Weak and strong regularity, compactness, and approximation of polynomials

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Abstract. Let X be an inner product space, let G be a group of orthogonal transformations of X, and let R be a bounded G-stable subset of X. We define very weak and very strong regularity for such pairs (R,G) (in the sense of Szemerédi's regularity lemma), and prove that these two properties are equivalent.

Moreover, these properties are equivalent to the compactness of the space $(B(H),d_R)/G$. Here H is the completion of X (a Hilbert space), $B(H)$ is the unit ball in H, d_R is the metric on H given by $d_R(x,y) := \sup_{r\in R} |\langle r, x-y\rangle|$, and $(B(H),d_R)/G$ is the orbit space of $(B(H),d_R)$ (the quotient topological space with the G-orbits as quotient classes).

As applications we give Szemerédi's regularity lemma, a related regularity lemma for partitions into intervals, and a low rank approximation theorem for homogeneous polynomials.

1. Equivalence of very weak regularity, very strong regularity, and compactness

This paper is inspired by Szemerédi's regularity lemma ([7]) and subsequent work on graph limits by Lovász and Szegedy ([3,4]) (cf. also [5]).

Let X be an inner product space and let R be a bounded subset of X spanning X. (So each element of X is a linear combination of finitely many elements of R.) Let G be a group of orthogonal transformations π of X with $\pi(R) = R$. Let $B(X)$ denote the unit ball in X. For any k, let $R_k := \{\pm r_1 \pm \cdots \pm r_k \mid r_1,\ldots,r_k \in R\}$. Let H be the completion of X, which is a Hilbert space. Then G naturally acts on H. For $x,y \in H$, define

$$(1) \qquad d_R(x,y) := \sup_{r\in R} |\langle r, x-y\rangle|.$$

The space $(B(H),d_R)/G$ is the orbit space of $(B(H),d_R)$, i.e., the quotient topological space of $(B(H),d_R)$ taking the G-orbits as classes.

Theorem 1. *The following are equivalent:*
(i) (R,G) *is* ***very weakly regular****: for each k there exists a finite set $Z \subseteq X$ such that for each $x \in R_k$ there exist $z \in Z$ and $\pi \in G$ satisfying $\langle r, x - z^\pi\rangle^2 \le 1$ for each $r \in R$;*
(ii) (R,G) *is* ***weakly regular****: for each $\varepsilon > 0$ there exists a finite set $Z \subseteq B(X)$ such that for each $x \in B(X)$ there exist $z \in Z$ and $\pi \in G$ satisfying $|\langle r, x - z^\pi\rangle| < \varepsilon$ for each $r \in R$;*
(iii) (R,G) *is* ***very strongly regular****: for each $\varepsilon > 0$ and $f : X \to \{1,2,\ldots\}$ there exists a finite set $Z \subseteq B(X)$ such that for each $x \in B(X)$ there exist $z \in Z$ and $\pi \in G$ satisfying*[2]

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[2] $\|.\|_p$ is the L^p-norm, here for the finite-dimensional space $\mathbb{R}^{f(z)}$.

$$\sum_{i=1}^{f(z)} |\langle r_i, x - z^\pi\rangle|^t \leq \varepsilon(1 + \|(\|r_1\|^t, \ldots, \|r_{f(z)}\|^t)\|_p), \tag{2}$$

for all $t \in [\varepsilon, 2]$, where $p := 2/(2-t)$, and for all orthogonal $r_1, \ldots, r_{f(z)} \in R$;
(iv) *the space $(B(H), d_R)/G$ is compact.*

Proof. (iii)⇒(ii) follows by taking $f(x) = 1$ for each $x \in X$ and $t = 1$. (ii)⇒(i) follows by observing that $\frac{1}{t}R_k \subseteq B(X)$ for some t, and taking $\varepsilon := 1/t$. So it suffices to prove (i)⇒(ii), (ii)⇒(iv), and (iv)⇒(iii).

For all $x, y \in B(H)$ define

$$\delta_R(x, y) := \inf_{\pi \in G} d_R(x, y^\pi). \tag{3}$$

Then δ_R is a pseudometric, and the space $(B(H), \delta_R)$ is topologically homeomorphic to the orbit space $(B(H), d_R)/G$.

Observe that (i) implies that the space (R_k, δ_R) is totally bounded[3]. Indeed, choose $\varepsilon > 0$. Let $t := \lceil \varepsilon^{-1} \rceil$. Then R_{kt} can be covered by finitely many δ_R-balls of radius 1. As $R_k \subseteq \frac{1}{t}R_{kt}$, R_k can be covered by finitely many δ_R-balls of radius $1/t \leq \varepsilon$.

So we can assume, by scaling, that $\|r\| \leq 1$ for each $r \in R$.

(i)⇒(ii): We saw above that (i) implies that (R_k, δ_R) is totally bounded for each k. Now define, for each k,

$$S_k := \{\lambda_1 r_1 + \cdots + \lambda_k r_k \mid r_1, \ldots, r_k \in R, \lambda_1, \ldots, \lambda_k \in [-1, +1]\}. \tag{4}$$

Then also (S_k, δ_R) is totally bounded. Indeed, choose $\varepsilon > 0$, and define $t := k\lceil \varepsilon^{-1} \rceil$. Then each $x \in S_k$ has Hilbert distance less than ε to $\frac{1}{t}R_{kt}$. By the above, (R_{kt}, δ_R) is totally bounded, hence so is $(\frac{1}{t}R_{kt}, \delta_R)$. So (S_k, δ_R) is totally bounded.

Next we show that for each k:

$$B(X) \subseteq B_{d_R}(S_k, 1/\sqrt{k}). \tag{5}$$

To see this, choose $a \in B(X)$. Let $a_0 := a$. If a_i has been found, and $d_R(a_i, 0) > 1/\sqrt{k}$, choose r with $\langle r, a_i\rangle > 1/\sqrt{k}$. Let $a_{i+1} := a_i - \langle r, a_i\rangle r$. Then by induction on i, as $\|r\| \leq 1$,

$$\begin{aligned}\|a_{i+1}\|^2 = \|a_i\|^2 - 2\langle r, a_i\rangle^2 + \langle r, a_i\rangle^2\|r\|^2 \leq \|a_i\|^2 - \langle r, a_i\rangle^2 \leq \|a_i\|^2 - 1/k \leq \\ 1 - i/k - 1/k = 1 - (i+1)/k.\end{aligned} \tag{6}$$

So the process terminates for some $i \leq k$, and we have (5), since $a - a_i \in S_k$ and hence $d_R(a, S_k) \leq d_R(a, a - a_i) = d_R(a_i, 0) \leq 1/\sqrt{k}$.

As each (S_k, δ_R) is totally bounded, (5) implies that $(B(X), \delta_R)$ is totally bounded.

[3] A pseudometric space is *totally bounded* if for each $\varepsilon > 0$ it can be covered by finitely many balls of radius ε (cf. [1]).

(ii)⇒(iv): By (ii), the space $(B(H), \delta_R)$ is totally bounded. So it suffices to show that $(B(H), \delta_R)$ is complete. Let $x_1, x_2, \ldots$ be a Cauchy sequence in $(B(H), \delta_R)$. We show that it is convergent. We can assume that $\delta_R(x_n, x_{n+1}) < 2^{-n}$ for each n. Let π_1 be the identity in G. For each $n \geq 1$, we can choose $\pi_{n+1} \in G$ such that $d_R(x_n^{\pi_n}, x_{n+1}^{\pi_{n+1}}) < 2^{-n}$. Replacing x_n by $x_n^{\pi_n}$, we can assume that $x_1, x_2, \ldots$ is a Cauchy sequence in $(B(H), d_R)$. As $B(H)$ is weakly compact, $x_1, x_2, \ldots$ has a subsequence that converges to some $a \in B(H)$ in the weak topology on $B(H)$. Then $\lim_{n\to\infty} d_R(x_n, a) = 0$. Indeed, $d_R(x_n, a) \leq 2^{-n+2}$ for each n. Otherwise, $|\langle r, x_n - a\rangle| > 2^{-n+2}$ for some $r \in R$. As a is weak limit of some subsequence of $x_1, x_2, \ldots$, there is an $m \geq n$ with $|\langle r, x_m - a\rangle| < 2^{-n+1}$. As $|\langle r, x_n - x_m\rangle| \leq d_R(x_n, x_m) < 2^{-n+1}$, this gives a contradiction.

(iv)⇒(iii): Choose $\varepsilon > 0$ and $f : X \to \{1, 2, \ldots\}$. For any k, consider the function $\phi_k : X \to \mathbb{R}$ defined by

$$\phi_k(x) := \sup_{t\in[\varepsilon,2]} \sup_{\substack{\text{orthogonal} \\ r_1,\ldots,r_k\in R}} \frac{\sum_{i=1}^k |\langle r_i, x\rangle|^t}{(1 + \|(\|r_1\|^t, \ldots, \|r_k\|^t)\|_p)} \tag{7}$$

for $x \in X$, where $p = (1 - t/2)^{-1}$. Then ϕ_k is continuous with respect to the d_R-topology on $B(H)$. To see this, let $x, y \in B(H)$ with $d_R(x, y) \leq 1$. Then $|\langle r, x\rangle|^t - |\langle r, y\rangle|^t \leq 2|\langle r, x - y\rangle|^\varepsilon \leq 2d_R(x, y)^\varepsilon$ for each $r \in R$ and $t \in [\varepsilon, 2]$.[4] This gives, by considering any t and $r_1, \ldots, r_k$ in the suprema for x, that $\phi_k(y) \geq \phi_k(x) - 2kd_R(x, y)^\varepsilon$ (using that the denominator in (7) is at least 1). So ϕ_k is continuous in the d_R-topology on $B(H)$.

Define for each $z \in B(X)$:

$$U_z := \{x \in B(H) \mid \phi_{f(z)}(x - z) < \varepsilon\}. \tag{8}$$

So U_z is open in de d_R-topology. Moreover, the U_z for $z \in B(X)$ cover $B(H)$. Indeed, for any $x \in B(H)$ there exists $z \in B(X)$ with $\|x - z\| < \varepsilon^{1/\varepsilon}$. Then $x \in U_z$, since $\phi_k(x - z) < \varepsilon$ for any k, which follows from the following inequality. Let $t \in [\varepsilon, 2]$ and $r_1, \ldots, r_k \in R$ be orthogonal and nonzero, for some $k \geq 1$. Define $s_i := r_i/\|r_i\|$ for each i. So $s_1, \ldots, s_k$ are orthonormal. Denote $\rho := \|(\|r_1\|^t, \ldots, \|r_k\|^t)\|_p$, with $p := 2/(2 - t)$. Then one has for any $y \in B(H)$, using the Hölder inequality, and setting $q := 2/t$ (so that $p^{-1} + q^{-1} = 1$):

$$\begin{aligned}\sum_{i=1}^k |\langle r_i, y\rangle|^t &= \sum_{i=1}^k \|r_i\|^t \cdot |\langle s_i, y\rangle|^t \leq (\sum_{i=1}^k \|r_i\|^{tp})^{1/p} \cdot (\sum_{i=1}^k |\langle s_i, y\rangle|^{tq})^{1/q} = \\ &\rho(\sum_{i=1}^k \langle s_i, y\rangle^2)^{1/q} \leq \rho\|y\|^{2/q} = \rho\|y\|^t \leq (1 + \rho)\|y\|^\varepsilon.\end{aligned} \tag{9}$$

So $\phi_{f(z)}(x - z) \leq \|x - z\|^\varepsilon < \varepsilon$, and hence $x \in U_z$.

As $(B(H), \delta_R)$ is compact by (iv), there is a finite set $Z \subseteq X$ such that voor each $x \in X$ there exist $z \in Z$ and $\pi \in G$ such that $x \in U_{z^\pi}$. This gives (iii). ∎

[4] This follows from the fact that if $0 \leq b \leq a \leq 1$, then for each $t \in [1, 2]$: $a^t - b^t \leq a^t - b^t + (a^{2-t} - b^{2-t})(ab)^{t-1} = (a - b)(a^{t-1} + b^{t-1}) \leq 2(a - b) \leq 2(a - b)^\varepsilon$, and for each $t \in [\varepsilon, 1)$, by the concavity of the function x^t: $a^t - b^t \leq (a - b)^t \leq (a - b)^\varepsilon$.

2. Applications

Since R spans X, X is fully determined by the positive semidefinite $R \times R$ matrix giving the inner products of pairs from R. Then G is given by a group of permutations of R that leave the matrix invariant. It is convenient to realize that R is weakly regular if (but not only if) the orbit space R^k/G is compact for each k.

1. Szemerédi's regularity lemma [7]. Let R be the collection of sets $I \times J$, with I and J each being a union of finitely many subintervals of $[0,1]$, with inner product equal to the measure of the intersection. Let G be the group of permutations of the intervals of any partition of $[0,1]$ into intervals. Then G acts on R.

Let Π be the collection of partitions of $[0,1]$ into finitely many sets, each being a union of finitely many intervals. For $P, Q \in \Pi$, $P \leq Q$ if and only if P is a refinement of Q. This gives a lattice; let $\wedge$ be the meet.

For any $P \in \Pi$, let L_P be subspace of X spanned by the elements $I \times J$ with $I, J \in P$. For any $x \in X$, let x_P be the orthogonal projection of x onto L_P.

Lemma 1. *For each $x \in X$ and $\varepsilon > 0$ there exists $t_{\varepsilon,x}$ such that for each $N \in \Pi$ there is a $P \geq N$ such that $\|x_N - x_P\| < \varepsilon$ and $|P| \leq t_{\varepsilon,x}$.*

Proof. Let Y be the set of those x for which the statement holds for all $\varepsilon > 0$. Then Y is a linear space. Indeed, if $x \in Y$ and $\lambda \neq 0$ then $\lambda x \in Y$, as we can take $t_{\varepsilon,\lambda x} := t_{|\lambda^{-1}|\varepsilon,x}$. If $x, y \in Y$ then $x + y \in Y$, as we can take $t_{\varepsilon,x+y} := t_{\varepsilon/2,x} t_{\varepsilon/2,y}$, since if $\|x_N - x_P\| < \varepsilon/2$ and $\|y_N - y_Q| < \varepsilon/2$ for some $P, Q \geq N$, then $\|(x+y)_N - x_P - y_Q\| < \varepsilon$, hence $\|(x+y)_N - (x+y)_{P\wedge Q}\| \leq \varepsilon$, since $x_P + y_Q \in L_{P\wedge Q}$ and $((x+y)_N)_{P\wedge Q} = (x+y)_{P\wedge Q}$ (since $L_{P\wedge Q} \subseteq L_N$). Note that $|P \wedge Q| \leq |P||Q|$.

So Y is a linear space, and hence it suffices to show that $R \subseteq Y$. Let $x \in R$ and $\varepsilon > 0$. We claim that $t_{\varepsilon,x} := (1 + 2/\varepsilon)^2$ will do. Indeed, let $N \in \Pi$. Then

$$x_N = \sum_{I,J \in N} \alpha_I \beta_J (I \times J) \tag{10}$$

for some $\alpha, \beta : N \to [0,1]$. Let α' and β' be obtained from α and β by rounding down the values to an integer multiple of $\varepsilon/2$. Let $P \geq N$ be such that two classes I and J of N are contained in the same class of P if and only if $\alpha'_I = \alpha'_J$ and $\beta'_I = \beta'_J$. As the pairs (α'_I, β'_I) take at most $(1 + 2/\varepsilon)^2$ different values, we have $|P| \leq (1 + 2\varepsilon^{-1})^2$. Define

$$y := \sum_{I,J \in N} \alpha'_I \beta'_J (I \times J). \tag{11}$$

Then $y \in L_P$. Hence, since $x_P = (x_N)_P$ (as $L_P \subseteq L_N$), implying that x_P is the point on L_P closest to x_N:

$$\|x_N - x_P\|^2 \leq \|x_N - y\|^2 \leq \sum_{I,J \in N} (\alpha_I \beta_J - \alpha'_I \beta'_J)^2 \mu(I \times J) \leq \varepsilon^2 \sum_{I,J \in N} \mu(I \times J) = \varepsilon^2. \tag{12}$$

Here $\mu(I \times J)$ is the measure of $I \times J$. ∎

Call a collection P of sets *balanced* if all sets in P have the same cardinality. Call a partition P of a finite set V *ε-balanced* if $P \setminus P'$ is balanced for some $P' \subseteq P$ with $|\bigcup P'| \leq \varepsilon |V|$.

Lemma 2. *Let $\varepsilon > 0$. Then each partition P of a finite set V has an ε-balanced refinement Q with $|Q| \leq (1 + 1/\varepsilon)|P|$.*

Proof. Define $t := \varepsilon|V|/|P|$. Split each class of P into classes, each of size $\lceil t \rceil$, except for at most one of size less than t. This gives Q. Then $|Q| \leq |P| + |V|/t = (1 + 1/\varepsilon)|P|$. Moreover, the union of the classes of Q of size less than t has size at most $|P|t = \varepsilon|V|$. So Q is ε-balanced. ∎

Given a graph $H = (V, E)$ and $C, D \subseteq V$, then $e(C, D)$ is the number of adjacent pairs of vertices in $C \times D$. If $C, D \neq \emptyset$, let $d(C, D) := e(C, D)/|C||D|$.

Theorem 2 (Szemerédi's regularity lemma)**.** *For each $\varepsilon > 0$ and $p \in \mathbb{N}$ there exists $k_{p,\varepsilon} \in \mathbb{N}$ such that for each graph $H = (V, E)$ and each partition P of V with $|P| = p$ there is an ε-balanced refinement Q of P with $|Q| \leq k_{p,\varepsilon}$ and*

$$(13) \qquad \sum_{A,B \in Q} \max_{\substack{\emptyset \neq C \subseteq A \\ \emptyset \neq D \subseteq B}} (|C||D| \cdot |d(C, D) - d(A \times B)|)^2 < \varepsilon |V|^2.$$

Proof. Let R and G be as above. It is easy to check that R^k/G is compact for each k, hence (R, G) is very weakly regular. So, by Theorem 1, (R, G) is very strongly regular.

Fix $\varepsilon > 0$ and $p \in \mathbb{N}$. For each $x \in X$, define $f(x) := ((1 + 1/\varepsilon)pt_{\varepsilon/4,x})^2$, where $t_{\varepsilon/4,x}$ is as given in Lemma 1.

By the very strong regularity of (R, G), there exists a finite set $Z \subseteq X$ such that for each $x \in B(X)$ there exist $z \in Z$ and $\pi \in G$ satisfying

$$(14) \qquad \sum_{j=1}^{f(z)} \langle r_j, x - z^\pi \rangle^2 < \varepsilon^2/16 \text{ for all orthogonal } r_1, \ldots, r_{f(z)} \in R.$$

Let $k_{p,\varepsilon} := \max\{f(z) \mid z \in Z\}$. We show that $k_{p,\varepsilon}$ is as required.

Let $H = ([n], E)$ be a graph. Let N be the partition of $[0, 1]$ into n equal consecutive intervals $I_1, \ldots, I_n$, and let $x := \sum_{i,j \in [n] \text{ adjacent}} I_i \times I_j$ (the corresponding *graphon*).

By the above there exists a $z \in Z$ and a $\pi \in G$ satisfying (14). By Lemma 1, there is a partition $U \in \Pi$ with $U \geq N$ such that $|U| \leq t_{\varepsilon/4,z}$ and $\|z_N - z_U\| \leq \varepsilon/4$. Let $S := P \wedge U$. So $|S| \leq |P||U| \leq pt_{\varepsilon/4,z}$. By Lemma 2, there is an ε-balanced refinement Q of S with $N \leq Q \leq S$ and $|Q| \leq (1 + 1/\varepsilon)|S| \leq \sqrt{f(z)} \leq k_{p,\varepsilon}$. We show that this Q gives the partition of the theorem.

For each $A, B \in Q$, choose $r \in R$ with $r \subseteq A \times B$, such that $r \in L_N$ and such that $|\langle r, x - z_Q \rangle|$ is maximized. This implies for each $r' \in R$ with $r' \subseteq A \times B$ and $r' \in L_N$:

$$(15) \qquad |\langle r', x - x_Q \rangle| \leq |\langle r', x - z_Q \rangle| + |\langle r', x_Q - z_Q \rangle| \leq |\langle r', x - z_Q \rangle| + |\langle A \times B, x_Q - z_Q \rangle| =$$

$$|\langle r', x - z_Q\rangle| + |\langle A \times B, x - z_Q\rangle| \leq 2|\langle r, x - z_Q\rangle|.$$

Let $r_1, \ldots, r_t$ be the chosen elements. So $t = |Q|^2 \leq f(z)$. Hence, noting that $\langle r_i, z\rangle = \langle r_i, z_N\rangle$, since $r_i \in L_N$,

$$(16) \qquad \begin{aligned}&(\sum_{i=1}^{t}\langle r_i, x - z_Q\rangle^2)^{1/2} \leq (\sum_{i=1}^{t}\langle r_i, x - z_N\rangle^2)^{1/2} + \|z_N - z_Q\| \leq \\ &(\sum_{i=1}^{t}\langle r_i, x - z\rangle^2)^{1/2} + \varepsilon/4 \leq \varepsilon/2.\end{aligned}$$

For the graph H, (15) and (16) give (13). ∎

To interpret (13), for $A, B \in Q$, let $m_{A,B}$ denote the maximum described in (13). Let Q' be such that $Q \setminus Q'$ is balanced and $|\bigcup Q'| \leq \varepsilon|V|$. Set $Q'' := Q \setminus Q'$, and let Z be the collection of pairs $(A, B) \in Q'' \times Q''$ with $m_{A,B} \geq \sqrt{\varepsilon}|A||B|$. Then (13) implies

$$(17) \qquad \sum_{(A,B)\in Z} |A||B| \leq \sum_{(A,B)\in Z} \varepsilon^{-1/2} m_{A,B} \leq \sqrt{\varepsilon}|V|^2.$$

Moreover, as $|\bigcup Q'| < \varepsilon|V|$,

$$(18) \qquad \sum_{A,B\in Q''} |A||B| \geq \sum_{A,B\in Q} |A||B| - 2\varepsilon|V|^2 = (1 - 2\varepsilon)|V|^2.$$

Hence, assuming $\varepsilon < 1/4$, $|Z| \leq \sqrt{\varepsilon}(1 - 2\varepsilon)^{-1}|Q''|^2 < 2\sqrt{\varepsilon}|Q''|^2$. For each $(A, B) \in (Q'' \times Q'') \setminus Z$ one has $m_{A,B} < \sqrt{\varepsilon}|A||B|$, implying that for each rectangle $R \subseteq A \times B$ with $|R|/|A \times B| \geq \sqrt[4]{\varepsilon}$ one has $|d(R) - d(A \times B)| < \sqrt[4]{\varepsilon}$. In other words, $A \times B$ is $\sqrt[4]{\varepsilon}$-regular.

2. "Interval regularity". Let R be the collection of sets $I \times J$, with I and J subintervals of $[0, 1]$, with inner product given by the measure of the intersection. Then Theorem 1 gives an "interval regularity theorem" for graphs (it can also be proved with Szemerédi's classical combinatorial method):

Theorem 3. *For each $\varepsilon > 0$ and $p \in \mathbb{N}$ there exists $k_{p,\varepsilon} \in \mathbb{N}$ such that for each n, each graph $H = ([n], E)$ and each partition P of $[n]$ into intervals with $|P| \leq p$, P has a refinement to a partition Q into at most $k_{p,\varepsilon}$ intervals such that all intervals in Q have the same size except for some of them covering $\leq \varepsilon n$ vertices and such that*

$$(19) \qquad \sum_{A,B\in Q} \max_{\substack{I\subseteq A, J\subseteq B\\ I,J \text{ intervals}}} |I||J||d(I, J) - d(A, B)| < \varepsilon n^2.$$

Here $d(I, J)$ and $d(A, B)$ are the densities of the corresponding subgraphs of H.

This can be derived similarly as (in fact, easier than) Szemerédi's regularity lemma above.

3. Polynomial approximation. Let $k \leq n$. Each polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ can be uniquely written as $p = \sum_\mu \mu p_\mu$, where μ ranges over the set M of all monomials in $\mathbb{R}[x_1, \ldots, x_k]$ and where $p_\mu \in \mathbb{R}[x_{k+1}, \ldots, x_n]$. If p is homogeneous of degree d, we say that p is *ε-concentrated on the first k variables* if

$$\sum_{\substack{\mu \in M \\ \deg(\mu) < d}} \max_{\substack{x \in \mathbb{R}^{n-k} \\ \|x\|=1}} p_\mu(x)^2 \leq \varepsilon \|p\|^2, \tag{20}$$

where $\|p\|$ is the square root of the sum of the squares of the coefficients of p.

Theorem 4. *For each $\varepsilon > 0$ and $d \in \mathbb{N}$ there exists $k_{d,\varepsilon}$ such that for each n, each homogeneous polynomial of degree d in n variables is ε-concentrated on the first k variables* ***after some orthogonal transformation of*** *$\mathbb{R}^n$, for some $k \leq k_{d,\varepsilon}$.*

This can be derived by setting R to be the set of all polynomials $(a^\mathsf{T} x)^d$, with $a \in \mathbb{R}^n$ and $\|a\| = 1$ for some n (setting $x = (x_1, x_2, \ldots)$), taking the inner product of $(a^\mathsf{T} x)^d$ and $(b^\mathsf{T} x)^d$ equal to $(a^\mathsf{T} b)^d$. (This corollary strengthens a 'weak regularity' result of Fernandez de la Vega, Kannan, Karpinski, and Vempala [2].) For details, we refer to [6].

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