Observation and Evolution of Finite-dimensional Markov Systems

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1 Markov systems

A system $S$ is an entity that can be in one of several states. Let $S$ be the set of states of $S$. An $n$-dimensional Markov representation is an injective map $\rho : S \to Q$ onto an affine hyperplane $Q$ of an $n$-dimensional Hilbert space $H$ over $\mathbb{R}$. We denote the inner product in $H$ by $\langle x|y \rangle$ and assume

$$Q = \{ x \in H \mid \tau(x) = 1 \},$$

where $\tau : H \to \mathbb{R}$ is a linear functional. Given the representation $\rho$, we identify $S$ with $Q$ and speak of $Q$ as the collection of (Markov) states of $S$.

An $n$-dimensional Markov system $S$ admits a standard representation $\sigma : S \to Q$ into the euclidean coordinate space $\mathbb{R}^n$ with inner product

$$\langle x|y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i \quad \text{for all } x^T = (x_1, \ldots, y_n), y^T = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$  

and, with $1^T := (1,1,\ldots,1)$, the affine hyperplane

$$Q = \{ x \in \mathbb{R}^n \mid \tau(x) = 1^T x = x_1 + \ldots + x_n = 1 \}.$$  

However, also other representations are of interest to the mathematical modeler:

1.1 Quantum Markov systems

Motivated by the classical model of $m$-dimensional quantum systems, consider the (complex) Hilbert space $\mathbb{C}^{m \times m}$ of complex $(m \times m)$-matrices with inner product

$$\langle C|D \rangle = \text{tr}(D^* C),$$

where $D^*$ is the conjugate transpose of $D$ and $\text{tr}(A)$ denotes the trace of a matrix $A$. Recall that a matrix $C$ is self-adjoint (or hermitian) if $C = C^*$ and let $\mathcal{H}$ denote the collection of all self-adjoint $(m \times m)$-matrices $C$. It is not difficult to see that $\mathcal{H}$ forms a real(!) Hilbert space of dimension $n = m^2$. Letting $I$ denote the identity matrix of $\mathbb{C}^{m \times m}$, we call the members of the hyperplane

$$\mathcal{D} = \{ D \in \mathcal{H} \mid \text{tr}(D) = \langle D|I \rangle = 1 \}$$

Markov density matrices and refer to a system with states corresponding to Markov density matrices a Markov quantum system.
1.2 Quantum activity systems and quantum bits

While classical computation is based on boolean bits, quantum computation (see, e.g., [8]) models activities by quantum bits ("qbits"), where one qbit has the form

\[ q = \alpha |0\rangle + \beta |1\rangle \quad \text{with } \alpha, \beta \in \mathbb{C} \text{ s.t. } |\alpha|^2 + |\beta|^2 = 1. \]

The qbit \( q \) has the interpretation that \( |0\rangle \) is observed with probability \( |\alpha|^2 \geq 0 \) and \( |1\rangle \) with probability \( |\beta|^2 = 1 - |\alpha|^2 \geq 0 \).

An \( n \)-dimensional quantum activity system is the \( n \)-fold tensor product \( A = A_1 \otimes \cdots \otimes A_n \) of 1-dimensional quantum activity systems \( A_i \). An \( n \)-dimensional quantum activity state ("\( n \)-qbit") is therefore of the form

\[ q = \sum_{k \in \{0, 1\}^n} \alpha_k |k\rangle \quad \text{with } \alpha_k \in \mathbb{C} \text{ and } \sum_k |\alpha_k|^2 = 1 \]

(1)

and corresponds to the parameter vector \( v = (\alpha_k |k\rangle \in \mathbb{C}^{2^n} \) with (squared) norm

\[ ||v||^2 = v^* v = |\alpha_1|^2 + \ldots + |\alpha_n|^2 = 1. \]

Note that an \( n \)-qbit \( q \) in the form (1) cannot directly be interpreted a Markov state in standard form. The associated matrix \( Q = vv^* \) is self-adjoint with trace

\[ \text{tr}(vv^*) = v^* v = |\alpha_1|^2 + \ldots + |\alpha_n|^2 = 1 \]

and hence a Markov density (in fact, a classical quantum density).

1.3 Pseudo-boolean functions and cooperative games

A real-valued set function \( v : 2^N \to \mathbb{R} \) is a pseudo-boolean function (see [6]). Identifying the subsets \( K \subseteq N \) with their associated boolean states \( |k\rangle \), a pseudo-boolean function \( v \) can be viewed as a formal linear combination

\[ v = \sum_{k \in \{0, 1\}^n} \alpha_k |k\rangle \]

with the coefficients \( \alpha_k = v(K) \).

From a game theoretic point of view, the pair \( \Gamma = (N, v) \) is a cooperative game with characteristic function \( v \). The parameter \( v(K) \) is thought to reflect the "value" of the coalition \( K \subseteq N \) in a given economic context. It is reasonable to assume that the game \( \Gamma \) is scaling-invariant. So we might equally well study the normalized game \((N, \tilde{v})\), where

\[ \tilde{v} = \begin{cases} 0 & \text{if } v \equiv 0 \\ v/\|v\|^2 & \text{if } \|v\|^2 = \sum_{K \subseteq N} v(K)^2 \neq 0 \end{cases} \]

and think of a non-trivial cooperative game as a qbit with real coefficients.

**Remark 1.1.** The Hadamard transformation \( H \) of a a 1-qbit is the linear transformation

\[ |k_1 \ldots k_n\rangle \to H|k_1\rangle \otimes \cdots \otimes H|k_n\rangle \quad \text{for } (k_1 \ldots k_n) \in \{0, 1\}^2. \]

The Hadamard coefficients \( \hat{\alpha}_k \) of \( v \) correspond to the Banzhaf indices (see [2]), well-known in social choice theory. (See, e.g., [7] for more applications of the Hadamard transformation to social choice problems and [5] for more on interaction indices).
2 Observables and measurements

Returning to the general Markov state model with the $n$-dimensional Hilbert space $\mathcal{H}$ and $Q = \{v \in \mathcal{H} \mid \tau(v) = 1\}$ relative to the system $\mathcal{S}$, let us fix a particular basis $B \subseteq Q$.

**Remark 2.1.** We think of $B$ as the set of representatives of the "ground states" of $\mathcal{S}$.

We call a function $X : B \rightarrow \{0, 1\}$ an *information function*. So $X$ models a "property" ground states $b \in B$ may or may not have. Extending $X$ linearly to all of $\mathcal{H}$, $X$ corresponds to an element $x \in \mathcal{H}$ such that

$$\langle x | b \rangle = X(b) \quad \text{for all } b \in B.$$ 

Assume that $\mathcal{S}$ happens to be in the Markov state $q = \sum_{b \in B} q_b b$ and define

$$\pi^q(r) = \sum_{b \in B : X(b) = r} q_b \quad (r = 0, 1).$$ 

We call $X$ (statistically) observable in the state $q$ if $\pi^q(r) \geq 0$ holds for $r = 0, 1$.

3 Evolution of Markov systems

A *Markov (evolution) operator* relative to the Markov system $\mathcal{S}$, represented as the hyperplane $Q$ of the Hilbert space $\mathcal{H}$ is a linear transformation $\mu : \mathcal{H} \rightarrow \mathcal{H}$ such that $\mu(q) \in Q$ holds for all $q \in Q$.

A *(generalized) Markov chain* is a pair $(\mu, q^{(0)})$ where $\mu$ is a Markov operator and $q$ a Markov state. The pair $(\mu, q^{(0)})$ stands short for the Markov evolution of states in discrete time when the Markov system $\mathcal{S}$ is in state $q^{(0)}$ at time $t = 0$:

$$q^{(t)} = \mu(q^{(t-1)}) = \mu^t(q^{(0)}) \quad \text{for } t = 1, 2, \ldots.$$ 

Examples of Markov chains relative to the standard representation are, of course, classical Markov chains, where $\mu$ is represented by a probability transition matrix.

Other examples arise from the Schrödinger wave evolution in quantum activity systems.

3.1 Evolution and measurement

The concept of a measurement can be naturally be put into context with evolution. We call a family $X = \{\mu_r \mid r \in R\}$ of linear operators $\mu_r : \mathcal{H} \rightarrow \mathcal{H}$ a *Markov measurement* with (finite) scale $R$ iff

$$\mu_X := \sum_{r \in R} \mu_a \quad \text{is a Markov operator.} \quad (3)$$

In light of (3), we write $(X, q)$ as a unifying notation for both a Markov measurement $X$ and an associated Markov chain $(\mu_X, q)$ and refer to it as a *Markov measurement chain*. A Markov measurement chain is *invariant* if $\mu_X(q) = q$.

Now consider concatenating measurements $(w := r_1\ldots r_n)$

$$\mu_w(q) := \mu_{r_n}(\ldots(\mu_{r_1}(q))\ldots)$$
and observe that, by multinomial expansion, $\mu'_X = \sum_{w \in \mathcal{R}} \mu_w$. We call a Markov measurement chain $(X, q)$ (statistically) observable iff

$$\tau(\mu_w(q)) \geq 0 \quad \text{for all} \quad w \in \mathcal{R}^*.$$ 

### 3.2 Equivalence and minimality of Markov measurements

We call two Markov measurement chains $X_1 = (\{\mu_r : \mathcal{H}_1 \to \mathcal{H}_1 \mid r \in \mathcal{R}\}, q_1)$ and $X_2 = (\{\rho_r : \mathcal{H}_2 \to \mathcal{H}_2 \mid r \in \mathcal{R}\}, q_2)$ where, possibly, $\dim \mathcal{H}_1 \neq \dim \mathcal{H}_2$, equivalent iff

$$\tau_1(\mu_{\bar{r}}(q_1)) = \tau_2(\mu_{\bar{r}}(q_2)) \quad \text{for all} \quad \bar{r} \in \mathcal{R}^* = \sum_{t \geq 0} \mathcal{R}^t.$$ 

We write

$$(X_1, q_1) \sim (X_2, q_2)$$

in that case.

We call a Markov measurement chain $(X, q)$ on $\mathcal{H}$ minimal iff $\dim \mathcal{H}$ is minimal among all Markov measurement chains that are equivalent to $(X, q)$. (See also [4] for details on how to perform equivalence tests efficiently.)

### 3.3 Decomposition of Markov measurements

We present the following new theorem:

**Theorem 3.1** (Decomposition of invariant Markov measurement chains). Let $X = (\{\mu_r : \mathcal{H} \to \mathcal{H} \mid r \in \mathcal{R}\}, q)$ be a minimal, observable, invariant Markov measurement chain. Let $d := \dim(\text{Eig}_{\mu_X}(1))$. Then there are minimal, observable, invariant Markov measurement chains

$$(i) \quad q = q_1 + \ldots + q_d$$

$$(ii) \quad (X, q) \sim (X_1, q_i)$$

$$(iii) \quad \dim(\text{Eig}_{\mu_{X_i}}(1)) = 1.$$ 

$$(iv) \quad \mathcal{H} \cong \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_d.$$ 

**Remark 3.2.** $\dim \text{Eig}_{\mu_X}(1) \geq 1$, see [3].

One may perceive this theorem as a building block for a unifying theory of classification for, for example, hidden Markov processes, quantum random walks and action-based cooperation systems emerging from game theory [10].
References


