Weak and strong regularity, compactness, and approximation of polynomials

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Abstract. Let $X$ be an inner product space, let $G$ be a group of orthogonal transformations of $X$, and let $R$ be a bounded $G$-stable subset of $X$. We define very weak and very strong regularity for such pairs $(R, G)$ (in the sense of Szemerédi's regularity lemma), and prove that these two properties are equivalent.

Moreover, these properties are equivalent to the compactness of the space $(B(H), d_R)/G$. Here $H$ is the completion of $X$ (a Hilbert space), $B(H)$ is the unit ball in $H$, $d_R$ is the metric on $H$ given by $d_R(x, y) := \sup_{r \in R} |\langle r, x - y \rangle|$, and $(B(H), d_R)/G$ is the orbit space of $(B(H), d_R)$ (the quotient topological space with the $G$-orbits as quotient classes).

As applications we give Szemerédi’s regularity lemma, a related regularity lemma for partitions into intervals, and a low rank approximation theorem for homogeneous polynomials.

1. Equivalence of very weak regularity, very strong regularity, and compactness

This paper is inspired by Szemerédi’s regularity lemma ([7]) and subsequent work on graph limits byLovász and Szegedy ([3,4]) (cf. also [5]).

Let $X$ be an inner product space and let $R$ be a bounded subset of $X$ spanning $X$. (So each element of $X$ is a linear combination of finitely many elements of $R$.) Let $G$ be a group of orthogonal transformations $\pi$ of $X$ with $\pi(R) = R$. Let $B(X)$ denote the unit ball in $X$. For any $k$, let $R_k := \{\pm r_1 \pm \cdots \pm r_k \mid r_1, \ldots, r_k \in R\}$. Let $H$ be the completion of $X$, which is a Hilbert space. Then $G$ naturally acts on $H$. For $x, y \in H$, define

\[
d_R(x, y) := \sup_{r \in R} |\langle r, x - y \rangle|.
\]

The space $(B(H), d_R)/G$ is the orbit space of $(B(H), d_R)$, i.e., the quotient topological space of $(B(H), d_R)$ taking the $G$-orbits as classes.

**Theorem 1.** The following are equivalent:

(i) $(R, G)$ is very weakly regular: for each $k$ there exists a finite set $Z \subseteq X$ such that for each $x \in R_k$ there exist $z \in Z$ and $\pi \in G$ satisfying $|\langle r, x - z \pi \rangle|^2 \leq 1$ for each $r \in R$;

(ii) $(R, G)$ is weakly regular: for each $\varepsilon > 0$ there exists a finite set $Z \subseteq B(X)$ such that for each $x \in B(X)$ there exist $z \in Z$ and $\pi \in G$ satisfying $|\langle r, x - z \pi \rangle| < \varepsilon$ for each $r \in R$;

(iii) $(R, G)$ is very strongly regular: for each $\varepsilon > 0$ and $f : X \to \{1, 2, \ldots\}$ there exists a finite set $Z \subseteq B(X)$ such that for each $x \in B(X)$ there exist $z \in Z$ and $\pi \in G$ satisfying $\|f\|_p$ is the $L^p$-norm, here for the finite-dimensional space $R^{d(x)}$.}
for all $t \in [ε, 2]$, where $p := 2/(2 - t)$, and for all orthogonal $r_1, \ldots, r_{f(z)} \in R$; (iv) the space $(B(H), d_R)/G$ is compact.

**Proof.** (iii)⇒(ii) follows by taking $f(x) = 1$ for each $x \in X$ and $t = 1$. (ii)⇒(i) follows by observing that $\frac{1}{t}R_k \subseteq B(X)$ for some $t$, and taking $ε := 1/t$. So it suffices to prove (i)⇒(ii), (ii)⇒(iv), and (iv)⇒(iii).

For all $x, y \in B(H)$ define

\( \delta_R(x, y) := \inf_{p \in G} d_R(x, y^p). \)

Then $δ_R$ is a pseudometric, and the space $(B(H), δ_R)$ is topologically homeomorphic to the orbit space $(B(H), d_R)/G$.

Observe that (i) implies that the space $(R_k, δ_R)$ is totally bounded. Indeed, choose $ε > 0$. Let $t := [ε^{-1}]$. Then $R_{kt}$ can be covered by finitely many $δ_R$-balls of radius 1. As $R_k \subseteq \frac{1}{t}R_{kt}$, $R_k$ can be covered by finitely many $δ_R$-balls of radius $1/t \leq ε$.

So we can assume, by scaling, that $∥r∥ \leq 1$ for each $r \in R$.

(i)⇒(ii): We saw above that (i) implies that $(R_k, δ_R)$ is totally bounded for each $k$. Now define, for each $k$,

\( S_k := \{λ_1r_1 + \cdots + λ_kr_k \mid r_1, \ldots, r_k \in R, λ_1, \ldots, λ_k \in [-1, +1]\}. \)

Then also $(S_k, δ_R)$ is totally bounded. Indeed, choose $ε > 0$, and define $t := k[ε^{-1}]$. Then each $x \in S_k$ has Hilbert distance less than $ε$ to $\frac{1}{t}R_{kt}$. By the above, $(R_{kt}, δ_R)$ is totally bounded, hence so is $(\frac{1}{t}R_{kt}, δ_R)$. So $(S_k, δ_R)$ is totally bounded.

Next we show that for each $k$:

\( B(X) \subseteq B_{d_R}(S_k, 1/\sqrt{k}). \)

To see this, choose $a \in B(X)$. Let $a_0 := a$. If $a_i$ has been found, and $d_R(a_i, 0) > 1/\sqrt{k}$, choose $r$ with $⟨r, a_i⟩ > 1/\sqrt{k}$. Let $a_{i+1} := a_i - ⟨r, a_i⟩r$. Then by induction on $i$, as $∥r∥ \leq 1$,

\( \|a_{i+1}\|^2 = \|a_i\|^2 - 2⟨r, a_i⟩ + (r, a_i)^2∥r∥^2 \leq \|a_i\|^2 - ⟨r, a_i⟩^2 \leq \|a_i\|^2 - 1/k \leq 1 - i/k - 1/k = 1 - (i + 1)/k. \)

So the process terminates for some $i \leq k$, and we have \( S_k \), since $a - a_i \in S_k$ and hence $d_R(a, S_k) \leq d_R(a, a - a_i) = d_R(a_i, 0) \leq 1/\sqrt{k}$.

As each $(S_k, δ_R)$ is totally bounded, \( S_k \) implies that $(B(X), δ_R)$ is totally bounded.
(ii)$\Rightarrow$(iv): By (ii), the space $(B(H),\delta_R)$ is totally bounded. So it suffices to show that $(B(H),\delta_R)$ is complete. Let $x_1,x_2,\ldots$ be a Cauchy sequence in $(B(H),\delta_R)$. We show that it is convergent. We can assume that $\delta_R(x_n,x_{n+1}) < 2^{-n}$ for each $n$. Let $\pi_1$ be the identity in $G$. For each $n \geq 1$, we can choose $\pi_{n+1} \in G$ such that $d_R(x_n^{\pi_{n+1}},x_{n+1}^{\pi_{n+1}}) < 2^{-n}$. Replacing $x_n$ by $x_n^{\pi_n}$, we can assume that $x_1,x_2,\ldots$ is a Cauchy sequence in $(B(H),d_R)$. As $B(H)$ is weakly compact, $x_1,x_2,\ldots$ has a subsequence that converges to some $a \in B(H)$ in the weak topology on $B(H)$. Then $\lim_{n \to \infty} d_R(x_n,a) = 0$. Indeed, $d_R(x_n,a) \leq 2^{-n+2}$ for each $n$. Otherwise, $|\langle r,x_n-a \rangle| > 2^{-n+2}$ for some $r \in R$. As $a$ is a weak limit of some subsequence of $x_1,x_2,\ldots$, there is an $m \geq n$ with $|\langle r,x_m-a \rangle| < 2^{-n+1}$. As $|\langle r,x_n-x_m \rangle| \leq d_R(x_n,x_m) < 2^{-n+1}$, this gives a contradiction.

(iv)$\Rightarrow$(iii): Choose $\varepsilon > 0$ and $f : X \to \{1,2,\ldots\}$. For any $k$, consider the function $\phi_k : X \to \mathbb{R}$ defined by

$$\phi_k(x) := \sup_{t \in [\varepsilon,2]} \sup_{r_1,\ldots,r_k \in R} \frac{\sum_{i=1}^{k} |\langle r_i,x \rangle|^t}{(1 + \|\langle r_1^t,\ldots,r_k^t \rangle\|_p)}$$

for $x \in X$, where $p = (1-t/2)^{-1}$. Then $\phi_k$ is continuous with respect to the $d_R$-topology on $B(H)$. To see this, let $x,y \in B(H)$ with $d_R(x,y) \leq 1$. Then $|\langle r,x \rangle|^t - |\langle r,y \rangle|^t \leq 2|\langle r,x-y \rangle|^\varepsilon \leq 2d_R(x,y)^\varepsilon$ for each $r \in R$ and $t \in [\varepsilon,2]$. This gives, by considering any $t$ and $r_1,\ldots,r_k$ in the suprema for $x$, that $\phi_k(y) \geq \phi_k(x) - 2kd_R(x,y)^\varepsilon$ (using that the denominator in (7) is at least 1). So $\phi_k$ is continuous in the $d_R$-topology on $B(H)$.

Define for each $z \in B(X)$:

$$U_z := \{x \in B(H) \mid \phi_{f(z)}(x-z) < \varepsilon\}.$$

So $U_z$ is open in the $d_R$-topology. Moreover, the $U_z$ for $z \in B(X)$ cover $B(H)$. Indeed, for any $x \in B(H)$ there exists $z \in B(X)$ with $\|x-z\| < \varepsilon^{1/\varepsilon}$. Then $x \in U_z$, since $\phi_k(x-z) < \varepsilon$ for any $k$, which follows from the following inequality. Let $t \in [\varepsilon,2]$ and $r_1,\ldots,r_k \in R$ be orthogonal and nonzero, for some $k \geq 1$. Define $s_i := r_i/\|r_i\|$ for each $i$. So $s_1,\ldots,s_k$ are orthonormal. Denote $\rho := \|(\|r_1^t\|,\ldots,\|r_k^t\|)\|_p$, with $p := 2/(2-t)$. Then one has for any $y \in B(H)$, using the Hölder inequality, and setting $q := 2/t$ (so that $p^{-1} + q^{-1} = 1$):

$$\sum_{i=1}^{k} |\langle r_i,y \rangle|^t = \sum_{i=1}^{k} \|r_i\|^t \cdot |\langle s_i,y \rangle|^t \leq (\sum_{i=1}^{k} \|r_i\|^p)^{1/p} \cdot (\sum_{i=1}^{k} |\langle s_i,y \rangle|^q)^{1/q} = \rho (\sum_{i=1}^{k} |\langle s_i,y \rangle|^2)^{1/2} \leq \rho \|y\|^{2/q} = \rho \|y\|^t \leq (1 + \rho)\|y\|^\varepsilon.$$

So $\phi_{f(z)}(x-z) \leq \|x-z\|^\varepsilon < \varepsilon$, and hence $x \in U_z$.

As $(B(H),\delta_R)$ is compact by (iv), there is a finite set $Z \subseteq X$ such that for each $x \in X$ there exist $z \in Z$ and $\pi \in G$ such that $x \in U_{z,\pi}$. This gives (iii).
2. Applications

Since $R$ spans $X$, $X$ is fully determined by the positive semidefinite $R \times R$ matrix giving the inner products of pairs from $R$. Then $G$ is given by a group of permutations of $R$ that leave the matrix invariant. It is convenient to realize that $R$ is weakly regular if (but not only if) the orbit space $R^k/G$ is compact for each $k$.

1. Szemerédi’s regularity lemma [7]. Let $R$ be the collection of sets $I \times J$, with $I$ and $J$ each being a union of finitely many subintervals of $[0,1]$, with inner product equal to the measure of the intersection. Let $G$ be the group of permutations of the intervals of any partition of $[0,1]$ into intervals. Then $G$ acts on $R$.

Let $\Pi$ be the collection of partitions of $[0,1]$ into finitely many sets, each being a union of finitely many intervals. For $P, Q \in \Pi$, $P \leq Q$ if and only if $P$ is a refinement of $Q$. This gives a lattice; let ∧ be the meet.

For any $P \in \Pi$, let $L_P$ be subspace of $X$ spanned by the elements $I \times J$ with $I, J \in P$. For any $x \in X$, let $x_P$ be the orthogonal projection of $x$ onto $L_P$. 

**Lemma 1.** For each $x \in X$ and $\varepsilon > 0$ there exists $t_{\varepsilon,x}$ such that for each $N \in \Pi$ there is a $P \geq N$ such that $\|x_N - x_P\| < \varepsilon$ and $|P| \leq t_{\varepsilon,x}$.

**Proof.** Let $Y$ be the set of those $x$ for which the statement holds for all $\varepsilon > 0$. Then $Y$ is a linear space. Indeed, if $x, y \in Y$ and $\lambda \neq 0$ then $\lambda x, y \in Y$, as we can take $t_{\varepsilon,\lambda x} := t_{|\lambda|^{-1}\varepsilon,x}$. If $x, y \in Y$ then $x + y \in Y$, as we can take $t_{\varepsilon,x+y} := t_{\varepsilon/2,x} t_{\varepsilon/2,y}$, since if $\|x_N - x_P\| < \varepsilon/2$ and $\|y_N - y_Q\| < \varepsilon/2$ for some $P, Q \geq N$, then $\|(x + y)_N - x_P - y_Q\| < \varepsilon$, hence $\|(x + y)_N - x_P - y_Q\| < \varepsilon$, since $x_P + y_Q \in L_P \cap Q$ and $(x + y)_N \in L_P \cap Q$ (since $L_P \cap Q \subseteq L_N$). Note that $|P \cap Q| \leq |P||Q|/2$.

So $Y$ is a linear space, and hence it suffices to show that $Y \subseteq Y$. Let $x \in R$ and $\varepsilon > 0$. We claim that $t_{\varepsilon,x} := (1 + 2/\varepsilon)^2$ will do. Indeed, let $N \in \Pi$. Then

\begin{equation}
(10) \quad x_N = \sum_{I,J \in N} \alpha_I \beta_J (I \times J)
\end{equation}

for some $\alpha, \beta : N \to [0,1]$. Let $\alpha'$ and $\beta'$ be obtained from $\alpha$ and $\beta$ by rounding down the values to an integer multiple of $\varepsilon/2$. Let $P \geq N$ be such that two classes $I$ and $J$ of $N$ are contained in the same class of $P$ if and only if $\alpha'_I = \alpha'_J$ and $\beta'_I = \beta'_J$. As the pairs $(\alpha'_I, \beta'_I)$ take at most $(1 + 2/\varepsilon)^2$ different values, we have $|P| \leq (1 + 2\varepsilon^{-1})^2$. Define

\begin{equation}
(11) \quad y := \sum_{I,J \in N} \alpha'_I \beta'_J (I \times J).
\end{equation}

Then $y \in L_P$. Hence, since $x_P = (x_N)_P$ (as $L_P \subseteq L_N$), implying that $x_P$ is the point on $L_P$ closest to $x_N$:

\begin{equation}
(12) \quad \|x_N - x_P\|^2 \leq \|x_N - y\|^2 \leq \sum_{I,J \in N} (\alpha_I \beta_J - \alpha'_I \beta'_J)^2 \mu(I \times J) \leq \varepsilon^2 \sum_{I,J \in N} \mu(I \times J) = \varepsilon^2.
\end{equation}
Here $\mu(I \times J)$ is the measure of $I \times J$.

Call a collection $P$ of sets balanced if all sets in $P$ have the same cardinality. Call a partition $P$ of a finite set $V$ $\varepsilon$-balanced if $P \setminus P'$ is balanced for some $P' \subseteq P$ with $|\bigcup P'| \leq \varepsilon|V|$.

**Lemma 2.** Let $\varepsilon > 0$. Then each partition $P$ of a finite set $V$ has an $\varepsilon$-balanced refinement $Q$ with $|Q| \leq (1 + 1/\varepsilon)|P|$.

**Proof.** Define $t := \varepsilon|V|/|P|$. Split each class of $P$ into classes, each of size $\lfloor t \rfloor$, except for at most one of size less than $t$. This gives $Q$. Then $|Q| \leq |P| + |V|/t = (1 + 1/\varepsilon)|P|$. Moreover, the union of the classes of $Q$ of size less than $t$ has size at most $|P|t = \varepsilon|V|$. So $Q$ is $\varepsilon$-balanced.

Given a graph $H = (V, E)$ and $C, D \subseteq V$, then $e(C, D)$ is the number of adjacent pairs of vertices in $C \times D$. If $C, D \neq \emptyset$, let $d(C, D) := e(C, D)/|C||D|$.

**Theorem 2** (Szemerédi’s regularity lemma). For each $\varepsilon > 0$ and $p \in \mathbb{N}$ there exists $k_{p, \varepsilon} \in \mathbb{N}$ such that for each graph $H = (V, E)$ and each partition $P$ of $V$ with $|P| = p$ there is an $\varepsilon$-balanced refinement $Q$ of $P$ with $|Q| \leq k_{p, \varepsilon}$ and

$$
\sum_{A, B \in Q} \max_{0 \neq C, D \subseteq A} (|C||D| \cdot |d(C, D) - d(A \times B)|^2 < \varepsilon|V|^2.
$$

**Proof.** Let $R$ and $G$ be as above. It is easy to check that $R^k/G$ is compact for each $k$, hence $(R, G)$ is very weakly regular. So, by Theorem 1, $(R, G)$ is very strongly regular.

Fix $\varepsilon > 0$ and $p \in \mathbb{N}$. For each $x \in X$, define $f(x) := ((1 + 1/\varepsilon)p t_{\varepsilon/4,x})^2$, where $t_{\varepsilon/4,x}$ is as given in Lemma 1.

By the very strong regularity of $(R, G)$, there exists a finite set $Z \subseteq X$ such that for each $x \in B(X)$ there exist $z \in Z$ and $\pi \in G$ satisfying

$$
\sum_{j=1}^{f(z)} \langle r_j, x - z \pi^2 \rangle^2 < \varepsilon^2/16 \text{ for all orthogonal } r_1, \ldots, r_{f(z)} \in R.
$$

Let $k_{p, \varepsilon} := \max\{f(z) \mid z \in Z\}$. We show that $k_{p, \varepsilon}$ is as required.

Let $H = ([n], E)$ be a graph. Let $N$ be the partition of $[0,1]$ into $n$ equal consecutive intervals $I_1, \ldots, I_n$, and let $x := \sum_{i,j \in [n]}$ adjacent $I_i \times I_j$ (the corresponding graphon).

By the above there exists a $z \in Z$ and a $\pi \in G$ satisfying (12). By Lemma 1, there is a partition $U \in \Pi$ with $U \geq N$ such that $|U| \leq t_{\varepsilon/4.x}$ and $\|z_N - z_U\| \leq \varepsilon/4$. Let $S := P \cap U$. So $|S| \leq |P||U| \leq pt_{\varepsilon/4,x}$. By Lemma 2, there is an $\varepsilon$-balanced refinement $Q$ of $S$ with $N \leq Q \leq S$ and $|Q| \leq (1 + 1/\varepsilon)|S| \leq \sqrt{f(z)} \leq k_{p, \varepsilon}$. We show that this $Q$ gives the partition of the theorem.

For each $A, B \in Q$, choose $r \in R$ with $r \subseteq A \times B$, such that $r \in L_N$ and such that $|\langle r, x - z_Q \rangle|$ is maximized. This implies for each $r' \in R$ with $r' \subseteq A \times B$ and $r' \in L_N$:

$$
|\langle r', x - z_Q \rangle| \leq |\langle r, x - z_Q \rangle| + |\langle r', x_Q - z_Q \rangle| \leq |\langle r', x - z_Q \rangle| + |\langle A \times B, x_Q - z_Q \rangle| =
$$
\[ |\langle r', x - z_Q \rangle| + |\langle A \times B, x - z_Q \rangle| \leq 2|\langle r, x - z_Q \rangle|. \]

Let \( r_1, \ldots, r_t \) be the chosen elements. So \( t = |Q|^2 \leq f(z) \). Hence, noting that \( \langle r, z \rangle = \langle r_i, z_N \rangle \), since \( r_i \in L_N \),

\[
\begin{align*}
(\sum_{i=1}^{t} & \langle r_i, x - z_Q \rangle^2)^{1/2} \leq (\sum_{i=1}^{t} \langle r_i, x - z_N \rangle^2)^{1/2} + \|z_N - z_Q\| \leq \\
& (\sum_{i=1}^{t} \langle r_i, x - z \rangle^2)^{1/2} + \varepsilon/4 \leq \varepsilon/2.
\end{align*}
\]

For the graph \( H \), (13) and (16) give (13).

To interpret (13), for \( A, B \in Q \), let \( m_{A,B} \) denote the maximum described in (13). Let \( Q' \) be such that \( Q \setminus Q' \) is balanced and \( |\bigcup Q'| \leq \varepsilon|V| \). Set \( Q'' := Q \setminus Q' \), and let \( Z \) be the collection of pairs \( (A, B) \in Q'' \times Q'' \) with \( m_{A,B} \geq \sqrt{\varepsilon}|A||B| \). Then (13) implies

\[
\sum_{(A,B) \in Z} |A||B| \leq \sum_{(A,B) \in Z} \varepsilon^{-1/2} m_{A,B} \leq \sqrt{\varepsilon}|V|^2.
\]

Moreover, as \( |\bigcup Q'| \leq \varepsilon|V| \),

\[
\sum_{A,B \in Q''} |A||B| \geq \sum_{A,B \in Q} |A||B| - 2\varepsilon|V|^2 = (1 - 2\varepsilon)|V|^2.
\]

Hence, assuming \( \varepsilon < 1/4 \), \( |Z| \leq \sqrt{\varepsilon}(1 - 2\varepsilon)^{-1}|Q''|^2 < 2\sqrt{\varepsilon}|Q''|^2 \). For each \( (A, B) \in (Q'' \times Q'') \setminus Z \) one has \( m_{A,B} < \sqrt{\varepsilon}|A||B| \), implying that for each rectangle \( R \subseteq A \times B \) with \( |R|/|A \times B| \geq \sqrt{\varepsilon} \) one has \( d(R) - d(A \times B) < \sqrt{\varepsilon} \). In other words, \( A \times B \) is \( \sqrt{\varepsilon} \)-regular.

2. “Interval regularity”. Let \( R \) be the collection of sets \( I \times J \), with \( I \) and \( J \) subintervals of \([0,1]\), with inner product given by the measure of the intersection. Then Theorem 1 gives an “interval regularity theorem” for graphs (it can also be proved with Szemerédi’s classical combinatorial method):

**Theorem 3.** For each \( \varepsilon > 0 \) and \( p \in \mathbb{N} \) there exists \( k_{p,\varepsilon} \in \mathbb{N} \) such that for each \( n \), each graph \( H = ([n], E) \) and each partition \( P \) of \([n]\) into intervals with \( |P| \leq p \), \( P \) has a refinement to a partition \( Q \) into at most \( k_{p,\varepsilon} \) intervals such that all intervals in \( Q \) have the same size except for some of them covering \( \leq \varepsilon n \) vertices and such that

\[
\sum_{A,B \in Q} \max_{I \subseteq A, J \subseteq B} |I||J||d(I,J) - d(A,B)| < \varepsilon n^2.
\]

Here \( d(I,J) \) and \( d(A,B) \) are the densities of the corresponding subgraphs of \( H \).

This can be derived similarly as (in fact, easier than) Szemerédi’s regularity lemma above.
3. Polynomial approximation. Let $k \leq n$. Each polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ can be uniquely written as $p = \sum_{\mu} \mu p_{\mu}$, where $\mu$ ranges over the set $M$ of all monomials in $\mathbb{R}[x_1, \ldots, x_k]$ and where $p_{\mu} \in \mathbb{R}[x_{k+1}, \ldots, x_n]$. If $p$ is homogeneous of degree $d$, we say that $p$ is $\varepsilon$-concentrated on the first $k$ variables if

$$
\sum_{\mu \in M} \max_{x \in \mathbb{R}^{n-k}} \frac{p_{\mu}(x)^2}{\|x\|^2} \leq \varepsilon \|p\|^2,
$$

where $\|p\|$ is the square root of the sum of the squares of the coefficients of $p$.

**Theorem 4.** For each $\varepsilon > 0$ and $d \in \mathbb{N}$ there exists $k_{d,\varepsilon}$ such that for each $n$, each homogeneous polynomial of degree $d$ in $n$ variables is $\varepsilon$-concentrated on the first $k$ variables after some orthogonal transformation of $\mathbb{R}^n$, for some $k \leq k_{d,\varepsilon}$.

This can be derived by setting $R$ to be the set of all polynomials $(a^T x)^d$, with $a \in \mathbb{R}^n$ and $\|a\| = 1$ for some $n$ (setting $x = (x_1, x_2, \ldots)$), taking the inner product of $(a^T x)^d$ and $(b^T x)^d$ equal to $(a^T b)^d$. (This corollary strengthens a ‘weak regularity’ result of Fernandez de la Vega, Kannan, Karpinski, and Vempala [2].) For details, we refer to [6].

**References**


