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A Calculus of Transition Systems (towards Universal Coalgebra)

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ABSTRACT. By representing transition systems as coalgebras, the three main ingredients of their theory: coalgebra, homomorphism, and bisimulation, can be seen to be in a precise correspondence to the basic notions of universal algebra: Σ-algebra, homomorphism, and substitutive relation (or congruence). In this paper, some standard results from universal algebra (such as the three isomorphism theorems and facts on the lattices of subalgebras and congruences) are reformulated (using the afore mentioned correspondence) and proved for transition systems.

1 Introduction

A transition system is usually defined as a set together with a relation on that set. It is a simple observation, possibly first made in Kent 1987 and Aczel 1988, that—equivalently—a transition system can be represented as a *coalgebra* by viewing its relation as a (nondeterministic) function. This representation gives rise to a natural (and standard) notion of *homomorphism* of transition systems. Moreover, a *bisimulation relation* simply turns out to be a coalgebra with some special properties (Aczel and Mendler 1989).

The general definition of coalgebra is dual to that of *algebra*, which has many well-known instances such as groups, rings, etc. The features common to all of these examples are subject of a renowned field of mathematics called *universal algebra*. The central notions there are Σ-*algebra*, homomorphism of Σ-algebras, and *congruence*. It has been observed in Rutten and Turi 1994 that, on the coalgebra side, the corresponding notions are: transition system, homomorphism of transition systems, and bisimulation equivalence. (More generally, the notion of *substitutive relation* corresponds to that of bisimulation relation; hence congruences, which

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are substitutive *equivalence* relations, correspond to bisimulation equivalences.) More about the precise nature of this correspondence is to be found in Section 11.

The aim of this paper can be summarized as an attempt to understand how much of universal algebra can be done for transition systems. Our approach has been, rather naively, to take a textbook on universal algebra (actually two: Cohn 1981 and Meinke and Tucker 1992); reformulate definitions and theorems there by replacing everywhere: Σ-algebra, homomorphism of Σ-algebras, and congruence, by: transition system, homomorphism of transition systems, and bisimulation equivalence, respectively; and see whether the resulting statements could actually be proved.

As a result, many of the familiar facts on Σ-algebras turn out to be valid (in their translated version) for transition systems as well. For certain notions sometimes more and sometimes less can be stated and proved. (Examples are *simple* transition systems and the lattice of bisimulations, respectively.) Many of the facts thus found are well-known as theorems in the literature (cf. Sifakis 1984, Badouel 1993, Rutten and Turi 1994), others are typically folklore, and some seem to be new. As in universal algebra, most proofs are easy and consequently often omitted.

This programme is, to some extent, first carried out for one particular family of transition systems: unlabelled and nondeterministic (also called frames). (The absence of labels is just for convenience; all what follows can straightforwardly be adapted for labelled systems.) After mentioning some other examples of transition systems (in Section 9), a modest attempt is made (in Section 10) to generalize the results on transition systems in such a way that they apply to these other examples as well. It is argued that a general theory of transition systems (as coalgebras) has to be categorical, because different examples involve different *functors*.

Deep insights about groups are not obtained by studying universal algebra. Nor will universal coalgebra lead to difficult theorems about (specific types of) transition systems. Like universal algebra, its possible merit consists of the fact that it '... tidies up a mass of rather trivial detail, allowing us to concentrate our powers on the hard core of the problem.' (Cohn 1981).

2 Basic definitions and basic facts

Let S be any set. A *coalgebra structure* or *transition structure* on S is a mapping $\alpha_S : S \to \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the collection of all subsets of S: $\mathcal{P}(S) = \{V \mid V \subseteq S\}$. The pair (S, α_S) consisting of the set S and the transition structure α_S is called a *coalgebra* or *transition system*. The set S is called the *carrier*, also referred to as the set of *states*. For a state $s \in S$, the set $\alpha_S(s)$ consists of all states that are *reachable* from s. That transition systems in this sense are the usual (unlabelled) nondeterministic transition systems (also called *frames*) can be easily seen, by defining a corresponding *transition relation*: for any s and s' in S,

$$s \longrightarrow s' \text{ iff } s' \in \alpha_S(s).$$

This shorthand will be used throughout the paper.

Let (S, α_S) and (T, α_T) be two transition systems. A mapping $f : S \to T$ is called a *homomorphism* if $\alpha_T \circ f = \mathcal{P}(f) \circ \alpha_S$:

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ \Big\downarrow{\scriptstyle \alpha_S} & * & \Big\downarrow{\scriptstyle \alpha_T} \\ \mathcal{P}(S) & \xrightarrow[\mathcal{P}(f)]{} & \mathcal{P}(T), \end{array}$$

where $\mathcal{P}(f) : \mathcal{P}(S) \to \mathcal{P}(T)$ is defined, for any $V \subseteq S$, by

$$\mathcal{P}(f)(V) = \{t \in T \mid \exists v \in V,\ f(v) = t\} \quad (= f(V)).$$

Lemma 2.1 *Let (S, α_S) and (T, α_T) be transition systems and $f : S \to T$ any mapping. The following are equivalent.*

1. *f is a homomorphism.*
2. *For all $s \in S$,*
   a. *If $s \longrightarrow s'$, for some $s' \in S$, then $f(s) \longrightarrow f(s')$.*
   b. *If $f(s) \longrightarrow t$, for some $t \in T$, then there exists $s' \in S$ with $s \longrightarrow s'$ and $f(s') = t$.*

   (Note that in our notation we do not distinguish between the transition relations defined by α_S and α_T.)

Proof. Immediate from the observation that the inclusion $\mathcal{P}(f) \circ \alpha_S \subseteq \alpha_T \circ f$ and its reverse are equivalent to clauses (a) and (b), respectively. □

The above definition of homomorphism is an instance of a general (categorical) definition, see Section 9. It has been invented many times, with different names such as *saturating morphism*, *p-morphism*, *bounded morphism* and *functional bisimulation*.

The composition of two homomorphisms is again a homomorphism. The identity mapping on a transition system is a homomorphism. As a consequence, the class of all transition systems together with the homomorphisms between them is a category.

A homomorphism $f : S \to T$ with an inverse $f^{-1} : T \to S$ which is also a homomorphism is called an *isomorphism* between S and T. As usual, $S \cong T$ means that there exists an isomorphism between S and T.

Injective and surjective homomorphisms are called *monomorphisms* and *epimorphisms*, respectively.

Given transition systems S and T, we say that S can be *embedded* into T if there is a monomorphism from S to T. If there exists an epimorphism from S to T, T is called a *homomorphic image* of S.

Given transition systems S and T, we say that T is a *subsystem* of S if $T \subseteq S$ and α_T equals the function α_S restricted to T. So subsystems are subsets of S that are closed under (outgoing) transitions. A subset T of S is a subsystem if the inclusion mapping from T to S is a homomorphism. Any transition system has the empty set and itself as subsystems. A transition system S is called *minimal* if it does not have any proper subsystem (i.e., different from $\emptyset$ and S). (In the world of modal logic, subsystems are called *generated subframes*.)

The *direct sum* (or *coproduct*) of any collection of transition systems consists of the disjoint union of their carriers together with (the transition structure determined by) the disjoint union of their transition relations. In general, the *product* (in the category of transition systems) of two transition systems need not exist. For instance, let $S = \{0, 1, 2\}$ with $\alpha_S(0) = \{0, 1\}$, and $\alpha_S(1) = \alpha_S(2) = \emptyset$. There does not exist a product of (S, α_S) with itself. (Cf. the remark at the end of Section 4.)

A given set S can in general be supplied with different transition structures which usually are not isomorphic. The empty set together with the empty mapping is called the *trivial* transition system.

A *bisimulation relation* between two transition systems (S, α_S) and (T, α_T) is a set $R \subseteq S \times T$ for which there is a transition structure α_R, such that the projections $\pi_1 : R \to S$ and $\pi_2 : R \to T$ are homomorphisms. Graphically:

$$\begin{array}{ccccc} S & \xleftarrow{\ \pi_1\ } & R & \xrightarrow{\ \pi_2\ } & T \\ \Big\downarrow{\scriptstyle \alpha_S} & * & \Big\downarrow{\scriptstyle \alpha_R} & * & \Big\downarrow{\scriptstyle \alpha_T} \\ \mathcal{P}(S) & \xleftarrow[\mathcal{P}(\pi_1)]{} & \mathcal{P}(R) & \xrightarrow[\mathcal{P}(\pi_2)]{} & \mathcal{P}(T) \end{array}$$

This definition of bisimulation is equivalent to the usual one:

Lemma 2.2 *Let S and T be transition systems and let $R \subseteq S \times T$. Then the following are equivalent:*

1. *R is a bisimulation*
2. *For all $s \in S$ and $t \in T$ with $(s,t) \in R$:*
   a. *If $s \longrightarrow s'$, for some $s' \in S$, then $t \longrightarrow t'$ for some $t' \in T$ with $(s', t') \in R$.*

b. *If* $t \longrightarrow t'$, *for some* $t' \in T$, *then* $s \longrightarrow s'$ *for some* $s' \in S$ *with* $(s', t') \in R$.

Proof.
1 ⇒ 2: Let R be a bisimulation and $(s,t) \in R$, and suppose $s \longrightarrow s'$. Because $s = \pi_1((s,t))$ this implies $\pi_1((s,t)) \longrightarrow s'$, and because π_1 is a homomorphism, it follows from Lemma 2.1 that there is $(s'', t') \in R$ with $(s,t) \longrightarrow (s'', t')$ and $\pi_1((s'', t')) = s'$. Thus $(s', t') \in R$. Because π_2 is a homomorphism it follows, again by Lemma 2.1, that $t \longrightarrow t'$, which concludes the proof of clause (a). Clause (b) is proved similarly.
2 ⇒ 1: Suppose R satisfies clauses (a) and (b). Define $\alpha_R : R \to \mathcal{P}(R)$, for $(s,t) \in R$, by

$$\alpha_R((s,t)) = \{(s', t') \in R \mid s \longrightarrow s' \text{ and } t \longrightarrow t'\}.$$

It is immediate from clauses (a) and (b) that the projections from (R, α_R) to (S, α_S) and (T, α_T) are homomorphisms. (In general more than one choice can be made for α_R.) □

A bisimulation between a transition system S and itself is called a bisimulation *on* S. If R is moreover an equivalence relation, then it is called a *bisimulation equivalence*. On any transition system S, the diagonal Δ_S of S defined by $\Delta_S = \{(s, s') \in S \times S \mid s = s'\}$ is trivially a bisimulation equivalence.

Lemma 2.3 *Let S, T and U be transition systems, R a relation between S and T, and Q a relation between T and U. Let R^{-1} and $R \circ Q$ be defined by*

$$R^{-1} = \{(t,s) \in T \times S \mid (s,t) \in R\},$$
$$R \circ Q = \{(s,u) \in S \times U \mid \exists t \in T,\ (s,t) \in R \text{ and } (t,u) \in Q\}.$$

If R and Q are bisimulations then so are R^{-1} and $R \circ Q$. □

Let $f : S \to T$ be any mapping. The *image* $I(f)$, the *kernel* $K(f)$, and the *graph* $G(f)$ of f are defined as follows:

$$I(f) = \{t \in T \mid \exists s \in S,\ f(s) = t\},$$
$$K(f) = \{(s, s') \in S \times S \mid f(s) = f(s')\},$$
$$G(f) = \{(s,t) \in S \times T \mid f(s) = t\}.$$

For subsets $V \subseteq S$ and $W \subseteq T$, let

$$f(V) = \{t \in T \mid \exists s \in V,\ f(s) = t\},$$
$$f^{-1}(W) = \{s \in S \mid f(s) \in W\}.$$

(Note that $I(f) = f(S)$.)

Proposition 2.4 *Let S and T be two transition systems and $f : S \to T$ any mapping.*

1. *If f is a homomorphism and $V \subseteq S$ is a subsystem of S, then $f(V)$ is a subsystem of T. (In particular, $I(f)$ is a subsystem of T.)*
2. *If f is a homomorphism and $W \subseteq T$ is a subsystem of T, then $f^{-1}(W)$ is a subsystem of S.*
3. *f is a homomorphism if and only if $G(f)$ is a bisimulation between S and T.*
4. *If f is a homomorphism then $K(f)$ is a bisimulation equivalence on S.*

Proof. Statements 1 and 2 are immediate from the definition of subsystem and Lemma 2.1. Statement 3 is immediate by Lemma 2.1 and Lemma 2.2. The last statement follows from Lemma 2.3 and the observation that $K(f) = G(f) \circ G(f)^{-1}$. □

Because of 3, homomorphisms are sometimes called *functional bisimulations*.

Let S be any set and R an equivalence relation on S. Let the quotient set S/R be defined by $S/R = \{[s]_R \mid s \in S\}$, with $[s]_R = \{s' \in S \mid (s, s') \in R\}$. Let $\epsilon_R : S \to S/R$ be the surjective mapping sending each element s to its equivalence class $[s]_R$. It is called the *quotient map* of R.

Proposition 2.5 *Let (S, α_S) be a transition system and R a bisimulation equivalence on S. Define $\alpha_{S/R} : S/R \to \mathcal{P}(S/R)$, for all s and s' in S, by*

$$[s']_R \in \alpha_{S/R}([s]_R) \text{ iff } \exists t, t' \in S,\ (s,t) \in R \text{ and } (s',t') \in R \text{ and } t' \in \alpha_S(t).$$

Equivalently,

$$[s]_R \longrightarrow [s']_R \text{ iff } \exists t, t' \in S,\ (s,t) \in R \text{ and } (s',t') \in R \text{ and } t \longrightarrow t'.$$

Then $\alpha_{S/R}$ is the unique transition structure on S/R such that $\epsilon_R : S \to S/R$ is a homomorphism.

Proof. If $s \longrightarrow s'$ then $\epsilon_R(s) \longrightarrow \epsilon_R(s')$. Suppose $\epsilon_R(s) \longrightarrow [s']_R$, for some $s' \in S$. Then there are $t, t' \in S$ such that $t \longrightarrow t'$, $(s,t) \in R$, and $(s', t') \in R$. Because $t \longrightarrow t'$ and $(s, t) \in R$ it follows that there is $s'' \in S$ with $s \longrightarrow s''$ and $(s'', t') \in R$. Then $s \longrightarrow s''$ and $\epsilon_R(s'') = \epsilon_R(t') = \epsilon_R(s') = [s']_R$. Thus ϵ_R is a homomorphism (by Lemma 2.1). The fact that $\alpha_{S/R}$ is the unique transition structure with this property can be easily shown 'by hand', and also follows from more general considerations in Section 10. □

Let $f : S \to T$ be any mapping, P a relation on S, and Q a relation on T. Let P^f and Q_f be defined by

$$\begin{aligned} P^f &= \{(t,t') \in T \times T \mid \exists s, s' \in S,\ f(s) = t \text{ and} \\ &\qquad f(s') = t' \text{ and } (s,s') \in P\}, \\ Q_f &= \{(s,s') \in S \times S \mid (f(s), f(s')) \in Q\}. \end{aligned}$$

Proposition 2.6 *Let $f : S \to T$ be a homomorphism of transition systems. If P and Q are bisimulations on S and T, then P^f and Q_f are bisimulations on T and S, respectively.*

Proof. Immediate from Lemma 2.3 and the fact that $P^f = G(f)^{-1} \circ P \circ G(f)$ and $Q_f = G(f) \circ Q \circ G(f)^{-1}$. □

Theorem 2.7 *For transition systems S, T and U, and homomorphisms $f : U \to S$ and $g : U \to T$, there are a transition system V and homomorphisms $h : S \to V$ and $i : T \to V$ such that*

1. $h \circ f = i \circ g$
2. *For all transition systems V' and homomorphisms $h' : S \to V'$ and $i' : T \to V'$ such that $h' \circ f = i' \circ g$, there is a unique homomorphism $k : V \to V'$ such that $h' = k \circ h$ and $i' = k \circ i$.*

For transition systems S, T and U, and homomorphisms $f : S \to U$ and $g : T \to U$, there is a transition system V and homomorphisms $h : V \to S$ and $i : V \to T$ such that $f \circ h = g \circ i$.

These statements can be most easily proved (and their asymmetry best explained) categorically, see Section 10. For a direct proof of the first, let V be the quotient of the disjoint union of S and T with respect to the smallest equivalence relation generated by $\{(s,t) \in S \times T \mid \exists u \in U,\ f(u) = s \text{ and } g(u) = t\}$. In the latter statement, take $V = \{(s,t) \in S \times T \mid f(s) = g(t)\}$.

3 The lattice of subsystems

The collection of all subsystems of a transition system S is closed under arbitrary unions and intersections, and hence is a complete lattice. For a subset X of a transition system S let $\langle X \rangle$ denote the subsystem of S *generated by* X. It is defined as

$$\langle X \rangle = \bigcap \{T \subseteq S \mid T \text{ is a subsystem of } S \text{ and } X \subseteq T\}.$$

Equivalently, it is the least fixed point of an operator $\Psi_X : \mathcal{P}(S) \to \mathcal{P}(S)$ which takes any subset $V \subseteq S$ to

$$\Psi_X(V) = X \cup V \cup \{s \in S \mid \exists s' \in (X \cup V),\ s' \longrightarrow s\}.$$

A third description of $\langle X \rangle$ is

$$\langle X \rangle = \{s \in S \mid \exists x \in X,\ x \stackrel{*}{\longrightarrow} s\},$$

where $\stackrel{*}{\longrightarrow}$ is the reflexive and transitive closure of the transition relation $\longrightarrow$ on S. The transition relation on $\langle X \rangle$ is given by $\longrightarrow \cap\, (\langle X \rangle \times \langle X \rangle)$. If $S = \langle X \rangle$ for some subset X of S then S is said to be *generated by* X.

Proposition 3.1 *A transition system S is minimal if and only if for every non-empty subset $X \subseteq S$: $S = \langle X \rangle$.* □

Proposition 3.2 *Let S be a transition system, $X \subseteq S$, and R a bisimulation equivalence on S. If $S = \langle X \rangle$ then $S/R = \langle X/(R \cap (X \times X)) \rangle$.* □

Let S be a transition system generated by X and let $f : S \to T$ and $g : S \to T$ be two homomorphisms such that $f = g$ on X. In general f and g need not be equal on the whole of S. But we do have that $f(S) = g(S)$. It is an immediate consequence of the following simple fact.

Proposition 3.3 *Let $f : S \to T$ be a homomorphism of transition systems and $X \subseteq S$. If $S = \langle X \rangle$ then $f(S) = \langle f(X) \rangle$.* □

The operator $\langle \cdot \rangle : \mathcal{P}(S) \to \mathcal{P}(S)$ satisfies, for all $X \subseteq S$:

1. $X \subseteq \langle X \rangle$,
2. $\langle \langle X \rangle \rangle = \langle X \rangle$,
3. $\langle X \rangle = \bigcup \{ \langle \{s\} \rangle \mid s \in S \}$,

and is therefore called a *completely additive closure* operator. A subset $X \subseteq S$ with $X = \langle X \rangle$ is called *closed*. (Thus the closed subsets are precisely the subsystems.) The following theorem shows that all operators satisfying 1, 2 and 3 above, are obtained in this way. (It is a simple variation on the theorem by Birkhoff and Frink that any algebraic lattice is isomorphic to the lattice of subalgebras of some algebra.)

Theorem 3.4 *Let S be any set and $c : \mathcal{P}(S) \to \mathcal{P}(S)$ a completely additive closure operator. Then there is a transition structure $\alpha : S \to \mathcal{P}(S)$ such that the lattice of closed subsets of S coincides with the lattice of all subsystems of (S, α).*

Proof. Define $\alpha : S \to \mathcal{P}(S)$, for $s \in S$, by $\alpha(s) = c(\{s\})$. For any $X \subseteq S$ the set $c(X)$ is a subsystem of (S, α), because if $s \in c(X)$ and $s \longrightarrow s'$, for some $s' \in S$, then

$$s' \in c(\{s\}) \subseteq c(c(X)) = c(X).$$

Moreover, $X \subseteq c(X)$, hence $\langle X \rangle \subseteq c(X)$. On the other hand, $c(\{s\}) \subseteq \langle X \rangle$, for any $s \in X$, and $c(X) = \bigcup \{ c(\{s\}) \mid s \in X \}$ imply $c(X) \subseteq \langle X \rangle$. Thus $c(X) = \langle X \rangle$. □

4 The lattice of bisimulations

Let S and T be a transition systems. The collection of all bisimulations between S and T,

$$B(S,T) = \{ V \subseteq S \times T \mid V \text{ is a bisimulation } \},$$

can be seen to be a complete lattice $(B(S,T), \bigvee, \bigwedge)$ as follows. Since the union of bisimulations is again a bisimulation, we can take $\bigvee$ to be set union. Next consider, for an arbitrary relation $R \subseteq S \times T$, the function

$\Phi_R : \mathcal{P}(S \times T) \to \mathcal{P}(S \times T)$, defined for any $V \subseteq S \times T$, by

$$\begin{aligned}\Phi_R(V) = \{(s,t) \in R \quad | \quad & \forall s' \in S \text{ s.t. } s \longrightarrow s' \\ & \exists t' \in T \text{ s.t. } t \longrightarrow t' \text{ and } (s',t') \in V \\ & \text{and} \\ & \forall t' \in T \text{ s.t. } t \longrightarrow t' \\ & \exists s' \in S \text{ s.t. } s \longrightarrow s' \text{ and } (s',t') \in V\}.\end{aligned}$$

It follows from the definition of Φ_R that $V \subseteq R$ is a bisimulation if and only if $V \subseteq \Phi_R(V)$. The greatest bisimulation relation between S and T which is contained in R, is given by the greatest fixed point of Φ_R (which exists because $\mathcal{P}(S \times T)$ is a complete lattice and Φ_R is monotone). Now $\bigwedge$ can be defined, for an arbitrary collection of bisimulations $\{R_i\}_{i \in I}$ (for some index set I), by

$$\bigwedge \{R_i\}_{i \in I} = \text{gfp } \Phi_{\bigcap \{R_i\}_{i \in I}}.$$

The greatest bisimulation on a single transition system S, usually denoted by $\sim$, is equal to gfp $\Phi_{S \times S}$. Elements s and s' in S with $s \sim s'$ are called *bisimilar*. If (S, α_S) is *finitely branching*—i.e., $\alpha_S(s)$ is finite, for all s in S—then $\sim$ is obtained as the intersection of a sequence of approximations: Let $\sim_0 = S \times S$ and, given $\sim_n$ let $\sim_{n+1} = \Phi_{S \times S}(\sim_n)$. (Elements s and s' in S with $s \sim_n s'$ are called *bisimilar up to depth n*.) One readily checks that $\sim = \bigcap \{\sim_n | \; n \geq 0\}$.

We have seen that the product of two transition systems generally does not exist. However, for *deterministic* transitions systems: (S, α) such that for all s in S, $\alpha(s)$ contains at most one element, products *do* exist. The product of two deterministic transition systems S and T is given by the greatest bisimulation between them.

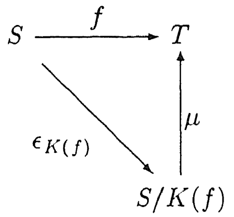
5 Three isomorphism theorems

Lemma 5.1 *Every bijective homomorphism is necessarily an isomorphism.*

Proof. By Proposition 2.4, the relation of a homomorphism is a bisimulation and by Lemma 2.3, so is its inverse. □

Lemma 5.2 *Let S, T, and U be transition systems, and $f : S \to T$, $g : S \to U$, and $h : U \to T$ any mappings. If $f = h \circ g$, g is surjective, and f and g are homomorphisms, then h is a homomorphism.* □

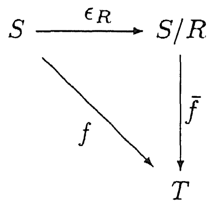
Theorem 5.3 *(First isomorphism theorem)*
Let $f : S \to T$ be a homomorphism of transition systems. Then there is a factorization $f = \mu \circ \epsilon_{K(f)}$ of f:



where $\epsilon_{K(f)}$ is the quotient map of the kernel $K(f)$ of f, and μ is a monomorphism. Moreover, $S/K(f)$ is isomorphic to $I(f)$, the image of f.

Proof. Putting $\mu([s]_{K(f)}) = f(s)$, for $s \in S$, defines an injective mapping μ with $\mu \circ \epsilon_{K(f)} = f$. Because $\epsilon_{K(f)}$ (by Proposition 2.5) and f are homomorphisms, so is μ (by Lemma 5.2). By Proposition 2.4, $I(f)$ is a subsystem of T. Defining $f' : S/K(f) \to I(f)$, for $s \in S$, by $f'([s]_{K(f)}) = f(s)$ yields a bijective homomorphism. By Lemma 5.1 it is an isomorphism. □

Theorem 5.4 *Let $f : S \to T$ be a homomorphism of transition systems and R a bisimulation equivalence on S which is contained in the kernel of f. Then there is a unique homomorphism $\bar{f} : S/R \to T$ such that $f = \bar{f} \circ \epsilon_R$:*



Proof. Putting $\bar{f}([s]_R) = f(s)$, for $s \in S$, uniquely defines a mapping $\bar{f} : S/R \to T$ for which $\bar{f} \circ \epsilon_R = f$. It follows from Lemma 5.2 that it is a homomorphism. □

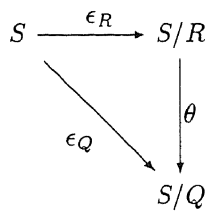
Theorem 5.5 *(Second isomorphism theorem)*
Let S be a transition system, T a subsystem of S, and R a bisimulation equivalence on S. Let T^R be defined by $T^R = \{s \in S \mid \exists t \in T,\ (s,t) \in R\}$. The following facts hold:

1. *T^R is a subsystem of S.*
2. *$Q = R \cap (T \times T)$ is a bisimulation equivalence on T.*
3. *$T/Q \cong T^R/R$.*

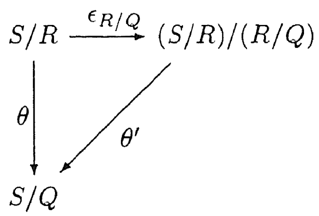
Proof. Since $T^R = \pi_1({\pi_2}^{-1}(T))$, it is a subsystem of S by Proposition 2.4. One readily verifies that Q is a bisimulation equivalence on T. Consider the quotient homomorphism $\epsilon_R : S \to S/R$, and let $\epsilon : T \to S/R$ be its restriction to T. Because $I(\epsilon) = \epsilon(T) = \epsilon_R(T^R) = T^R/R$, and $K(f) = Q$, it follows from Theorem 5.3 that $T/Q \cong T^R/R$. □

Let S be a transition system, T a subsystem of S, and R a bisimulation equivalence on S. If $R \cap (T \times T) = \Delta_T$ then R is said to *separate* T (because, equivalently: for all $t, t' \in T$, if $t \neq t'$ then $(t, t') \notin R$). In this case, the above theorem yields that $T \cong T^R/R$.

Theorem 5.6 *(Third isomorphism theorem)*
Let S be a transition system, and let R and Q be bisimulation equivalences on S such that $R \subseteq Q$. There is a unique homomorphism $\theta : S/R \to S/Q$ such that $\theta \circ \epsilon_R = \epsilon_Q$:



Let R/Q denote the kernel of θ: it is a bisimulation equivalence on S/R and induces an isomorphism $\theta' : (S/R)/(R/Q) \to S/Q$ such that $\theta = \theta' \circ \epsilon_{R/Q}$:



Proof. The existence of θ follows from Theorem 5.4. Because ϵ_Q is surjective also θ is surjective. The existence of the isomorphism θ' is now given by Theorem 5.3. □

6 Simple transition systems

Since the diagonal of a transition system is always a bisimulation equivalence, it follows from Proposition 2.5 that every transition system S has itself as a homomorphic image. If it has no others then it is said to be *simple*. In other words, S is simple if every epimorphism $f : S \to T$ is an isomorphism.

Theorem 6.1 *Let S be a transition system. The following are equivalent:*

1. *S is simple.*
2. *Δ_S is the only bisimulation equivalence on S.*
3. *For every bisimulation R on S, $R \subseteq \Delta_S$.*
4. *For any transition system T, and mappings $f : T \to S$ and $g : T \to S$: if f and g are homomorphisms then $f = g$.*

5. *The quotient homomorphism $\epsilon : S \to S/\!\sim$, where $\sim$ denotes the greatest bisimulation on S, is an isomorphism.*

Proof. $1 \Rightarrow 2$: Let R be a bisimulation equivalence on S and consider the quotient homomorphism $\epsilon_R : S \to S/R$. If S is simple then ϵ_R is an isomorphism. Thus $R = \Delta_S$.
$2 \Rightarrow 1$: Let $f : S \to T$ be an epimorphism. Since the kernel of f is a bisimulation equivalence, it follows from 2 that it is equal to Δ_S. By Theorem 5.3, $S/\Delta_S \cong T$, hence $S \cong T$. Thus S is simple.
$3 \Rightarrow 4$: Let T be a transition system, and let $f : T \to S$ and $g : T \to S$ be homomorphisms. Define

$$Q = \{(s, s') \in S \times S \mid \exists t \in T,\ s = f(t) \text{ and } s' = g(t)\}.$$

Since $Q = G(f)^{-1} \circ G(g)$, it is a bisimulation by Proposition 2.4 and Lemma 2.3. It follows from 3 that $Q \subseteq \Delta_S$. Thus $f = g$.
$4 \Rightarrow 3$: Let R be a bisimulation on S. By definition, its projections $\pi_1 : R \to S$ and $\pi_2 : R \to S$ are homomorphisms. It follows from 4 that $\pi_1 = \pi_2$, hence $R \subseteq \Delta_S$.
$2 \Leftrightarrow 3$: Immediate from the observation that the greatest bisimulation on S is an equivalence.
$1 \Rightarrow 5.$: Immediate.
$5. \Rightarrow 2$: Suppose that $\epsilon : S \to S/\!\sim$ is an isomorphism. Let R be a bisimulation equivalence on S. Because $R \subseteq\sim$ and $\sim$ is the kernel of ϵ, there exists by Theorem 5.4 a (unique) homomorphism $\bar{f} : S/R \to S/\!\sim$ such that $\bar{f} \circ \epsilon_R = \epsilon$. Since ϵ is an isomorphism this implies that ϵ_R is injective. Thus $R = \Delta_S$. □

Clauses 3 and 4 indicate that 'simplicity' can actually be interpreted as a proof principle. For instance, in order to show that two elements s and s' of a simple transition system S are equal, it is sufficient to establish the existence of a bisimulation R on S such that $(s, s') \in R$. This property is sometimes referred to as *strong extensionality* or *co-induction.* We shall see examples of its use in Section 8.

Proposition 6.2 *For every transition system S and bisimulation equivalence R on S, the quotient S/R is simple if and only if $R = \sim$.*

Proof.
$\Leftarrow$: Let Q be a bisimulation on $S/\!\sim$. We show that $Q \subseteq \Delta_{S/\sim}$. Then it follows from Theorem 6.1 that $S/\!\sim$ is simple. Consider $\epsilon : S \to S/\!\sim$. By Proposition 2.6, the relation

$$Q_\epsilon = \{(s, s') \in S \times S \mid ([s]_\sim, [s']_\sim) \in Q\}$$

is a bisimulation on S and hence is included in $\sim$. Thus for all s and s' in S, if $([s]_\sim, [s']_\sim) \in Q$ then $[s]_\sim = [s']_\sim$.

$\Rightarrow$: Let Q be a bisimulation on S. We show that $Q \subseteq R$. By definition the projections $\pi_1 : Q \to S$ and $\pi_2 : Q \to S$ are homomorphisms. Consider the compositions $\epsilon \circ \pi_1 : Q \to S/R$ and $\epsilon \circ \pi_2 : Q \to S/R$. By assumption, S/R is simple. It follows from Theorem 6.1 that $\epsilon \circ \pi_1 = \epsilon \circ \pi_2$, whence $Q \subseteq R$. Therefore $R = \sim$. □

By the above proposition, the quotient of any transition system with respect to its greatest bisimulation, is simple. For instance, $\{s, s'\}$ with $s \longrightarrow s'$ and $s' \longrightarrow s$ is not simple, but $\{t\}$ with $t \longrightarrow t$ is.

7 Initial and final transition systems

In the world of Σ-algebras, *initial* algebras are of particular interest; e.g., they are used in what is called *initial algebra semantics*. Similarly, *final* transition systems are of importance in the world of coalgebras. In this section, some properties of final transition systems are discussed, and a number of examples is given. For one of the examples, Section 8 will show how it gives rise to *final (coalgebra) semantics*, the coalgebraic counterpart of initial algebra semantics.

Let *TS* be the class of all transition systems, K a subclass of *TS*. A transition system S in K is *initial in* K if for any other transition system T in K there exists a unique homomorphism from S to T; S is called *final in* K if there exists a unique homomorphism from any other transition system in K to S. It is easy to prove that initial and final transition systems are unique up to isomorphism.

Initial transition systems are not very exiting: the trivial (empty) transition system is initial in every class of which it is a member. Somewhat disappointingly, there is also the following.

Theorem 7.1 *There is no final transition system in TS.* □

This follows from the fact that if S is final in TS then S is isomorphic with $\mathcal{P}(S)$, and the fact that such sets do not exist. (Cf. Rutten and Turi 1994.)

Nevertheless, it is worthwhile to look for subclasses of K that *do* include a final transition system, because final transition systems have various nice properties. For one thing, they are simple. More precisely: let us generalize the definition of Section 6 and call a transition system S simple *in* K if S is in K and any epimorphism $f : S \to T$ with T *in* K is an isomorphism. Now suppose that K is closed under taking bisimulations: that is, if S and T are in K and $R \subseteq S \times T$ is a bisimulation between S and T, then there exists a transition structure α_R on R such that (R, α_R) is in K. For such K, it follows from (the proof of) Theorem 6.1 that S is simple in K if and only if for all T in K and homomorphisms $f : T \to S$ and $g : T \to S$,

$f = g$. In other words, being simple amounts to one 'half' of the definition of being final (the uniqueness part). Which implies the following.

Theorem 7.2 *If K is closed under taking bisimulations and S is final in K, then S is simple in K.* □

As a consequence, final transition systems satisfy the proof principle of strong extensionality (co-induction), mentioned after the proof of Theorem 6.1, which will be used in Section 8.

This section is concluded with three examples of classes of transition systems that have a final element. Firstly, let S be a transition system and let $[S]$ be the equivalence class of S under the following equivalence relation: $S \approx T$ whenever there exists a (so-called *total*) bisimulation relation between S and T such that its projections are epimorphisms.

Theorem 7.3 *Let $\sim$ be the greatest bisimulation on S. Then $S/{\sim}$ is final in $[S]$.*

Proof. It follows from the fact that homomorphisms are (functional) bisimulations (Proposition 2.4.3) that $S/{\sim}$ is in $[S]$. Consider a transition system T in $[S]$. Let R be a bisimulation between S and T whose projections π_1 and π_2 are epimorphisms. Consider the quotient homomorphism $\epsilon : S \to S/{\sim}$. The composition $G(\pi_2)^{-1} \circ G(\pi_1) \circ G(\epsilon)$ is a bisimulation between T and $S/{\sim}$ which actually is a function. Hence it is a homomorphism from T to $S/{\sim}$. This proves the existence part. It follows from Proposition 6.2 and Theorem 6.1 that there is at most one such homomorphism. □

A second example is the following. A transition system (S, α_S) is *deterministic* if for all s, s', and s'' in S:

$$\text{if } s \longrightarrow s' \text{ and } s \longrightarrow s'' \text{ then } s' = s'',$$

or, equivalently, if $\alpha(s)$ contains at most one element. Let $\omega + 1$ be a transition system with states $\{0, 1, 2, \ldots\} \cup \{\omega\}$, and transitions $n + 1 \longrightarrow n$, for all $n \geq 0$, and $\omega \longrightarrow \omega$. For a deterministic transition system (S, α_S), there is precisely one homomorphism from S to $\omega + 1$: it maps a state s in S to the number (possibly ω) of steps that can be taken starting in s. Thus $\omega + 1$ is final in the class of all deterministic transition systems.

For our last example, let *FB* be the class of all *finitely branching* transition systems (S, α_S): for all s in S, $\alpha(s)$ is finite. For such transition systems, the transition structure is actually a mapping $\alpha_S : S \to \mathcal{P}_f(S)$, where $\mathcal{P}_f(S)$ is the collection of all *finite* subsets of S.

Theorem 7.4 *The class of all finitely branching transition systems has a final element.* □

The reader is referred to Barr 1993 and Rutten and Turi 1994 for a formal proof; here we only mention the main idea. A final transition system can be constructed as follows: consider the class of all *finitely branching, ordered trees*; define a notion of bisimulation on such trees by viewing them as transition systems; and take the collection of all equivalence classes of trees with respect to the greatest bisimulation relation. This collection, which turns out to be a set, can be easily supplied with a transition structure. The result, which for future reference is denoted by (P, π), is a finitely branching transition system that is final in FB.

For the reader with some background in modal logic, the following alternative way of obtaining a final transition system might help. Here we adopt for a moment the jargon of modal logic. Consider the *canonical frame* F^K for the *basic normal logic* K; next take the union of the images in F^K of all bounded morphisms from finitely branching frames to F^K. This defines a generated subframe of F^K that can be shown to be finitely branching. It follows immediately from the so-called *Truth lemma* (cf. Goldblatt 1987) that it is final in FB. Note that it must be isomorphic to (P, π), since final transition systems are unique up to isomorphism.

8 Final semantics

Transition systems are often used as a so-called operational semantics for programming languages, notably since the appearance of Plotkin 1981. Final transition systems are then of particular interest because their elements can be considered as canonical representatives of bisimulation equivalence classes as follows. Suppose F is final in a class K of transition systems, and S is in K. By finality, there is a unique homomorphism $f : S \to F$. It satisfies, for all s and s' in S,

$$s \sim s' \text{ if and only if } f(s) = f(s').$$

(The implication from left to right follows from Proposition 2.6, Theorem 7.2, and Theorem 6.1. The converse follows from Proposition 2.4.4.) Thus F has, for any S in K, a subsystem that is isomorphic to the quotient $S/\sim$.

Final transition systems are furthermore useful because they can be supplied, in addition to their transition structure, with an algebraic structure by exploiting the finality. This will be briefly illustrated here; for a more extensive treatment, the reader is referred to Rutten and Turi 1994.

Consider the class FB of finitely branching transition systems. There exists (Theorem 7.4) a finitely branching transition system (P, π) that is final in FB. We show how to define a binary operator $\|: P \times P \to P$, which models the *merge* or *interleaving* of pairs of states in P. To this end, a transition structure $\alpha_{\|} : (P \times P) \to \mathcal{P}_f(P \times P)$ is defined as follows: for p and q in P,

$$\alpha(\langle p, q\rangle) = \{\langle p', q\rangle \in P \times P \mid p' \in \pi(p)\} \;\cup\; \{\langle p, q'\rangle \in P \times P \mid q' \in \pi(q)\}.$$

By finality of (P, π), there exists a unique homomorphism $\|: (P \times P, \alpha_{\|}) \to$

(P, π). It follows from the definition of homomorphism, writing $p \parallel q$ for $\parallel (\langle p, q \rangle)$, that

$$\pi(p \parallel q) = \{p' \parallel q \in P \mid p' \in \pi(p)\} \quad \cup \quad \{p \parallel q' \in P \mid q' \in \pi(q)\}.$$

In terms of transition relations, this is equivalent to

$$p \parallel q \longrightarrow p' \parallel q' \text{ if and only if } (p \longrightarrow p' \text{ and } q = q') \text{ or } (q \longrightarrow q' \text{ and } p = p').$$

Thus finality can be used for *definitions* (of operators like $\parallel$). At the same time, finality enables one to *prove* certain properties. For instance, that the operator $\parallel$ is associative: Consider the following relation on P:

$$\{\langle (p \parallel q) \parallel r,\ p \parallel (q \parallel r) \rangle \in P \times P \mid\ p, q, r \in P\}.$$

It is easy to show that this is a bisimulation relation. Final transition systems are simple, by Theorem 7.2, and any simple transition system is strongly extensional, according to Theorem 6.1: that is, any bisimulation on P is contained in the diagonal Δ_P. Thus for all p, q, and r in P, $(p \parallel q) \parallel r = p \parallel (q \parallel r)$.

9 Other transition structures

A transition system (S, α_S), consisting of a set S and a function $\alpha_S : S \rightarrow \mathcal{P}(S)$ is an instance of the following categorical definition: let C be a category and $F : C \rightarrow C$ a functor. An *F-coalgebra* is a pair (c, α_c) consisting of an object c in C and an arrow $\alpha_c : c \rightarrow F(c)$ (cf. Mac Lane 1971). Thus transition systems are $\mathcal{P}$-coalgebras, where $\mathcal{P} : \mathit{Set} \rightarrow \mathit{Set}$ is the powerset functor from the category of sets and functions to itself.

Let (c, α_c) and (d, α_d) be two F-coalgebras. An arrow $f : c \rightarrow d$ is a *homomorphism of F-coalgebras* if $F(f) \circ \alpha_c = \alpha_d \circ f$. The collection of all F-coalgebras together with F-coalgebra homomorphisms is a category, which we denote by C_F. The class TS of transition systems together with all homomorphisms between them thus constitutes the category $\mathit{Set}_{\mathcal{P}}$.

Similarly, the notion of bisimulation can be defined for arbitrary functors (Aczel and Mendler 1989): A subobject R of the product of c and d (if it exists) is called an F-bisimulation if there exists an F-coalgebra structure $\alpha_R : R \rightarrow F(R)$ such that the projections from R to c and d are homomorphisms of F-coalgebras. (Thus bisimulations on transition systems are $\mathcal{P}$-bisimulations.)

Below a number of further examples of interesting F-coalgebras is given, each of which is obtained by making a particular choice for the category C and the functor F. For each of them, facts like the ones of the preceding sections hold. Rather than proving them again, we shall in the next section investigate how proofs can be given for arbitrary C and F.

1. *Labelled* transition systems are coalgebras of the following functor. Let A be a set of (action) labels. The functor $\mathcal{P}(A \times \cdot) : \mathit{Set} \rightarrow \mathit{Set}$

takes a set S to the collection of subsets of $A \times S$ (on functions it is defined as one would expect). Homomorphisms and bisimulations turn out to be the standard notions.

2. Let Φ be a given set of *atomic formulas*. Consider the functor $\mathcal{M}$ on the category *Set*, defined for a set S by

$$\mathcal{M}(S) = \mathcal{P}(S) \times \mathcal{P}(\Phi).$$

A function $f : S \to T$ is mapped to $\mathcal{M}(f) : \mathcal{M}(S) \to \mathcal{M}(T)$, taking a pair (V, W) in $\mathcal{P}(S) \times \mathcal{P}(\Phi)$ to $(f(V), W)$ in $\mathcal{P}(T) \times \mathcal{P}(\Phi)$. Now $\mathcal{M}$-coalgebras are the so-called Φ-*models* of modal logic, homomorphisms are *p-morphisms*, and bisimulations are *zig-zag* relations.

3. Let 1 denote a one element set. Consider the functor on *Set* which takes a set S to $1+(A\times S)$. Its coalgebras are *deterministic* transition systems (with labels in A). The set 1 is used to model termination. A variant would be the functor $A \times (\cdot)$ defined on the category of sets with *partial* maps. In this case, termination is modelled by the partiality of the functions.

4. Coalgebras of the functor which maps a set S to $A \to (S + 1)$—the collection of all functions from A to $S+1$—could be called *functionally deterministic* transition systems. For an element $s \in S$ of such a coalgebra (S, α_S), the possible transitions depend on the argument $a \in A$, with which the function $\alpha_S(s) : A \to (S + 1)$ has to be supplied.

5. A *metric space* is a pair (M, d) consisting of a set M and a *metric* or *distance function* d. Let *Met* be the category of metric spaces and non-expansive functions. Consider the metric powerdomain functor $\mathcal{P}_c : Met \to Met$, which maps a metric space (M, d) to the metric space of all its *compact* subsets (supplied with the so-called Hausdorff metric). For a (non-expansive) function $f : M \to M'$, $\mathcal{P}_c$ maps a compact subset V of M to the compact subset $f(V)$ of M'. Coalgebras of this functor are called *metric* transition systems. In a metric transition system $((M, d), \alpha_M : (M, d) \to \mathcal{P}_c((M, d)))$, the metric d on M expresses 'the amount of' bisimilarity: the smaller the distance between two elements, the 'more bisimilar' they are. This can be made more precise as follows. Let (S, α_S) be an (ordinary) transition system which is finitely branching. A natural candidate for a metric on S is defined, for s and s' in S, by

$$d_S(s, s') = \inf\{2^{-n} \mid s \sim_n s'\},$$

where $\sim_n$ is the bisimilarity-up-to-depth-n relation of Section 4. This does not define a metric yet, since if $s \sim s'$ then $d_S(s, s') = 0$, whereas s and s' may be different. If S, however, is simple then d_S is a metric indeed: $d_S(s, s') = 0$ if and only if $s \sim s'$ if and only if

$s = s'$. This definition is an immediate generalization of the metric on synchronization trees introduced in Golson and Rounds 1983. With this metric on S, α_S can be shown to be a non-expansive function $\alpha_S : (S, d_S) \to \mathcal{P}_c((S, d_S))$. Thus we have turned (S, α_S) into a metric transition system $((S, d_S), \alpha_S)$. (See Breugel 1994 for more observations on metric transition systems.)

10 Towards universal coalgebra

Some of the results of the preceding sections will be generalized to the category of coalgebras of arbitrary functors $F : C \to C$ on a category C. A categorical approach to universal algebra has been developed in Manes 1976, part of which has served as a guideline for this section.

Many proofs in the preceding sections consist of two parts: first results on sets and mappings are established, and next they are 'lifted' to transition systems and homomorphisms. This turns out to be a very general phenomenon: Let C be any category and $F : C \to C$ a functor. A typical way of proving facts about the category C_F of F-coalgebras is to see whether facts about the the underlying category C carry over to F-coalgebras.

In this section, we shall in particular investigate how the existence of colimits (such as sums and coequalizers) and limits (such as kernel pairs) *in* C_F is related to the existence of colimits and limits *in* C. The insight thus gained will next be helpful in a discussion of *image factorization systems*. As a result, we shall be able to find (generalized versions of) proofs of some of the theorems on transition systems (like the isomorphism theorems).

Starting with colimits, recall that the class of transition systems actually equals the category $Set_{\mathcal{P}}$ of $\mathcal{P}$-coalgebras. In this category, the *sum* of two transition systems (S, α_S) and (T, α_T) exists, because *in* Set the sum $S + T$ of their carriers exists (it is the disjoint union). This is a typical instance of the following general fact.

Theorem 10.1 *The functor $U : C_F \to C$ which maps an F-coalgebra (c, α_c) to its carrier c (thus 'forgetting' the coalgebra structure α_c) creates colimits. This means that if a certain type of colimit (like sum) exists in C, then it exists in C_F as well, and it is obtained by supplying the colimit in C (in a unique way) with an F-coalgebra structure.* □

Rather than giving an exact formulation and proof of this theorem (which would not be difficult, cf. Barr 1993), it will for the purpose of the present paper be more instructive to look at an example. Consider two arrows $f : c \to d$ and $g : c \to d$ in a category C. An arrow $h : d \to e$ is called a *coequalizer of f and g* if the following two conditions hold:

1. $h \circ f = h \circ g$

2. For all arrows $h' : d \to e'$ such that $h' \circ f = h' \circ g$, there exists a unique arrow $l : e \to e'$ with the property that $l \circ h = h'$.

In the category *Set*, equalizers always exist (we say: *Set has* coequalizers): given $f : S \to T$ and $g : S \to T$, the quotient of T with respect to the smallest equivalence relation on T that contains the set

$$\{(t, t') \in T \times T \mid \exists s \in S,\ t = f(s) \text{ and } t' = g(s)\},$$

is a coequalizer of f and g.

Now assume that C has coequalizers. We show that also C_F has coequalizers. Consider two homomorphisms of F-coalgebras $f : (c, \alpha_c) \to (d, \alpha_d)$ and $g : (c, \alpha_c) \to (d, \alpha_d)$. Since (per definition) f and g are arrows $f : c \to d$ and $g : c \to d$ in C, there exists a coequalizer $h : d \to e$ in C. Consider $F(h) \circ \alpha_d : d \to F(e)$. Because

$$\begin{aligned} F(h) \circ \alpha_d \circ f &= F(h) \circ F(f) \circ \alpha_c \\ &= F(h \circ f) \circ \alpha_c \\ &= F(h \circ g) \circ \alpha_c \\ &= F(h) \circ F(g) \circ \alpha_c \\ &= F(h) \circ \alpha_d \circ g, \end{aligned}$$

and $h : d \to e$ is a coequalizer, there exists a unique arrow $\alpha_e : e \to F(e)$ such that $\alpha_e \circ h = F(h) \circ \alpha_d$. Thus (e, α_e) is an F-coalgebra and h is a homomorphism $h : (d, \alpha_d) \to (e, \alpha_e)$ of F-coalgebras. One easily checks that it is a coequalizer in C_F.

Because the category *Set* has coequalizers, as a consequence also the category $Set_{\mathcal{P}}$ of transition systems has coequalizers. This yields an easy proof of Proposition 2.5: Consider a bisimulation equivalence (R, α_R) on a transition system (S, α_S). One can easily verify that the homomorphism $\epsilon_R : (S, \alpha_S) \to (S/R, \alpha_{S/R})$ of Proposition 2.5 can be obtained as a coequalizer of the projections π_1 and π_2 of R on S.

Also Theorem 2.7, which asserts the existence of another type of colimit: *push-out*, in the category of transition systems, is an immediate consequence of Theorem 10.1, since in the category *Set*, all push-outs exist. More generally, because in *Set all* colimits exist, all colimits exist in $Set_{\mathcal{P}}$ as well.

Summarizing the above, one can conclude that in C_F, colimits are as easy as they are in C. What about limits? It turns out that here the situation depends very much on properties of the functor F. Notably there is the following.

Theorem 10.2 *If $F : C \to C$ preserves a (certain type of) limit, then the functor $U : C_F \to C$ creates that (type of) limit.* □

Again, rather than being precise and general, we prove one particular instance of this theorem. For the theory of transition systems, the notion of *kernel* of a mapping is important (cf. the isomorphism theorems). It is an instance of the following categorical definition, which describes a special kind of limit. A *kernel pair* of an arrow $f : c \to d$ consists of (an object a together with) two arrows $k : a \to c$ and $l : a \to d$ with the following properties:

1. $f \circ k = f \circ l$
2. For every object a' and arrows $k' : a' \to c$ and $l' : a' \to c$ such that $f \circ k' = f \circ l'$, there exists a unique arrow $i : a' \to a$ such that $k' = k \circ i$ and $l' = l \circ i$.

For a mapping $f : S \to T$ between sets, it is easily verified that the kernel $K(f)$ together with its projections on S is a kernel pair in the above sense.

Now suppose that in C there exists a kernel pair for every arrow. Furthermore suppose that F preserves kernel pairs: that is, if a with (k, l) is a kernel pair of an arrow f, then $F(a)$ with $(F(k), F(l))$ is a kernel pair of $F(f)$. We show that kernel pairs exist in C_F as well. Let $f : (c, \alpha_c) \to (d, \alpha_d)$ be a homomorphism of F-coalgebras. Let a together with arrows $k : a \to c$ and $l : a \to c$ be a kernel pair of $f : c \to d$ in the category C. Since F preserves kernel pairs, $F(a)$ together with the arrows $F(k) : F(a) \to F(c)$ and $F(l) : F(a) \to F(c)$ is a kernel pair of $F(f) : F(c) \to F(d)$, again in C. Now consider the arrows $\alpha_c \circ k : a \to F(c)$ and $\alpha_c \circ l : a \to F(c)$. Because

$$\begin{aligned} F(f) \circ \alpha_c \circ k &= \alpha_d \circ f \circ k \\ &= \alpha_d \circ f \circ l \\ &= F(f) \circ \alpha_c \circ l, \end{aligned}$$

there is a unique arrow $\alpha_a : a \to F(a)$ such that $F(k) \circ \alpha_a = \alpha_c \circ k$ and $F(l) \circ \alpha_a = \alpha_c \circ l$. Thus $k : (a, \alpha_a) \to (c, \alpha_c)$ and $l : (a, \alpha_a) \to (c, \alpha_c)$ are homomorphisms of F-coalgebras, and one easily checks that (a, α_a) together with k and l is a kernel pair of f in C_F.

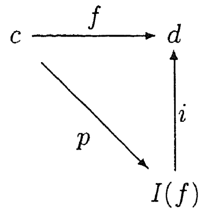
Unfortunately, the functor we have so far been interested in most: $\mathcal{P} : Set \to Set$, does *not* preserve kernel pairs. (Let 1 and 2 be a one and a two element set, and $f : 2 \to 1$ the only possible mapping between them. Then $\mathcal{P}$ does not preserve the kernel of f.) However, it 'almost' does: Consider a mapping $f : S \to T$ between sets. We saw that $K(f)$ together with the projections (π_1, π_2) on S is a kernel pair for f. Clearly, $\mathcal{P}(K(f))$ together with $(\mathcal{P}(\pi_1), \mathcal{P}(\pi_2))$ satisfies clause 1 of the definition of kernel pair. It is not difficult to prove that it satisfies clause 2 as well *but for* the unicity requirement. (The functor $\mathcal{P}$ is therefore said to preserve kernel pairs *weakly*.) A re-investigation of the little proof above (of an instance of Theorem 10.2) shows that there exists a coalgebra structure $\alpha : K(f) \to \mathcal{P}(K(f))$ (though not necessarily unique) such that the projections π_1 and

π_2 are homomorphisms from $(K(f), \alpha)$ to (S, α_S). Even though this does not mean that $(K(f), \alpha)$ together with (π_1, π_2) is a kernel pair in $Set_{\mathcal{P}}$, it *does* show that $(K(f), \alpha)$ is a bisimulation on S (cf. Proposition 2.4). In fact, this will be all we need in what follows.

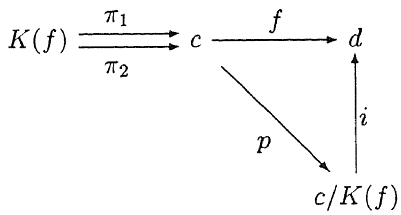
Our interest in colimits and limits, and more specifically, in coequalizers and kernel pairs of F-coalgebras, is mainly motivated by the role they play in the following.

The first isomorphism theorem states that every homomorphism of transition systems $f : S \to T$ factors through the image of f by means of an epimorphism and a monomorphism. This is called an *image factorization* of f (cf. Manes 1976). The existence of such a factorization is based on the fact that in the underlying category of sets, such factorizations exist. Following the same line of thought as above, we shall investigate next how the existence of image factorizations in an arbitrary category C is related to their existence in C_F.

To this end, it turns out to be convenient to look at one particular kind of image factorizations, which exists in many categories: a category *C has coequalizer-mono factorizations* if for every arrow $f : c \to d$ there is a unique (up to isomorphism) factorization $f = i \circ p$:



such that p is a coequalizer (of two arrows in C) and i is mono. If the category C moreover has all kernel pairs and all coequalizers, then it is easy to prove that such a coequalizer-mono factorization of $f : c \to d$ is in particular of the following form:



where $K(f)$ together with (π_1, π_2) is a kernel pair for f, p is (not just any coequalizer but) a coequalizer of π_1 and π_2, and i is (the unique arrow) given by the coequalizer property of p. In that case, we say that *C has image factorizations by means of kernels and coequalizers*.

For instance, the category *Set* is easily seen to have image factorizations by means of kernels and coequalizers.

Theorem 10.3 *If the category C has image factorizations by means of kernels and coequalizers, and if moreover F weakly preserves kernel pairs, then C_F has coequalizer-mono factorizations.*

Proof. Let $f : (c, \alpha_c) \to (d, \alpha_d)$ be a homomorphism of F-coalgebras. Let $\pi_1, \pi_2 : K(f) \to c$ be a kernel pair for f in C. Because F weakly preserves kernel pairs, there exists $\alpha : K(f) \to F(K(f))$ such that π_1 and π_2 are homomorphisms from $(K(f), \alpha)$ to (c, α_c). Since C has all coequalizers and $U : C_F \to C$ creates colimits, there exists a coequalizer $\epsilon : (c, \alpha_c) \to (e, \alpha_e)$ of π_1 and π_2 in C_F. Because $f \circ \pi_1 = f \circ \pi_2$ in C_F there exists a homomorphism $i : (e, \alpha_e) \to (d, \alpha_d)$ such that $i \circ \epsilon = f$. Because C has coequalizer-mono factorizations, this i is mono in C and hence mono in C_F. □

Since $\mathcal{P} : Set \to Set$ weakly preserves kernel pairs, the first isomorphism theorem is an immediate corollary of the above. Also (the categorical generalizations of) the other two isomorphism theorems can be proved with the use of the theorem above.

We have treated the categorical versions of only a few of the theorems on transition systems and, clearly, much remains to be done (cf. Section 12).

11 Comparison with algebras

Let C be a category and $F : C \to C$ a functor. The relation between the category of *algebras* and the category of coalgebras of F is slightly more complicated than one might expect at first sight.

An *F-algebra* is a pair (c, α_c) consisting of an object c in C and an arrow $\alpha_c : F(c) \to c$. Let (c, α_c) and (d, α_d) be two F-algebras. An arrow $f : c \to d$ is a *homomorphism of F-algebras* if $f \circ \alpha_c = \alpha_d \circ F(f)$. The collection of all F-algebras together with F-algebra homomorphisms is a category, which we denote by C^F. A subobject R of the product of c and d (if it exists) is called an *F-substitutive relation* if there exists an F-algebra structure $\alpha_R : F(R) \to R$ such that the projections from R to c and d are homomorphisms of F-algebras. (Somewhat confusingly, F-substitutive relations are called *F-congruences* in Manes 1976, Rutten and Turi 1994.)

It can be easily shown that Σ-algebras are the F-algebras for a particular functor F on the category of sets (see, e.g., Asperti and Longo 1990, Rutten and Turi 1994).

Although the notion of F-algebra is dual to that of F-coalgebra, the category C^F of F-algebras is *not* dual to the category C_F of F-coalgebras. Informally speaking, this can be explained by the following two diagrams (of a homomorphism of F-algebras and a homomorphism of F-coalgebras):

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(d) \\ {\scriptstyle \alpha_c}\downarrow & * & \downarrow{\scriptstyle \alpha_d} \\ c & \xrightarrow[f]{} & d \end{array} \qquad \begin{array}{ccc} c & \xrightarrow{f} & d \\ {\scriptstyle \alpha_c}\downarrow & * & \downarrow{\scriptstyle \alpha_d} \\ F(c) & \xrightarrow[F(f)]{} & F(d), \end{array}$$

and the observation that the second diagram is obtained from the first one by reversing the vertical arrows only (rather than *all* arrows). This process of *reversing vertical arrows* is essentially what underlies the translation mentioned in the introduction: F-algebra becomes F-coalgebra; homomorphism of F-algebras becomes homomorphism of F-coalgebras; and F-substitutive relation becomes F-bisimulation. (Note, however, that for Σ-algebras and transition systems different functors are used.)

The precise relationship between the categories C_F and C^F can be expressed as follows: $C_F \cong (C^{F^{op}})^{op}$. The opposite functor $F^{op} : C^{op} \to C^{op}$ acts on objects as F does, and maps an arrow f^{op} to $(F(f))^{op}$. Cf. Mac Lane 1971.

As we have seen in Section 10, many theorems hold in C_F because similar theorems are true in C. The same applies to F-algebras.

Theorem 11.1 *Let $V : C^F \to C$ be the functor which maps an F-algebra (c, α_c) to its carrier c. Similarly, the functor $U : C_F \to C$ maps an F-coalgebra (c, α_c) to its carrier c.*

1. *The functor V creates all limits and those colimits that are preserved by F.*
2. *(Theorems 10.1 and 10.2:) The functor U creates all colimits and all limits that are preserved by F.* □

On the basis of the above theorem, it should be possible to characterize a family of statements that are valid for F-algebras, and for which the translation described above yields a valid statement on F-coalgebras.

12 Much remains to be done

Both for our basic example of unlabelled, nondeterministic transition systems and for the general case of coalgebras of an arbitrary functor, there is still much left to do.

The lattices of subsystems and bisimulations deserve further study, and so do the notions of initiality and finality. The simple observation that the composition of bisimulations is again a bisimulation has not been dealt with on the categorical level. The category C should be suitable for reasoning about subobjects and relations. (The family of *regular* categories seems to be a good candidate.) Then there is the notion of coalgebras of a *comonad* (a comonad is a functor together with some natural transformations, see Mac Lane 1971). Such coalgebras have more structure in the sense that the coalgebra arrow is required to satisfy some conditions, and their relevance in terms of transition systems is still to be investigated.

We have not touched upon closure properties. For instance, a class of Σ-algebras is called a *variety* if it is closed under the construction of subalgebras, homomorphic images and products. Equivalently, it is an equationally defined class (by Birkhoff's variety theorem). What would be an appropriate definition of a variety of transition systems? A possible candidate might be a class that is closed under subsystems, homomorphic images and *sums* (instead of products of transition systems, which generally do not exists). With this definition, the class of, e.g., finitely branching transition systems would, and the class of finite transition systems would not be a variety. In the definability theory of modal logic, these constructions have received much attention. A well-known result is for instance, that a first-order definable class of transition systems is *modally* definable if and only if it is closed under the constructions mentioned above and its complement is closed under so-called ultrafilter extensions (Goldblatt and Thomason 1975). The latter result can be obtained rather easily by exploiting the duality of the category of Boolean algebras with operators (Jónsson and Tarski 1951) and transition systems: it is a translation of Birkhoff's variety theorem mentioned above. (See Goldblatt 1989; a pleasant introduction to this 'algebraizing' of modal logic is given in Blackburn et al. 1994. Also in Malacaria 1995, algebraic tools are used in the analysis of transition systems.) It will be interesting to see to what extent these results on frames can be generalized to (classes of) arbitrary coalgebras.

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