n^2 - \tfrac{1}{2}n, \\
\sum_{i=1}^{n-1} i^2 &= \tfrac{1}{3}n^3 - \tfrac{1}{2}n^2 + \tfrac{1}{6}n, \\
\sum_{i=1}^{n-1} i^3 &= \tfrac{1}{4}n^4 - \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2, \\
\sum_{i=1}^{n-1} i^4 &= \tfrac{1}{5}n^5 - \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n.
\end{aligned}$$

Nowadays we write for $p=0,1,2,\ldots,\ n=1,2,3,\ldots$ (putting $0^0=1$)

$$\sum_{i=0}^{n-1} i^p = \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} B_k n^{p+1-k},$$

where the coefficient of the linear term n equals the p–th Bernoulli number.

In this way the numbers were mentioned (without using their present names and notation) by Jakob I. Bernoulli in his posthumous *Ars conjectandi* of 1713. In fact he gave the above general formula, observing that the numbers also occur in the coefficients of the other powers of n. Euler also tackled the problem of summing powers and in 1755 he published a proof of the Bernoulli forms based on the calculus of finite differences, christening the coefficients of n the *Bernoulli numbers* in honour of Jakob.

Next we give some general definitions through generating functions. The *generalized Bernoulli polynomials* $B_n^\mu(z)$ are defined for all complex numbers z and μ by the expansion

$$\sum_{n=0}^{\infty} B_n^\mu(z) \frac{t^n}{n!} = e^{zt} \left(\frac{t}{e^t - 1}\right)^\mu, \quad |t| < 2\pi. \tag{1.1}$$

An immediate consequence of this definition is the representation of the generalized Bernoulli polynomials as a Cauchy type integral:

$$B_n^\mu(z) = \frac{n!}{2\pi i} \int_C e^{zt} \left(\frac{t}{e^t - 1}\right)^\mu \frac{dt}{t^{n+1}}, \tag{1.2}$$

where the contour C is a circle with radius less than 2π around the origin.

There are several reductions for this general definition.

- When $\mu = 1$ we have the *Bernoulli polynomials* $B_n(z)$.
- When $z = 0$ we have the *generalized Bernoulli numbers* B_n^μ.
- When $\mu = 1$ and $z = 0$, we have the classical *Bernoulli numbers* B_n.

The quantities $B_n^\mu(z)$ are polynomials of degree n in both μ and z; μ is called the order. The classical numbers B_n occur in practically every field of mathematics, in particular in combinatorial theory, finite difference calculus, numerical analysis, analytical number theory, and probability theory. For the polynomials the same remarks apply, although in several occurrences the polynomials give just a convenient method of notation instead of giving insight or possibilities to further manipulate analytical expressions.

In this paper we consider two problems on the generalized Bernoulli polynomials $B_n^\mu(z)$. One is connected with defining functions $B_\nu(z)$ where ν is a complex variable. We derive a functional equation that generalizes the well-known property $B_n(1-z) = (-1)^n B_n(z)$, and that gives information how th interpret $B_\nu(x)$ when $x < 0$. In the second problem we are concerned with the asymptotic behaviour of $B_n^\mu(z)$ when the degree n tends to infinity. We consider this problem in connection with our earlier results for Stirling numbers and discuss some other results from the literature. Finally we give new asymptotic representations.

2. BERNOULLI FUNCTIONS

We consider the problem of generalizing $B_n(z)$ by making n a complex variable. A motivation for this is given by the wish to generalize a fundamental difference relation of the Bernoulli numbers:

$$B_n(z+1) = B_n(z) + nz^{n-1}, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

to a relation that also holds when n is replaced by a complex parameter ν. A further step then is to interpret such a generalization for negative values of z. When we now how to interpret another fundamental property:

$$B_n(1-z) = (-1)^n B_n(z) \tag{2.2}$$

when n is complex and z is negative the problem can completely be solved.

A second motivation comes from the recent set of papers [4]–[7] by BUTZER *et al.* in which Bernoulli numbers and polynomials (and related quantities) are generalized. It seems that Butzer et al. have overlooked several rather old papers (for instance JONQUIÈRE [9] and BÖHMER [3]), in which generalizations of Bernoulli polynomials are considered. Part of our analysis is based on these two classical papers.

The difference relation (2.1) is the heart of *difference calculus*, the branch of mathematics that became so important in solving problems from numerical analysis, in particular in solving differential equations. Further information on classical difference calculus can be found in JORDAN [10], NÖRLUND [13], and MILNE-THOMSON [12].

One of the striking occurrences of Bernoulli numbers in special functions is the relation

$$\zeta(2m) = \tfrac{1}{2}(-1)^{m+1}(2\pi)^{2m}\frac{B_{2m}}{(2m!}, \quad m = 1, 2, 3, \ldots, \tag{2.3}$$

where $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$, the Riemann zeta function. This relation was given by Euler (1735/1739), and Ramanujan used it to define *signless* Bernoulli numbers of arbitrary index s by writing

$$B_s^* = 2\Gamma(s+1)\zeta(s)(2\pi)^{-s} \tag{2.4}$$

(see BERNDT [2, pp. 125f, 151f]), 'signless' meaning that

$$B_{2m}^* = (-1)^{m+1}B_{2m} > 0, \quad m = 1, 2, 3 \ldots .$$

In fact, as BERNDT [2, p.125) remarks, already Euler made a very first attempt to introduce (signless) Bernoulli numbers of arbitrary index as above. Apparently, he made no significant use of his idea. The relation (2.2) gave hope to many mathematicians that it would be possible to find values of $\zeta(2m+1)$, and the numbers B_s^* defined in (2.3) might be a convenient starting point for this when relevant new properties of B_s^* could be found. Until now this approach to identify $\zeta(2m+1)$ in terms of simple quantities has not been successful.

In this section we consider a different way of generalizing, by taking another explicit representation. When we generalize this by taking n complex we write (1.2) in the form

$$B_\nu^\mu(z) = \frac{\Gamma(\nu+1)}{2\pi i}\int_C e^{zt}\left(\frac{t}{e^t-1}\right)^\mu \frac{dt}{t^{\nu+1}}, \tag{2.5}$$

where $\Re z > 0$. Because of the algebraic singularity of $t^{\nu+1}$ at the origin we assume now that the contour of integration C runs from $-\infty, \arg t = -\pi$, encircles the origin in positive direction (that is, anti-clockwise) terminates at $-\infty$, now with $\arg t = +\pi$. We assume that all zeros of $e^t - 1$ (except $t = 0$) are not enclosed by the contour, and, initially, that the many-valued function t^ν is real for real values of ν and $t > 0$.

2.1. Bernoulli functions $B_\nu(z)$ in the complex plane

In this subsection we first consider the analytic continuation of $B_\nu(z)$ up to the negative z-axis. Originally the branch cut of the many-valued function t^ν in (2.5) runs from 0 to $-\infty$. However, this choice is by convention. When $\arg z \geq 0$, we may turn the loop C in clockwise direction into the upper half plane. In this way we redefine the location of the branch cut in the t-plane. Turning around a positive angel δ, we have at one side of the cut $\arg t = \pi + \delta$, and on the other side $\arg t = -\pi + \delta$. When we take $\delta \in [0, \frac{1}{2}\pi)$, the integral remains convergent when we allow $\arg z$ ranging in the interval $[0, \pi)$. A similar method can be used for z in the lower half plane. This gives the analytic continuation of $B_\nu(z)$ defined in (2.5) to the sector $|\arg z| < \pi$, for any complex value of ν.

By using (2.5) it follows easily that the basic *difference property* (2.1) of the Bernoulli polynomials remains valid for the Bernoulli functions:

$$B_\nu(z+1) = B_\nu(z) + \nu z^{\nu-1}, \quad \nu \in \mathbb{C}, \quad |\arg z| < \pi. \tag{2.6}$$

Also the *derivative property*

$$\frac{d}{dz}B_\nu(z) = \nu B_{\nu-1}(z), \quad \nu \in \mathbb{C}, \quad |\arg z| < \pi \tag{2.7}$$

is easily verified by using (2.5). Observe that the analytic continuation of $B_\nu(z)$ from the half plane $\Re z > 0$ into the left half of the complex plane also follows from (2.6). Also in this way we cannot reach the negative z-axis.

Next we want to verify how relation (2.4) transforms when n becomes a complex parameter. This will give a quite non-trivial property. To obtain information on $B_\nu(1-z)$ we replace z with $-z$ in (2.6). To take into account the many-valuedness of the function z^ν and the condition $|\arg z| < \pi$, we change in (2.6) z into $ze^{-i\pi}$ when z is in the upper half plane $\Im z > 0$ and change z into $ze^{+i\pi}$ when z is in the lower half plane. The result is when $\Im z > 0$:

$$B_\nu(1-z) = B_\nu(-z) - \nu e^{-i\pi\nu} z^{\nu-1}.$$

Combining this with (2.6) and eliminating $z^{\nu-1}$ we obtain the relation

$$e^{i\pi\nu} B_\nu(1-z) - B_\nu(z) = e^{i\pi\nu} B_\nu(-z) - B_\nu(1+z), \quad \Im z > 0,$$

which says that the left-hand side is a periodic function of z with period 1. In other words,

$$B_\nu(z) = e^{i\pi\nu} B_\nu(1-z) + \omega_\nu^+(z), \quad \Im z > 0, \tag{2.8}$$

where $\omega_\nu^+(z)$ is a 1–periodic function in the upper half plane. In a similar way we obtain

$$B_\nu(z) = e^{-i\pi\nu} B_\nu(1-z) + \omega_\nu^-(z), \quad \Im z < 0, \tag{2.9}$$

where $\omega_\nu^-(z)$ is a 1–periodic function in the lower half plane.

The functions $\omega_\nu^\pm(z)$ can be obtained as follows. Consider (2.5) with $\mu = 1$ and $\Im z > 0$. As we did for the analytic continuation we can turn the path of integration C into the upper half plane, even across the poles at $t_k = 2\pi ik$, $k = 1, 2, 3, \ldots$, and pick up the residues. Summing the residues, which can be done when $\Im z > 0$, and taking into account the value of the phases of t at both sides of the cut when both branches of C pass the poles, we obtain

$$\begin{aligned} B_\nu(z) = \ & \Gamma(\nu+1)\left[e^{\frac{3}{2}\pi\nu i} - e^{-\frac{1}{2}\pi\nu i}\right] \sum_{k=1}^{\infty} \frac{e^{2\pi ikz}}{(2\pi k)^\nu} + \\ & \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{e^{zt}}{(e^t-1)t^\nu}\, dt, \end{aligned} \tag{2.10}$$

where C runs around the cut, which now occurs in the first quadrant of the t–plane. When $\frac{1}{2}\pi < \arg z \le \pi$ we can take the cut along the positive t–axis. At the upper part of the cut we have $\arg t = -2\pi$, at the lower side $\arg t = 0$. The contour starts at $+\infty$ (at the upper side of the cut) and encircles the origin in positive direction.

In this position of the contour we introduce a new variable of integration by writing $t = ve^{-i\pi}$. By using the relation

$$\frac{e^{-zv}}{e^{-v}-1} = -\frac{e^{(1-z)v}}{e^v-1}$$

and interpreting the new integral in terms of $B_\nu(1-z)$, we obtain the functional equation (2.8) with

$$\omega_\nu^+(z) = 2i \sin \pi\nu \, e^{\frac{1}{2}\pi\nu i}\Gamma(\nu+1) \sum_{k=1}^{\infty} \frac{e^{2\pi i k z}}{(2\pi k)^\nu}. \tag{2.11}$$

This relation holds for all values of z in the upper half plane, since all three terms in (2.8) are analytic functions with respect to z in this domain; ν may be any complex number.

Repeating the procedure for values of z in the lower half plane, we obtain (2.9) with

$$\omega_\nu^-(z) = -2i \sin \pi\nu \, e^{-\frac{1}{2}\pi\nu i}\Gamma(\nu+1) \sum_{k=1}^{\infty} \frac{e^{-2\pi i k z}}{(2\pi k)^\nu}, \tag{2.12}$$

a result as in (2.11), with all quantities i replaced by $-i$.

We can now define the Bernoulli function $B_\nu(x)$ for $x < 0$. This will depend on the way we approach the negative $z-$axis: from above or from below. Taking the average of the two values obtained so, we define

$$B_\nu^*(x) := \lim_{y\to 0} \frac{B_\nu(x+iy) + B_\nu(x-iy)}{2}, \quad x < 0. \tag{2.13}$$

It easily follows that we have

$$B_\nu^*(-x) = \cos \pi\nu B_\nu(x+1) + 2\Gamma(\nu+1) \sin \pi\nu \sum_{k=1}^{\infty} \frac{\sin(2\pi k x - \frac{1}{2}\nu\pi)}{(2\pi k)^\nu}, \tag{2.14}$$

where $x > 0, \Re\nu > 1$, the latter condition being needed to guarantee the convergence of the infinite series. Again, the series is a 1–periodic function on the real line. The function $B_\nu^*(x)$ satisfies the following difference property (compare this with (2.1)):

$$B_\nu(x+1) - B_\nu(x) = \begin{cases} \nu x^{\nu-1}, & \text{if x} \geq 0; \\ -\nu|x|^{\nu-1}\cos\pi\nu, & \text{if x} < 0,\ \Re\nu > 1. \end{cases} \tag{2.15}$$

The series in (2.5) is closely connected with the familiar Fourier series for the Bernoulli polynomials:

$$B_n(x) = -2n! \sum_{k=1}^{\infty} \frac{\cos(2\pi k x - \frac{1}{2}n\pi)}{(2\pi k)^n},$$

$n = 1, 2, 3, \ldots, x \in [0, 1)$.

In BUTZER *et al.* [6] a quite different approach and result is given for defining the value of $B_\nu(z)$ for negative values of z. Our approach, which leads to (2.5) and (2.6), is based on the crucial functional relations in (2.8) and (2.9), with (2.11) and (2.12). These relations are not available in the cited reference, and there the difference property (2.15) contains for $x < 0$ the factor $\cos \pi\nu - \sin \pi\nu$ instead of only $\cos \pi\nu$. In our approach the relation for $x < 0$ links up nicely with the original difference relation in (2.1), because in order to replace $(-1)^n$ we just take the average of $e^{\pm i\pi\nu}$.

2.2. Series in powers of z

We conclude by giving the Maclaurin series (in powers of z) and an asymptotic expansions (in negative powers of z) of $B_\nu(z)$. These expansions have received little or no attention in the literature.

The well-known property

$$B_n(z) = \sum_{k=0}^{n} B_k \binom{n}{k} z^{n-k} \tag{2.16}$$

holds for the Bernoulli functions in the form of an asymptotic expansion:

$$B_\nu(z) \sim \sum_{k=0}^{\infty} B_k \binom{\nu}{k} z^{\nu-k}, \quad \text{as} \quad z \to \infty \tag{2.17}$$

in the sector $|\arg z| < \pi$. This follows by taking in (2.5) $\mu = 1$ and expanding

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Interchanging the order of summation and integration, applying Watson's Lemma for loop integrals (see OLVER [15]), and using Hankel's contour integral for the reciprocal gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^t t^{-z}\, dt$$

(where $\mathcal{C}$ is the same as in (2.5)), we obtain (2.7).

It follows that, when $\Re\nu < 0$,

$$B_\nu(z) \to 0, \quad \text{as} \quad z \to \infty$$

in the sector $|\arg z| < \pi$. An application of this yields an interesting relation with the generalized zeta function, which is defined by

$$\zeta(s,t) = \sum_{n=0}^{\infty} (n+t)^{-s}, \quad \Re s > 1, \quad t \neq 0, -1, -2, \ldots, \tag{2.18}$$

and which reduces to the familiar Riemann zeta function when $t = 1$: $\zeta(s) = \zeta(s, 1)$. Observe that repeated application of (2.6) gives

$$B_\nu(z+m) = B_\nu(z) + \nu \sum_{k=0}^{m-1} (z+k)^{\nu-1}. \tag{2.19}$$

When m tends to infinity and $\Re\nu < 0$ the left-hand side vanishes. It follows that

$$B_\nu(z) = -\nu\zeta(1-\nu, z), \quad z \neq 0, -1, -2, \ldots \tag{2.20}$$

By using analytic continuation it follows that this relation holds for all complex values of ν. The function $\zeta(s,t)$ has a pole at $s = 1$, with residue 1. Hence, the right-hand side of (2.20) is well defined as $\nu \to 0$.

From the expansion

$$\zeta(s,t) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \Gamma(s+k)\zeta(s+k)\frac{(1-t)^k}{k!}, \quad |t-1| < 1,$$

which easily follows by expanding in (2.18)

$$(n+t)^{-s} = (n+1)^{-s}\left[1 + \frac{t-1}{n+1}\right]^{-s}$$

in powers of $(t-1)$, and using (2.1), we obtain

$$B_\nu(z) = -\nu z^{\nu-1} + \frac{1}{\Gamma(-\nu)} \sum_{k=0}^{\infty} \Gamma(k+1-\nu)\zeta(k+1-\nu)\frac{(-z)^k}{k!}, \quad |z| < 1.$$

This expansion reduces to the finite (polynomial) representation (2.7) when we take the limit $\nu \to n$ (integer).

Both expansions (2.16) and (2.17) are contained in one integral:

$$B_\nu(z+1) = \frac{1}{\Gamma(-\nu)\, 2\pi i} \int_{\mathcal{L}} \zeta(1-\nu-w)\Gamma(w)\Gamma(1-\nu-w) z^{-w}\, dw, \tag{2.21}$$

where $\Re\nu < -1$ and $\mathcal{L}$ is a vertical in the strip $0 < \Re w < -\nu$. This integral follows from the Mellin transform of $\zeta(s, t+1)$ with respect to t, which reads:

$$\int_0^\infty \zeta(s, t+1) t^{w-1}\, dt = \zeta(s-w) B(w, s-w), \quad 0\Re w < \Re s - 1,$$

where we have used the Beta integral

$$\int_0^\infty t^{x-1}(t+1)^{-x-y}\, dt = B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y), \quad \Re x, y > 0.$$

Upon inverting the Mellin transform we obtain (2.21).

The expansions (2.16) and (2.17) follow from (2.21) by shifting the contour $\mathcal{L}$ to the left, across the poles of the gamma function $\Gamma(w)$, and picking up the residues to obtain the Maclaurin expansion (2.7), and shifting to the right across the pole of $\zeta(1-\nu-w)$ at $w=-\nu$ and the poles of $\Gamma(1-\nu-w)$ at $w=k-\nu+1, k=0,1,2,\ldots$, to obtain the asymptotic expansion (2.17).

3. Asymptotics of B_ν^μ

Our current interest in the asymptotic behaviour of the generalized Bernoulli numbers B_ν^μ stems from our earlier research on *Stirling numbers*, as published recently in Temme [18]. Indeed, the quantities B_ν^μ are related with Stirling numbers. First we explain this relationship. Next we summarize the uniform asymptotic approximations of the Stirling numbers, and we give several asymptotic approximations for the Bernoulli numbers.

3.1. Stirling numbers are generalized Bernoulli numbers

The Stirling numbers of the first and second kind, respectively denoted by $S_n^{(m)}$ and $\mathfrak{S}_n^{(m)}$, are usually defined through the finite generating functions

$$x(x-1)\cdots(x-n+1)=\sum_{m=0}^{n} S_n^{(m)} x^m, \tag{3.1}$$

$$x^n=\sum_{m=0}^{n} \mathfrak{S}_n^{(m)} x(x-1)\cdots(x-m+1) \tag{3.2}$$

where we give the left-hand side of (3.1) the value 1 when $n=0$. Similarly, the factors on the right-hand side of (3.2) have the value 1 when $m=0$. This gives the 'boundary values'

$$S_n^{(n)}=\mathfrak{S}_n^{(n)}=1,\ n\geq 0, \quad \text{and} \quad S_n^{(0)}=\mathfrak{S}_n^{(0)}=0,\ n\geq 1.$$

Furthermore it is convenient to agree on $S_n^{(m)}=\mathfrak{S}_n^{(m)}=0$ if $m>n$.

Several other generating functions are available for Stirling numbers. We have

$$\frac{[\ln(x+1)]^m}{m!}=\sum_{n=m}^{\infty} S_n^{(m)}\frac{x^n}{n!}, \tag{3.3}$$

$$\frac{(e^x-1)^m}{m!}=\sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)}\frac{x^n}{n!}. \tag{3.4}$$

These two equations give the link with the generating functions of the generalized Bernoulli numbers given in (1.1). The relations are

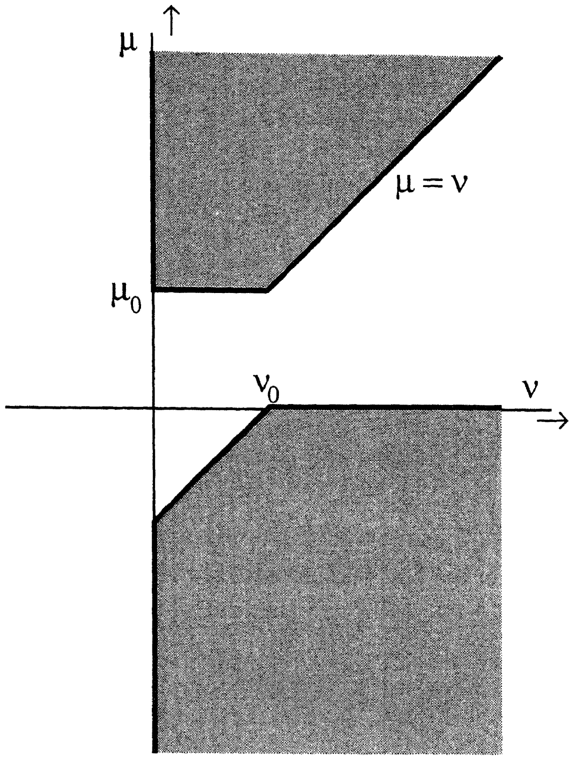

FIGURE 1. Parameter domains (shaded) for which the uniform asymptotic expansions of the Stirling numbers can be used to obtain a first order approximation for B_ν^μ; upper area via the Stirling numbers of the first kind, lower part via the Stirling numbers of the second kind.

$$S_n^{(m)} = \binom{n-1}{m-1} B_{n-m}^n, \quad \mathfrak{S}_n^{(m)} = \binom{n}{m} B_{n-m}^{-m}. \tag{3.5}$$

To explain this for the numbers of the first kind, we write

$$S_n^{(m)} = \frac{1}{2\pi i} \frac{n!}{m!} \int_C \frac{[\ln(z+1)]^m}{z^{n+1}}\, dz,$$

where C is a small circle around $z = 0$. Substituting $z = e^w - 1$ and integrating by parts gives an integral that is similar to (1.2). For the Stirling numbers of the second kind the relation with the numbers B_ν^μ follows quite easily by comparing (1.1) with (3.3).

3.2. Uniform asymptotics for Stirling numbers: summary of results

The problem is to find asymptotic expansions for the Stirling numbers as $n \to \infty$ that hold uniformly with respect to $m \in [0, n]$; we summarize the results of TEMME [17].

We start with the numbers of the second kind. The generating function (3.3) gives the Cauchy integral representation

$$\mathfrak{S}_n^{(m)} = \frac{n!}{m!} \frac{1}{2\pi i} \oint \frac{(e^x - 1)^m}{x^{n+1}} \, dx.$$

The contour is a small circle around the origin. We write this in the form

$$\mathfrak{S}_n^{(m)} = \frac{n!}{m!} \frac{1}{2\pi i} \oint e^{\phi(x)} \frac{dx}{x}, \tag{3.6}$$

where

$$\phi(x) = -n \ln x + m \ln(e^x - 1).$$

This integral can be estimated by using the saddle point method (for a general introduction to this topic see, for instance, OLVER [15] or WONG [19]). The saddle point is the solution of the equation $\phi'(x) = 0$. The real positive saddle point x_0 solves the equation

$$\frac{m}{n} x = 1 - e^{-x}. \tag{3.7}$$

The solution $x_0 = 0$ is not of interest, because the contour in (3.6) is not allowed to pass through the origin. The saddle point method is based on replacing $\phi(x)$ by a quadratic function, for instance by writing $\phi(x) - \phi(x_0) = t^2$, a local transformation near $x = x_0$. However, straightforward application of the saddle point method gives approximations that are less accurate when $m \to n$ (in that case the saddle point moves to the origin and the quadratic transformation becomes singular near the saddle point). To define a different transformation, we observe that when $x \to 0^+$, we have $\phi(x) \sim (m - n) \ln x$, and when $x \to \infty$, we have $\phi(x) \sim mx$. This suggests the transformation $x \to t(x)$ defined by

$$\phi(x) = mt + (m - n) \ln t + A, \tag{3.8}$$

where A is not depending on t. The derivative of the right-hand side vanishes at $t_0 = (n - m)/m$. We prescribe for the mapping in (3.8) the corresponding points

$$x = 0 \Longleftrightarrow t = 0, \quad x = x_0 \Longleftrightarrow t = t_0, \quad x = +\infty \Longleftrightarrow t = +\infty.$$

The quantity A follows from substitution of $x = x_0$, $t = t_0$ in (3.8), which gives

$$A = \phi(x_0) - mt_0 + (n - m) \ln t_0.$$

Transformation (3.8) brings (3.6) in the form

$$\mathfrak{S}_n^{(m)} = \frac{n!}{m!}\frac{e^A}{2\pi i}\int e^{mt} f(t)\frac{dt}{t^{n-m+1}}, \tag{3.9}$$

where

$$f(t) = \frac{t}{x}\frac{dx}{dt} = \frac{m(t-t_0)}{x\phi'(x)}. \tag{3.10}$$

A transformation like (3.8) is investigated earlier in TEMME [17]. The function f is analytic at $t = 0$ and at $t = t_0$, also when t_0 (that is, x_0) tends to zero.

A first approximation to $\mathfrak{S}_n^{(m)}$ is now obtained by replacing $f(t)$ in (3.9) with the value of this function at the saddle point t_0. This gives

$$\mathfrak{S}_n^{(m)} \sim e^A m^{n-m} f(t_0)\binom{n}{m}, \text{ as } n \to \infty \tag{3.11}$$

where

$$f(t_0) = \frac{1}{x_0}\sqrt{\frac{mt_0}{\phi''(x_0)}} = \sqrt{\frac{t_0}{(1+t_0)(x_0-t_0)}}.$$

When we compare the approximation in (3.11) for $n = 10$ with exact values we find that the maximal relative error is 0.0064, and occurs at $m = 3$. Similar computations with $n = 20, n = 30$ show the following: the maximal errors are 0.0031, 0.0021, and occur for $m = 7, m = 10$, respectively. The maximal errors do not occur at boundary values of m, but at about $m = \frac{1}{3}n$. Also larger values of n confirm the uniform character of the asymptotic estimate (3.11).

Next we consider the Stirling numbers of the first kind. From (3.1) it follows that

$$\begin{aligned}(-1)^{n-m} S_{n+1}^{(m+1)} &= \frac{1}{2\pi i}\oint \frac{(x+1)(x+2)\cdots(x+n)}{x^{m+1}}\,dx = \\ &\quad \frac{1}{2\pi i}\oint e^{\phi(x)}\frac{dx}{x},\end{aligned} \tag{3.12}$$

where

$$\phi(x) = \ln[(x+1)(x+2)\cdots(x+n)] - m\ln x.$$

When $1 \le m \le n-1$ there is one positive saddle point x_0. The behaviour of $\phi(x)$ on the positive real axis is: $\phi(x) \sim -m\ln x$ as $x \to 0$, and $\phi(x) \sim (n-m)\ln x$, as $x \to \infty$. Combining these two limiting cases, we observe that the function $n\ln(x+1) - m\ln x$ has (globally on $(0,\infty)$) the same graph as $\phi(x)$. This suggests the transformation

$$\phi(x) = n\ln(1+t) - m\ln t + B. \tag{3.13}$$

The derivative of the right-hand side vanishes at $t_0 = m/(n-m)$. We prescribe for the mapping in (3.13) the corresponding points

$$x = 0 \Longleftrightarrow t = 0,\ x = x_0 \Longleftrightarrow t = t_0,\ x = +\infty \Longleftrightarrow t = +\infty.$$

The quantity B follows from substitution of $x = x_0$, $t = t_0$ in (3.13), which gives

$$B = \phi(x_0) - n\ln(t_0+1) + m\ln t_0.$$

Transformation (3.13) brings (3.12) in the form

$$(-1)^{n-m} S_{n+1}^{(m+1)} = \frac{e^B}{2\pi i}\int \frac{(1+t)^n}{t^{m+1}} g(t)\, dt, \tag{3.14}$$

where, initially, the contour is a small circle around $t = 0$, and

$$g(t) = \frac{t}{x}\frac{dx}{dt} = \frac{(n-m)t - m}{(t+1)x\phi'(x)}.$$

A first approximation to $S_{n+1}^{(m+1)}$ is now obtained by replacing $g(t)$ in (3.14) with the value of this function at the saddle point t_0. This gives the one-term approximation

$$S_{n+1}^{(m+1)} \sim (-1)^{n-m} e^B g(t_0)\binom{n}{m}, \quad \text{as} \quad n \to \infty. \tag{3.15}$$

The quantity $g(t_0)$ is given by

$$g(t_0) = \frac{1}{x_0}\sqrt{\frac{m(n-m)}{n\phi''(x_0)}}.$$

For $n = 10$ the maximal relative error now occurs at $m = 3$, and is 0.0082. For $n = 20$, $n = 30$, the maximal errors are: 0.0063 and 0.0053, respectively; again they occur at $m = 3$. This confirms the uniform character with respect to m of the result in (3.15).

Although the Stirling numbers are defined for integer values of n, m, the results and methods can be interpreted for continuous variables. Considering the relations in (3.15), we observe that the uniform asymptotic results of the Stirling numbers numbers can be used for the generalized Bernoulli numbers B_ν^μ in the shaded areas of the (ν,μ)−plane, given in Figure 1. Here ν_0, μ_0 are large numbers, ν_0 indicating the large ν−domain $[\nu_0, \infty)$ for which the uniform approximations of the Stirling numbers can be used for the generalized Bernoulli numbers B_ν^μ. In the next subsections we concentrate on the asymptotic behaviour of B_ν^μ for (ν,μ) in the non-shaded area in the upper right half plane of Figure 1. That is, we assume that ν is large and $0 \le \mu \le \nu$. In fact our goal is to obtain a uniform approximation in this domain, as we obtained for the Stirling numbers in the shaded areas. However, the situation here is quite different from the Stirling case, as will be explained in §3.4. In §3.5 we consider a problem for $B_\nu(z)$ in which z is large and ν acts as a uniformity parameter on the real axis. First we summarize existing results from the literature.

3.3. *Nörlund's results*

In NÖRLUND [13] results are given for a parameter domain that corresponds to the neighbourhood of the diagonal $\nu = \mu$. In fact, Nörlund considered the polynomials $B_\nu^{\nu+\rho+1}(z)$, where ρ and z are fixed complex numbers (*fixed* means independent of ν). His result is

$$\frac{B_\nu^{\nu+\rho+1}(z)}{\nu!} \sim (-1)^\nu \frac{(\ln\nu)^\rho}{\nu^z}\left[\sum_{s=0}^{n-1}\binom{\rho}{s}\frac{(-1)^s}{(\ln\nu)^s}A_s(z) + \mathcal{O}\left((\ln\nu)^{-n}\right)\right], \quad (3.16)$$

as $\nu \to \infty$. The coefficients $A_s(z)$ are derivatives of the reciprocal gamma function:

$$A_s(z) = \frac{d^s}{dz^s}\frac{1}{\Gamma(1-z)}.$$

The asymptotic expansion (3.16) shows inverse powers of $\ln\nu$, giving a rather slow asymptotic convergence for computations, unless ν is very large. When $z = 0$, this is in fact the case $S_{\nu+\rho+1}^{(\rho+1)}$ of Stirling numbers of the first kind, the coefficients $A_s(z)$ reduce to the coefficients of the Maclaurin expansion of $1/\Gamma(1-z)$, which easily follow from those of $1/\Gamma(z)$ (see, for instance, ABRAMOWITZ AND STEGUN [1, p.256]).

When $\rho = 0, 1, 2, \ldots$, the series in (3.16) reduces to a finite number of terms (because the binomial coefficient vanishes when $s > \rho$). In particular, when $\rho = 0$, we have the simple case $B_\nu^{\nu+1}(z) = (z-1)(z-2)\cdots(z-\nu)$. That is,

$$\frac{B_\nu^{\nu+1}(z)}{\nu!} = (-1)^\nu \frac{\Gamma(\nu+1-z)}{\Gamma(1-z)\Gamma(\nu+1)} \sim \sum_{n=0}^{\infty}\frac{B_n^{1-z}}{n!\,\Gamma(1-z-n)}\frac{(-1)^n}{\nu^{z+n}},$$

which is a well-known result for the ratio of two gamma functions.

We observe that this expansion is in negative powers of ν, because in (3.16) the expansion containing inverse powers of $\ln\nu$ completely vanishes. What remains was hidden in the $\mathcal{O}$–symbol of (3.16) and shows up when $\rho = 0$ (quantities that are asymptotically negligible with respect to all negative powers of $\ln\nu$ occurring in the series and the $\mathcal{O}$–term in (3.16)). This is a nice example in which 'exponentially small terms' become important when a parameter changes a critical value (in this case: when $\rho = 0$).

For fixed values of μ Nörlund gives the expansion

$$\frac{B_\nu^\mu(z)}{\nu!} = \nu^{\mu-1}\frac{2\cos\pi(2z+\mu-\frac{1}{2}\nu)}{\Gamma(\mu)(2\pi)^\nu}\left[1+\mathcal{O}\left(\nu^{-1}\right)\right].$$

3.4. Saddle point methods for B_ν^μ

We now discuss asymptotic properties of the generalized Bernoulli numbers B_ν^μ in connection with the results for the Stirling numbers given in §3.2. Consider (1.2) with $z=0$ and $\nu \neq \mu$, and integrate by parts. It follows that

$$B_\nu^\mu = \frac{\mu}{\mu-\nu}\frac{\Gamma(\nu+1)}{2\pi i}\int_{\mathcal{C}}\frac{t^{\mu-\nu}e^t}{(e^t-1)^{\mu+1}}\,dt.$$

This integral has better convergence properties when we deform the contour $\mathcal{C}$ into a path that extends to $-\infty$. We write

$$B_\nu^\mu = \frac{\mu}{\mu-\nu}\frac{\Gamma(\nu+1)}{2\pi i}\int_{\mathcal{C}} e^{\phi(t)}\frac{e^t\,dt}{e^t-1}, \tag{3.17}$$

where

$$\phi(t) = (\mu-\nu)\ln t - \mu\ln\left(e^t-1\right).$$

To calculate the saddle points we have to solve the equation $\frac{d}{dt}\phi(t)=0$, which is equivalent to solving

$$1-e^{-t} = \lambda t, \quad \lambda = \frac{\mu}{\mu-\nu}. \tag{3.18}$$

The solution $t=0$ is not of interest, because the contour $\mathcal{C}$ is not allowed to pass through the origin. To keep the discussion surveyable we assume that ν is large and positive, and that μ is a real parameter.

We can distinguish three $\mu-$domains of interest, which correspond with the three domains indicated in Figure 1.

- $\mu < 0 \Rightarrow 0 < \lambda < 1$; in this case (3.18) has a real positive solution;
- $0 < \mu < \nu \Rightarrow \lambda < 0$; in this case (3.18) has no real solutions;
- $\mu > \nu \Rightarrow \lambda > 1$; in this case (3.18) has a real negative solution.

We conclude that in the two shaded areas of Figure 1 (both 'Stirling cases') there is a real saddle point, and that in the area that has to be done there is no real saddle point. It will turn out that in the latter case, that is, when $\lambda < 0$, equation (3.18) has an infinite number of complex solutions, which occur in complex conjugated pairs, and one pair $\{t^+, t^-\}$ can be used for the saddle point method. The absolute values of the imaginary parts of $t^\pm$ belong to the interval $(\pi, 2\pi)$.

Equation (3.18) is equivalent to the equation

$$we^w = x, \quad \text{where} \quad w = t - \tfrac{1}{\lambda}, \quad x = -\tfrac{1}{\lambda}e^{-\frac{1}{\lambda}}. \tag{3.19}$$

When λ ranges through the interval $(-\infty, 0)$ the quantity x ranges through the interval $(0, +\infty)$. The trivial solution $w = -\frac{1}{\lambda}$ (corresponding to $t = 0$) is not of interest. The equation $we^w = x$ has received quite some attention in the literature. MAPLE, the package for symbolic computations, has the solution $w(x)$ as a standard function. To give more insight on the location of the complex solutions of this equation, we give a few steps in solving the equation for real positive x.

We write $w = u + iv$, with u, v real, and see that the equation $we^w = x$ is equivalent to

$$v = -xe^{-u}\sin v, \quad u = -v\cot v.$$

For positive values of x solutions occur in the $v-$intervals

$$(\pi, 2\pi), (3\pi, 4\pi), \ldots$$

and in similar negative $v-$intervals. When x is small, that is, $-\lambda$ is a large positive number, a conjugate pair of saddle points $t^{\pm}$ has imaginary parts near $\pm\pi$ and the real parts satisfy $\Re t^{\pm} \sim -\ln(-\lambda)$. Because of the convergence of the integral in (3.4) at $t = \pm\infty$, the contour $\mathcal{C}$ can be deformed into two separate conjugate paths $\mathcal{C}^{\pm}$, $\mathcal{C}^{-}$ running from $-\infty$ to $+\infty$ with $\Im t \in (-\pi, -2\pi)$, and $\mathcal{C}^{+}$ from $+\infty$ to $-\infty$ with $\Im t \in (\pi, 2\pi)$, such that $\mathcal{C}^{\pm}$ run through the saddle points $t^{\pm}$.

The method used for the Stirling numbers is based on replacing the function $\phi(t)$ in (3.6) and (3.12) by a simpler function that has the same properties with respect to saddle points and singularities, in particular when saddle points and singularities coalesce. Morerover, the resulting integrals (3.9) and (3.14) can be evaluted in closed form when f and g are constant. In the present case we have not found such a simpler function, and we proceed by applying the standard saddle point method.

Locally at $t = t^{\pm}$ we can approximate $\phi(t)$ up to the quadratic term of its Maclaurin expansion, and we obtain the asymptotic result

$$B_\nu^\mu \sim \frac{\lambda\Gamma(\nu+1)}{2\pi i} \sum_{(+,-)} \frac{e^{\phi(t^{\pm})}}{\lambda t^{\pm}} \int_{\mathcal{C}^{\pm}} e^{\frac{1}{2}\phi''(t^{\pm})(t-t^{\pm})^2}\, dt.$$

That is,

$$B_\nu^\mu \sim \frac{\Gamma(\nu+1)}{\sqrt{2\pi}\, i}\left[\frac{e^{\phi(t^-)}}{t^-\sqrt{-\phi''(t^-)}} - \frac{e^{\phi(t^+)}}{t^+\sqrt{-\phi''(t^+)}}\right]. \tag{3.20}$$

We have for the second derivative of ϕ:

$$\phi''(t) = -\frac{\mu-\nu}{t^2} + \frac{\mu e^t}{(e^t-1)^2}. \tag{3.21}$$

Evaluating this at the saddle points, using $1 - \lambda t^{\pm} = e^{-t^{\pm}}$, see (3.5), we have

$$\phi''(t^{\pm}) = \frac{\nu-\mu}{\lambda t^{\pm}}[\lambda(1+t^{\pm}) - 1].$$

These quantities have negative real parts when $-\lambda$ is a large positive number.

The first approximation given in (3.20) can be supplied with more terms by using standard techniques of the saddle point method, but we omit the details here. Also, it is possible to repeat the analysis for the generalized Bernoulli polynomials $B_\nu^\mu(z)$, and to compare the results with Nörlund's results. All this is outside the scope of the present paper, because the elaborations are rather technical and complicated. Moreover, further investigations are needed to determine the range of the parameters for which the expansion holds. We expect that (3.20) will be uniformly valid for $\lambda = \mu/(\mu-\nu)$ belonging to compact sets of the interval $(-\infty, 0)$, and $\nu \to +\infty$. When indeed this is true, we can fill a large part of the unshaded area in the first quadrant of Figure 1.

3.5. Uniform asymptotics for large values of z

We return to $B_\nu(z)$ and consider the problem of obtaining an expansion for large values of z and ν. When ν is fixed the expansion in (2.17) is applicable. In this subsection we give two expansions, one holding uniformly with respect to $\nu \in [0, \infty)$, and a similar expansion holding uniformly with respect to $\nu \in (-\infty, 0]$. The approach is based on earlier work discussed in TEMME [16].

The asymptotic problem in that paper is to obtain an expansion of the integral

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt, \quad z, \lambda > 0, \tag{3.22}$$

that holds uniformly with respect to $\lambda \in [0, \infty)$. Laplace integrals can be expanded by invoking Watson's Lemma (see OLVER [15] or WONG [19]): expand f at the origin and interchange summation and integration. That is,

$$f(t) = \sum_{n=0}^\infty c_n t^n \quad \Rightarrow \quad F_\lambda(z) \sim \sum_{n=0}^\infty c_n \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} z^{-n-\lambda}$$

as $z \to \infty$, λ fixed. When λ is not fixed (say, λ is depending on z) this becomes invalid. It is better to expand at $t = \kappa := \lambda/z$, the saddle point of the dominant part $t^\lambda e^{-zt}$ of the integrand. We have

$$f(t) = \sum_{n=0}^\infty a_n(\kappa)\,(t-\kappa)^n \quad \Rightarrow \quad F_\lambda(z) \sim z^{-\lambda} \sum_{n=0}^\infty a_n(\kappa) P_n(\lambda)\, z^{-n}, \tag{3.23}$$

where

$$P_n(\lambda) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-z\,t} (t-\kappa)^n \, dt. \tag{3.24}$$

That is,

$$P_0(\lambda) = 1, \quad P_1(\lambda) = 0, \quad P_2(\lambda) = \lambda, \quad P_3(\lambda) = 2\lambda, \ldots\ .$$

It is quite easy to obtain the recursion:

$$P_{n+1}(\lambda) = n\left[P_n(\lambda) + \lambda P_{n-1}(\lambda)\right],$$

and the estimate

$$P_n(\lambda) = \mathcal{O}\left(\lambda^{[n/2]}\right), \quad \lambda \to \infty. \tag{3.25}$$

This expansion is, under mild conditions on $a_n(\kappa)$, that is, on f, uniformly valid with respect to $\lambda \in [0,\infty)$. For instance, when $f(t) = 1/(t+1)$, the coefficients $a_n(\kappa)$ are given by

$$\frac{1}{t+1} = \sum_{n=0}^{\infty} a_n(\kappa)\ (t-\kappa)^n, \quad a_n(\kappa) = \frac{(-1)^n}{(1+\kappa)^{n+1}}, \tag{3.26}$$

and we see from (3.25) that, in this case, the terms $a_n(\kappa)\ P_n(\lambda)\ z^{-n}$ in the expansion of $F_\lambda(z)$ given in (3.23) do not lose their asymptotic character when λ runs through the domain $[0,\infty)$. This is not a proof of the asymptotic nature of the expansion, but an indication that the expansion has some robustness with respect to large values of λ. For a proof we refer to TEMME [16].

We can apply this method by writing $B_\nu(z)$ in the form (3.6). This is possible when ν is negative. We observe that in that case we can integrate in (2.5) along both sides of the negative real axis (using different phases $\pm\pi i$ of t), and obtain

$$B_\nu(z) = \frac{1}{\Gamma(-\nu)} \int_0^\infty t^{-\nu-1} e^{-z\,t} f(t)\, dt, \quad f(t) = \frac{t}{1-e^{-t}}. \tag{3.27}$$

We define $\kappa = -\nu/z$ and expand f at $t = \kappa$ as in (3.23). In this case the expansion has a different asymptotic character than in the example with $f(t) = 1/(t+1)$. To explain this, we have in the latter case the lucky situation that $\{a_n\}$ constitute an asymptotic scale as $\kappa \to \infty$. That is,

$$a_{n+1}/a_n = \mathcal{O}(1/\kappa) \quad \text{as} \quad \kappa \to \infty.$$

In fact, when this is the case, the expansion in of $F_\lambda(z)$ in (3.23) has a double asymptotic property: it is also valid when $\lambda \to \infty$, uniformly with respect to $z \in [z_0,\infty)$, where z_0 is a fixed positive number.

Let us now consider f defined in (3.27) for the case of the Bernoulli functions. We have, as $t \to \infty$,

$$f(t) = t\left(1 + e^{-t} + e^{-2t} + \ldots\right)$$

and

$$f'(t) = 1 + \mathcal{O}\left(te^{-t}\right), \quad f^{(n)}(t) = \mathcal{O}\left(te^{-t}\right), \quad n = 2, 3, 4, \ .$$

Hence, for $n \geq 2$, the coefficients $a_n(\kappa)$ are asymptotically small. The only snag is that the coefficients do not constitute an asymptotic scale.

We conclude with giving a similar expansion for positive values of ν. The starting point is the contour integral (2.5)

$$B_\nu(z) = \frac{\Gamma(\nu+1)}{2\pi i}\int_C e^{zt} f(t) \frac{dt}{t^{\nu+1}}\, dt,$$

with

$$f(t) = \frac{t}{e^t - 1}.$$

Again, there is a saddle point at $t = \kappa := \nu/z$ and we obtain

$$f(t) = \sum_{n=0}^{\infty} b_n(\kappa)\,(t-\kappa)^n \quad \Rightarrow \quad B_\nu(z) \sim z^{-\nu}\sum_{n=0}^{\infty} b_n(\kappa)\, Q_n(\nu)\, z^{-n}, \quad (3.28)$$

where

$$Q_n(\nu) = \frac{\Gamma(\lambda+1)}{2\pi i}\int_C e^{zt}(t-\kappa)^n \frac{dt}{t^{\nu+1}}\, dt.$$

It is easily verified that

$$Q_n(\nu) = (-1)^n P_n(-\nu), \quad n = 0, 1, 2, \ldots,$$

where the polynomials P_n are given in (3.7), and that

$$a_0(\kappa) = b_0(\kappa) + \kappa, \quad a_1(\kappa) = b_1(\kappa) + 1, \quad a_n(\kappa) = b_n(\kappa), \quad n \geq 2.$$

That's why I call the expansions in (3.23) and (3.28) quite similar. Also, the expansion for $B_\nu(z)$ for positive values of ν has the same asymptotic nature as the one for negative values of ν given in (3.23). When $n \geq 2$ the coefficients $b_n(\kappa)$ are exponentially small when κ is large, and do not constitute an asymptotic scale.

where

$$P_n(\lambda) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} (t-\kappa)^n \, dt. \tag{3.24}$$

That is,

$$P_0(\lambda) = 1, \quad P_1(\lambda) = 0, \quad P_2(\lambda) = \lambda, \quad P_3(\lambda) = 2\lambda, \ldots .$$

It is quite easy to obtain the recursion:

$$P_{n+1}(\lambda) = n\left[P_n(\lambda) + \lambda P_{n-1}(\lambda)\right],$$

and the estimate

$$P_n(\lambda) = \mathcal{O}\left(\lambda^{[n/2]}\right), \quad \lambda \to \infty. \tag{3.25}$$

This expansion is, under mild conditions on $a_n(\kappa)$, that is, on f, uniformly valid with respect to $\lambda \in [0, \infty)$. For instance, when $f(t) = 1/(t+1)$, the coefficients $a_n(\kappa)$ are given by

$$\frac{1}{t+1} = \sum_{n=0}^{\infty} a_n(\kappa)\, (t-\kappa)^n, \quad a_n(\kappa) = \frac{(-1)^n}{(1+\kappa)^{n+1}}, \tag{3.26}$$

and we see from (3.25) that, in this case, the terms $a_n(\kappa)\ P_n(\lambda)\ z^{-n}$ in the expansion of $F_\lambda(z)$ given in (3.23) do not lose their asymptotic character when λ runs through the domain $[0, \infty)$. This is not a proof of the asymptotic nature of the expansion, but an indication that the expansion has some robustness with respect to large values of λ. For a proof we refer to TEMME [16].

We can apply this method by writing $B_\nu(z)$ in the form (3.6). This is possible when ν is negative. We observe that in that case we can integrate in (2.5) along both sides of the negative real axis (using different phases $\pm\pi i$ of t), and obtain

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$${}_{+1}/a_n = \mathcal{O}(1/\kappa) \quad \text{as} \quad \kappa \to \infty.$$

n this is the case, the expansion in of $F_\lambda(z)$ in (3.23) has a double roperty: it is also valid when $\lambda \to \infty$, uniformly with respect to where z_0 is a fixed positive number.

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