

# On Tree-Constrained Matchings and Generalizations<sup>\*</sup>

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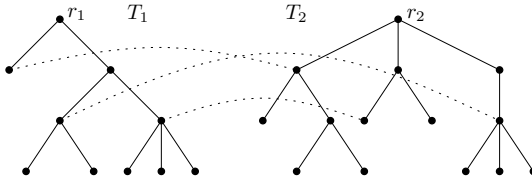
**Abstract.** We consider the following TREE-CONSTRAINED BIPARTITE MATCHING problem: Given two rooted trees  $T_1 = (V_1, E_1)$ ,  $T_2 = (V_2, E_2)$  and a weight function  $w : V_1 \times V_2 \mapsto \mathbb{R}_+$ , find a maximum weight matching  $\mathcal{M}$  between nodes of the two trees, such that none of the matched nodes is an ancestor of another matched node in either of the trees. This generalization of the classical bipartite matching problem appears, for example, in the computational analysis of live cell video data. We show that the problem is  $\mathcal{APX}$ -hard and thus, unless  $\mathcal{P} = \mathcal{NP}$ , disprove a previous claim that it is solvable in polynomial time. Furthermore, we give a 2-approximation algorithm based on a combination of the local ratio technique and a careful use of the structure of basic feasible solutions of a natural LP-relaxation, which we also show to have an integrality gap of  $2 - o(1)$ . In the second part of the paper, we consider a natural generalization of the problem, where trees are replaced by partially ordered sets (posets). We show that the local ratio technique gives a  $2k\rho$ -approximation for the  $k$ -dimensional matching generalization of the problem, in which the maximum number of incomparable elements below (or above) any given element in each poset is bounded by  $\rho$ . We finally give an almost matching integrality gap example, and an inapproximability result showing that the dependence on  $\rho$  is most likely unavoidable.

## 1 Introduction

This paper contains both approximation and hardness results for the TREE-CONSTRAINED BIPARTITE MATCHING (TCBM) problem, a natural generalization of the classical maximum matching problem in bipartite graphs. The input of TCBM consists of a weighted bipartite graph  $G = (V_1, V_2, E)$  and two rooted trees  $T_1$  and  $T_2$ . The vertex set of  $T_i$  is  $V_i$  for  $i = 1, 2$ . The objective is to find a maximum weight matching such that the matched vertices in each tree are not

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<sup>\*</sup> A full version of this paper is available as the technical report [5].



**Fig. 1.** Example of a feasible tree-constrained bipartite matching. For each matched pair of vertices, indicated by dotted lines, neither of their descendants are matched.

comparable; that is, if  $u, v \in V_i$  are matched then  $u$  cannot be  $v$ 's ancestor or vice-versa. Figure 1 illustrates the definition.

TCBM arises naturally in the computational analysis of live cell video data. Studying cell motility using live cell video data helps understand important biological processes, such as tissue repair, the analysis of drug performance, and immune system responses. Segmentation based methods for cell tracking typically follow a two stage approach (see [14] for a survey): The goal of the first *detection* step is to identify individual cells in each frame of the video independently. In a second step, the linkage of consecutive frames, and thus the *tracking* of a cell, is achieved by assigning cells identified in one frame to cells identified in the next frame. However, limited contrast and noise in the video sequence often leads to *over-segmentation* in the first stage: a single cell is comprised of several segments. A major challenge in this application domain is therefore the ability to distinguish biological cell division from over-segmentation.

Mosig et al. [10] and Xiao et al. [12] address this problem by proposing a novel approach for the *linkage* stage. As opposed to previous methods, they match *sets of segments* between neighboring frames rather than singletons, where the segment sets correspond to the nodes of an agglomerative hierarchical clustering tree. A subsequent bipartite matching between the nodes of the clustering trees corresponding to neighboring frames integrates the identification of a cell as a set of segments and the tracking of the cell between two different video frames. Since segment sets representing different cells in the same frame must be disjoint, no two nodes on any root-to-leaf path can be matched at the same time, leading to an instance of TCBM. To assess the quality of such a tree-constrained matching, Mosig et al. consider the relative overlap of the convex hulls of matched segment sets. This *cosegmentation* via TCBM promises to be useful also in other bioimaging applications, for example in protein-colocalization studies [12].

To track cells not only between two consecutive frames but across a whole video sequence, the bipartite graphs have to be concatenated to a *cell connection graph*, as introduced in [13]. In [10] this is done by solving a standard maximum weight bipartite matching problem for each frame  $i$ , which is at the intersection between the tree-constrained alignment of frames  $i - 1$  and  $i$ , and the alignment of frames  $i$  and  $i + 1$ . Concluding, the authors mention post-processing the cell connection graph as a promising improvement. Therefore, the generalization of

bipartite tree-constrained matching to a tree-constrained matching in a  $k$ -partite  $k$ -uniform hypergraph is an important problem for this application. By linking more than two frames at the same time, over-segmentation and cell-division can be distinguished by taking into account the cell behavior over a larger time-scale.

Another natural generalization of the problem is obtained by replacing trees by partially ordered sets (posets), because they permit the representation of alternative clustering hierarchies. For example, various meaningful distance measures between (sets of) segments could make it necessary to assign them to multiple parent clusters. In particular, noise in the video data may make it difficult to determine a unique tree.

Mosig et al. [10] present a linear programming formulation for TCBM and claim that the constraint matrix is totally unimodular, which would imply that the problem is solvable in polynomial time [11]. We disprove this statement by showing an instance with a fractional vertex and proving that the problem is in fact  $\mathcal{NP}$ -hard and even hard to approximate within a constant. Thus, conditional on  $\mathcal{P} \neq \mathcal{NP}$ , there is no polynomial time algorithm for our problem.

TCBM and its generalization to  $k$  trees are special cases of the maximum weighted independent set (MWIS) problem on 2-interval graphs and  $k$ -interval graphs, respectively. The connection is given by ordering the leaves of the trees by depth-first search and identifying each node with the interval of leaves below it. In fact, TCBM captures precisely the subclass of 2-union graphs (the first interval of a 2-interval cannot intersect the second interval of another 2-interval) where the two interval families are *laminar* (any two intervals are either disjoint or one is nested in the other). In [3] the fractional local ratio technique was developed and applied to MWIS in  $k$ -interval graphs to get a  $2k$ -approximation algorithm. This result immediately implies a 4-approximation for TCBM.

## 1.1 Our Results

In this paper, we give a 2-approximation algorithm for TCBM, improving upon the 4-approximation that follows from the work of Bar-Yehuda et al. [3]. Our method is based on a combination of the local ratio technique and a careful use of the structure of basic feasible solutions of a natural LP-relaxation. In Section 2.1 we show a 3-approximation based on fractional local ratio and prove that this is the best guarantee the fractional local ratio technique alone can deliver when rounding one coordinate at a time. The main difference between our approach and that of Bar-Yehuda et al. [3] is that we round basic feasible solutions. This allows us to exploit their structure in the analysis in order to get better approximation guarantees. In Section 2.2, we show how to get a 2-approximation and give an instance for which our LP-relaxation has an integrality gap of  $2 - o(1)$ . In Section 2.3, we show that the problem is  $\mathcal{APX}$ -hard. Our results imply that the MWIS problem on 2-union graphs in which both families of intervals are *laminar*, is still  $\mathcal{APX}$ -hard, but can be approximated within a factor of 2.

In Section 3, we consider the  $k$ -dimensional generalization of the problem to posets. In this case, the natural LP-relaxation has an exponential number of constraints, but admits an alternative linear-size LP-formulation. Even though the

result of Bar-Yehuda et al. [3] does not apply directly to the poset case, we show that the fractional local ratio technique yields a  $2 \sum_{i=1}^k \rho(\mathcal{P}_i)$ -approximation here, where  $\rho(\mathcal{P}_i)$  is the maximum number of incomparable elements below (or above) any given element in poset  $\mathcal{P}_i$ . We also give an example which shows that the integrality gap of the LP-relaxation is tight within almost a factor of 2. Finally, Section 3.2 gives a reduction from *Maximum Label Cover* showing that the 2-dimensional matching problem with poset constraints is hard to approximate within a factor of  $2^{\log^{1-\epsilon} \rho}$ , for any  $\epsilon > 0$ , where  $\rho = \max\{\rho(\mathcal{P}_1), \rho(\mathcal{P}_2)\}$ , unless  $\mathcal{NP} \subseteq \text{DTIME}(n^{\text{polylog} n})$ . Note that the  $k$ -dimensional version of TCBM includes as a special case the  $k$ -dimensional matching problem, and hence [8] is NP-hard to approximate within a factor of  $O(k/\log k)$ .

## 2 Matching Trees

In this section we focus on the basic TCBM problem, formally defined as follows:

**Definition 1 (Tree-constrained bipartite matching problem, TCBM).**

Given two rooted trees  $T_1 = (V_1, E_1)$ ,  $T_2 = (V_2, E_2)$  with roots  $r_1 \in V_1$  and  $r_2 \in V_2$ , and a weight function  $w : V_1 \times V_2 \mapsto \mathbb{R}_+$ , find a maximum weight matching  $\mathcal{M}$  in the complete bipartite graph  $G = (V_1, V_2, E)$  with edge weights induced by  $w$ , such that  $(u, v) \in \mathcal{M}$  implies  $(u', v') \notin \mathcal{M}$ , if  $u'$  is a descendant of  $u$  or  $v'$  is a descendant of  $v$ .

Consider an instance  $(T_1, T_2, w)$  of TCBM with  $T_1 = (V_1, E_1)$  and  $T_2 = (V_2, E_2)$ . Let  $r_1 \in V_1$  and  $r_2 \in V_2$  denote the roots and  $\mathcal{L}_1 \subset V_1$  and  $\mathcal{L}_2 \subset V_2$  denote the set of leaves of the trees, respectively. For two vertices  $p$  and  $q$  denote by  $[p, q]$  the path in  $T_1$  (or  $T_2$ ) between  $p$  and  $q$ . For each  $p_1 \in V_1$  and  $p_2 \in V_2$  let  $x_{p_1, p_2} \in \{0, 1\}$  be a variable which takes value 1 if and only if  $p_1$  is matched to  $p_2$ , i.e.  $(p_1, p_2) \in \mathcal{M}$ . Consider the following LP-relaxation of the problem.

$$\begin{aligned}
 \max \quad & \sum_{p_1 \in V_1, p_2 \in V_2} w_{p_1, p_2} x_{p_1, p_2} & (P) \\
 \text{s.t.} \quad & \sum_{p_1 \in [r_1, \ell], p_2 \in V_2} x_{p_1, p_2} \leq 1 & \text{for all } \ell \in \mathcal{L}_1 & (1) \\
 & \sum_{p_1 \in V_1, p_2 \in [r_2, \ell]} x_{p_1, p_2} \leq 1 & \text{for all } \ell \in \mathcal{L}_2 & (2) \\
 & x_{p_1, p_2} \geq 0 & \text{for all } p_1 \in V_1, p_2 \in V_2 .
 \end{aligned}$$

For the purpose of the analysis of the algorithm we will consider a more general LP formulation  $(P_b)$ , in which the right hand sides of constraints (1) and (2) are replaced by some  $b_\ell$ ,  $0 \leq b_\ell \leq 1$ , for all  $\ell \in \mathcal{L}_1 \cup \mathcal{L}_2$ . We will use  $x(F)$  to denote  $\sum_{e \in F} x_e$  for a subset of the edges  $F \subseteq E$  in the complete bipartite graph  $G$  and  $\delta(p)$  to denote the set of edges in  $E$  incident to a vertex  $p$ .

*Overview of the technique.* Our main tool will be the *fractional local ratio technique*, applied (recursively) to an optimal *basic* feasible solution (BFS)  $x^*$  of the above LP, where the base case of the recursion involves the computation of maximum weight independent sets in interval graphs. As long as there is a pair  $(p, q) \in V_1 \times V_2$  with "small fractional ratio", that is the total contribution  $\sum_{(p', q')} x_{p', q'}^*$ , over all pairs  $(p', q')$  that are in conflict with  $(p, q)$ , is at most 2, we can take this pair  $(p, q)$  into the solution and recurse on a new instance with reduced weights. Otherwise (if there is no such pair), we can use a structural result about the BFS's (Lemmas 1 and 3) to reduce the problem to computing maximal independent sets in interval graphs. In the next subsection, we prove this structural result, and show the fractional local ratio approach alone can give a 3-approximation, but not more. We then extend this to a 2-approximation in Section 2.2.

### 2.1 A 3-Approximation by Fractional Local Ratio

For a feasible fractional solution  $\mathbf{x}$  to linear program  $(P_{\mathbf{b}})$  and  $i \in \{1, 2\}$  we denote by  $\mathcal{L}_i(\mathbf{x})$  the set of nodes  $\ell \in V_i$  with  $x(\delta(\ell)) > 0$  and  $x(\delta(p)) = 0$  for all descendants  $p$  of  $\ell$  in  $T_i$ .

**Lemma 1.** *For any basic feasible solution  $\mathbf{x}$  to linear program  $(P_{\mathbf{b}})$  one of the following holds:*

- (a) *there exist nodes  $\ell_1 \in \mathcal{L}_1(\mathbf{x})$  and  $\ell_2 \in \mathcal{L}_2(\mathbf{x})$  such that  $x_{\ell_1, \ell_2} > 0$ , or*
- (b) *for every  $x_{p_1, p_2} > 0$  either  $p_1 \in \mathcal{L}_1(\mathbf{x})$  and  $x_{p_1, p'_2} = 0$ , for all  $p'_2 \neq p_2$ , or  $p_2 \in \mathcal{L}_2(\mathbf{x})$  and  $x_{p'_1, p_2} = 0$ , for all  $p'_1 \neq p_1$ .*

*Proof.* Assume (a) does not hold, i.e.  $x_{\ell_1, \ell_2} = 0$  for all  $\ell_1 \in \mathcal{L}_1(\mathbf{x})$  and  $\ell_2 \in \mathcal{L}_2(\mathbf{x})$ . For an arbitrary node  $p_1 \in V_1$ , any two constraints (1) for leaves in the subtree rooted at  $p_1$  linearly depend on the set of non-negativity constraints for variables  $x_{p'_1, p_2}$ , with  $p_2 \in V_2$  and  $p'_1$  being a descendant of  $p_1$ . Thus, every  $\ell_1 \in \mathcal{L}_1(\mathbf{x})$  implies at most one linear independent tight constraint (1) for one of the leaves in its subtree. Since a symmetric argument applies to nodes  $\ell_2 \in \mathcal{L}_2(\mathbf{x})$ , the number of linearly independent tight constraints (1), (2) for a given basic feasible solution  $\mathbf{x}$  is at most  $|\mathcal{L}_1(\mathbf{x})| + |\mathcal{L}_2(\mathbf{x})|$ .

On the one hand, since  $x(\delta(\ell)) > 0$  for all  $\ell \in \mathcal{L}_1(\mathbf{x}) \cup \mathcal{L}_2(\mathbf{x})$  and since no positive edge between two leaves exists, for every node in  $\mathcal{L}_1(\mathbf{x}) \cup \mathcal{L}_2(\mathbf{x})$  there is at least one distinct non-zero edge incident to it. On the other hand, in a basic feasible solution the number of non-zero variables is at most the number of linearly independent tight constraints (1), (2), which in turn is at most  $|\mathcal{L}_1(\mathbf{x})| + |\mathcal{L}_2(\mathbf{x})|$ . Therefore, exactly one distinct non-zero edge is incident to every node in  $\mathcal{L}_1(\mathbf{x}) \cup \mathcal{L}_2(\mathbf{x})$  and all other edges must be 0.  $\square$

Based on this property of basic feasible solutions to  $(P_{\mathbf{b}})$  we next show that there always exists an edge with local ratio at most 3. For this, let  $N[(p, q)]$  be the set of edges  $(p', q') \in E$  that are in conflict with edge  $(p, q)$  (including  $(p, q)$  itself), i.e.,

$$N[(p, q)] = \{(p', q') \in E \mid p' \in [r_1, p] \vee p \in [r_1, p'] \vee q' \in [r_2, q] \vee q \in [r_2, q']\} .$$

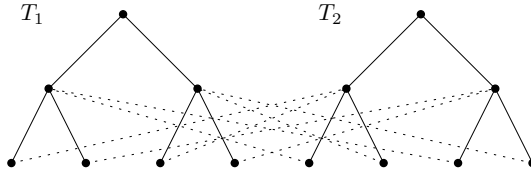


Fig. 2. Tight example for the fractional local ratio of an edge,  $k = 3$

**Lemma 2.** *Let  $\mathbf{x}$  be a basic feasible solution to  $(P_{\mathbf{b}})$ . There exists an edge  $(p, q) \in E$  with  $x_{p,q} > 0$  such that  $\sum_{(p',q') \in N[(p,q)]} x_{p',q'} \leq 3$ .*

The following lemma shows that we can apply Lemma 2 inductively in a local ratio framework.

**Lemma 3.** *Let  $\mathbf{x}$  be a basic feasible solution to  $(P_{\mathbf{b}})$ . For an edge  $(p, q)$  with  $x_{p,q} > 0$ , let  $\mathbf{b}'$  be such that  $b'_\ell = b_\ell - x_{p,q}$  if  $p \in [r_1, \ell]$ , respectively  $q \in [r_2, \ell]$ , and  $b'_\ell = b_\ell$  otherwise. Then  $\mathbf{x}'$  with  $x'_{p,q} = 0$  and  $x'_{e'} = x_{e'}$  for  $e' \neq (p, q)$ , is a basic feasible solution to  $(P_{\mathbf{b}'})$ .*

*Proof.* Since the right hand side of constraints in  $(P_{\mathbf{b}'})$  for all root to leaf paths through  $p$ , respectively  $q$ , are decreased by exactly  $x_{p,q}$ ,  $\mathbf{x}'$  is still feasible for  $(P_{\mathbf{b}'})$ . Suppose that  $\mathbf{x}'$  can be expressed as a convex combination of two feasible solutions  $\mathbf{y}$  and  $\mathbf{z}$  (of  $(P_{\mathbf{b}'})$ ). Then setting  $y_{p,q} = x_{p,q}$  and  $z_{p,q} = x_{p,q}$  would yield a convex combination of  $\mathbf{x}$  (in  $(P_{\mathbf{b}})$ ), a contradiction. Thus,  $\mathbf{x}'$  must be a basic feasible solution to  $(P_{\mathbf{b}'})$ .  $\square$

Now by applying the fractional local ratio technique of [3] we immediately obtain the following result.

**Theorem 1.** *There is a 3-approximation algorithm for TCBM.*

We next give an example instance that shows that using the fractional local ratio method, our approximation guarantee is tight.

**Lemma 4.** *There exists an instance of TCBM such that the optimal solution to  $(P)$  is unique and the fractional local ratio of every edge is at least  $3 - o(1)$ .*

*Proof.* Let  $T_1$  and  $T_2$  be trees of height two (i.e., they have three levels) where each internal node has  $k - 1$  children. The edges  $E$  connecting the nodes of  $T_1$  and  $T_2$  are as follows. Let  $x$  be a leaf of  $T_1$ . If  $x$  is the  $i$ th child of its parent in  $T_1$ , we connect  $x$  with the  $i$ th child of the root of  $T_2$ . We connect in a similar fashion the leaves of  $T_2$  with the children of the root of  $T_1$ . All edges have a weight of 1. Figure 2 illustrates the construction for  $k = 3$ .

It can be verified that the optimal fractional solution must set the value of every edge to  $\frac{1}{k}$ . Let  $(u, v)$  be an edge where  $u$  is a leaf and  $v$  is an internal node. Notice that  $(u, v)$  is in conflict with  $k - 1$  edges incident on children of  $v$ ,  $k - 2$  edges incident on  $v$  and  $k - 1$  edges incident on  $u$ 's parent. Since each edge carries a fractional contribution of  $\frac{1}{k}$  their combined fractional value is  $3 - \frac{4}{k}$ .  $\square$

**Algorithm 1.** Tree-Matching( $F, \mathbf{w}$ )

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**Require:** A BFS  $\mathbf{x}$  to  $(P_{\mathbf{b}})$ ,  $0 \leq \mathbf{b} \leq 1$ , edge set  $F$ , and weights on the edges  $\mathbf{w}$ .

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1: if  $F = \emptyset$  then
2:   return  $\emptyset$ 
3: Define  $F_0 = \{e \in F \mid w_e \leq 0\}$ 
4: if  $F_0 \neq \emptyset$  then
5:   return Tree-Matching( $F \setminus F_0, w$ )
6: if  $\forall e \in F : x(N[e]) > 2$  then
7:    $\mathcal{I}_1 \leftarrow$  MWIS in  $\text{IG}(T_1)$  w.r.t. weights  $\bar{w}_p^1 = \max_{(p,q) \in F} w_{p,q}$ 
8:    $\mathcal{I}_2 \leftarrow$  MWIS in  $\text{IG}(T_2)$  w.r.t. weights  $\bar{w}_q^2 = \max_{(p,q) \in F} w_{p,q}$ 
9:   if  $\bar{w}^1(\mathcal{I}_1) \geq \bar{w}^2(\mathcal{I}_2)$  then
10:    return  $\bigcup_{p \in \mathcal{I}_1} \{\text{argmax}_{(p,q) \in F} w_{p,q}\}$ 
11:   else
12:    return  $\bigcup_{q \in \mathcal{I}_2} \{\text{argmax}_{(p,q) \in F} w_{p,q}\}$ 
13: else
14:   Let  $e' \in F$  be s.t.  $x(N[e']) \leq 2$ 
15:   Decompose  $\mathbf{w}$  by  $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2$  where  $w_e^1 := \begin{cases} w_{e'} & \text{if } e \in N[e'], \\ 0 & \text{otherwise.} \end{cases}$ 
16:    $\mathcal{M}' \leftarrow$  Tree-Matching( $F, \mathbf{w}^2$ )
17:   if  $N[e'] \cap \mathcal{M}' = \emptyset$  then
18:    return  $\mathcal{M} = \mathcal{M}' \cup \{e'\}$ 
19:   else
20:    return  $\mathcal{M} = \mathcal{M}'$ 

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**2.2 A 2-Approximation**

The algorithm requires a basic feasible solution  $\mathbf{x}$  to (P) and is initially called with an edge set  $F$ , in which all edges  $e$  with  $x_e = 0$  have been removed. The idea is to recurse in a local ratio manner as long as we can find an edge with local ratio at most two. If this is not possible anymore, we exploit the specific structure of basic feasible solutions by computing *maximum weight independent sets* (MWIS) in the interval graphs  $\text{IG}(T_1)$  and  $\text{IG}(T_2)$  induced by the two trees: For  $i \in \{1, 2\}$ ,  $\text{IG}(T_i)$  is the interval graph obtained by ordering the leaves of  $T_i$  by depth-first search and identifying each node of  $T_i$  with the interval of leaves below it. As usual, we define the maximum (in lines 7-8) over an empty set to be 0. It is not difficult to see that the matching  $\mathcal{M}$  returned by the algorithm is feasible for (P). It remains to assess the quality of this solution.

**Theorem 2.** *Let  $\mathbf{x}$  be a basic feasible solution to linear program  $(P_{\mathbf{b}})$ . The matching  $\mathcal{M}$  returned by Algorithm 1 satisfies  $w(\mathcal{M}) \geq \frac{1}{2} \cdot \mathbf{w} \cdot \mathbf{x}$ .*

*Proof.* The proof is by induction on the number of edges having positive weight. In the base case either there is no edge with positive weight (lines 1-2) or no edge  $e'$  in line 14 exists. In the former case, the induction hypothesis clearly holds. In the latter case, according to Lemma 1 the non-zero edges can be partitioned into sets  $F_1$  and  $F_2$ , containing edges with one endpoint in  $\mathcal{L}_1(\mathbf{x})$ , respectively one endpoint in  $\mathcal{L}_2(\mathbf{x})$ . For  $i \in \{1, 2\}$ , let

$$\mathcal{P}_{IG}(T_i) := \left\{ \mathbf{y} \in \mathbb{R}_+^{V_i} : \sum_{p \in V_i: p \in [r_i, \ell]} y_p \leq 1, \text{ for all } \ell \in \mathcal{L}_i \right\}$$

be the fractional independent set polytope in the interval graph represented by tree  $T_i$ . It is well-known (see e.g. [7]) that  $\mathcal{P}_{IG}(T_i)$  is integral. Given the basic feasible solution  $\mathbf{x}$  of  $(P_{\mathbf{b}})$ , define  $\mathbf{y}^i \in \mathbb{R}^{V_i}$ , for  $i \in \{1, 2\}$ , as follows:  $y_p^i = \sum_{q:(p,q)} x_{p,q}$ , for  $p \in V_i$ . The feasibility of  $\mathbf{x}$  to  $(P_{\mathbf{b}})$  implies that  $\mathbf{y}^i \in \mathcal{P}_{IG}(T_i)$ , for  $i \in \{1, 2\}$ . Let  $\mathcal{I}_1$  and  $\mathcal{I}_2$  be the independent sets computed in steps 7 and 8 of the algorithm, and  $\mathcal{M}'$  be the matching computed in step 10 or 12. Then

$$\begin{aligned} \mathbf{w} \cdot \mathbf{x} &= \sum_{(p,q) \in F_1} w_{p,q} x_{p,q} + \sum_{(p,q) \in F_2} w_{p,q} x_{p,q} \\ &\leq \sum_{p \in V_1} \bar{w}_p^1 \sum_{(p,q) \in F_1} x_{p,q} + \sum_{q \in V_2} \bar{w}_q^2 \sum_{(p,q) \in F_2} x_{p,q} \end{aligned} \tag{3}$$

$$= \sum_{p \in V_1} \bar{w}_p^1 y_p + \sum_{q \in V_2} \bar{w}_q^2 y_q \leq \max_{\mathbf{y}' \in \mathcal{P}_{IG}(T_1)} \bar{\mathbf{w}}^1 \cdot \mathbf{y}' + \max_{\mathbf{y}' \in \mathcal{P}_{IG}(T_2)} \bar{\mathbf{w}}^2 \cdot \mathbf{y}' \tag{4}$$

$$= \bar{w}(\mathcal{I}_1) + \bar{w}(\mathcal{I}_2) \leq 2 \cdot \max\{\bar{w}(\mathcal{I}_1), \bar{w}(\mathcal{I}_2)\} = 2 \cdot w(\mathcal{M}'). \tag{5}$$

Inequality (3) follows from the definition of the weights  $\bar{\mathbf{w}}^1$  and  $\bar{\mathbf{w}}^2$  in lines 7, 8; inequalities (4) and (5) follow respectively from the fact that  $\mathbf{y}^i \in \mathcal{P}_{IG}(T_i)$ , and the integrality of  $\mathcal{P}_{IG}(T_i)$ , for  $i \in \{1, 2\}$ ; and the last equality is due to the choice the algorithm makes in line 9. Note that the matchings constructed in lines 10 respectively 12 are feasible solutions to the TCBM problem. Indeed, due to the structure of a basic feasible solution (Lemma 1 which will also continue to hold inductively by Lemma 3), the edges that induce an independent set in one tree end in leaves of the second tree and therefore do not conflict.

We next prove the inductive step (the rest of the argument is the same as in [2]; we include it for completeness). If  $F_0$  is non-empty in step 4, extending  $\mathbf{w}$  in the induction hypothesis by the non-positive components that were deleted in line 5 cannot make the inequality invalid. Let  $\mathbf{y}'$  and  $\mathbf{y}$  be the characteristic vectors of matchings  $\mathcal{M}'$  and  $\mathcal{M}$ , obtained in lines 16 and lines 18-20, respectively. Let  $e'$  be the edge chosen in line 14. By the decomposition of  $\mathbf{w}$  in line 15,  $\mathbf{w}^2$  implies a smaller number of edges with positive weight than  $\mathbf{w}$ . By the induction hypothesis,  $\mathbf{w}^2 \cdot \mathbf{y}' \geq \frac{1}{2} \cdot \mathbf{w}^2 \cdot \mathbf{x}$ . From  $w_{e'}^2 = 0$  it also follows that  $\mathbf{w}^2 \cdot \mathbf{y} \geq \frac{1}{2} \cdot \mathbf{w}^2 \cdot \mathbf{x}$ . Since at least one edge in  $N[e']$  is in  $\mathcal{M}$  and  $x(N[e']) \leq 2$  (line 14), it also holds that  $\mathbf{w}^1 \cdot \mathbf{y} \geq \frac{1}{2} \cdot \mathbf{w}^1 \cdot \mathbf{x}$ . The claim follows.  $\square$

We conclude this section by giving an example showing that the integrality gap of (P) matches the approximation factor attained by our algorithm.

**Lemma 5.** *The integrality gap of (P) is  $2 - o(1)$ .*

*Proof.* Our bad instance consists of two stars of height 1 (i.e., they have two levels) where each internal node has  $k - 1$  children. The leaf nodes of one star are connected to the root node of the other star. An integral solution can pick at most one edge. However, a fractional solution that sets the value of every edge to  $\frac{1}{k}$  is feasible and has value  $2 - \frac{2}{k}$ .  $\square$



### 2.3 Hardness and Inapproximability Results

In this section we prove hardness of tree-constrained bipartite matching even if the weights in the matching are restricted to the values zero and one. Subsequently, we show by an approximation-preserving reduction from a restricted MAX SAT version that TCBM does not admit a PTAS.

**Theorem 3.** *For an instance  $I = (T_1, T_2, w)$  of TCBM, with  $w : V_1 \times V_2 \mapsto \{0, 1\}$ , and an integer  $k$ , it is  $\mathcal{NP}$ -complete to decide whether there exists a tree-constrained bipartite matching of weight at least  $k$ .*

*Proof.* Clearly, the problem is in  $\mathcal{NP}$ . To prove that it is  $\mathcal{NP}$ -hard, we devise a polynomial-time reduction  $\tau$  from SAT, the problem of deciding whether a Boolean formula has a satisfying assignment. Given a CNF formula  $\phi$  with  $m$  clauses over  $n$  variables, we construct a TCBM instance  $I = (T_1, T_2, w)$ , such that  $\phi$  is satisfiable if and only if  $I$  admits a matching of weight  $n + m$ .

Tree  $T_1$  is the star  $S_{n+m}$ , with one leaf per variable and clause. Depth-2 tree  $T_2$  has for each literal occurring in  $\phi$  a node at level 1. Such a node, corresponding to some literal  $l_k$ , has a child node for each occurrence of literal  $l_k$  in  $\phi$ . What remains is the definition of the weight function  $w$ . For each leaf  $u$  in  $T_1$ , representing some variable  $x_i$ , we define  $w(u, v) := 1$  and  $w(u, v') := 1$ , where  $v, v'$  are level-1 nodes in  $T_2$  that correspond to literals  $x_i$  and  $\neg x_i$ , respectively. For each leaf  $u$  in  $T_1$ , representing some clause  $C_i$  of  $\phi$ , we set, for all literals  $l_j$  occurring in  $C_i$ ,  $w(u, v) := 1$ , where  $v$  is a child node of a level-1 node in  $T_2$  that corresponds to literal  $l_j$ . Hereby we pick distinct child nodes  $v$  in  $T_2$ , such that each level-2 node is incident to exactly one edge of weight 1. All remaining weights are set to 0. See Figure 3 for an illustrative example of this construction.

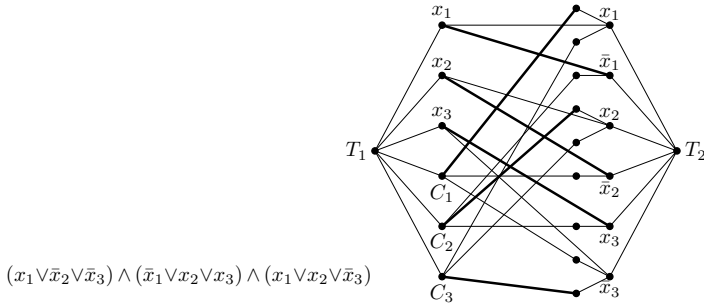
Finally, it can be shown that  $\phi$  is satisfiable if and only if  $\tau(\phi)$  admits a matching  $\mathcal{M}$  of weight  $m + n$ .  $\square$

We next prove that TCBM is  $\mathcal{APX}$ -hard. The reduction is made from 3-OCC-MAX 2SAT, a restricted form of MAX SAT, where each clause contains two literals and each variable occurs at most three times.

**Theorem 4** ([4]). *For any  $\epsilon > 0$  it is  $\mathcal{NP}$ -hard to decide whether an instance of 3-OCC-MAX 2SAT with  $2016n'$  clauses (and  $1344n'$  variables) has a truth assignment that satisfies at least  $(2012 - \epsilon)n'$  clauses, or at most  $(2011 + \epsilon)n'$ .*

**Theorem 5.** *For any  $\epsilon > 0$ , it is  $\mathcal{NP}$ -hard to approximate TCBM within factor  $6044/6043 - \epsilon$ .*

*Proof.* Our reduction  $\tau'$  from 3-OCC-MAX 2SAT to TCBM differs from reduction  $\tau$  described in the proof of Theorem 3 only in the definition of the weight function  $w$ . For leaves  $u$  in  $T_1$  representing some variable  $x_i$  and level-1 nodes  $v$  in  $T_2$  corresponding to literal  $x_i$  or  $\neg x_i$ , we set  $w(u, v) := 3$ , and leave  $w$  unchanged otherwise. Then it is easy to see that the maximum number of satisfiable clauses in  $\phi$  is  $k$  if and only if the maximum weight of a tree-constrained matching in  $\tau'(\phi)$  is  $3n + k$ . Since the instance constructed in [4] uses  $1344n'$  variables,  $k = (2012 - \epsilon)n'$  and  $k = (2011 + \epsilon)n'$  correspond to tree constrained matchings of weight  $(6044 - \epsilon)n'$  and  $(6043 + \epsilon)n'$  respectively.  $\square$



**Fig. 3.** Left: SAT instance in conjunctive normal form. Right: Transformed tree-constrained bipartite matching instance, only edges with unit weight are shown. A maximum weight tree-constrained bipartite matching, which corresponds to a satisfying truth assignment, is shown in bold.

### 3 Matching Posets

**Definition 2 (Poset-constrained  $k$ -partite matching problem,  $k$ -PCM).** Given  $k$  posets  $\mathcal{P}_i = (V_i, \leq_i)$ ,  $1 \leq i \leq k$ , and a weight function  $w : V_1 \times V_2 \times \dots \times V_k \mapsto \mathbb{R}_+$ , find a maximum weight  $k$ -dimensional matching  $\mathcal{M}$  in the complete  $k$ -partite  $k$ -uniform hypergraph  $H = (V_1, \dots, V_k, E)$  with hyper-edge weights induced by  $w$ , such that  $(p_1, \dots, p_k) \in \mathcal{M}$  implies  $(q_1, \dots, q_k) \notin \mathcal{M}$  if there exists  $1 \leq i \leq k$  with  $q_i$  being comparable to  $p_i$  in  $\mathcal{P}_i$ .

#### 3.1 A Fractional Local Ratio Algorithm

Unlike the tree case, this problem cannot be directly reduced to MWIS in  $k$ -interval graphs, and therefore, the  $2k$ -approximation of Bar-Yehuda et al. [3] does not readily apply. However, we show that the fractional local ratio technique can still be used to solve the poset case. We work with the following linear programming formulation.

$$\begin{aligned}
 & \max \sum_{p \in E} w_p x_p && \text{(MP)} \\
 \text{s.t.} \quad & \sum_{p: p_i \in C} x_p \leq 1 && \forall \text{ chain } C \text{ in } \mathcal{P}_i, i = 1, \dots, k && (6) \\
 & x_p \geq 0 && \forall p \in E .
 \end{aligned}$$

We remark that even though the above linear program is exponentially large, there is a simple separation oracle based on the polynomial time algorithm for computing a longest path in an acyclic directed graph. In fact, there is an alternative linear-size LP formulation, see[5].

As we did before in the tree case, the crux of the analysis is to show that there is an edge  $p \in E$  with low fractional local ratio. Our bound will depend on the *maximum upward independence number* of the individual posets.

**Definition 3.** For a given poset  $\mathcal{P} = (V, \preceq)$ , the maximum upward independence number of  $\mathcal{P}$ , denoted by  $\rho(\mathcal{P})$  is defined as

$$\max_{v \in V} \text{size of largest antichain in } (\{u \in V : v \preceq u\}, \preceq).$$

We show that the fractional local ratio of any feasible solution is bounded by the maximum upward independence number of the posets in our instance.

**Lemma 6.** Let  $\mathbf{x}$  be a feasible solution to (MP). There is some  $p \in E$  such that

$$\sum_{q : \exists i \bullet q_i \preceq_i p_i \vee p_i \preceq_i q_i} x_q \leq 2 \sum_i \rho(\mathcal{P}_i).$$

Notice that if we consider the poset  $\mathcal{P}$  induced by some tree  $T$ , the maximum upward independence number is 1. This is because for any vertex  $v$  of  $T$ , the poset induced by  $\{u \in T : v \preceq u\}$  is a total order; namely, the path from  $v$  to the root of the tree. Using the fractional local ratio framework of Bar-Yehuda et al. [3] we immediately obtain the following result.

**Theorem 6.** There is a  $2 \sum_i \rho(\mathcal{P}_i)$  approximation algorithm for  $k$ -PCM.

It can be shown that the dependency on  $\sum_i \rho(\mathcal{P}_i)$  in the approximation ratio is necessary for any algorithm based on the linear program (MP).

**Lemma 7.** There are instances of  $k$ -PCM where the integrality gap of (MP) is  $(1 - \frac{1}{k}) \sum_i \rho(\mathcal{P}_i)$  for arbitrary large  $k$ .

### 3.2 Hardness

In this section we show that the dependence of the approximation factor on the maximum poset width  $\rho(\mathcal{P})$  is unavoidable, by showing that, under plausible complexity assumptions, 2-PCM is hard to approximate within  $2^{\log^{1-\epsilon} \rho}$  for any  $\epsilon > 0$ , where  $\rho$  is the maximum width of the posets.

We will use a reduction from the *maximum label-cover* problem [1]. For convenience we use the following equivalent definition [9].

**Definition 4 (MAXREP).** Given a bipartite graph  $G = (A, B, E)$ , with a partition of  $A$  and  $B$  into  $k$  disjoint sets  $A_1, \dots, A_k$  and  $B_1, \dots, B_k$ , respectively, find subsets of vertices  $A' \subseteq A$  and  $B' \subseteq B$ , such that,  $|A' \cap A_i| \leq 1$  and  $|B' \cap B_i| \leq 1$ , for  $i = 1, \dots, k$ , so as to maximize the number of edges

$$E(A', B') := \{\{a, b\} \in E : A' \cap A_i = \{a\} \text{ and } B' \cap B_j = \{b\} \text{ for some } i, j\}.$$

**Theorem 7 ([6, 9]).** MAXREP on a graph with  $|A| = |B| = n$  cannot be approximated within a factor of  $2^{\log^{1-\epsilon} n}$ , for any  $\epsilon > 0$ , unless  $NP \subseteq DTIME(n^{\text{polylog } n})$ .

**Theorem 8.** For any  $\epsilon > 0$  and  $\rho := \max\{\rho(\mathcal{P}_1), \rho(\mathcal{P}_2)\}$  there is no  $2^{\log^{1-\epsilon} \rho}$ -factor approximation for 2-PCM unless  $NP \subseteq DTIME(n^{\text{polylog } n})$ .

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