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Multiparameter Quantum Groups and Multiparameter R-Matrices

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Abstract. There exists an $\binom{n}{2}+1$ parameter quantum group deformation of GL_n which has been constructed independently by several (groups of) authors. In this note, I give an explicit R-matrix for this multiparameter family. This gives additional information on the nature of this family and facilitates some calculations. This explicit R-matrix satisfies the Yang–Baxter equation. The centre of the paper is Section 3 which describes all solutions of the YBE under the restriction $r^{ab}_{cd}=0$ unless $\{a,b\}=\{c,d\}$. One kind of the most general constituents of these solutions precisely corresponds to the $\binom{n}{2}+1$ parameter quantum group mentioned above. I describe solutions which extend to an enhanced Yang–Baxter operator and, hence, define link invariants. The paper concludes with some preliminary results on these link invariants.

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0. Introduction and Statement of Main Results

This paper is concerned with multiparameter R-matrices and corresponding quantum groups and knot and link invariants. The starting point is an $\binom{n}{2}+1$ parameter deformation of the bialgebra of polynomials on the $n\times n$ matrices

$$K[t^1_1,t^1_2,t^1_3,\ldots,t^2_1,t^2_2,\ldots,t^n_1,\ldots,t^n_n]=K[t],\quad t^i_j\longmapsto t^i_k\otimes t^k_j,\quad \varepsilon(t^i_j)=\delta^i_j,$$

where δ^i_j is the Kronecker delta. Here K is an arbitrary ground field and the Einstein summation convention is in force, i.e. $t^i_k\otimes t^k_j$ stands for $\sum_{k=1}^n t^i_k\otimes t^k_j$. This $\binom{n}{2}+1$ parameter deformation has apparently been independently constructed in various ways by many (groups of) authors, published and unpublished, all more or less in the winter of 1990/1991. I know of several (including myself) and the construction is so natural that quite likely there are more, [1, 3, 5, 9, 11, 14–21]. (Not all these papers deal with the full family and [3], in fact, describes a quantum group which does not fit in this family at all.)

Perhaps the most natural point of view is to take two 'most general' n-dimensional quantum spaces

$$\mathbb{A}=K\langle X^1,\ldots,X^n\rangle/(X^iX^j=q^{ij}X^jX^i),$$

$$\mathbb{B} = K\langle Y_1, \ldots, Y_n\rangle/(Y_iY_j = q_{ij}Y_jY_i).$$

Here the $q^{ij} = (q^{ji})^{-1}$, $q^{ii} = 1$, $q_{ji} = (q_{ij})^{-1}$, $q_{ii} = 1$ are arbitrary parameters (viewed as elements of K or as (Laurent) variables). Now look for a maximal quotient $K\langle t\rangle/I$, of $K\langle t\rangle$, $t^i_j \mapsto t^i_k \otimes t^k_j$, to co-act on the left on $\mathbb{A}$ and on the right on $\mathbb{B}$ by the standard formulas

$$X^i \longmapsto t^i_k \otimes X^k, \qquad Y_j \longmapsto Y_k \otimes t^k_j.$$

For the resulting bialgebra $K\langle t\rangle/I$ to be nice, in the sense that the underlying algebra is PBW (Poincaré–Birkhoff–Witt), certain relations must hold between the q^{ij} and q_{kl}, viz. that after a possible permutation of the $1, \ldots, n$ (a renumbering of the variables), $q^{ij}q_{ij} = \rho \neq -1$, for all $i < j$. This material, which can also be found in [1] and other papers, is recalled in Sections 1 and 2 below.

The heart of the paper is Section 3. In it I consider the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \quad R = (r^{ab}_{cd}) \tag{0.1}$$

and describe all invertible solutions which satisfy the additional condition

$$r^{ab}_{cd} = 0 \text{ unless } \{a, b\} = \{c, d\}. \tag{0.2}$$

These solutions are constructed with blocks, consisting of several components, which are fitted together in certain not entirely trivial ways, cf. Theorem 3.35 for a precise discription. For instance, a block consisting of three components with only one element looks as follows:

$$\begin{array}{c|ccc|ccc|ccc|}
 & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
11 & \rho_1 & & & & & & & & \\
12 & & x_{21}^{-1}z & & y & & & & & \\
13 & & & x_{31}^{-1}z & & & & y & & \\
\hline
21 & & & & x_{21} & & & & & \\
22 & & & & & \rho_2 & & & & \\
23 & & & & & & x_{32}^{-1}z & & y & \\
\hline
31 & & & & & & & x_{31} & & \\
32 & & & & & & & & x_{32} & \\
33 & & & & & & & & & \rho_3
\end{array} \tag{0.3}$$

where the ρ_1, ρ_2, ρ_3 are all three solutions of $X^2 = yX + z$ (but not necessarily all three are equal). Given any $n^2 \times n^2$ matrix R, there is a natural bialgebra $K\langle t\rangle/I(R)$, $t^i_j \mapsto t^i_k \otimes t^k_j$. Here, $I(R)$ is the ideal generated by the fundamental commutation relations (FCR) of [6]

$$RT_1T_2 = T_2T_1R, \tag{0.4}$$

where $T = (t^i_j)$, $T_1 = T \otimes I_n$, $T_2 = I_n \otimes T$.

Multiplying a solution of (0.1) with an invertible scalar, produces another solution and does not affect the relations defined by (0.4). Thus, the parameter z in (0.3) (or rather its $n^2 \times n^2$ generalization) can be normalized to 1 (by multiplying with $(\sqrt{z})^{-1}$). The two roots of $X^2 = yX + z$ are then q and $-q^{-1}$. If all the ρ_i are now equal to q, the invertible $n^2 \times n^2$ matrix like (0.3) precisely defines the $\binom{n}{2} + 1$-parameter deformation of Sections 1 and 2. This is the main result of Section 4. Having an explicit invertible R-matrix that satisfies the YBE (0.1), for this $\binom{n}{2}+1$-parameter quantum matrix algebra has a number of considerable advantages. For instance, it immediately follows that the rewriting rules (0.4) are confluent which greatly simplifies the proof that this $\binom{n}{2} + 1$ parameter quantum matrix algebra is a PBW algebra. It also helps with the matter of defining a quantum determinant and the definition of an antipode on the bialgebra obtained by making the quantum determinant invertible, thus obtaining an $\binom{n}{2} + 1$ parameter quantum group. This is not further explored here, but see [4, 12, 13, 6].

It also seems from (0.3) that $\binom{n}{2}$ parameters of the $\binom{n}{2}+1$, viz. the x_{ij}, $i > j$, are rather trivial and that there is only one real parameter viz. y (or q; $y = q - q^{-1}$ if $z = 1$). This does not mean that the general quantum matrix algebra ($z = 1$, x_{ij} arbitrary) and the classical one ($z = 1 = x_{ij}$) are isomorphic; they are not. All the same, the x_{ij} do seem less basic than q. I do not know how to make this intuition more precise except in the case of the link invariants defined by the enhanced Yang–Baxter operator that is associated to (0.3), cf. below.

Each block of a solution of (0.1) (assuming (0.2)) defines a scalar. If all those scalars are equal (and only then) the solution gives rise to an enhanced Yang–Baxter operator $(\tau R, \nu, \alpha, \beta)$ in the sense of [22] and, hence, gives rise to a link invariant. In this setting, the $\binom{n}{2}$ extra parameters x_{ij}, $i > j$, are indeed trivial. They do not show up in the link invariant in the sense that if the $n^2 \times n^2$ generalization of (0.3) (even with both q and $-q^{-1}$ occurring for the ρ_i; we are taking $z = 1$) is extended to an enhanced Yang–Baxter operator, which can always be done, than the resulting link invariant is the same as one obtained with all $x_{ij} = 1 = z$ (but possibly a different n). This 'triviality of the x_{ij}' result only applies to 'one type II block' solutions of (0.1). Even in the case of a two size 1 block solution of (0.1), nontrivially fitted together, a nontrivial link invariant appears. Though, of course, the two constituents themselves give nothing. (An $n = 1$ solution of (0.1) always defines a trivial link invariant.) Mixing and fitting together different blocks of both different and the same types seems to promise a rich collection of probably new link invariants. This matter remains to be explored.

As indicated above, the general solution of the Yang–Baxter equation under condition (0.2) consists of blocks which are fitted together in certain ways, each block consisting of several components. In an earlier preprint version of this paper, I mistakenly concluded that each component would be of size one or that a whole block would consist of just one component. This oversight was spotted and corrected by Dr Nico van de Hijligenberg. I am most grateful to him for this and for the considerable amount of work he did in checking the whole manuscript in his characteristic thorough way and the work he put in towards the necessary corrections. In essence, the correction means that in the 'S-formulation' (see Section 5) certain diagonal scalars (those with all four upper and lower indices equal) in the general solution according to the original preprint, can be replaced by scalar matrices (that same scalar multiplying an identity matrix).

1. Generalized Quantum Space $\mathbb{A}_q^n$

The coordinate ring is $K\langle X^1, X^2, \ldots, X^n\rangle / I_n$, where I_n is the ideal generated by the elements

$$X^a X^b - q^{ab} X^b X^a, \tag{1.1}$$

where $q^{ab} = (q^{ba})^{-1}$ and $q^{aa} = 1$ for all $a, b \in \{1, \ldots, n\}$. Thus, depending on one's point of view, $\mathbb{A}_q^n$ is a family of algebras parametrized by $\binom{n}{2}$ parameters or an algebra over $K[q^{ab}, (q^{ab})^{-1}; a > b]$, the ring of commutative Laurent polynomials in $\binom{n}{2}$ variables q^{ab}, $a > b$.

If $q^{ab} = 1$, for all a, b, one refinds the coordinate ring $K[X^1, X^2, \ldots, X^n]$. The algebra $\mathbb{A}_q^n$ is graded and it is a graded deformation of $\mathbb{A}_0^n = K[X^1, \ldots, X^n]$ in the sense that $\dim(\mathbb{A}_q^n)_m = \dim(\mathbb{A}_0^n)_m$ for all q where a lower m indicates the homogeneous part of degree m. Also $\mathbb{A}_q^n$ is a PBW algebra in the sense that the monomials

$$(X^1)^{i_1} \cdots (X^n)^{i_n}, \quad i_j \in \mathbb{N} \cup \{0\} \tag{1.2}$$

form a basis of $\mathbb{A}_q^n$. Indeed it is obvious from (1.1) that every element can be written as a sum of elements of the form (1.2); to prove the other half, it suffices by the diamond lemma, [2], to prove that all the 'overlaps'

$$X^a(X^b X^c), \quad (X^a X^b) X^c$$

are confluent, i.e. give the same results when using the rewriting rules (1.1). Now

$$\begin{aligned} X^a(X^b X^c) &= q^{bc}(X^a X^c) X^b = q^{bc} q^{ac} X^c (X^a X^b) = q^{bc} q^{ac} q^{ab} X^c X^b X^a, \\ (X^a X^b) X^c &= q^{ab} X^b (X^a X^c) = q^{ab} q^{ac} (X^b X^c) X^a = q^{ab} q^{ac} q^{bc} X^c X^b X^a. \end{aligned}$$

So this is indeed the case.

2. Generalized Matrix Quantum Algebras

Consider the left-coaction of

$$K\langle t\rangle = K\langle t_1^1, \ldots, t_n^1; \ldots; t_1^n, \ldots, t_n^n\rangle$$

on

$$K\langle X\rangle = K\langle X^1, \dots, X^n\rangle$$

given by the usual formula

$$X^i \longmapsto t^i_k \otimes X^k \tag{2.1}$$

(summation implied).

Now look at what relations are needed between the t's in order that this becomes a co-action of some quotient of $K\langle t\rangle$ on $\mathbb{A}^n_q$. This means that the relations $X^aX^b = q^{ab}X^bX^a$ must be preserved. The image of $X^aX^b - q^{ab}X^bX^a$ under (2.1) is

$$t^a_{r_1}t^b_{r_2} \otimes X^{r_1}X^{r_2} - q^{ab}t^b_{s_1}t^a_{s_2} \otimes X^{s_1}X^{s_2} \tag{2.2}$$

The coefficient of X^rX^r in (2.2) is

$$t^a_rt^b_r - q^{ab}t^b_rt^a_r \tag{2.3}$$

and the coefficient of X^rX^s, $r < s$ in (2.2) is

$$t^a_rt^b_s - q^{ab}t^b_rt^a_s + (q^{rs})^{-1}t^a_st^b_r - (q^{rs})^{-1}q^{ab}t^b_st^a_r. \tag{2.4}$$

Let us count the number of independent relations.

(i) For $a = b$ no relations arise from (2.3).

(ii) If $a \neq b$, then the relations (2.3) fall in groups of two

$$t^a_rt^b_r = q^{ab}t^b_rt^a_r, \qquad t^b_rt^a_r = q^{ba}t^a_rt^b_r, \tag{2.5}$$

which are equivalent because $q^{ba} = (q^{ab})^{-1}$. Thus, there are precisely

$$n\binom{n}{2} = \frac{1}{2}n^2(n-1)$$

relations resulting from (2.3). And these are independent.

(iii) If $a = b$ in (2.4), no relations result.

(iv) If $r = s$ in (2.4), the relations (2.4) are implied by (2.3).

(v) For $a \neq b$, $r \neq s$, the relations (2.4) fall into groups of four (or groups of two if one takes $r < s$), viz.

$$\begin{aligned}
t^a_rt^b_s - q^{ab}t^b_rt^a_s + (q^{rs})^{-1}t^a_st^b_r - q^{ab}(q^{rs})^{-1}t^b_st^a_r &= 0,\\
t^b_rt^a_s - q^{ba}t^a_rt^b_s + (q^{rs})^{-1}t^b_st^a_r - q^{ba}(q^{rs})^{-1}t^a_st^b_r &= 0,\\
t^a_st^b_r - q^{ab}t^b_st^a_r + (q^{sr})^{-1}t^a_rt^b_s - q^{ab}(q^{sr})^{-1}t^b_rt^a_s &= 0,\\
t^b_st^a_r - q^{ba}t^a_st^b_r + (q^{sr})^{-1}t^b_rt^a_s - q^{ba}(q^{sr})^{-1}t^a_rt^b_s &= 0.
\end{aligned} \tag{2.6}$$

These four relations are all the same, e.g., the second is obtained from the first by multiplication of the first by $-q^{ba}$ and the fourth results from the first by multiplication of the first by $(-q^{ba})(q^{sr})^{-1}$. These relations only involve the four products $t_r^a t_s^b$, $t_r^b t_s^a$, $t_s^a t_r^b$, $t_s^b t_r^a$ and they are the only relations in which these four (for given a, b, r, s) are involved. Thus, there are precisely

$$\frac{n^2(n-1)^2}{4}$$

independent relations of this type. In total we therefore have

$$\tfrac{1}{2}n^2(n-1) + \tfrac{1}{4}n^2(n-1)^2 = \tfrac{1}{4}n^2(n^2-1)$$

quadratic relations.

To make the dimension of the degree two part of $K\langle t\rangle/I$ equal to that of the degree two part of $K[t]$, we need

$$n^4 - \left(n^2 + \frac{n^2(n^2-1)}{2}\right) = \tfrac{1}{2}n^2(n^2-1)$$

relations, so that precisely half of them are missing. There are a variety of ways to add the missing relations. An extremely elegant one is to make $K\langle t\rangle/I$ also act on the right on the dual of the quantum space $\mathbb{A}_q^n$, [16]. This, however, does not result in the most general quantum matrix algebra. To obtain that, consider a second, a priori completely different, quantum space

$$\mathbb{B}_q^n = K\langle X_q, \ldots, X_n\rangle/(X_b X_a = q_{ba} X_a X_b,\ a, b \in \{1, \ldots, n\}) \tag{2.7}$$

on which a suitable quotient of $k\langle t\rangle$ is supposed to act on the right by

$$X_i \longmapsto X_j \otimes t_i^j, \tag{2.8}$$

where, of course, $q_{ba} = q_{ab}^{-1}$, $q_{aa} = 1$, $q_{ab} \neq 0$.

(NB, the q_{ba} are a second set of parameters, which have, a priori, nothing to do with the q^{ab}.) The requirement that the action (2.8) be compatible with the commutation relations $X_b X_a = q_{ba} X_a X_b$ of $\mathbb{B}_q^n$, gives necessary relations on the t_j^i which are completely analogous to those produced by having $k\langle t_j^i\rangle$ act on the left on $K\langle X^a\rangle$ as above. They are

$$t_a^r t_b^r = q_{ab} t_b^r t_a^r, \tag{2.9}$$

$$t_a^r t_b^s - q_{ab} t_b^r t_a^s + (q_{rs})^{-1} t_a^s t_b^r - q_{ab}(q_{rs})^{-1} t_b^s t_a^r = 0. \tag{2.10}$$

In case $q_{ab} = -(q^{ab})^{-1}$, relations (2.4) and (2.10) coincide. But generically they are independent.

2.11. LEMMA. *Let I_L in $K\langle t\rangle$ be the two sided-ideal generated by the elements* (2.3) *and* (2.4), *let I_R be the two-sided ideal generated by the relations* (2.9) *and* (2.10). *Both I_L and I_R are bialgebra ideals in $K\langle t\rangle$ and, hence, so is I, the two sided ideal generated by I_L and I_R together.*

The proof of this is contained in Appendix 1.

Remark. There is also a more elegant way to see that I_L and I_R are bialgebra ideals. Let $A = \mathbb{A}_q^n$. The dual space is $A^! = K\langle X_1, \dots, X_n\rangle/J$, where J is generated by X_j^2, $X_iX_j = -q^{ij}X_jX_i$. It is now a simple mater to check that $A^! \bullet A$, as defined in [16], is precisely $K\langle t\rangle/I_L$. Now $A^! \bullet A$ is always a bialgebra ([16, Section 5]), for any quadratic algebra A. The results above now brings the additional bit of information that $A^! \bullet A$ is, in fact, the largest quotient of $K\langle t\rangle$ which co-acts on the left on $\mathbb{A}_q^n$.

Assume from now on that $q^{ab} + q_{ba}^{-1} \neq 0$ for all a, b. Then the relations (2.4) and (2.10) combine to give

$$\begin{aligned} t_s^b t_r^a &= (q^{sr} + q_{sr}^{-1})^{-1}(q^{ba}q^{sr} - q_{sr}^{-1}q_{ba}^{-1})t_s^a t_r^b + \\ &\quad + (q^{sr} + q_{sr}^{-1})^{-1}(q^{ba} + q_{ba}^{-1})t_r^a t_s^b. \end{aligned} \tag{2.12}$$

Now order the t_b^a as follows. Choose an ordering on the set of indices $\{1, \dots, n\}$ and define

$$t_b^a < t_d^c \iff \begin{cases} a < c, \\ \text{or } a = c \text{ and } b < d. \end{cases} \tag{2.13}$$

Then it follows from $t_r^a t_s^a = q_{rs} t_s^a t_r^a$ and (2.12) that every monomial in $K\langle t\rangle$ can be written modulo I in the form

$$t_{j_1}^{i_1} t_{j_2}^{i_2} \dots t_{j_m}^{i_m} \quad t_{j_1}^{i_1} \leqslant t_{j_2}^{i_2} \leqslant \dots \leqslant t_{j_m}^{i_m}. \tag{2.14}$$

2.15. DEFINITION. An algebra A over K is a PBW algebra if there are elements $x_1, \dots, x_m$ in A such that the monomials

$$x_1^{r_1} x_2^{r_2} \dots x_m^{r_m}, \quad r_i \in \mathbb{N} \cup \{0\}$$

form a basis of A over K.

It does not yet follow that $K\langle t\rangle/I$ is a PBW algebra. All we know so far is that (for any ordering of the indices $a, b, \dots$) the monomials (2.14) generate the algebra and that the monomials of degree 2

$$t_{j_1}^{i_1} t_{j_2}^{i_2} \quad t_{j_1}^{i_1} \leqslant t_{j_2}^{i_2}$$

are independent (as they should be for a PBW algebra).

2.16. EXAMPLE OF A PBW ALGEBRA. Let $\mathfrak{g}$ be a Lie algebra over K and $U\mathfrak{g}$ its universal enveloping algebra. Let $x_1, \dots, x_m$ be a basis over K for $g \subset U\mathfrak{g}$ (as a vector space). Then by the PBW-theorem (Poincaré–Birkhoff–Witt). The

$$x_1^{r_1} \dots x_m^{r_m}, \quad r_i \in \mathbb{N} \cup \{0\}$$

are a basis for $U\mathfrak{g}$ over K. Thus, $U\mathfrak{g}$ is a PBW algebra. This is, of course, the result which suggested the phrase 'PBW-algebra'. If $\mathfrak{g}$ is Abelian, then $U\mathfrak{g} = S\mathfrak{g}$ the symmetric algebra of $\mathfrak{g}$ over K, viz.

$$S\mathfrak{g} = K[x_1, \dots, x_m]$$

2.17. THEOREM [1]. *Let K, q_{ab}, q^{ab}, t, I be as before, then $K\langle t\rangle/I$ is a PBW algebra with generators t^i_j, $i, j = 1, \dots, n$ if and only if $q^{ab} + q_{ab}^{-1} \neq 0$ for all a, b and there is a total ordering on the index set I (possibly different from $1 < 2 < \cdots < n$) such that*

$$q^{ab}/q_{ba} = q^{cd}/q_{dc} = \rho \neq -1 \quad \text{for all} \quad a < b,\ c < d. \tag{2.18}$$

Thus, we get an $\binom{n}{2} + 1$ parameter family of PBW deformations of the polynomial algebra $K[t^1_1, \dots, t^n_n]$. Note that I is a graded ideal so that $M_q = K\langle t\rangle/I$ is also graded. Give the t^i_j degree 1, then

$$\begin{aligned}\dim(M_q)_r &= \#\left\{(r_1, \dots, r_m) \mid r_i \in \mathbb{N} \cup \{0\},\ \sum_{i=1}^{m} r_i = r\right\}\\ &= \dim K[t^1_1, \dots, t^n_n]_r,\end{aligned}$$

where $m = n^2$, and A_r denotes the homogeneous component of degree r of a graded algebra A.

The Hilbert–Poincaré series of a graded algebra A is by definition equal to

$$H_A(t) = \sum_{r=1}^{\infty} \dim(A_r) t^r. \tag{2.19}$$

Thus, the Hilbert–Poincaré series of every $K\langle t\rangle/I$ satisfying (2.18) is equal to that of the polynomial algebra $K[t]$ and the $M_q = K\langle t\rangle/I$ are a deformation of the graded algebra $K[t]$ in the sense of graded algebras.

2.20. *Proof of the necessity of* (2.18). By the remark just below 2.10, we already know that we must have $q^{ab} + q_{ab}^{-1} \neq 0$ to get the right amount of linear independent monomials of degree 2.

Take $s = a$, $r = b$ in (2.12) to get

$$t^b_a t^a_b = q^{ba} q_{ab} t^a_b t^b_a. \tag{2.21}$$

Now use (2.21) and (2.12) and $t^a_r t^b_r = q^{ab} t^b_r t^a_r$, $t^a_r t^a_s = q_{rs} t^a_s t^a_r$ to calculate $t^c_a t^b_a t^a_b$ in two ways for $a \neq b \neq c \neq a$

$$\begin{aligned}t^c_a(t^b_a t^a_b) &= q^{ba} q_{ab} (t^c_a t^a_b) t^b_a\\ &= q^{ba} q_{ab} (q^{ab} + q_{ab}^{-1})^{-1} (q^{ca} q^{ab} - q_{ab}^{-1} q_{ca}^{-1}) t^a_a (t^c_b t^b_a) +\\ &\quad + q^{ba} q_{ab} (q^{ab} + q_{ab}^{-1})^{-1} (q^{ca} + q_{ca}^{-1}) t^a_b (t^c_a t^b_a)\end{aligned}$$

$$\begin{aligned}
&= q^{ba}q_{ab}(q^{ab}+q_{ab}^{-1})^{-1}(q^{ca}q^{ab}-q_{ab}^{-1}q_{ca}^{-1})(q^{ba}+q_{ba}^{-1})^{-1}\times\\
&\quad\times(q^{cb}q^{ba}-q_{cb}^{-1}q_{ba}^{-1})t_a^a t_b^b t_a^c+\\
&\quad+q^{ba}q_{ab}(q^{ab}+q_{ab}^{-1})^{-1}(q^{ca}q^{ab}-q_{ab}^{-1}q_{ca}^{-1})\times\\
&\quad\times(q^{ba}+q_{ba}^{-1})^{-1}(q^{cb}+q_{cb}^{-1})t_a^a t_a^b t_b^c+\\
&\quad+q^{ba}q_{ab}(q^{ab}+q_{ab}^{-1})^{-1}(q^{ca}+q_{ca}^{-1})q^{cb}t_b^a t_a^b t_a^c.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(t_a^c t_a^b)t_b^a &= q^{cb}t_a^b(t_a^c t_b^a)\\
&= q^{cb}(q^{ab}+q_{ab}^{-1})^{-1}(q^{ca}q^{ab}-q_{ca}^{-1}q_{ab}^{-1})(t_a^b t_a^a)t_b^c+\\
&\quad+q^{cb}(q^{ab}+q_{ab}^{-1})^{-1}(q^{ca}+q_{ca}^{-1})(t_a^b t_b^a)t_a^c\\
&= q^{cb}(q^{ab}+q_{ab}^{-1})^{-1}(q^{ca}q^{ab}-q_{ca}^{-1}q_{ab}^{-1})q^{ba}t_a^a t_a^b t_b^c+\\
&\quad+q^{cb}(q^{ab}+q_{ab}^{-1})^{-1}(q^{ca}+q_{ca}^{-1})q^{ba}q_{ab}t_b^a t_a^b t_a^c.
\end{aligned}$$

It follows that the coefficient of $t_a^a t_b^b t_a^c$ must be zero, which gives

$$q^{ca}q^{ab}-q_{ab}^{-1}q_{ca}^{-1}=0 \quad\text{or}\quad q^{cb}q^{ba}-q_{cb}^{-1}q_{ba}^{-1}=0. \tag{2.22}$$

Let $\rho_{ab}=q^{ab}q_{ab}=q^{ab}q_{ba}^{-1}$. Then (2.22) says

$$\rho_{ab}=\rho_{ac} \quad\text{or}\quad \rho_{ab}=\rho_{cb} \tag{2.23}$$

(This holds for all triples $a\neq b\neq c\neq a$.) Choose a fixed i,j say $i=1$, $j=2$ and let $\rho=\rho_{ij}$. Then (2.23) implies

$$\rho_{ab}=\rho \quad\text{or}\quad \rho_{ab}=\rho^{-1}, \quad \text{for all } a,b \tag{2.24}$$

(but (2.24) is strictly weaker than (2.23)).

If $\rho=\rho^{-1}$ (i.e. $\rho=\pm 1$), then for all a,b, $\rho_{ab}=q^{ab}/q_{ba}=\rho$ and any ordering works. If $\rho\neq\rho^{-1}$ define

$$i>j \iff \rho_{ij}=\rho \tag{2.25}$$

Then $i>j$, $j>k \Rightarrow \rho_{ij}=\rho$ and $\rho_{jk}=\rho$, so that by (2.23) (with $a=i$, $b=k$, $c=j$) $\rho_{ik}=\rho$, i.e. $i>k$, proving that the order defined by (2.25) is transitive. For this order, we have

$$\frac{q^{ij}}{q_{ji}}=\rho_{ij}=\rho \quad \text{for } i>j.$$

This finishes the proof of the necessity of Theorem 2.17. The sufficiency can now be handled by the Diamond lemma [2], which says, in this case, that if all the overlaps $(t_b^a t_s^r)t_v^u - t_b^a(t_s^r t_v^u)$ are zero, then the monomials (2.14) are a basis. Though there is a good deal of symmetry which can be exploited, this still involves quite a number of cases and rather lengthy calculations for each case. We shall use a different approach, cf. Corollary 4.25.

3. A Rather General Candidate R-Matrix

Let $R = (r^{ab}_{cd})$ be an $n^2 \times n^2$ matrix over K. In this section, we examine a fairly general R-matrix whose form is inspired by the kind of commutation relations of Section 2 and study when it satisfies the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3.1}$$

Here, R: $V \otimes V \to V \otimes V$, where V has basis $e^1, \dots, e^n$, is given by

$$R(e^i \otimes e^j) = r^{ij}_{kl} e^k \otimes e^l,$$

$$R_{12} = R \otimes \mathrm{Id}, \qquad R_{23} = \mathrm{Id} \otimes R,$$

and

$$R_{13}(e^i \otimes e^j \otimes e^k) = r^{ik}_{mn} e^m \otimes e^j \otimes e^n.$$

In terms of the entries r^{ab}_{cd} of R, Equation (3.1) says

$$r^{ab}_{k_1k_2} r^{k_1c}_{uk_3} r^{k_2k_3}_{vw} = r^{bc}_{l_1l_2} r^{al_2}_{l_3w} r^{l_3l_1}_{uv}, \tag{3.2}$$

for all $a, b, c, u, v, w \in \{1, 2, \dots, n\}$.

Now consider a general R-matrix with the requirement that

$$r^{ab}_{cd} = 0 \quad \text{unless } \{a, b\} = \{c, d\}. \tag{3.3}$$

Thus, the only possibly nonzero entries are of the form r^{ab}_{ab}, r^{ab}_{ba}, r^{aa}_{aa} (and r^{ba}_{ba}, r^{ba}_{ab}), $a \neq b$.

This is more or less inspired by the commutation relations of Section 2 and, as we shall see in Section 4, it is possible to choose the r^{ab}_{cd} such that the commutation relations of Section 2 are reproduced. It is somewhat remarkable that the requirement that an R-matrix of type (3.3) satisfy YB is practically (but not quite) equivalent to the requirement that it gives the right number of relations in degree 2 and that then these are precisely the commutation relations of Section 2 above.

The following lemma drastically reduces the number of equations (3.2) that must be examined (from n^6 to $6n^3$).

3.4. LEMMA. *Let R be an $n^2 \times n^2$ matrix satisfying* (3.3). *Then both sides of* (3.2) *are zero unless* $\{a, b, c\} = \{u, v, w\}$.

Proof. If a term on the left-hand side of (3.2) is nonzero we must have $\{a, b\} = \{k_1, k_2\}$, $k_3 \in \{k_1, c\}$ so $\{k_1, k_2, k_3\} \subset \{a, b, c\}$. Further $u \in \{k_1, c\}$,

$v, w \in \{k_2, k_3\}$ so $\{u, v, w\} \subset \{k_1, c, k_2, k_3\} = \{k_1, k_2, k_3\} \subset \{a, b, c\}$. Similarly $\{k_2, k_3\} = \{v, w\}$, $k_1 \in \{u, k_3\}$ so $\{k_1, k_2, k_3\} \subset \{u, v, w\}$; $\{a, b\} = \{k_1, k_2\}$, $c \in \{u, k_3\}$ so $\{a, b, c\} \subset \{k_1, k_2, k_3, u\} \subset \{u, v, w\}$.

The argument that for a nonzero term on the right-hand side we must have $\{a, b, c\} = \{u, v, w\} = \{l_1, l_2, l_3\}$ is quite similar. Indeed $\{b, c\} = \{l_1, l_2\}$, $l_3 \in \{a, l_2\}$ so $\{l_1, l_2, l_3\} \subset \{a, b, c\}$; $\{u, v\} = \{l_1, l_3\}$, $w \in \{a, l_2\}$, so $\{u, v, w\} \subset \{l_1, l_2, l_3, a\} \subset \{a, b, c\}$; and $\{l_1, l_3\} = \{u, v\}$, $l_2 \in \{l_3, w\}$, so $\{l_1, l_2, l_3\} \subset \{u, v, w\}$; $\{b, c\} = \{l_1, l_2\}$, $a \in \{l_3, w\}$ so $\{a, b, c\} \subset \{l_1, l_2, l_3, w\} \subset \{u, v, w\}$.

3.5. LEMMA. *Let R be an $n^2 \times n^2$ matrix satisfying* (3.3). *Then*

$$\det(R) = \prod_{i=1}^{n} r_{ii}^{ii} \prod_{i<j} (r_{ij}^{ij} r_{ji}^{ji} - r_{ji}^{ij} r_{ij}^{ji}).$$

Proof. Immediate.

3.6. THE R-EQUATIONS

Many of the equations (3.2), assuming (3.3), are automatically satisfied. Take, for example, $a \neq b \neq c \neq a$, $u = a$, $v = b$, $w = c$. Then the nonzero left-hand terms must have $k_1 = a = u$, $k_3 = c$ and, hence, $k_2 = b$ so the LHS is equal to $r_{ab}^{ab} r_{ac}^{ac} r_{bc}^{bc}$. For the RHS, we must have $l_3 = a$, $l_2 = c$, hence $l_1 = b$ and so the RHS is $r_{bc}^{bc} r_{ac}^{ac} r_{ab}^{ab}$ and so this equation is automatically satisfied. As it turns out, there remain the following equations

$$r_{bc}^{bc}(r_{ba}^{ab} r_{ca}^{ac}) = r_{bc}^{bc}(r_{ba}^{ab} r_{cb}^{bc} + r_{ca}^{ac} r_{bc}^{cb}) \quad (a \neq b \neq c \neq a, u = b, v = c, w = a), \tag{R1}$$

$$r_{ab}^{ab}(r_{ca}^{ac} r_{ab}^{ba} + r_{ba}^{ab} r_{cb}^{bc}) = r_{ab}^{ab}(r_{cb}^{bc} r_{ca}^{ac}) \quad (a \neq b \neq c \neq a, u = c, v = a, w = b), \tag{R2}$$

$$r_{ab}^{ab} r_{ba}^{ba} r_{ca}^{ac} + r_{ba}^{ab} r_{ba}^{ab} r_{cb}^{bc} = r_{bc}^{bc} r_{cb}^{cb} r_{ca}^{ac} + r_{cb}^{bc} r_{cb}^{bc} r_{ba}^{ab} \quad (a \neq b \neq c \neq a, u = c, v = b, w = a), \tag{R3}$$

$$r_{ac}^{ac} r_{ca}^{ac} r_{ac}^{ca} = 0 \quad (a = b \neq c, u = a, v = c, w = a), \tag{R4}$$

$$r_{aa}^{aa} r_{aa}^{aa} r_{ca}^{ac} = r_{aa}^{aa} r_{ca}^{ac} r_{ca}^{ac} + r_{ac}^{ac} r_{ca}^{ac} r_{ca}^{ca} \quad (a = b \neq c, u = c, v = w = a), \tag{R5}$$

$$r_{aa}^{aa} r_{aa}^{aa} r_{ac}^{ca} = r_{aa}^{aa} r_{ac}^{ca} r_{ac}^{ca} + r_{ca}^{ca} r_{ac}^{ac} r_{ac}^{ca} \quad (a \neq b = c, u = b, v = b, w = a), \tag{R6}$$

$$r_{ba}^{ab} r_{ab}^{ba} r_{ba}^{ab} = r_{ba}^{ab} r_{ab}^{ba} r_{ab}^{ba} \quad (a = c \neq b, u = a, v = b, w = a) \tag{R7}$$

All the other cases either give nothing or give back one of these seven types of equations. For the complete detailed analysis, cf. Appendix 2.

3.7. A SOLUTION FAMILY

Take

$$r_{ij}^{ij} = x_{ij} \quad \text{for } i > j, \qquad r_{ij}^{ij} = x_{ji}^{-1} \lambda^u \lambda_d \quad \text{for } i < j,$$

$$r_{ii}^{ii} = \lambda^u, \qquad r_{ji}^{ij} = \lambda^u - \lambda_d \quad \text{if } i < j, \qquad r_{ji}^{ij} = 0 \quad \text{for } i > j.$$

It is a straightforward matter to check that these r's satisfy (R1)–(R7).

There are $\binom{n}{2}$ parameters x_{ij}, $i < j$ and two more parameters λ^u, λ_d. One of these can be eliminated by dividing all parameters by an arbitrary number.

Thus, we have here an $\binom{n}{2} + 1$ parameter family and this is, in fact, the $\binom{n}{2} + 1$ parameter family of Section 2 above. The connections are

$$q^{ab} = x_{ab}^{-1} \lambda^u, \qquad q_{ba} = x_{ab}^{-1}, \quad a > b. \tag{3.8}$$

3.9. 'PARTIAL ORDERING' $\{1, \dots, n\}$

We assume that R is invertible. Define for $a, b \in \{1, \dots, n\}$:

$$a \leqslant b \iff r_{ba}^{ab} \neq 0. \tag{3.10}$$

3.11. LEMMA. *The relation defined by* (3.10) *is a 'partial order'.*

Proof. We have to show transitivity. Let $r_{ba}^{ab} \neq 0 \neq r_{cb}^{bc}$, i.e. $a \leqslant b$, $b \leqslant c$ and we have to show $r_{ca}^{ac} \neq 0$ (which is $a \leqslant c$).

By (R7), there are four cases to be considered

$$r_{ba}^{ab} \neq 0, \qquad r_{ab}^{ba} = 0, \qquad r_{cb}^{bc} \neq 0, \qquad r_{bc}^{cb} = 0, \tag{3.11.1}$$

$$r_{ba}^{ab} \neq 0, \qquad r_{ab}^{ba} = 0, \qquad r_{cb}^{bc} = r_{bc}^{cb} \neq 0, \tag{3.11.2}$$

$$r_{ba}^{ab} = r_{ab}^{ba} \neq 0, \qquad r_{cb}^{bc} \neq 0, \qquad r_{bc}^{cb} = 0, \tag{3.11.3}$$

$$r_{ba}^{ab} = r_{ab}^{ba} \neq 0, \qquad r_{cb}^{bc} = r_{bc}^{cb} \neq 0. \tag{3.11.4}$$

In case (1) by the invertibility of R (cf. Lemma 3.5), also $r_{ab}^{ab} \neq 0 \neq r_{ba}^{ba}$. Hence, by (R2), $r_{ba}^{ab} r_{cb}^{bc} = r_{cb}^{bc} r_{ca}^{ac}$ and, hence, $r_{ca}^{ac} = r_{ba}^{ab} \neq 0$.

In case (2), also $r_{ab}^{ab} \neq 0 \neq r_{ba}^{ba}$ and using (R2) with a and b interchanged gives $r_{cb}^{bc} r_{ba}^{ab} = r_{ca}^{ac} r_{cb}^{bc}$ so that again $r_{ca}^{ac} = r_{ba}^{ab} \neq 0$.

In case (3), by the invertibility of R, $r_{bc}^{bc} \neq 0 \neq r_{cb}^{cb}$ and hence by (R1) $r_{ba}^{ab} r_{ca}^{ac} = r_{ba}^{ab} r_{cb}^{bc}$ and, hence, $r_{ca}^{ac} = r_{cb}^{bc} \neq 0$.

In case (4), suppose that $r_{ca}^{ac} = 0$. Then, by invertibility of R, $r_{ac}^{ac} \neq 0 \neq r_{ca}^{ca}$.

Now use (R3) with b and c interchanged to obtain

$$r_{ac}^{ac} r_{ca}^{ca} r_{ba}^{ab} + r_{ca}^{ac} r_{ca}^{ac} r_{bc}^{cb} = r_{cb}^{cb} r_{bc}^{bc} r_{ba}^{ab} + r_{bc}^{cb} r_{bc}^{cb} r_{ca}^{ac}.$$

By (R4), $r_{cb}^{cb} = r_{bc}^{bc} = 0$ (because $r_{cb}^{bc} r_{bc}^{cb} \neq 0$); hence this would give

$$r_{ac}^{ac} r_{ca}^{ca} r_{ba}^{ab} = 0, \quad \text{i.e. } r_{ba}^{ab} = 0,$$

a contradiction. Hence, $r_{ca}^{ac} \neq 0$, concluding the proof of the lemma. □

We note that the relation $\leqslant$ does not satisfy the antisymmetry, i.e. it does not satisfy: $a \leqslant b$ and $b \leqslant a$ implies $a = b$. For this reason, we wrote 'partial ordering', the consequences of this will be examined in more detail in Section 3.15.

3.12. Blocks

Still assuming that R is invertible, define two indices $a, b \in \{1, \dots, n\}$ to be *connected* (notation $\sim$) if $a \leqslant b$ or $b \leqslant a$ in the ordering of (3.9) above.

3.13. LEMMA. *Connectedness is an equivalence relation.*

Remark. This is not immediately implied by Lemma 3.11. It adds information, e.g., to the case $a \leqslant b, a \leqslant c$, by stating that then b and c are connected.

Proof of Lemma 3.13. Suppose that $a \sim b$ and $b \sim c$, we prove that $a \sim c$. There are four cases to consider

$$r_{ba}^{ab} \neq 0,\ r_{cb}^{bc} \neq 0. \text{ Then } a \leqslant b, b \leqslant c, \text{ hence } a \leqslant c \text{ and } r_{ca}^{ac} \neq 0, \tag{3.13.1}$$

$$r_{ab}^{ba} \neq 0, r_{bc}^{cb} \neq 0. \text{ Then } b \leqslant a, c \leqslant b, \text{ hence } c \leqslant a \text{ and } r_{ac}^{ca} \neq 0. \tag{3.13.2}$$

The other two cases involve more work:

$$r_{ba}^{ab} \neq 0, \qquad r_{bc}^{cb} \neq 0. \tag{3.13.3}$$

As in the case of the proof of Lemma 3.11, there are (by (R7)) four possible subcases to consider.

$$r_{ba}^{ab} \neq 0, \qquad r_{ab}^{ba} = 0, \qquad r_{bc}^{cb} \neq 0, \qquad r_{cb}^{bc} = 0, \tag{3.13.3.1}$$

$$r_{ba}^{ab} \neq 0, \qquad r_{ab}^{ba} = 0, \qquad r_{bc}^{cb} = r_{cb}^{bc} \neq 0, \tag{3.13.3.2}$$

$$r_{ba}^{ab} = r_{ab}^{ba} \neq 0, \qquad r_{bc}^{cb} \neq 0, \qquad r_{cb}^{bc} = 0, \tag{3.13.3.3}$$

$$r_{ba}^{ab} = r_{ab}^{ba} \neq 0, \qquad r_{bc}^{cb} = r_{cb}^{bc} \neq 0. \tag{3.13.3.4}$$

In the last three subcases, Lemma 3.11 is immediately applicable. It remains to deal with (3.13.3.1). In this case, $r^{bc}_{bc} \neq 0$ by invertibility. □

Now use (R2) after the permutation $b \mapsto c \mapsto b$, $a \mapsto a$ to find

$$r^{ac}_{ac}(r^{ab}_{ba}r^{ca}_{ac} + r^{ac}_{ca}r^{cb}_{bc}) = r^{ac}_{ac}(r^{cb}_{bc}r^{ab}_{ba}). \tag{3.13.3.5}$$

Now if $r^{ac}_{ca} = r^{ca}_{ac} = 0, r^{ac}_{ac} \neq 0$ by invertibility. Hence, the RHS of (3.13.3.5) is not equal to zero so that also r^{ca}_{ac} or r^{ac}_{ca} must be nonzero, yielding a contradiction. By consequence, $a \sim c$.

The final case is

$$r^{ba}_{ab} \neq 0, \qquad r^{bc}_{cb} \neq 0, \tag{3.13.4}$$

Again there are four subcases

$$r^{ba}_{ab} \neq 0, \qquad r^{ab}_{ba} = 0, \qquad r^{bc}_{cb} \neq 0, \qquad r^{cb}_{bc} = 0, \tag{3.13.4.1}$$

$$r^{ba}_{ab} \neq 0, \qquad r^{ab}_{ba} = 0, \qquad r^{bc}_{cb} = r^{cb}_{bc} \neq 0, \tag{3.13.4.2}$$

$$r^{ba}_{ab} = r^{ab}_{ba} \neq 0, \qquad r^{bc}_{cb} \neq 0, \qquad r^{cb}_{bc} = 0, \tag{3.13.4.3}$$

$$r^{ba}_{ab} = r^{ab}_{ba} \neq 0, \qquad r^{bc}_{cb} = r^{cb}_{bc} \neq 0. \tag{3.13.4.4}$$

Again, Lemma 3.11 immediately takes care of (3.13.4.2)–(3.13.4.4) and only (3.13.4.1) remains. In this case if $r^{ac}_{ca} = r^{ca}_{ac} = 0$, $r^{ac}_{ac} \neq 0$, which by (R1) (with a and b interchanged) would imply $r^{ba}_{ab}r^{bc}_{cb} = 0$, contradicting (3.13.4.1). Hence, $r^{ac}_{ca} \neq 0$ or $r^{ca}_{ac} \neq 0$ and we are done.

3.14. DEFINITION. An equivalence class $B \subset \{1, \dots, n\}$ under the equivalence relation of connectedness will be called a block.

3.15. Structure of Blocks I

In this subsection and the next, the structure of blocks is examined. More precisely, if B is a block, the submatrix $R_B = (r^{ab}_{cd})_{a,b,c,d \in B}$ is determined. After that, we will examine how blocks can fit together.

A block is a totally ordered subset of $\{1, \dots, n\}$. However, due to the lack of the antisymmetry property of the ordering relation $\leqslant$, it is possible that inside a block elements a and b exist that cannot be separated. By this, we mean that there may be elements a and b that satisfy the condition $a \leqslant b$ and $b \leqslant a$. In this case, we will say that a and b are *strongly connected* (notation $a \simeq b$).

3.16. DEFINITION. An equivalence class $C \subset B$ under the equivalence relation of strong connectedness will be called a component.

The first step in constructing the general R-matrix is determining the submatrix $R_C = (r^{ab}_{cd})_{a,b,c,d \in C}$, where C is a component of a block B.

3.17. PROPOSITION. *Let C be a component of a block B, then there is a $\lambda \neq 0$ such that for all $a, b \in C$ $(a \neq b)$:*

$$r^{aa}_{aa} = r^{bb}_{bb} = r^{ab}_{ba} = r^{ba}_{ab} = \lambda, \qquad r^{ab}_{ab} = r^{ba}_{ba} = 0. \tag{3.18}$$

Proof. By assumption $r^{ab}_{ba} \neq 0 \neq r^{ba}_{ab}$. Hence, $r^{ab}_{ab} = r^{ba}_{ba} = 0$ by (R4), and $\lambda = r^{ab}_{ba} = r^{ba}_{ab}$ by (R6). Putting this in (R5) gives

$$r^{aa}_{aa} r^{aa}_{aa} r^{ab}_{ba} = r^{aa}_{aa} r^{ab}_{ba} r^{ab}_{ba}. \tag{3.19}$$

By invertibility of R (cf. Lemma 3.5), $r^{aa}_{aa} \neq 0$. Hence, $r^{aa}_{aa} = r^{ab}_{ba} = \lambda$ and switching a, b, also $r^{bb}_{bb} = \lambda$. Hence, (3.18) holds for these particular $a, b \in C$. Now let $c \in C, a \neq c \neq b$. The same argument as given above can be applied with c substituted for b which proves the proposition.

3.20. STRUCTURE OF BLOCKS II

Let B be a block, it consists of several components $C_1, C_2, \ldots, C_p$. Since all elements of B are connected, we may assume that the components are numbered such that $C_1 < C_2 < \cdots < C_p$, i.e. $i < j$, $a \in C_i$ and $b \in C_j$ implies $a < b$. Here $a < b$ stands for $a \leqslant b$ and not $b \leqslant a$. The structure of the submatrices R_{C_j} follows from the preceding proposition, the next proposition describes the structure of the submatrix R_B.

3.21. PROPOSITION. *Let B be a block with components $C_1 < C_2 < \cdots < C_p$ and let λ_j be the scalar that corresponds to the submatrix R_{C_j} according to Proposition* 3.17 *(for all $1 \leqslant j \leqslant p$), then there are scalars $y \neq 0$ and $z \neq 0$ such that for all $i < j$, $a \in C_i$ and $b \in C_j$:*

$$r^{ba}_{ab} = 0, \qquad r^{ab}_{ba} = y \quad \textit{and} \quad r^{ab}_{ab} r^{ba}_{ba} = z. \tag{3.22}$$

Furthermore, the scalars λ_j satisfy the quadratic equation

$$(\lambda_j)^2 = y\lambda_j + z. \tag{3.23}$$

Proof. According to Proposition 3.17, we already know that for all $a, b \in C_j$

$$r^{aa}_{aa} = r^{bb}_{bb} = r^{ab}_{ba} = r^{ba}_{ab} = \lambda_j, \qquad r^{ab}_{ab} = r^{ba}_{ba} = 0.$$

We take elements a and b of C_i and c of C_j $(i < j)$, then $a \simeq b < c$ so

$$r^{ac}_{ca} \neq 0 \neq r^{bc}_{cb}, \qquad r^{ca}_{ac} = 0 = r^{cb}_{bc}. \tag{3.24}$$

It follows from (3.24) that

$$r_{bc}^{bc} \neq 0 \neq r_{cb}^{cb}, \qquad r_{ac}^{ac} \neq 0 \neq r_{ca}^{ca}. \tag{3.25}$$

Now use (R1) to see that

$$r_{ca}^{ac} = r_{cb}^{bc} = y_{i,j} \quad \text{(defining } y_{i,j}\text{)}. \tag{3.26}$$

Consider (R5)

$$(\lambda_i)^2 y_{i,j} = \lambda_i (y_{i,j})^2 + r_{ac}^{ac} r_{ca}^{ca} y_{i,j}, \tag{3.27}$$

and similarly with a and b interchanged to find

$$(\lambda_i)^2 y_{i,j} = \lambda_i (y_{i,j})^2 + r_{bc}^{bc} r_{cb}^{cb} y_{i,j} \tag{3.28}$$

which gives us the definition of $z_{i,j}$ as $z_{i,j} = r_{ac}^{ac} r_{ca}^{ca} = r_{bc}^{bc} r_{cb}^{cb}$. Take $a \in C_i$, $b \in C_j$ and $c \in C_k$ with $i < j < k$, then by using (R1) and (R2), it follows that

$$r_{ba}^{ab} = y_{i,j} = r_{cb}^{bc} = y_{j,k} = r_{ca}^{ac} = y_{i,k}. \tag{3.29}$$

By this y is well defined. Using this in (R3) gives

$$r_{ab}^{ab} r_{ba}^{ba} y + y^3 = z_{i,j} y + y^3 = r_{bc}^{bc} r_{cb}^{cb} y + y^3 = z_{j,k} y + y^3 \tag{3.30}$$

and, hence, as $y \neq 0$, $z_{i,j} = z_{j,k}$. Switching b and c in (R3) now gives $z_{i,k} = z_{j,k}$ and this establishes the first part of the proposition. The last part of Proposition 3.21 now follows directly from (R5) and (R6).

3.31. PROPOSITION. *Let $B_1, \ldots, B_m$ be the blocks of $\{1, \ldots, n\}$, then there are z_{st}, $s, t \in \{1, \ldots, m\}$, $z_{st} = z_{ts}$, such that*

$$r_{ab}^{ab} r_{ba}^{ba} = z_{st} \quad \textit{for all} \quad a \in B_s,\ b \in B_t\ (s \neq t). \tag{3.32}$$

Proof. Choose $c \in B_s, d \in B_t$ and set

$$z_{st} = r_{cd}^{cd} r_{dc}^{dc}. \tag{3.33}$$

If $\#B_s = \#B_t = 1$ there is nothing more to prove. If $\#B_s = 1$, $\#B_t > 1$, let $b \in B_t$, $b \neq d$. Then $r_{db}^{bd} \neq 0$ or $r_{bd}^{db} \neq 0$ and in both cases (R3) gives

$$r_{bc}^{bc} r_{cb}^{cb} = r_{cd}^{cd} r_{dc}^{dc}, \tag{3.34}$$

establishing the result in this case. The case $\#B_s > 1$, $\#B_t = 1$ goes the same. Finally, if $a \neq c$, $a \in B_s$, $b \neq d$, $b \in B_t$, then we get again $r_{ba}^{ab} = r_{bc}^{cb}$ and also because $r_{ca}^{ac} \neq 0$ or $r_{ac}^{ca} \neq 0$

$$r_{ab}^{ab} r_{ba}^{ba} = r_{bc}^{bc} r_{cb}^{cb},$$

which combined with (3.33) gives (3.32).

It will now turn out that the various properties which have been derived are, in fact, also sufficient to guarantee a solution of the YBE. This leads to the following description of all invariable solutions of the YBE under the restriction $r_{cd}^{ab} = 0$, unless $\{a, b\} = \{c, d\}$.

3.35. THEOREM. *Divide the set of indices* $\{1, \dots, n\}$ *into blocks and divide these blocks into components. Further choose numbers* $\in K$ *as follows*:

(i) *For each block* B_s *consisting of a single component* C *choose* $\lambda_s \in K$, $\lambda_s \neq 0$.
(ii) *For each block* B_s *with more than one component, choose* $y_s \in K$, $z_s \in K$, $z_s \neq 0$, $y_s \neq 0$ *and for each component* C_j^s *in* B_s *choose a* λ_j^s *satisfying* $(\lambda_j^s)^2 = \lambda_j^s y_s + z_s$.
(iii) *For each two blocks* B_s, B_t, $s \neq t$ *choose* $z_{st} \in K$, $z_{st} \neq 0$, $z_{st} = z_{ts}$.
(iv) *For each* $a, b \in B_s$ *with* $a > b$ *choose* $x_{ab} \in K$, $x_{ab} \neq 0$.
(v) *For each* $a \in B_s$ *and* $b \in B_t$ *with* $s > t$ *choose* $x_{ab} \in K$, $x_{ab} \neq 0$.

Now define the r_{cd}^{ab} *as follows*

(vi) *If* $a, b \in C_j^s \subset B_s$, $a \neq b$, $r_{aa}^{aa} = r_{bb}^{bb} = r_{ba}^{ab} = r_{ab}^{ba} = \lambda_j^s$, $r_{ab}^{ab} = r_{ba}^{ba} = 0$.
(vii) *If* $a, b \in B_s$, $a < b$, $r_{ba}^{ab} = y_s$, $r_{ab}^{ba} = 0$, $r_{ab}^{ab} = z_s x_{ba}^{-1}$, $r_{ba}^{ba} = x_{ba}$.
(viii) *If* $a \in B_s$, $b \in B_t$, $s < t$, $r_{ab}^{ab} = x_{ab}$, $r_{ba}^{ba} = z_{st} x_{ab}^{-1}$, $r_{ba}^{ab} = r_{ab}^{ba} = 0$.
(ix) $r_{cd}^{ab} = 0$ *unless* $\{a, b\} = \{c, d\}$.

Then the r_{cd}^{ab} *thus specified constitute a solution of the* YBE.

Moreover, up to a permutation of $\{1, \dots, n\}$ *(nonunique as a rule) every solution satisfying* (ix) *is thus obtained.*

Proof. After a permutation of indices, if necessary, the 'partial order' defined by $a \leqslant b \Leftrightarrow r_{ba}^{ab} \neq 0$ is compatible with the natural order of $\{1, \dots, n\}$. The statement that all solutions under the restriction (ix) are obtained by the recipe (i)–(viii) above is now the content of the lemmas and formulas (3.10)–(3.34). It remains to show that if $R = (r_{cd}^{ab})$ is constructed by this recipe, then it is indeed a solution. This is a fairly straightforward verification of (R1)–(R7).

The six equations (R1). If a, b, c do not all belong to the same block, at most one of the three pairs r_{ba}^{ab}, r_{ab}^{ba}; r_{cb}^{bc}, r_{bc}^{cb}; r_{ca}^{ac}, r_{ac}^{ca} can be nonzero. As each term in an (R1) equation involves a product of elements from different pairs, all terms in an (R1) equation are zero in this case. It remains to check the case that a, b, c all belong to the same block. If they all belong to the same component, then $r_{bc}^{bc} = 0$ and both sides are zero. If they belong to different components, then if $a < b < c$, $r_{bc}^{cb} = 0$ and $r_{ca}^{ac} = r_{cb}^{bc} = y_s$; if $a < c < b$, $r_{cb}^{bc} = 0$ and $r_{ba}^{ab} = y_s = r_{bc}^{cb}$; if $b < a < c$, $r_{ba}^{ab} = 0 = r_{bc}^{cb}$; if $b < c < a$, $r_{ca}^{ac} = 0 = r_{ba}^{ab}$; if $c < a < b$, $r_{ca}^{ac} = 0 = r_{cb}^{bc}$; if $c < b < a$, $r_{ba}^{ab} = 0 = r_{ca}^{ac}$; so (R1) holds in all six cases. If two of them are in the same component, then there also are six cases to be investigated: if $a \simeq b < c$ $r_{ca}^{ac} = r_{cb}^{bc} = y_s$; if $a \simeq c < b$ $r_{ba}^{ab} = r_{bc}^{cb} = y_s$; if

$b < a \simeq c$ $r^{cb}_{bc} = r^{ab}_{ba} = 0$; if $c < a \simeq b$ $r^{ac}_{ca} = r^{bc}_{cb} = 0$; if $b \simeq c < a$ or $a < b \simeq c$ then $r^{bc}_{bc} = 0$.

The six equations (R2). As in the case of (R1) if a, b, c, $a \neq b \neq c \neq a$, do not all belong to the same block, all terms are zero, and, also again, if a, b, c all belong to the same component, then $r^{ab}_{ab} = 0$. If two of them are in the same component, then if $a \simeq b$ (R2) is trivial since $r^{ab}_{ab} = 0$. If $a \simeq c < b$, $r^{ba}_{ab} = r^{bc}_{cb} = 0$; if $b \simeq c < a$, $r^{ab}_{ba} = r^{ac}_{ca} = 0$; if $a < b \simeq c$, $r^{ab}_{ba} = r^{ac}_{ca} = y_s$ and if $b < a \simeq c$ $r^{ba}_{ab} = r^{bc}_{cb} = y_s$. It remains to deal with the case that a, b, c all belong to a block B_s and to different components. If $a < b < c$, $r^{ba}_{ab} = 0$ and $r^{ab}_{ba} = y_s = r^{ac}_{ca}$; if $a < c < b$, $r^{bc}_{cb} = 0 = r^{ba}_{ab}$; if $b < a < c$, $r^{ab}_{ba} = 0$, $r^{ba}_{ab} = y_s = r^{bc}_{cb}$; if $b < c < a$, $r^{ac}_{ca} = 0 = r^{ab}_{ba}$; if $c < a < b$, $r^{ac}_{ca} = 0 = r^{bc}_{cb}$; if $c < b < a$; $r^{ac}_{ca} = 0 = r^{ab}_{ba}$. Thus, (R2) holds in all cases.

The six equations (R3). If a, b, c do not belong to the same block, both the second term on the left and the second term on the right are equal to zero. Take $a \in B_s$, $b \in B_t$, $c \in B_u$, if $s \neq u$ then (R3) is trivial since $r^{ac}_{ca} = 0$ and if $t \neq s = u$, then $r^{ab}_{ab}r^{ba}_{ba} = r^{bc}_{bc}r^{cb}_{cb} = z_{st}$. What remains is the case $s = t = u$. If a, b and c belong to the same component C^s_j both sides are equal to $(\lambda^s_j)^3$ since $r^{ab}_{ab} = r^{bc}_{bc} = 0$. If two of them are in the same component, then again there are six cases to be considered: if $a \simeq b < c$, $y_s(\lambda^s_j)^2 = y_s z_s + \lambda^s_j(y_s)^2$; if $c < a \simeq b$, $r^{ac}_{ca} = r^{bc}_{cb} = 0$; if $c \simeq a < b$, $r^{bc}_{cb} = 0$ and $r^{ab}_{ab}r^{ba}_{ba} = r^{bc}_{bc}r^{cb}_{cb} = z_s$; if $b < c \simeq a$, $r^{ab}_{ba} = 0$ and $r^{ab}_{ab}r^{ba}_{ba} = r^{bc}_{bc}r^{cb}_{cb} = z_s$; if $a < b \simeq c$, $y_s z_s + \lambda^s_j(y_s)^2 = y_s(\lambda^s_j)^2$; if $b \simeq c < a$, $r^{ac}_{ca} = r^{ab}_{ba} = 0$. Finally, if a, b, c all belong to different components of a block B_s the first term on the left and the first term on the right are either equal to zero ($c < a$) or equal to $z_s y_s$ ($a < c$). The other terms are zero unless $a < b < c$ and then both are equal to $(y_s)^3$. By this (R3) holds in all cases.

The two equations (R4). If a and c are not in the same block $r^{ac}_{ca} = 0$. If they are in the same component of a block, $r^{ac}_{ac} = 0$; if they are in the same block but in different components $r^{ac}_{ca}r^{ca}_{ac} = 0$.

The two equations (R5) If a and c are not both in the same block $r^{ac}_{ca} = 0$ and all terms are zero. If a and c are in the same component of a block B_s, $r^{aa}_{aa} = \lambda_s = r^{ac}_{ca}$ and $r^{ac}_{ac} = r^{ca}_{ca} = 0$ so that (R5) holds. Finally, if a and c are in different components of B_s, all terms are zero unless $a > c$ and then $r^{aa}_{aa} = \lambda^s_j$; $r^{ac}_{ca} = y_s$, $r^{ac}_{ac}r^{ca}_{ca} = z_s$ by (viii) and (R5) holds because λ^s_j solves $X^2 = X y_s + z_s$.

The two equations (R6). Exactly the same argument as (R5).

The two equations (R7). $r^{ab}_{ba}r^{ba}_{ab} = 0$ unless a and b belong to the same component of a block B_s and then $r^{ab}_{ba} = r^{ba}_{ab} = \lambda^s_j$.

3.36. SOME EXAMPLES

In case of a solution consisting of only one block we speak of an *irreducible* solution, a solution consisting of several blocks is called *reducible*. There are two kinds of blocks which are rather special. The first is the one that consists of only one component and the second one is build from components that contain only one element, we shall denote these blocks by blocks of type I and type II, respectively.

	11	12	13	21	22	23	31	32	33
11	λ								
12		zx_{21}^{-1}		y					
13			zx_{31}^{-1}				y		
21				x_{21}					
22					λ				
23						zx_{32}^{-1}		y	
31							x_{31}		
32								x_{32}	
33									μ

$n = 3$; one block of type II
($\lambda^2 = \lambda y + z;\ \mu^2 = \mu y + z;\ \lambda, \mu, x_{ij}, z \neq 0;\ p = 5$)

	11	12	13	21	22	23	31	32	33
11	λ								
12				λ					
13							λ		
21		λ							
22					λ				
23								λ	
31			λ						
32						λ			
33									λ

$n = 3$; one block of type I
($\lambda \neq 0;\ p = 1$)

	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	λ_1															
12		$z_1x_{21}^{-1}$			y_1											
13			$z_{12}x_{31}^{-1}$													
14				$z_{12}x_{41}^{-1}$												
21					x_{21}											
22						λ_1										
23							$z_{12}x_{32}^{-1}$									
24								$z_{12}x_{42}^{-1}$								
31									x_{31}							
32										x_{32}						
33											λ_2					
34												$z_2x_{43}^{-1}$			y_2	
41													x_{41}			
42														x_{42}		
43															x_{43}	
44																μ_2

$n = 4$; two blocks of type II of size 2
($\lambda_1^2 = \lambda_1 y_1 + z_1$; $\lambda_2^2 = \lambda_2 y_2 + z_2$; $\mu_2^2 = \mu_2 y_2 + z_2$; x_{ij}, λ_i, μ_2, z_i, $z_{12} \neq 0$; $p = 11$)

	11	12	13	21	22	23	31	32	33
11	λ_1								
12		$z_{12}x_{21}^{-1}$							
13			$z_{13}x_{31}^{-1}$						
21				x_{21}					
22					λ_2				
23						$z_{23}x_{32}^{-1}$			
31							x_{31}		
32								x_{32}	
33									λ_3

$n = 3$, three blocks of type I of size 1
($p = 9$, all parameters $\neq 0$; if all blocks are of size 1, R is simply any invertible diagonal matrix)

	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	λ_1															
12					λ_1											
13			zx_{31}^{-1}						y							
14				zx_{41}^{-1}									y			
21		λ_1														
22						λ_1										
23							zx_{32}^{-1}			y						
24								zx_{42}^{-1}						y		
31									x_{31}							
32										x_{32}						
33											λ_2					
34												zx_{43}^{-1}			y	
41													x_{41}			
42														x_{42}		
43															x_{43}	
44																λ_3

$n = 4$, one block of size 4 with three components, two of size 1 and one of size 2 ($\lambda_i^2 = \lambda_i y + z$; λ_i, y, z, $x_{ij} \neq 0$; $p = 7$)

	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	λ_1															
12					λ_1											
13			zx_{31}^{-1}						y							
14				zx_{41}^{-1}									y			
21		λ_1														
22						λ_1										
23							zx_{32}^{-1}			y						
24								zx_{42}^{-1}						y		
31									x_{31}							
32										x_{32}						
33											λ_2					
34															λ_2	
41													x_{41}			
42														x_{42}		
43												λ_2				
44																λ_2

$n = 4$, one block with two components of size 2 ($\lambda_i^2 = \lambda_i y + z$; $\lambda_i, z, y, x_{ij} \neq 0$; $p = 6$)

In the examples above, p is the number of parameters that are present in the R-matrix. An irreducible solution has $p = 1$ in case of type I and $p = \binom{n}{2} + 2$ in case of type II, where n is the size of the block. In the reducible cases, the number of parameters can increase drastically to a maximum of n^2; in that case there are n blocks of size 1 and R is simply any invertible diagonal matrix. This is, in a way, the most degenerate case.

3.37. Concluding comments for Section 3

Any solution of the YBE, in fact any $n^2 \times n^2$ matrix R, can be used to define a bialgebra by commutation relations $RT_2T_2 = T_2T_1R$, cf. below. The 'standard' quantum group of type A_{n-1} corresponds to the case of one block of type II of size n with $y = q - q^{-1}$, $r^{aa}_{aa} = \lambda = q$ for all a, $z = 1$, $x_{ab} = 1$ for all $a > b$.

As we shall see, the irreducible case of type II, with r^{aa}_{aa} for all a equal to the same solution λ of $X^2 = yX + z$ corresponds to the $\binom{n}{2} + 1$ multiparameter quantum group of Section 2. In this case, there are $p = \binom{n}{2} + 2$ parameters, but one is superfluous because multiplication by a scalar is irrelevant both for the YBE and for the commutation relations defined by an R.

The structure of the R-matrix for the $\binom{n}{2} + 1$ parameter quantum group is illuminating. There are $\binom{n}{2}$ 'diagonal parameters' and these define what in several ways seems to be a rather nonessential (though definitely not trivial in the technical sense) deformation of the matrix algebra. The phrase 'rather nonessential' here is intuitive and should be given precise meaning. One fact in this direction is that the extra $\binom{n}{2}$ parameters (the x_{ij}) do not appear to give any more sensitive Turaev-type knot invariants; they simple drop out of the defining trace formula, even though the relevant braid group representations are different.

The irreducible type II R-matrix with mixed r^{aa}_{aa}, meaning that some of the r^{aa}_{aa} are equal to one solution of $X^2 = (q - q^{-1})X + 1$ and some to the other one, give rise to bialgebras with nilpotents (so not quantum groups in the accepted sense of the word); they also give the same polynomial Turaev-type knot invariants (for a lower size R-matrix).

The known classical R-matrices of type B^1, C^1, D^1, A^2 do not arise as special cases of those of Theorem 3.35. These classical R-matrices do, however, satisfy a very similar condition to the one considered here. Let σ be the involution on $\{1, \dots, n\}$ given by $\sigma(i) = n + 1 - i$. Then these R matrices of type B^1, C^1, D^1, A^2 satisfy

$$r^{ab}_{cd} = 0 \quad \text{unless} \quad \{a, b\} = \{c, d\} \quad \text{or} \quad b = \sigma(a),\ d = \sigma(c). \tag{3.38}$$

It looks possible to extend the analysis of this section to the case of all solutions of the YBE satisfying (3.38).

It seems likely that the $\binom{n}{2}+1$ parameter quantum R-matrix is maximal though this remains to be proved. Possibly it will thus be possible to find the maximal families for type B^1, C^1, C^1, A^2 as well.

Work on all these matters is in progress.

4. The R-Matrix Bialgebras Defined by the Fairly General R-Matrix of Section 3

Let R again be any matrix satisfying

$$R^{ab}_{cd} = 0 \quad \text{unless} \quad \{a,b\} = \{c,d\}. \tag{4.1}$$

We investigate the commutation relations defined by

$$RT_1T_2 = T_2T_1R, \tag{4.2}$$

where

$$T = \begin{pmatrix} t^1_1 & \cdots & t^1_n \\ \vdots & & \vdots \\ t^n_1 & \cdots & t^n_n \end{pmatrix}, \quad T_1 = T \otimes I_n, \quad T_2 = I_n \otimes T.$$

Then the relations (4.2) written out become

$$r^{ab}_{i_1i_2} t^{i_1}_c t^{i_2}_d = r^{j_1j_2}_{cd} t^b_{j_2} t^a_{j_1}. \tag{4.3}$$

Let $I(R)$ be the two-sided ideal in $K\langle t\rangle$ generated by the relations (4.3). Then $I(R)$ is a bialgebra ideal, cf., e.g., [10].

4.4. THEOREM. *Let R be a solution of the YBE consisting of one type II block of size n such that, moreover, r^{aa}_{aa} = constant for all $a \in \{1,\dots,n\}$, then R defines a multiparameter quantum matrix algebra as described in Section 2 above.*

Proof. Recall that the quantum matrix algebra in question arises by taking the maximal quotient of $K\langle t^1_1,\dots,t^n_n\rangle$ that acts from the left on a quantum space $K\langle X^1,\dots,X^n\rangle$, $X^iX^j = q^{ij}X^jX^i$ by the usual matrix action and from the right on a quantum space $K\langle Y_1,\dots,Y_n\rangle$, $Y_kY_l = q_{kl}Y_lY_k$, where $q^{ii} = 1$, $q^{ij} = (q^{ji})^{-1}$, $q_{kk} = 1$, $q_{kl} = (q_{lk})^{-1}$ and the q^{ij} and q_{kl} are related by

$$q^{ij}q_{ij} = \rho \neq -1 \quad (i<j) \tag{4.5}$$

and the relations defining the quantum matrix algebra are

$$\begin{aligned} t^r_a t^r_b &= q_{ab} t^r_b t^r_a, & (4.6)\\ t^r_a t^s_b &- q_{ab} t^r_b t^s_a + (q_{rs})^{-1} t^s_a t^r_b - q_{ab}(q_{rs})^{-1} t^s_b t^r_a = 0, & (4.7)\\ t^r_a t^s_a &= q^{rs} t^s_a t^r_a, & (4.8)\\ t^r_a t^s_b &- q^{rs} t^s_a t^r_b + (q^{ab})^{-1} t^r_b t^s_a - (q^{rs})(q^{ab})^{-1} t^s_b t^r_a = 0. & (4.9) \end{aligned}$$

Choose y, z, x_{ij}, $i < j$, as in Theorem 3.35. Let λ^u, $-\lambda_d$ be the two solutions of $X^2 = Xy + z$ and take

$$r_{aa}^{aa} = \lambda^u, \qquad r_{ab}^{ab} = x_{ab} \quad \text{for } a > b,$$
$$r_{ba}^{ba} = \lambda^u \lambda_d x_{ab}^{-1} \quad \text{for } a > b, \tag{4.10}$$
$$r_{ba}^{ab} = \lambda^u - \lambda_d \quad \text{for } a < b, \qquad r_{ba}^{ab} = 0 \quad \text{for } a > b,$$

as described by Theorem 3.35. (One can also take $r_{aa}^{aa} = -\lambda_d$ for all a; that gives an isomorphic matrix algebra.)

The nontrivial relations resulting from 4.3 are

$$a = b, \quad c = d, \quad r_{aa}^{aa} t_c^a t_c^a = r_{cc}^{cc} t_c^a t_c^a, \tag{4.11}$$
$$a = b, \quad c \neq d, \quad r_{aa}^{aa} t_c^a t_d^a = r_{cd}^{cd} t_d^a t_c^a + r_{cd}^{dc} t_c^a t_d^a, \tag{4.12}$$
$$a \neq b, \quad c = d, \quad r_{ab}^{ab} t_c^a t_c^b + r_{ba}^{ab} t_c^b t_c^a = r_{cc}^{cc} t_c^b t_c^a, \tag{4.13}$$
$$a \neq b, \quad c \neq d, \quad r_{ab}^{ab} t_c^a t_d^b + r_{ba}^{ab} t_c^b t_d^a = r_{cd}^{cd} t_d^b t_c^a + r_{cd}^{dc} t_c^b t_d^a. \tag{4.14}$$

Because $r_{aa}^{aa} = r_{cc}^{cc} = \lambda^u$, (4.11) holds. Now take

$$q^{ab} = x_{ab}(\lambda^u)^{-1}, \qquad q_{ba} = x_{ab}\lambda_d^{-1} \quad \text{for } a < b. \tag{4.15}$$

Notice that indeed $q^{ab}q_{ab} = x_{ab}(\lambda^u)^{-1}(x_{ab}^{-1}\lambda_d) = \lambda_d(\lambda^u)^{-1} = \rho =$ constant.

Substituting the values of (4.10) in (4.12), we obtain for $d < c$

$$\lambda^u t_c^a t_d^a = x_{cd} t_d^a t_c^a + (\lambda^u - \lambda_d) t_c^a t_d^a$$

so that indeed

$$t_c^a t_d^a = \lambda_d^{-1} x_{cd} t_d^a t_c^a = \lambda_d^{-1} x_{dc} t_d^a t_c^a = q_{cd} t_d^a t_c^a, \tag{4.16}$$

which is (4.6). And for $c < d$, we get

$$\lambda^u t_c^a t_d^a = \lambda^u \lambda_d x_{cd}^{-1} t_d^a t_c^a,$$

which gives

$$t_c^a t_d^a = \lambda_d x_{cd}^{-1} t_d^a t_c^a = q_{dc}^{-1} t_d^a t_c^a = q_{cd} t_c^a t_d^a,$$

which is the same as (4.16).

Now substitute the values of (4.10) in (4.13). There are again two cases to consider.

If $a < b$ we find

$$\lambda_d \lambda^u x_{ab}^{-1} t_c^a t_c^b + (\lambda^u - \lambda_d) t_c^b t_c^a = \lambda^u t_c^b t_c^a,$$

which gives (using 4.15)

$$t_c^a t_c^b = (\lambda^u)^{-1} x_{ab} t_c^b t_c^a = q^{ab} t_c^b t_c^a,$$

which is (4.8).

If $a > b$ we find

$$x_{ab}t_c^a t_c^b = \lambda^u t_c^b t_c^a$$

which gives

$$t_c^b t_c^a = (\lambda^u)^{-1} x_{ab} t_c^a t_c^b = q^{ba} t_c^a t_c^b.$$

Finally substitute the values of (4.10) in (4.14). Note that (4.14) really embodies four equations between the $t_c^a t_d^b$, $t_d^a t_c^b$, $t_c^b t_d^a$, $t_d^b t_c^a$; namely, the one written down and the three obtained by switching a and b, switching c and d, and switching both.

Taking $a < b$, $c < d$, we find

$$\lambda^u \lambda_d x_{ab}^{-1} t_c^a t_d^b + (\lambda^u - \lambda_d) t_c^b t_d^a = \lambda^u \lambda_d x_{cd}^{-1} t_d^b t_c^a. \tag{4.17}$$

Switching a and b in (4.14) and then substituting gives

$$x_{ab} t_c^b t_d^a = \lambda^u \lambda_d x_{cd}^{-1} t_d^a t_c^b, \tag{4.18}$$

$$\lambda^u \lambda_d x_{ab}^{-1} t_d^a t_c^b + (\lambda^u - \lambda_d) t_d^b t_c^a = x_{cd} t_c^b t_d^a + (\lambda^u - \lambda_d) t_d^b t_c^a. \tag{4.19}$$

Finally, switching both a, b and c, d and then substituting gives

$$x_{ab} t_d^b t_c^a = x_{cd} t_c^a t_d^b + (\lambda^u - \lambda_d) t_d^a t_c^b. \tag{4.20}$$

Observe that (4.18) and (4.19) are identical. It is easily checked that

$$x_{ab}(\lambda^u \lambda_d)^{-1}(4.17) + (x_{cd}^{-1})(4.20) - (\lambda_d^{-1} - (\lambda^u)^{-1})(4.18)$$

has equal left- and right-hand sides. Thus (4.17)–(4.20) are equivalent to (4.17)–(4.18).

Multiply (4.17) by $x_{ab}(\lambda^u \lambda_d)^{-1}$ to find

$$t_c^a t_d^b + x_{ab} \lambda_d^{-1} t_c^b t_d^a - x_{ab} \lambda_u^{-1} t_c^b t_d^a - x_{ab} x_{cd}^{-1} t_d^b t_c^a = 0 \tag{4.21}$$

and now use (4.18) to rewrite the third term to find

$$t_c^a t_d^b + x_{ab} \lambda_d^{-1} t_c^b t_d^a - \lambda_d x_{cd}^{-1} t_d^a t_c^b - x_{ab} x_{cd}^{-1} t_d^b t_c^a = 0. \tag{4.22}$$

Because $a < b$, $c < d$, we have by (4.15) that

$$q_{ab}^{-1} = q_{ba} = x_{ab} \lambda_d^{-1}, \qquad q_{cd} = (q_{dc})^{-1} = (x_{cd} \lambda_d^{-1})^{-1} = \lambda_d x_{cd}^{-1},$$

$$q_{ab}^{-1} q_{cd} = x_{ab} \lambda_d^{-1} \lambda_d x_{cd}^{-1} = x_{ab} x_{cd}^{-1},$$

so that (4.22) is identical with (4.7).

Now use (4.18) to rewrite the second term in (4.21). This gives

$$t_c^a t_d^b + \lambda^u x_{cd}^{-1} t_d^a t_c^b - x_{ab}\lambda_u^{-1} t_c^b t_d^a - x_{ab} x_{cd}^{-1} t_d^b t_c^a = 0. \tag{4.23}$$

Again, as $a < b$, $c < d$, we have by (4.15) that

$$\begin{aligned} &q^{ab} = (\lambda^u)^{-1} x_{ab}, \qquad (q^{cd})^{-1} = ((\lambda^u)^{-1} x_{cd})^{-1} = \lambda^u x_{cd}^{-1}, \\ &q^{ab}(q^{cd})^{-1} = (\lambda^u)^{-1} x_{ab} \lambda^u x_{cd}^{-1}, \end{aligned} \tag{4.24}$$

so that (4.23) is identical with (4.9).

This finishes the proof of the theorem. (Though not necessary, given what has been shown about the rank of the various groups of relations involved, it is in fact now not difficult to show that inversely the groups of relations (4.7)–(4.9) imply the group (4.14), i.e. (4.17)–(4.20).)

4.25. COROLLARY. *Let $M_q^{n\times n}$ be the multiparameter quantum matrix algebra of Section 2, i.e. $M_q^{n\times n} = K\langle t\rangle / I$ when I is the ideal of the relations* (4.6)–(4.9). *Then $M_q^{n\times n}$ is a PBW algebra with the same Hilbert–Poincaré series as $K[t_1^1, \ldots, t_n^n]$.*

Proof. We already know that the dimension of the degree 2 part is exactly right viz. $n^2 + \binom{n^2}{2}$. The commutation relations are of the form

$$T_1 T_2 = R^{-1} T_2 T_1 R.$$

Now R satisfies the YBE, i.e.

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{4.26}$$

Now for the triple product $T_1 T_2 T_3$,

$$T_1 = T \otimes I \otimes I, \qquad T_2 = I \otimes T \otimes I, \qquad T_3 = I \otimes I \otimes T,$$

we have that

$$\begin{aligned} T_1(T_2T_3) &= T_1 R_{23}^{-1} T_3 T_2 R_{23} = R_{23}^{-1}(T_1T_3)T_2R_{23} = R_{23}^{-1}R_{13}^{-1}T_3T_1R_{13}T_2R_{23} \\ &= R_{23}^{-1}R_{13}^{-1}T_3(T_1T_2)R_{13}R_{23} = R_{23}^{-1}R_{13}^{-1}T_3R_{12}^{-1}T_2T_1R_{12}R_{13}R_{23} \\ &= R_{23}^{-1}R_{13}^{-1}R_{12}^{-1}T_3T_2T_1R_{12}R_{13}R_{23}. \end{aligned} \tag{4.27}$$

(Note that $R_{ij}T_k = T_kR_{ij}$ if $i \neq j \neq k \neq i$ because R_{ij} only affects factors i and j where T_k is the identity.) We also have

$$\begin{aligned} (T_1T_2)T_3 &= R_{12}^{-1}T_2T_1R_{12}T_3 = R_{12}^{-1}T_2(T_1T_3)R_{12} = R_{12}^{-1}T_2R_{13}^{-1}T_3T_1R_{13}R_{12} \\ &= R_{12}^{-1}R_{13}^{-1}(T_2T_3)T_1R_{13}R_{12} = R_{12}^{-1}R_{13}^{-1}R_{23}^{-1}T_3T_2R_{23}T_1R_{13}R_{12} \\ &= R_{12}^{-1}R_{13}^{-1}R_{23}^{-1}T_3T_2T_1R_{23}R_{13}R_{12}. \end{aligned} \tag{4.28}$$

The end products of (4.27) and (4.28) are the same proving the confluence conditions of the diamond lemma, [2], and the result follows. This argument: YBE $\Rightarrow$ confluence condition of diamond lemma has been observed before [6].

4.29. Comments on the Other Solutions of the YBE

The solutions consisting of one block of type I gives, as is easily checked, no relations at all among the t^i_j. The solutions consisting of one block with several components with mixed parameters λ_j give rise to a bialgebra $K\langle t\rangle/I(R)$ with nilpotent elements. Indeed if, say, $a \in C_1$ and $b \in C_2$ and $\lambda_1 \neq \lambda_2$, then by (4.11)

$$\lambda_1 t^a_b t^a_b = \lambda_2 t^a_b t^a_b, \tag{4.30}$$

so that $(t^a_b)^2 = 0$. These are, of course, perfectly good solutions of the YBE and as such are of potential use in, for example, the business of constructing link invariants (cf. Section 5 below) but the bialgebras they define are not quantum groups in the (more or less) accepted sense of the word. (There is no consensus and some authors equate the concepts Hopf algebra and quantum group; I would be inclined to reserve the phrase quantum group for a Hopf algebra that is a PBW algebra and is a deformation of the function algebra of a linear algebraic group.) Let me also remark that in spite of nilpotents, these bialgebras are still pretty nice in the sense that its defining rewriting rules (commutation relations) are confluent (so that it is easy to write down a basis and a version of Gröbner basis theory probably applies).

4.30. Quantum Groups

Let again R be a single block solution of the YBE with constant parameter λ_j defining a multiparameter quantum matrix algebra $M_q = K\langle t\rangle/I(R)$. As is shown in, e.g., [1], for the case of a single type II block there is an element d in M_q (a quantum determinant) such that the localization $M_q[d^{-1}]$ admits an antipode and thus becomes a Hopf algebra.

By the work of [6, 12, 13], cf. also [4], the fact that M_q comes from a solution of the YBE is useful in establishing such facts.

5. Yang–Baxter Operators and Link Invariants

For this section the Yang–Baxter equation takes the form

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23}. \tag{5.1}$$

If $S = (s^{ab}_{cd})$, then in terms of the entries of S, this works out as

$$s^{ab}_{kl}s^{lc}_{mw}s^{km}_{uv} = s^{ai}_{uk}s^{bc}_{ij}s^{kj}_{vw}. \tag{5.2}$$

There is a simple relation between (5.1) and the YBE (3.1): if $R = (r^{ab}_{cd})$ solves (3.1), then both

$$S = (s^{ab}_{cd}), \quad s^{ab}_{cd} = r^{ab}_{dc}, \qquad S' = (s'^{ab}_{cd}), \quad s'^{ab}_{cd} = r^{ba}_{cd} \tag{5.3}$$

solve (5.1) (and vice versa). Let's check that for S. Putting (5.3) in the LHS of (5.2) gives

$$r^{ab}_{lk} r^{lc}_{wm} r^{km}_{vu}, \tag{5.4}$$

which is the LHS of (3.2) with uvw replaced by wvu; now put (5.3) in the RHS of (5.2) to find

$$r^{ai}_{ku} r^{bc}_{ji} r^{kj}_{wv} = r^{bc}_{ji} r^{ai}_{ku} r^{kj}_{wv}, \tag{5.5}$$

which is the RHS of (3.2) also with uvw replaced by wvu. The proof for S' is as easy (except that now RHS and LHS switch).

5.6. DEFINITION ([22]). A Yang–Baxter operator consists of a quadruple (S, ν, α, β), where S is an $n^2 \times n^2$ matrix satisfying the YBE in the form (5.1), ν is an $n \times n$ matrix, and α, β are invertible scalars which are related to S by the conditions (5.7)–(5.9)

$$\nu \otimes \nu \text{ commutes with } S, \tag{5.7}$$

$$\mathrm{Tr}_2(S \circ (\nu \otimes \nu)) = \alpha\beta\nu, \tag{5.8}$$

$$\mathrm{Tr}_2(S^{-1} \circ (\nu \otimes \nu)) = \alpha^{-1}\beta\nu. \tag{5.9}$$

Here if $M = (m^{ij}_{kl})$ is an $n^2 \times n^2$ matrix (with the usual ordering $11, \ldots, 1n; 21, \ldots, 2n; \ldots; n1, \ldots, nn$ of rows and columns), then $\mathrm{Tr}_2(M) = N$ is the $n \times n$ matrix with entries

$$n^i_j = m^{i1}_{j1} + \ldots + m^{in}_{jn}, \tag{5.10}$$

i.e. if M is written as an $n \times n$ matrix of $n \times n$ blocks, then N is constructed by replacing each block of M by its trace. If ν is invertible, then (5.8) and (5.9) are equivalent to

$$\mathrm{Tr}_2(S^{\pm 1} \circ (I_n \otimes \nu)) = \alpha^{\pm 1}\beta I_n \tag{5.11}$$

(where I_n is the $n \times n$ identity matrix).

Given a YB operator (S, ν, α, β), Turaev's formula

$$T_S(\xi) = \alpha^{-w(\xi)}\beta^{-m}\mathrm{Tr}(\rho_S(\xi) \circ \nu^{\otimes m}) \tag{5.12}$$

defines a link invariant. Here $\xi \in B_m$, the braid group on m letters, $w(\xi) = \Sigma\varepsilon_i$ if $\xi = \sigma^{\varepsilon_1}_{i_1} \ldots \sigma^{\varepsilon_r}_{i_r}$, where the σ_i are the standard generators of B_m, and ρ_S is the representation of the braid group (in $(K^n)^{\otimes m}$) defined by S, $\sigma_i \mapsto S_{ii+1}$; $T_S(\xi)$ is then independent of the particular braid that gives rise to a link ξ by closure of the braid.

Now, given the solutions of the YBE described in Section 3, it is natural to investigate whether these extend to Yang–Baxter operators in the sense of Turaev

(Definition 5.12), and, if so, what the resulting link and knot invariants bring. Here I report some preliminary results only. Further work is in progress.

5.13. *Remarks.* Both the constants α and β can be normalized to 1. Indeed if (S,ν,α,β) is a Yang–Baxter operator then $(\alpha^{-1}S,\beta^{-1}\nu,1,1)$ is another one. However, for the formulas below it is convenient to keep α (but β will always be 1). As Turaev observes, if ν is diagonal, then (5.8) implies that $\bar{S}\bar{\nu}=\bar{\alpha}$ where $\bar{S}$ is the $n\times n$ matrix $\bar{s}^i_j=s^{ij}_{ij}$, $\bar{\nu}$ is the column vector $(\nu_1,\dots,\nu_n)^T$ and $\bar{\alpha}$ is the column vector $\alpha(1,1,\dots,1)^T$. Thus, assuming ν is diagonal, it is unique if $\bar{S}$ is invertible.

5.14. THEOREM. *Let R be a solution of the YBE (as described in Theorem 3.35) consisting of a single block (with components $C_1,C_2,\dots,C_p$ $(p\geqslant 2)$) with parameters y and z and let μ and λ be the two solutions of the equation $X^2=yX+z$. Let $S=\tau R$ be the associated solution of (5.1), then S extends to a Yang–Baxter operator with the scalar α such that*

$$\alpha^2=(-1)^{p-1}\lambda^{k_\lambda-k_\mu+1}\mu^{k_\mu-k_\lambda+1}, \tag{5.15}$$

where k_λ (resp., k_μ) is the number of components C_j with $\lambda_j=\lambda$ (resp., $\lambda_j=\mu$).

Proof. For the moment regard R, R^{-1} and S, S^{-1} as $n\times n$ matrices made up of blocks that are also $n\times n$ matrices. Observe that the diagonals of all the off-diagonal blocks are zero. Take $\nu=\operatorname{diag}(\nu_1,\dots,\nu_n)$, the diagonal $n\times n$ matrix with diagonal entries $\nu_1,\dots,\nu_n$. Because ν is diagonal and $s^{ab}_{cd}=0$, unless $\{a,b\}=\{c,d\}$, (5.7) holds. It also follows (cf. (5.10)) that the conditions (5.8), (5.9) only involve the diagonal blocks of S and S^{-1}. As is easily checked, the inverse R^{-1} of R is also a solution of the YBE and has the same structure as R. One can easily verify that R^{-1} is equal to

$$\begin{array}{lll}(R^{-1})^{ab}_{ba} &= \lambda^{-1}+\mu^{-1} & \text{if } a<b,\\ (R^{-1})^{ab}_{ba} &= (R^{ab}_{ba})^{-1} & \text{if } a\simeq b,\\ (R^{-1})^{ab}_{ab} &= z^{-1}x_{ba} & \text{if } a<b,\\ (R^{-1})^{ab}_{ab} &= x^{-1}_{ab} & \text{if } a>b,\\ (R^{-1})^{aa}_{aa} &= (R^{aa}_{aa})^{-1} & (=\lambda^{-1}\ (\text{resp.}, \mu^{-1})).\end{array} \tag{5.16}$$

Indeed if $a<b$, $\{a,b\}\neq\{c,d\}$

$$(RR^{-1})^{ab}_{cd}=R^{ab}_{ij}(R^{-1})^{ij}_{cd}=R^{ab}_{ab}(R^{-1})^{ab}_{cd}+R^{ab}_{ba}(R^{-1})^{ba}_{cd}=0.$$

Further, if $a<b$, $a=c$, $b=d$

$$(RR^{-1})^{ab}_{ab}=R^{ab}_{ab}(R^{-1})^{ab}_{ab}+R^{ab}_{ba}(R^{-1})^{ba}_{ab}=zx^{-1}_{ba}z^{-1}x_{ba}+0=1$$

and if $a<b$, $a=d$, $b=c$

$$\begin{aligned}(RR^{-1})^{ab}_{ba} &= R^{ab}_{ab}(R^{-1})^{ab}_{ba}+R^{ab}_{ba}(R^{-1})^{ba}_{ba}\\ &= zx^{-1}_{ba}(\lambda^{-1}+\mu^{-1})+(\lambda+\mu)x^{-1}_{ba}=0\end{aligned}$$

because $z = -\lambda\mu$.

The other cases $a \simeq b$, $a > b$ are even easier to check.

Switching λ and μ if necessary, we can assume that $\lambda_1 = \lambda$. Let the pattern of λ's and μ's be the following

$$\lambda_1 = \ldots = \lambda_{d_1} = \lambda; \qquad \lambda_{d_1+1} = \cdots = \lambda_{d_1+d_2} = \mu;$$

$$\lambda_{d_1+d_2+1} = \cdots = \lambda_{d_1+d_2+d_3} = \lambda; \ \ldots$$

Let r be the number of switches $(d_1, d_1+1), \ldots, (d_r, d_r+1)$, so that $\lambda_p = \lambda$ if r even and $\lambda_p = \mu$ if r is odd. We define a diagonal $p \times p$ matrix T =diag$(T_1, T_2, \ldots, T_p)$, where T_j is equal to the trace of ν with respect to the jth component, i.e. $T_j = \sum_{i \in C_j} \nu_i$.

It is now easy to see that Equations (5.8), (5.9) (with $\beta = 1$) amount to the following: (where the equations resulting from (5.8) constitute the upper block and those from (5.9) form the lower block. Here, as in the above, to follow the calculations, it is useful to keep the first example of (3.36) in front of one).

$$\begin{aligned}
&\lambda T_1 = \alpha \\
&\lambda T_2 + (\mu+\lambda)T_1 = \alpha \\
&\qquad\vdots \\
&\lambda T_{d_1-1} + (\mu+\lambda)(T_1 + \cdots + T_{d_1-2}) = \alpha \\
&\lambda T_{d_1} + (\mu+\lambda)(T_1 + \cdots + T_{d_1-1}) = \alpha \\
&\mu T_{d_1+1} + (\mu+\lambda)(T_1 + \cdots + T_{d_1}) = \alpha \\
&\mu T_{d_1+2} + (\mu+\lambda)(T_1 + \cdots + T_{d_1+1}) = \alpha \\
&\qquad\vdots \\
&\mu T_{d_1+d_2-1} + (\mu+\lambda)(T_1 + \cdots + T_{d_1+d_2-2}) = \alpha \\
&\mu T_{d_1+d_2} + (\mu+\lambda)(T_1 + \cdots + T_{d_1+d_2-1}) = \alpha \\
&\lambda T_{d_1+d_2+1} + (\mu+\lambda)(T_1 + \cdots + T_{d_1+d_2}) = \alpha \\
&\lambda T_{d_1+d_2+2} + (\mu+\lambda)(T_1 + \cdots + T_{d_1+d_2+1}) = \alpha \\
&\qquad\vdots \\
&\kappa T_p + (\mu+\lambda)(T_1 + \cdots + T_{p-1}) = \alpha
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\lambda}T_1 + (\lambda^{-1} + \mu^{-1})(T_2 + \cdots + T_p) = \frac{1}{\alpha} \\
&\frac{1}{\lambda}T_2 + (\lambda^{-1} + \mu^{-1})(T_3 + \cdots + T_p) = \frac{1}{\alpha} \\
&\qquad\vdots
\end{aligned}$$

$$\frac{1}{\lambda}T_{d_1-1} + (\lambda^{-1}+\mu^{-1})(T_{d_1}+\cdots+T_p) = \frac{1}{\alpha}$$
$$\frac{1}{\lambda}T_{d_1} + (\lambda^{-1}+\mu^{-1})(T_{d_1+1}+\cdots+T_p) = \frac{1}{\alpha}$$
$$\frac{1}{\mu}T_{d_1+1} + (\lambda^{-1}+\mu^{-1})(T_{d_1+2}+\cdots+T_p) = \frac{1}{\alpha}$$
$$\frac{1}{\mu}T_{d_1+2} + (\lambda^{-1}+\mu^{-1})(T_{d_1+3}+\cdots+T_p) = \frac{1}{\alpha}$$
$$\vdots$$
$$\frac{1}{\mu}T_{d_1+d_2-1} + (\lambda^{-1}+\mu^{-1})(T_{d_1+d_2}+\cdots+T_p) = \frac{1}{\alpha}$$
$$\frac{1}{\mu}T_{d_1+d_2} + (\lambda^{-1}+\mu^{-1})(T_{d_1+d_2+1}+\cdots+T_p) = \frac{1}{\alpha}$$
$$\frac{1}{\lambda}T_{d_1+d_2+1} + (\lambda^{-1}+\mu^{-1})(T_{d_1+d_2+2}+\cdots+T_p) = \frac{1}{\alpha}$$
$$\frac{1}{\lambda}T_{d_1+d_2+2} + (\lambda^{-1}+\mu^{-1})(T_{d_1+d_2+3}+\cdots+T_p) = \frac{1}{\alpha}$$
$$\vdots$$
$$\frac{1}{\kappa}T_p = \frac{1}{\alpha},$$

where $\kappa = \lambda$ (resp., μ) depending on whether r is even (resp., odd). Now observe that subtracting the $(i+1)$th from the ith equation in both the upper and lower blocks results in the same relation between T_{i+1} and T_i viz. $T_{i+1} = -\lambda^{-1}\mu T_i$, or $T_{i+1} = -\mu^{-1}\lambda T_i$, or $T_{i+1} = -T_i$. This results in the following recipe for the T's

$$T_1 = \lambda^{-1}\alpha,$$
$$T_i = \begin{cases} (-\lambda^{-1}\mu)T_{i-1} & \text{if } \lambda_i = \lambda = \lambda_{i-1}, \\ -T_{i-1} & \text{if } \lambda_i = \lambda, \lambda_{i-1} = \mu, \\ (-\mu^{-1}\lambda)T_{i-1} & \text{if } \lambda_i = \mu = \lambda_{i-1}, \\ -T_{i-1} & \text{if } \lambda_i = \mu, \lambda_{i-1} = \lambda, \end{cases} \tag{5.17}$$
$$T_p = \begin{cases} \lambda\alpha^{-1} & \text{if } r \text{ is even}, \\ \mu\alpha^{-1} & \text{if } r \text{ is odd}. \end{cases}$$

It follows that, depending on the number, r, of switches from λ to μ or vice versa

if r is even
$$T_p = (-1)^{p-1}\lambda^{k_\mu - k_\lambda + 1}\mu^{k_\lambda - k_\mu - 1}T_1, \quad T_1 = \lambda^{-1}\alpha,\ T_p = \lambda\alpha^{-1},$$

if r is odd
$$T_p = (-1)^{p-1}\lambda^{k_\mu - k_\lambda}\mu^{k_\lambda - k_\mu}T_1, \quad T_1 = \lambda^{-1}\alpha,\ T_p = \mu\alpha^{-1}, \tag{5.18}$$

where k_λ is the number of i's for which $\lambda_i = \lambda$ and k_μ the number of i's for which $\lambda_i = \mu$, $k_\lambda + k_\mu = p$. In both cases, it follows that

$$\alpha^2 = (-1)^{p-1}\lambda^{k_\lambda - k_\mu + 1}\mu^{k_\mu - k_\lambda + 1} \tag{5.19}$$

and for both α's solving (5.19) (taking, if necessary, a quadratic extension of K) (5.17) then specifies $T_1, \dots, T_p$ such that (5.8), (5.9) are satisfied (with $\beta = 1$). This concludes the proof of the theorem.

5.20. *Remark.* Both choices for α in (5.19) give up to a sign the same link invariant, cf. [22, 3.3]. As for the uniqueness of the Yang–Baxter operator it is evident that the solution of $T_1, \dots, T_p$ is unique, hence the solution of $\nu_1, \dots, \nu_n$ is unique if and only if all components consist of one element, i.e. the block is of type II. This can also be seen from the fact that the matrix $\bar{S}$ satisfies $\bar{s}^i_j = s^{ij}_{ij} = r^{ji}_{ij} = \lambda_k$ if $i \simeq j$, y if $j < i$ and 0 if $i < j$, so it is invertible if and only if we are dealing with a type II block.

5.21. COROLLARY. *Let R be any solution of the YBE as described by Theorem* 3.35 *and $S = \tau R$ the corresponding solution of* (5.1). *Then S extends to a Yang–Baxter operator $(S, \nu, \alpha, 1)$ if any only if for all blocks*

$$\alpha^2 = (-1)^{p_i - 1}\lambda_i^{k_{\lambda_i} - k_{\mu_i} + 1}\mu_i^{k_{\mu_i} - k_{\lambda_i} + 1} \tag{5.22}$$

for a block with $p_i \geqslant 2$ components,

$$\alpha^2 = \lambda_i^2 \tag{5.23}$$

for a block with one component.

Proof. Take ν diagonal. From the form of S (and S^{-1} which has the same form), one easily sees that (5.8) and (5.9) only involve the separate blocks and the ν's with corresponding indices. It is trivial to check (5.23). Finally, (5.7) holds because $s^{ab}_{cd} = 0$ unless $\{a, b\} = \{c, d\}$ and ν is diagonal. □

The next result is perhaps a disappointment. With $\binom{n}{2}$ extra variables in an $n^2 \times n^2$ single type II block solution of (5.1) it might be hoped (even expected) that these will give some extra information when employed to define link invariants via Turaev's formula (5.12). This is not the case, and using both solutions λ and μ of $X^2 = yX + z$ (instead of just 1) for the $\rho_a = s^{aa}_{aa}$ also gives nothing new.

5.24. PROPOSITION. *Let S be a single type II block solution of* (5.1). *Let μ occur m times as a ρ_a, $m \leqslant \frac{1}{2}n$. Then the link invariant T_S defined by S by formula* (5.12) *using the extended YB operator $(S, \nu, \alpha, 1)$ defined by Theorem* 5.14 *is the same as the one defined by the single type II block solution S_1 of size $(n-2m)^2 \times (n-2m)^2$, $x_{ij} = 1 = z$ for all i, j, same y as S (i.e. it is one of the 'classical' A_s invariants of Turaev).*

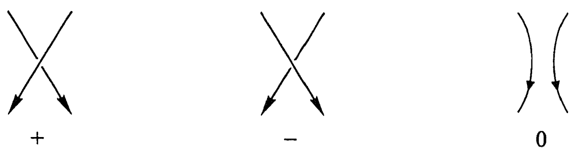
Proof. It follows immediately from (5.12) that (S, ν, α, β) and $(\rho S, \nu, \rho\alpha, \beta)$ define the same link invariant. We can therefore assume $z = 1$, i.e. $\lambda = q$, $\mu = -q^{-1}$. Then, by (5.15), $\alpha = \pm q^{n-2m}$. A simple check now shows that S satisfies the relation

$$S - S^{-1} = (q - q^{-1})I_{n^2} \tag{5.25}$$

and this also satisfied by S_1. It follows that the link invariants T and T_1 defined by S and S_1 (or $-S_1$ which does not matter by 5.20) both satisfy, [22], the same skein relation.

$$q^{n-2m}T_S(L_+) - q^{2m-n}T_S(L_-) = (q - q^{-1})T_S(L_0), \tag{5.26}$$

where L_+, L_-, and L_0 are three oriented links which are identical except for one crossing where they look, respectively, like



By repeated changing of $+$ crossings to $-$ crossings any link can be turned into an unlink. Thus, the value of T_S is uniquely determined by the skein relation (5.26) and its values on k-component unlinks. The latter are equal to $(\nu_1 + \cdots + \nu_n)^k$. Finally one checks that

$$(\nu_1 + \cdots + \nu_n) = (\bar{\nu}_1 + \cdots + \bar{\nu}_{n-2m}),$$

where $(S_1, \bar{\nu}, \alpha, 1)$ is the YB operator belonging to S_1. This is (with induction) seen as follows. If d_i is the shortest run of λ's or μ's, then if $i = 1$, the pattern $d_2 - d_1, d_3, \ldots, d_{r+1}$ gives the same trace value of ν as the original (because $\nu_{d_1+1} = -\nu_{d_1}$, $\nu_{d_1+i} = -\nu_{d_1-i+1}$, $i = 1, \ldots, d_1$) and similarly if $i > 1$, the pattern $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{r+1}$ gives the same trace value of ν as the original. This proves the proposition.

5.27. *Remark.* This result (Proposition 5.24), illustrates the previous remark (cf. (3.37)) that the $\binom{n}{2}$ extra diagonal parameters in the general on type II block solution of the YBE, i.e. the x_{ij} and z, play in some sense a trivial role, while there is but one essential parameter, viz. y (or q). On the other hand, the corresponding quantum groups, the general $\binom{n}{2} + 1$ parameter one, and the classical 1 parameter one are not isomorphic.

5.28. INVARIANTS FROM DIAGONAL SOLUTIONS

On the other hand, perhaps surprisingly, the diagonal solutions of the YBE can give rise to nontrivial knot invariants. Take, for example, the $n = 2$, 2 blocks of size 1 solution:

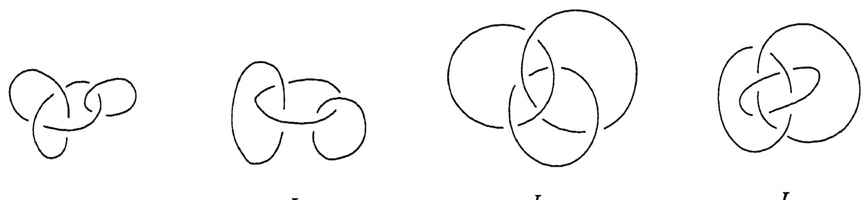
$$R = \begin{pmatrix} x_{11} & & & \\ & zx_{21}^{-1} & & \\ & & x_{21} & \\ & & & x_{22} \end{pmatrix}, \qquad R^{-1} = \begin{pmatrix} x_{11}^{-1} & & & \\ & z^{-1}x_{21}^{-1} & & \\ & & x_{21}^{-1} & \\ & & & x_{22}^{-1} \end{pmatrix} \tag{5.29}$$

with corresponding solutions of (5.1)

$$S = \begin{pmatrix} x_{11} & & & \\ & 0 & x_{21} & \\ & zx_{21}^{-1} & 0 & \\ & & & x_{22} \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} x_{11}^{-1} & & & \\ & 0 & z^{-1}x_{21} & \\ & x_{21}^{-1} & 0 & \\ & & & x_{22}^{-1} \end{pmatrix}. \tag{5.30}$$

This S, for $x_{11} = x_{22}$, extends to a Yang–Baxter operator (S, ν, α, β) with $\nu = I_2$, if $\alpha = x_{11} = x_{22}$, $\beta = 1$ and gives rise to a link invariant that takes the following values on the following links

L_0 L_1 L_2 L_3 L_4 (trefoil)



L_5 L_6 L_7 L_8

$$T(L_0) = 2, \qquad T(L_1) = 4, \qquad T(L_2) = 2 + 2\gamma,$$
$$T(L_3) = 2 + 2\gamma^2, \qquad T(L_4) = 2, \qquad T(L_5) = 2 + 2\gamma,$$
$$T(L_6) = 2(1+\gamma)^2, \qquad T(L_7) = 2 + 6\gamma^2, \qquad T(L_8) = 6 + 2\gamma^2. \tag{5.31}$$

Here $\gamma = r_{12}^{12} r_{21}^{21} = z$. Thus, this invariant counts components, can detect various ways in which components are linked but does not distinguish between, e.g., trefoil and unknot (L_0, and L_4; cf. also L_2 and L_5). The two size 1 blocks themselves give only the trivial invariant, thus this example shows conclusively that putting two blocks nontrivially together can definitely give nontrivial extra information.

5.32. *Remark.* The representations of the Braid group on k strings B_k defined by S and S_1 in Proposition 5.24 are different (even if $m = 0$), but this difference does not show up in the trace formula (5.12). This can also be seen directly in cases where there is no relation like (5.25), which is important in dealing with solutions S which do not consist of a single block. Indeed:

5.33. THEOREM. *R be an invertible $n^2 \times n^2$ matrix with diagonal entries x_{ij} and possibly nonzero diagonal entries $q_{ij} = r_{ij}^{ij}$, $i < j$, and no other nonzero entries. Let $S = \tau R$. Let $w = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_m}^{\varepsilon_m}$, $\varepsilon_i \in \{1, -1\}$ be an element of the braid group B_k of braids on k strings. Let $S_i = I_n^{\otimes i-1} \otimes S \otimes I_n^{\otimes k-i-1}$ and let $S_w = S_{i_1}^{\varepsilon_1} \dots S_{i_m}^{\varepsilon_m}$. Then the diagonal elements of S_w are Laurent polynomials in the q_{ij}, the x_{ii}, and the products $x_{ij}x_{ji} = z_{ij}$.*

Proof. The only off-diagonal elements of S are of the form

$$s_{ji}^{ij} = r_{ij}^{ij} = x_{ij}, \qquad s_{ij}^{ji} = r_{ji}^{ji} = x_{ji} = x_{ij}^{-1} z_{ij}. \tag{5.34}$$

The off-diagonal elements of R^{-1} are equal to $-q_{ij} x_{ji}^{-1} x_{ij}^{-1}$, $i < j$. It follows that the diagonal elements of $S^{-1} = R^{-1}\tau$ are of the form

$$x_{ii}^{-1}, \qquad -q_{ij}(x_{ij}x_{ij})^{-1} = -q_{ij} z_{ij}^{-1} \tag{5.35}$$

and that the off-diagonal elements of S^{-1} are of the form

$$(S^{-1})_{ji}^{ij} = x_{ji}^{-1} = z_{ij}^{-1} x_{ij}, \qquad (S^{-1})_{ij}^{ji} = x_{ij}^{-1}. \tag{5.36}$$

Now consider a diagonal element of S_w. Such an element is a sum of products of the form

$$t_{i_1(2)\dots i_n(2)}^{i_1(1)\dots i_n(1)} t_{i_1(3)\dots i_n(3)}^{i_1(2)\dots i_n(2)} \dots t_{i_1(m)\dots i_n(m)}^{i_1(m-1)\dots i_n(m-1)} \tag{5.37}$$

with $i_l(m) = i_l(1)$, $l = 1, \dots, n$, and $r_{i_1(l+1)\dots\dots i_n(l+1)}^{i_1(l)\dots i_n(l)}$ an element of $S_{i_l}^{\varepsilon_l}$. Because of (5.34)–(5.36) each product (5.37) is zero unless all the permutations

$$\begin{pmatrix} i_1(l) \dots i_n(l) \\ i_1(l+1) \dots i_n(l+1) \end{pmatrix}$$

are of the form identity or τ_k, where τ_k is the transposition $(k\ k+1)$ that interchanges the kth and $(k+1)$th entries and leaves all others in place. The identity permutations produce diagonal entries from S_{i_l} or $S_{i_l}^{-1}$ and by (5.34)–(5.36), these are of the desired form. The remaining permutations in (5.37) form a word ω in the $\tau_1, \ldots, \tau_{n-1}$ that is equal to the identity in the permutation group Π_n on n-letters. The relations between the generators $\tau_1, \ldots, \tau_{n-1}$ of Π_n are the following

$$\begin{aligned}
&\tau_k^2 = 1, \\
&\tau_k\tau_{k+1}\tau_k\tau_{k+1}^{-1}\tau_k^{-1}\tau_{k+1}^{-1} = 1, \\
&\tau_k\tau_l\tau_k^{-1}\tau_l^{-1} = 1, \quad \text{if} \quad |k-l| \geqslant 2.
\end{aligned} \tag{5.38}$$

It follows that somewhere in the word ω one of the three left hand sides of (5.38) occurs and by induction (in the length of ω) it follows that if suffices to check that in all three cases, the corresponding factors in (5.37) combine to give a monomial of the desired form. Observe that S and S^{-1} have the same off-diagonal entries except for a factor z_{ij}. Thus, replacing each S_l^{-1} with S_l only changes things by monomials in the z_{ij} and we may assume that all ε_l are 1.

In the first case we obtain a product

$$t_{\alpha_1 ba\alpha_2}^{\alpha_1 ab\alpha_2} t_{\alpha_1 ab\alpha_2}^{\alpha_1 ba\alpha_2}$$

which is equal to $x_{ab}x_{ba} = z_{ab}$. Here and below, the α_i stand for strings of indices that remain unchanged.

In the case of the second type of relation of (5.38) we obtain a product

$$t_{\alpha_1 bac\alpha_2}^{\alpha_1 abc\alpha_2} t_{\alpha_1 bca\alpha_2}^{\alpha_1 bac\alpha_2} t_{\alpha_1 cba\alpha_2}^{\alpha_1 bca\alpha_2} t_{\alpha_1 cab\alpha_2}^{\alpha_1 cba\alpha_2} t_{\alpha_1 acb\alpha_2}^{\alpha_1 cab\alpha_2} t_{\alpha_1 abc\alpha_2}^{\alpha_1 acb\alpha_2}$$

which is equal to $x_{ab}x_{ac}x_{bc}x_{ba}x_{ca}x_{cb} = z_{ab}z_{bc}z_{ac}$.

Finally, in the case of the third type of relation of (5.38), we obtain a product

$$t_{\alpha_1 ba\alpha_2 cd\alpha_3}^{\alpha_1 ab\alpha_2 cd\alpha_3} t_{\alpha_1 ba\alpha_2 dc\alpha_3}^{\alpha_1 ba\alpha_2 cd\alpha_3} t_{\alpha_1 ab\alpha_2 dc\alpha_3}^{\alpha_1 ba\alpha_2 dc\alpha_3} t_{\alpha_1 ab\alpha_2 cd\alpha_3}^{\alpha_1 ab\alpha_2 dc\alpha_3}$$

which is equal to $x_{ab}x_{cd}x_{ba}x_{dc} = z_{ab}z_{cd}$. This concludes the proof. □

5.39. COROLLARY. *Let R be any one of the solutions of the YBE described in Theorem* 3.35 *and suppose conditions* (5.22), (5.23) *of Corollary* 5.21 *hold* (*so that there is an YB operator* $(\tau R, \nu, \alpha, \beta)$). *Then the corresponding link invariant is a Laurent polynomial in the* λ_i, z_i, z_{ij}.

Proof. If there are no blocks of type I present this is an immediate corollary of Theorem 5.33. The presence of a block of type I changes very little (essentially on extra scalar multiple of the identity block in S) and the result remains true.

5.40. NEW INVARIANTS FROM MIXED SOLUTIONS

We already know from 5.28 that putting together several blocks (in a nontrivial way can give real extra information. In the case of $n = 4$ and 2 (different) type

II blocks of size 2 the resulting link invariant will be a Laurent polynomial in λ_1, λ_2, z_1, z_2, z_{12}. One of the z's, say z_1, can be normalized away (or absorbed into α which is the same thing) so that the result is a Laurent polynomial in four variables (with one nontrivial relation given by (5.22) between them and there does not seem to be any obvious way to write this polynomial in terms of known 'classical' ones. In particular, there is in general (e.g., for $\lambda_1 \neq \lambda_2$) no relation like (5.25). Just what this polynomial and all the other ones arising from Theorem 3.35 via Corollary 5.21 bring in terms of new invariants remains to be explored.

Appendix 1

Direct proof that the ideal generated by the elements (2.3), (2.4) *is a Hopf algebra ideal in* $K\langle t\rangle$.

Let I be the ideal in $K\langle t\rangle$ generated by the elements (2.3), (2.4). Under the comultiplication of $K\langle t\rangle$, we have

$$t_r^a t_r^b - q^{ab} t_r^b t_r^a \longmapsto t_{i_1}^a t_{i_2}^b \otimes t_r^{i_1} t_r^{i_2} - q^{ab} t_{j_1}^b t_{j_2}^a \otimes t_r^{j_1} t_r^{j_2}. \tag{A1.1}$$

First consider the terms on the right of (A1.1) with $i_1 = i_2$ and $j_1 = j_2$. These balance in pairs:

$$\begin{aligned} &t_i^a t_i^b \otimes t_r^i t_r^i - q^{ab} t_i^b t_i^a \otimes t_r^i t_r^i \\ &\quad = (t_i^a t_i^b - q^{ab} t_i^b t_i^a) \otimes t_r^i t_r^i \in I \otimes K\langle t\rangle. \end{aligned} \tag{A1.2}$$

The remaining terms on the right-hand side of (A1.1) are treated in groups of four ($i \neq j$).

$$\begin{aligned} &t_i^a t_j^b \otimes t_r^i t_r^j - q^{ab} t_i^b t_j^a \otimes t_r^i t_r^j + t_j^a t_i^b \otimes t_r^j t_r^i - q^{ab} t_j^b t_i^a \otimes t_r^j t_r^i \\ &\quad \equiv (t_i^a t_j^b - q^{ab} t_i^b t_j^a + (q^{ij})^{-1} t_j^a t_i^b - (q^{ab})(q^{ij})^{-1} t_j^b t_i^q) \otimes t_r^i t_r^j \\ &\quad \equiv 0 \quad \mathrm{mod}(I \otimes K\langle t\rangle + K\langle t\rangle \otimes I) \end{aligned} \tag{A1.3}$$

(where the first congruence is in fact $\mathrm{mod}(K\langle t\rangle \otimes I)$ and the second $\mathrm{mod}\, I \otimes (K\langle t\rangle))$.

The elements (2.4) are twice as complicated to treat. Under the comultiplication, (2.4) goes to

$$\begin{aligned} &t_{i_1}^a t_{i_2}^b \otimes t_r^{i_1} t_s^{i_2} - q^{ab} t_{j_1}^b t_{j_2}^a \otimes t_r^{j_1} t_s^{j_2} + \\ &\quad + (q^{rs})^{-1} t_{k_1}^a t_{k_2}^b \otimes t_s^{k_1} t_r^{k_2} - (q^{ab})(q^{rs})^{-1} t_{l_1}^b t_{l_2}^a \otimes t_s^{l_1} t_r^{l_2}. \end{aligned} \tag{A1.4}$$

The terms with $i_1 = i_2$ fit with those with $j_1 = j_2$ for the same value $(i_1 = i_2 = j_1 = j_2)$:

$$t_i^a t_i^b \otimes t_r^i t_s^i - q^{ab} t_i^b t_i^a \otimes t_r^i t_s^i = (t_i^a t_i^b - q^{ab} t_i^b t_i^a) \otimes t_r^i t_s^i \in I \otimes K\langle t\rangle.$$

Similarly, the terms with $k_1 = k_2$ fit with those of $l_1 = l_2$ for the same value.

Recall that if $a = b$ the element (2.4) is zero. So $a \neq b$ in (A1.4). The remaining terms of (A1.4) are dealt with in groups of eight as follows:

$$\begin{aligned}
&t_i^a t_j^b \otimes t_r^i t_s^j + t_j^a t_i^b \otimes t_r^j t_s^i - q^{ab} t_i^b t_j^a \otimes t_r^i t_s^j - q^{ab} t_j^b t_i^a \otimes t_r^j t_s^i + \\
&\quad + (q^{rs})^{-1} t_i^a t_j^b \otimes t_s^i t_r^j + (q^{rs})^{-1} t_j^a t_i^b \otimes t_s^j t_r^i - \\
&\quad - (q^{ab})(q^{rs})^{-1} t_i^b t_j^a \otimes t_s^i t_r^j - (q^{ab})(q^{rs})^{-1} t_j^b t_i^a \otimes t_s^j t_r^i \\
&= (t_i^a t_j^b - q^{ab} t_i^b t_j^a + (q^{ij})^{-1} t_j^a t_i^b - (q^{ab})(q^{ij})^{-1} t_j^b t_i^a) \otimes t_r^i t_s^j - \\
&\quad - (q^{ij})^{-1} t_j^a t_i^b \otimes (t_r^i t_s^j - q^{ij} t_r^j t_s^i + (q^{rs})^{-1} t_s^i t_r^j - q^{ij} (q^{rs})^{-1} t_s^j t_r^i) + \\
&\quad + (q^{ab})(q^{ij})^{-1} t_j^b t_i^a \otimes (t_r^i t_s^j - q^{ij} t_r^j t_s^i + (q^{rs})^{-1} t_s^i t_r^j - q^{ij} (q^{rs})^{-1} t_s^j t_r^i) + \\
&\quad + (t_i^a t_j^b - q^{ab} t_i^b t_j^a + (q^{ij})^{-1} t_j^a t_i^b - (q^{ab})(q^{ij})^{-1} t_j^b t_i^a) \otimes (q^{rs})^{-1} t_s^i t_r^j,
\end{aligned}$$

which is in $I \otimes K\langle t\rangle + K\langle t\rangle \otimes I$. Above the RHS differs from the LHS only in regrouping and the insertion of the four terms

$$(q^{ij})^{-1} t_j^a t_i^b \otimes t_r^i t_s^j, \qquad q^{ab} (q^{ij})^{-1} t_j^b t_i^a \otimes t_r^i t_s^j,$$

$$(q^{ij})^{-1} (q^{rs})^{-1} t_j^a b_i^b \otimes t_s^i t_r^j, \qquad q^{ab} (q^{ij})^{-1} (q^{rs})^{-1} t_j^b t_i^a \otimes t_s^i t_r^j,$$

each both with a plus and a minus sign.

This proves that I_L is a bialgebra ideal. The proof for I_R is completely analogous.

Appendix 2

Derivation of the R-equations (R1)–(R7) *of Subsection* 3.6 *and proof that these are all equations.*

The general equation is (cf. (3.2))

$$r_{k_1 k_2}^{ab} r_{u k_3}^{k_1 c} r_{vw}^{k_2 k_3} = r_{l_1 l_2}^{bc} r_{l_3 w}^{a l_2} r_{uv}^{l_3 l_1}. \tag{A2.1}$$

By Lemma (3.4), we know that under the condition

$$r_{cd}^{ab} = 0 \quad \text{unless} \quad \{a, b\} = \{c, d\} \tag{A2.2}$$

both sides of (A2.1) are zero unless $\{a, b, c\} = \{u, v, w\}$.

CASE 1. $a = b = c = u = v = w$.

Then the LHS of (A2.1) is nonzero iff $k_1 = k_2 = k_3 = a$ and then is equal to

$(r^{aa}_{aa})^3$. Similarly, the RHS of (A2.1) is nonzero iff $l_1 = l_2 = l_3 = a$ and then it is also equal to $(r^{aa}_{aa})^3$. No extra equation results from this case.

CASE 2. $a \neq b \neq c \neq a$.
There are six subcases to be considerd, namely how the u, v, w match up with the a, b, c.

Subcase 2.1. $u = a,\ v = b,\ w = c$.
For a nonzero term on the LHS we need $k_1 = a,\ k_2 = b,\ k_3 = c$ giving a term $r^{ab}_{ab} r^{ac}_{ac} r^{bc}_{bc}$.
For a nonzero term on the RHS we need $l_1 = b,\ l_2 = c,\ l_3 = a$ giving a term $r^{bc}_{bc} r^{ac}_{ac} r^{ab}_{ab}$.
Thus, always LHS = RHS in this subcase and no extra equation results.

Subcase 2.2. $u = a,\ v = c,\ w = b$.
For a nonzero term on the LHS we need $k_1 = a,\ k_2 = b,\ k_3 = c$ giving a term $r^{ab}_{ab} r^{ac}_{ac} r^{bc}_{cb}$.
For a nonzero term on the RHS we need $l_1 = c,\ l_2 = b,\ l_3 = a$ giving a term $r^{bc}_{cb} r^{ab}_{ab} r^{ac}_{ac}$.
Thus, always LHS = RHS in this subcase and no extra equation results.

Subcase 2.3. $u = b,\ v = a,\ w = c$.
For a nonzero term on the left hand side we need $k_1 = b,\ k_2 = a,\ k_3 = c$ giving a term $r^{ab}_{ba} r^{bc}_{bc} r^{ac}_{ac}$.
For a nonzero term on the RHS we need $l_1 = b,\ l_2 = c,\ l_3 = a$ giving a term $r^{bc}_{bc} r^{ac}_{ac} r^{ab}_{ba}$.
Thus, always LHS = RHS in this subcase and no extra equation results.

Subcase 2.4. $u = b,\ v = c,\ w = a$.
For a nonzero term on the LHS we need $k_1 = b,\ k_2 = a,\ k_3 = c$ giving a term $r^{ab}_{ba} r^{bc}_{bc} r^{ac}_{ca}$.
For a nonzero term on the RHS we need $l_1 = b,\ l_2 = c$ or $l_2 = c,\ l_1 = b$ and $l_3 = l_2$ giving the terms $r^{bc}_{bc} r^{ac}_{ca} r^{cb}_{bc}$ and $r^{bc}_{cb} r^{ab}_{ba} r^{bc}_{bc}$.
Thus, LHS = RHS in this subcase holds iff

$$r^{bc}_{bc}(r^{ab}_{ba} r^{ac}_{ca}) = r^{bc}_{bc}(r^{ac}_{ca} r^{cb}_{bc} + r^{bc}_{cb} r^{ab}_{ba}). \tag{R1}$$

Subcase 2.5. $u = c,\ v = a,\ w = b$.
For a nonzero term on the LHS we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$ and $k_3 = k_1$ giving the terms $r^{ab}_{ab} r^{ac}_{ca} r^{ba}_{ab}$ and $r^{ab}_{ba} r^{bc}_{cb} r^{ab}_{ab}$.
For a nonzero term on the RHS we need $l_1 = c,\ l_2 = b,\ l_3 = a$ giving a term $r^{bc}_{cb} r^{ab}_{ab} r^{ac}_{ca}$.
Thus, LHS = RHS in this subcase holds iff

$$r^{ab}_{ab}(r^{ac}_{ca} r^{ba}_{ab} + r^{ab}_{ba} r^{bc}_{cb}) = r^{ab}_{ab}(r^{bc}_{cb} r^{ac}_{ca}). \tag{R2}$$

Subcase 2.6. $u = c,\ v = b,\ w = a.$
For a nonzero term on the LHS we need $k_1 = a$, $k_2 = b$ or $k_1 = b$, $k_2 = a$ and $k_3 = k_1$ giving the terms $r^{ab}_{ab}r^{ac}_{ca}r^{ba}_{ba}$ and $r^{ab}_{ba}r^{bc}_{cb}r^{ab}_{ba}$.
For a nonzero term on the RHS we need $l_1 = b$, $l_2 = c$ or $l_1 = c$, $l_2 = b$ and $l_3 = l_2$ giving the terms $r^{bc}_{bc}r^{ac}_{ca}r^{cb}_{cb}$ and $r^{bc}_{cb}r^{ab}_{ba}r^{bc}_{cb}$.
Thus, LHS = RHS in this subcase holds iff

$$r^{ab}_{ab}r^{ba}_{ba}r^{ac}_{ca} + r^{ab}_{ba}r^{avb}_{ba}r^{bc}_{cb} = r^{bc}_{bc}r^{cb}_{cb}r^{ac}_{ca} + r^{ab}_{ba}r^{ba}_{cb}r^{bc}_{cb}. \tag{R3}$$

CASE 3. $a = b \neq c$.
Again there are a number of subcases to consider depending on how the u, v, w match up with the a, b, c. The six possibilities a priori coincide in pairs giving three subcases.

Subcase 3.1. $u = v = a = b,\ w = c.$
For a nonzero term on the LHS we need $k_1 = a$, $k_2 = a$, $k_3 = c$ giving the term $r^{aa}_{aa}r^{ac}_{ac}r^{ac}_{ac}$.
For a nonzero term on the RHS we need $l_1 = a$, $l_2 = c$, $l_3 = a$ giving a term $r^{ac}_{ac}r^{ac}_{ac}r^{aa}_{aa}$.
Thus, always LHS = RHS in this subcase and no extra equation results.

Subcase 3.2. $u = w = a = b, v = c.$
For a nonzero term on the LHS we need $k_1 = k_2 = a$, $k_3 = c$ giving a term $r^{aa}_{aa}r^{ac}_{ac}r^{ac}_{ca}$.
For a nonzero term on the RHS we need $l_1 = a$, $l_2 = c$ or $l_1 = c$, $l_2 = a$ and $l_3 = l_2$ giving the terms $r^{ac}_{ac}r^{ac}_{ca}r^{ca}_{ac}$ and $r^{ac}_{ca}r^{aa}_{aa}r^{ac}_{ac}$.
Thus, LHS = RHS in this subcase iff

$$r^{ac}_{ac}r^{ac}_{ca}r^{ca}_{ac} = 0. \tag{R4}$$

Subcase 3.3. $u = c,\ v = w = a = b.$
For a nonzero term on the LHS we need $k_1 = k_2 = k_3 = a$ giving a term $r^{aa}_{aa}r^{ac}_{ca}r^{aa}_{aa}$.
For a nonzero term on the RHS we need $l_1 = a$, $l_2 = c$ or $l_1 = c$, $l_2 = a$ and $l_3 = l_2$ giving the terms $r^{ac}_{ac}r^{ac}_{ca}r^{ca}_{ca}$ and $r^{ac}_{ca}r^{aa}_{aa}r^{ac}_{ca}$.
Thus, LHS = RHS in this subcase iff

$$r^{aa}_{aa}r^{aa}_{aa}r^{ac}_{ca} = r^{aa}_{aa}r^{ac}_{ca}r^{ac}_{ca} + r^{ac}_{ac}r^{ca}_{ca}r^{ac}_{ca}. \tag{R5}$$

CASE 4. $a \neq b = c$.
As in case 3, there are three subcases to consider

Subcase 4.1. $u = a,\ v = w = b = c.$
For a nonzero term on the LHS we need $k_1 = a$, $k_2 = k_3 = b$ giving a term $r^{ab}_{ab}r^{ab}_{ab}r^{bb}_{bb}$.
For a nonzero term on the RHS we need $l_1 = l_2 = b$, $l_3 = a$ giving a term

$r^{bb}_{bb}r^{ab}_{ab}r^{ab}_{ab}$.
Thus, always LHS = RHS in this subcase and no extra equation results.

Subcase 4.2. $u = b = w = c,\ v = a$.
For a nonzero term on the LHS we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$ and $k_3 = k_1$ giving the terms $r^{ab}_{ab}r^{ab}_{ba}r^{ba}_{ab}$ and $r^{ab}_{ba}r^{bb}_{bb}r^{ab}_{ab}$.
For a nonzero term on the RHS we need $l_1 = l_2 = b,\ l_3 = c$ giving a term $r^{bb}_{bb}r^{ab}_{ab}r^{ab}_{ba}$.
Thus, LHS = RHS in this subcase iff

$$r^{ab}_{ab}r^{ab}_{ba}r^{ba}_{ab} = 0$$

giving (R4) for the second time.

Subcase 4.3. $u = v = b = c,\ w = a$.
For a nonzero term on the LHS we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$ and $k_3 = k_1$ giving the terms $r^{ab}_{ab}r^{ab}_{ba}r^{ba}_{ba}$ and $r^{ab}_{ba}r^{bb}_{bb}r^{ab}_{ba}$.
For a nonzero term on the RHS we need $l_1 = l_2 = l_3 = b$ giving a term $r^{bb}_{bb}r^{ab}_{ba}r^{bb}_{bb}$.
Thus, RHS = LHS in this subcase iff

$$r^{bb}_{bb}r^{bb}_{bb}r^{ab}_{ba} = r^{bb}_{bb}r^{ab}_{ba}r^{ab}_{ba} + r^{ab}_{ab}r^{ba}_{ba}r^{ab}_{ba}. \tag{R6}$$

Note that this is not the same equation as (R5) (also after changing b to a, a to c).

CASE 5. $a = c \neq b$.
As in Cases 3 and 4, there are three subcases to consider.

Subcase 5.1. $u = w = a = c,\ v = b$.
For a nonzero term in the LHS we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$ and $k_3 = k_1$ giving the terms $r^{ab}_{ab}r^{aa}_{aa}r^{ba}_{ba}$ and $r^{ab}_{ba}r^{ba}_{ab}r^{ab}_{ba}$.
For a nonzero term on the RHS we need $l_1 = b,\ l_2 = a$ or $l_1 = a,\ l_2 = b$ and $l_3 = l_2$ giving the terms $r^{ba}_{ba}r^{aa}_{aa}r^{ba}_{ab}$ and $r^{ba}_{ab}r^{ab}_{ba}r^{ba}_{ab}$.
Thus, LHS = RHS in this subcase iff

$$r^{ab}_{ba}r^{ba}_{ab}r^{ab}_{ba} = r^{ab}_{ba}r^{ba}_{ab}r^{ba}_{ab}. \tag{R7}$$

Subcase 5.2. $u = v = a = c,\ w = b$.
For a nonzero term on the LHS we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = a$ and $k_3 = k_1$ giving the terms $r^{ab}_{ab}r^{aa}_{aa}r^{ba}_{ab}$ and $r^{ab}_{ba}r^{ba}_{ab}r^{ab}_{ab}$.
For a nonzero term on the RHS we need $l_1 = a,\ l_2 = b,\ l_3 = a$ giving a term $r^{ba}_{ab}r^{ab}_{ab}r^{aa}_{aa}$.
Thus, LHS = RHS in this subcase iff

$$r^{ab}_{ab}r^{ab}_{ba}r^{ba}_{ab} = 0$$

giving (R4) for the third time.

Subcase 5.3. $u = b,\ v = w = a = c$.
For a nonzero term on the LHS we need $k_1 = a,\ k_2 = b$ or $k_1 = b,\ k_2 = k_3 = a$ giving a term $r^{ab}_{ba} r^{ba}_{ba} r^{aa}_{aa}$.
For a nonzero term on the RHS we need $l_1 = b,\ l_2 = a$ or $l_1 = a,\ l_2 = b$ and $l_3 = l_2$ giving the terms $r^{ba}_{ba} r^{aa}_{aa} r^{ab}_{ba}$ and $r^{ba}_{ab} r^{ab}_{ba} r^{ba}_{ba}$.
Thus, LHS = RHS in this subcase iff

$$r^{ba}_{ba} r^{ab}_{ba} r^{ba}_{ab} = 0$$

giving (R4) for the fourth time.

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