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A NOTE ON DISJOINT-OCCURRENCE INEQUALITIES FOR MARKED POISSON POINT PROCESSES

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Abstract

For (marked) Poisson point processes we give, for increasing events, a new proof of the analog of the BK inequality. In contrast to other proofs, which use weak-convergence arguments, our proof is 'direct' and requires no extra topological conditions on the events. Apart from some well-known properties of Poisson point processes, the proof is self-contained.

DISJOINT OCCURRENCE; CONTINUUM PERCOLATION; POISSON POINT PROCESSES

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1. Introduction

The BK inequality for product measures on $\{0, 1\}^n$ has become a basic tool in lattice percolation theory (see, e.g., Grimmett (1989), Ch. 2). This inequality says that the probability that two increasing events 'occur disjointly' is smaller than or equal to the product of the two individual probabilities. We will refer to this inequality as the 'standard BK inequality' to distinguish it from generalizations which we will now discuss.

Firstly, van den Berg and Kesten (1985) conjectured that the inequality not only holds for increasing events but for *all* events (see also van den Berg and Fiebig (1987)). This conjecture remained open for about ten years, until it was recently proved by Reimer (1994).

Further, for several years continuum percolation models have been the subject of attention (see, e.g., Menshikov and Sidorenko (1987), Roy (1990), Penrose (1991) and Meester and Roy (1994)). In these models the points are no longer the vertices of a regular lattice, but the points in a stationary point process, often a Poisson point process. This development has led to a need to obtain BK-like inequalities for (marked) Poisson point processes. This need also came from a different but somewhat related field, namely, interacting particle systems. In this field many relevant problems can be formulated as percolation-like problems involving the so-called space-time diagram of the process.

Bezuidenhout and Grimmett (1991, Section 2) pointed out that this generalization to Poisson point processes can be done quite easily by general weak-convergence arguments. It seems that these arguments are sufficiently flexible to obtain also some Poisson-point-

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process analog of Reimer's extension mentioned above. However, it also seems inevitable in such arguments that an extra topological condition on the events is required (namely, that the boundaries of the events, with respect to a certain topology on the space of all realizations of the point process, have probability 0). Independently of Bezuidenhout and Grimmett, Roy and Sarkar (1992) and Sarkar (1994) also obtained a generalization for Poisson point processes. Their proof is given only for certain connection events (a special class of increasing events; see Remark (iii) in Section 2 below) and requires a topological condition of the same type as mentioned above.

In the present paper we give, for the class of increasing events (which is one of the most important classes of events in the fields of application mentioned above), a more direct proof of the inequality, which requires no extra technical condition on the events.

2. Definitions and statement of results

Consider a, possibly inhomogeneous, marked Poisson point process on $\mathbb{R}^d$. Intuitively, a realization of this process is obtained as follows. First generate a realization of the point process. Then assign to each point, independent of the others, a 'mark', which is drawn from the mark space according to a certain distribution (which may depend on the position of the point). A realization ω of the process can be (and will be in this paper) represented as a set $\{(x_i, s_i) : i \in \mathbb{N}\}$ of pairs, where the x_i form a discrete subset of $\mathbb{R}^d$ (here *discrete* means that for every element of this subset there exists an $\varepsilon > 0$ such that the distance to every other element of the subset is larger than ε) and each s_i is the mark assigned to x_i. The set of all realizations is denoted by Ω. For a formal construction of the process, see e.g. Daley and Vere-Jones (1988). Here we only remark that the sigma-algebra for the process is generated by the collection of all subsets of Ω of the form $\{\omega$: the number of points in V with a mark in M equals $n\}$, where V is a bounded Borel subset of $\mathbb{R}^d$, n a non-negative integer, and M a measurable subset of the mark space.

Before we state our theorem we need more definitions and notation. By a 'region' we will always denote a Borel set in $\mathbb{R}^d$. The distribution of the process will be denoted by μ. The restriction of a realization ω to a region U is denoted by ω_U. More formally (recall that we consider ω as a set of pairs),

$$\omega_U := \{(x, s) : (x, s) \in \omega \text{ and } x \in U\}. \tag{1}$$

This leads naturally to definitions of Ω_U and μ_U (the marginal distribution of the marked point process restricted to U). We also define the *cylinder event* $[\omega_U] := \{\omega' : \omega'_U = \omega_U\}$. We say that $\omega \leqq \omega'$ if ω is contained in ω'. An event A is called *increasing* if $\omega \in A$ and $\omega \leqq \omega'$ implies $\omega' \in A$. For an event A and a region U we say that 'A lives on U' if $\omega \in A$ implies $[\omega_U] \subset A$. Note that for increasing A this is equivalent to saying that $\omega \in A$ implies $\omega_U \in A$.

Let A and B be increasing events living on a bounded region. Analogously to the 'standard' case (i.e. the case of Bernoulli random variables) introduced by van den Berg and Kesten (1985), we say that A and B 'occur disjointly' if there exist two disjoint regions such that by looking only at the first (second) we are convinced that the realization is in A (B). More formally,

$$A \,\square\, B := \{\omega : \exists \text{ disjoint regions } K \text{ and } L \text{ with } [\omega_K] \subset A \text{ and } [\omega_L] \subset B\} \tag{2}$$

$$= \{\omega : \exists \text{ disjoint regions } K \text{ and } L \text{ with } \omega_K \in A \text{ and } \omega_L \in B\}, \tag{3}$$

where the equality of (2) and (3) follows easily from the fact that A and B are increasing.

Remarks.

(i) Since A and B are increasing, it is not difficult to see that, in the above definition, we can restrict to sets K and L which are finite unions of (hyper-)cubes with rational coordinates. This guarantees the measurability of $A \,\square\, B$.

(ii) In the remainder of this paper we always work with (3) as the definition of $A \,\square\, B$. Note that for general (i.e. not necessarily increasing) events (2) and (3) are usually not the same.

(iii) Another equivalent definition, explicitly using the definition of ω as a set, is $A \,\square\, B = \{\omega : \exists \omega' \subset \omega \text{ with } \omega' \in A \text{ and } \omega \backslash \omega' \in B\}$. Roy and Sarbar have used this form for the special case mentioned in the introduction. In that case each mark is a non-negative real number, interpreted as the (random) radius of a ball centered at the corresponding point, and A and B are of the form 'there exists a path of overlapping balls from one region to another'. More precisely,

$$A = \{\omega : \exists (x_1, r_1), \cdots, (x_n, r_n) \in \omega \text{ with } x_1 \in V_1, x_n \in W_1 \text{ and } x_2, \cdots, x_{n-1} \in U,$$
$$\text{and } \|x_{i+1} - x_i\| < r_{i+1} + r_i, \; i = 1, \cdots, n-1\},$$

and B is similar, but with V_1 and W_1 replaced by U_2 and W_2 respectively. Here U is a bounded region, and V_1, W_1, V_2 and W_2 are regions contained in U.

Our main result is the following theorem, which will be proved in Section 3.

Theorem. Let A and B be increasing events living on a bounded region. We have

$$\mu(A \,\square\, B) \leqq \mu(A)\mu(B). \tag{4}$$

We will finish this section by giving some background for Section 3. The proofs by Bezuidenhout and Grimmett and by Roy and Sarkar consist, roughly speaking, of partitioning space in small cells, applying the standard BK inequality to a collection of independent random variables indexed by these cells, and then taking limits as the cell size tends to zero. We too will partition space in cells, but instead of applying the standard BK inequality, we modify the method in one of the proofs which van den Berg and Kesten had obtained for the standard BK inequality (see e.g. van den Berg and Fiebig (1987, Sect. 4.4 and 5)). In this so-called *splitting method* the event B is changed step by step and eventually becomes an independent copy of the original B. Moreover, at each step the probability of $A \,\square\, B$ increases (or remains the same). In the Poisson point process case we can no longer prove this directly, but it appears that the 'total error' over all steps goes to 0 with the cell size, which makes the method still work.

3. Proof of the theorem

First we assume that the process is homogeneous (that is, the Poisson process has constant density λ and the mark distribution does not depend on location). The key of the proof is the proposition below, for which we need some more definitions. Let $x \in \mathbb{R}^d$ and V be a bounded region. Define $x+V := \{x+y : y \in V\}$. For a realization ω, define by $T_x^V(\omega)$ the realization obtained by exchanging the configurations on V and $x+V$. More precisely, $T_x^V(\omega)$ is defined by

$$(T_x^V(\omega))_{x+V} := \{(x+y, s) : (y, s) \in \omega_V\};$$

$$(T_x^V(\omega))_V := \{(y-x, s) : (y, s) \in \omega_{x+V}\};$$

$$(T_x^V(\omega))_{(V \cup (x+V))^c} := \omega_{(V \cup (x+V))^c}.$$

Finally, define for an event E, $T_x^V(E) := \{T_x^V(\omega) : \omega \in E\}$.

Proposition. Let E_1 and E_2 be increasing events living on a bounded region M. Let $V \subset M$ and $x \in \mathbb{R}^d$ be such that $M \cap (x+V) = \emptyset$. Then

$$\mu(E_1 \,\square\, E_2) \leqq \mu(E_1 \,\square\, T_x^V(E_2)) + \mu(\{\omega : |\omega_V| \geqq 2\}) + \mu(\{\omega : |\omega_V| \geqq 1\})^2, \tag{5}$$

where $|\omega_V|$ is the size (i.e. number of elements) of ω_V.

Remark. Note that without the last two terms in the r.h.s of (5) we would immediately have the theorem. Namely, with A and B as in the formulation of the theorem and U a bounded region on which A and B live, take V and M equal to U, $E_1 = A$ and $E_2 = B$. Then, since A and $T_x^U(B)$ live on disjoint regions, the first term in the r.h.s of (5) is $\mu(A)\mu(T_x^U(B))$ which, by homogeneity, equals $\mu(A)\mu(B)$.

Proof of the proposition. For $\alpha \in \Omega_{M \setminus V}$ let

$$E_1(\alpha) := \{\sigma \in \Omega_V : \exists \text{ disjoint regions } K' \text{ and } L' \subset V^c \text{ with } \alpha_{K'} \cup \sigma \in E_1 \text{ and } \alpha_{L'} \in E_2\}. \tag{6}$$

Analogously define $E_2(\alpha)$. We need the following lemma.

Lemma.

(a) If $\omega \in E_1 \,\square\, E_2$, then $\omega_V \in E_1(\omega_{M \setminus V})$ or $\omega_V \in E_2(\omega_{M \setminus V})$ or $|\omega_V| \geqq 2$.

(b) If $\omega_V \in E_1(\omega_{M \setminus V})$ or $\omega_{x+V} \in T_x^V(E_2(\omega_{M \setminus V}))$, then $\omega \in E_1 \,\square\, T_x^V(E_2)$.

The proof of this lemma is just a precise application of the definitions. As to part (a), if $\omega \in E_1 \,\square\, E_2$, then there exist disjoint regions K and L with $\omega_K \in E_1$ and $\omega_L \in E_2$. If, additionally, $|\omega_V| \leqq 1$, then $\omega_L \cap \omega_V = \emptyset$ or $\omega_K \cap \omega_V = \emptyset$. In the first case we have (taking $K' = K \setminus V$ and $L' = L \setminus V$)

(7) $$(\omega_{M\setminus V})_{L'} = \omega_{M\setminus V} \cap \omega_{L\setminus V} = \omega_{L\cap M} \in E_2,$$

(where the last equality above holds because $\omega_L \cap \omega_V = \emptyset$, and the result is in E_2 since $\omega_L \in E_2$ and E_2 lives on M), and

(8) $$(\omega_{M\setminus V})_{K'} \cup \omega_V = \omega_{(M\setminus V)\cap(K\setminus V)} \cup \omega_V = \omega_{(M\cap K)\cup V} \geqq \omega_{M\cap K} \in E_1,$$

where the last assertion is true because $\omega_K \in E_1$ and E_1 lives on M. By (7) and (8) we have $\omega_V \in E_1(\omega_{M\setminus V})$. Similarly, in the other case, we get $\omega_V \in E_2(\omega_{M\setminus V})$.

As to part (b) of the lemma: in the first case we have $\omega_V \in E_1(\omega_{M\setminus V})$. Hence there exist K' and L' outside V with $(\omega_{M\setminus V})_{K'} \cup \omega_V \in E_1$ and $(\omega_{M\setminus V})_{L'} \in E_2$. So we have

(9) $$\omega_{(M\cap K')\cup V} = (\omega_{M\setminus V})_{K'} \cup \omega_V \in E_1$$

and

(10) $$\omega_{M\cap L'} = T_x^V(\omega_{M\cap L'}) = T_x^V((\omega_{M\setminus V})_{L'}) \in T_x^V(E_2),$$

where, in the first equality of (10), we used $M\cap L' \cap V = \emptyset$ and $M \cap L' \cap (x+V) = \emptyset$. Hence, since $((M\cap K')\cup V)\cap(M\cap L') = \emptyset$, $\omega \in E_1 \,\square\, T_x^V(E_2)$ as required. The proof for the second case is similar. This completes the proof of the lemma.

We are now ready to prove the proposition. We compare the conditional probabilities of $E_1 \,\square\, E_2$ and $E_1 \,\square\, T_x^V(E_2)$, given $\omega_{M\setminus V}$. By part (a) of the lemma, the first is at most

(11) $$\mu_V(E_1(\omega_{M\setminus V})) + \mu_V(E_2(\omega_{M\setminus V})) + \mu_V(|\omega_V| \geqq 2).$$

Further, by part (b) of the lemma, we have the following: if $\emptyset \in E_1(\omega_{M\setminus V})$ or $\emptyset \in T_x^V(E_2(\omega_{M\setminus V}))$, then the second conditional probability is 1 (and hence not smaller than the first). Moreover, again by (b), the second conditional probability is always at least

(12) $$\mu_V(E_1(\omega_{M\setminus V})) + \mu_{x+V}(T_x^V(E_2(\omega_{M\setminus V}))) - \mu_V(E_1(\omega_{M\setminus V})) \times \mu_{x+V}(T_x^V(E_2(\omega_{M\setminus V}))),$$

and if the above does not hold (i.e. if $\emptyset \notin E_1(\omega_{M\setminus V})$ and $\emptyset \notin T_x^V(E_2(\omega_{M\setminus V}))$), this is at least (use also homogeneity)

(13) $$\mu_V(E_1(\omega_{M\setminus V})) + \mu_V(E_2(\omega_{M\setminus V})) - \mu_V(|\omega_V| \geqq 1)^2.$$

Now the proposition follows immediately.

Proof of the theorem. Without loss of generality (use the scaling properties of Poisson point processes), we may assume that A and B live on the unit cube $U \equiv [0, 1)^d$. Take $x = (2, 0, \cdots, 0)$. Fix a positive integer n and partition U in n^d cubes $S_1, \cdots, S_{n^d}$ of the form $[(i_1-1)/n, i_1/n) \times \cdots \times [(i_d-1)/n, i_d/n)$, with $1 \leqq i_1, \cdots, i_d \leqq n$. Define the events $B^{(k)}$, $k = 0, \cdots, n^d - 1$ as follows: $B^{(0)} = B$ and $B^{(k+1)} = T_x^{S_{k+1}}(B^{(k)})$, $k = 0, \cdots, n^d - 1$. Now apply the proposition n^d times (the kth time with $E_1 = A$, $E_2 = B^{(k-1)}$, $M = U \cup (x + (S_1 \cup \cdots \cup S_{k-1}))$, and $V = S_k$). Noting that, for each bounded region V, $|\omega_V|$ is Poisson distributed with parameter $\lambda|V|$, so that the sum of the last two terms in (5) is at most $2\lambda^2|V|^2$, we get

$$\begin{aligned} \mu(A \square B) &= \mu(A \square B^{(0)}) \\ &\leqq \mu(A \square B^{(1)}) + 2\lambda^2 |S_1|^2 \\ &\leqq \cdots \\ &\leqq \mu(A \square B^{(n^d)}) + 2\lambda^2|S_1|^2 + \cdots + 2\lambda^2|S_{n^d}|^2. \end{aligned} \tag{14}$$

Since $B^{(n^d)} = T_x^U(B)$ and each $|S_i|$ equals $1/n^d$, we get

$$\mu(A \square B) \leqq \mu(A)\mu(B) + \frac{2\lambda^2}{n^d}. \tag{15}$$

This holds for any n, completing the proof of the theorem for the homogeneous case.

As to the non-homogeneous case, note that neither side of the inequality in the theorem changes if we change the intensities and mark distributions outside U. So we may assume that, with x as before, for every $y \in U$, $\lambda_y = \lambda_{x+y}$ and the mark distribution for y is the same as for $x+y$. The proof now remains practically the same, the only difference being that λ in the last term in the r.h.s of (15) is replaced by $\sup_{y \in U} \lambda_y$. Since this is finite, the argument still works. This completes the proof of the theorem.

Remark. An event A is called *decreasing* if $\omega \in A$ and $\omega \geqq \omega'$ implies $\omega' \in A$. In the case of Bernoulli random variables there is complete symmetry between increasing and decreasing events. However, such symmetry is absent in Poisson point processes and we have not been able to adapt the proof of the above theorem for decreasing events. In particular, we do not know a useful analog of part (a) of the lemma in this section.

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References

van den Berg, J. and Fiebig, U. (1987) On a combinatorial conjecture concerning disjoint occurrences of events. *Ann. Prob.* **15**, 354–374.

van den Berg, J. and Kesten, H. (1985) Inequalities with applications to percolation and reliability. *J. Appl. Prob.* **22**, 556–569.

Bezuidenhout, C. and Grimmett, G. R. (1991) Exponential decay for subcritical contact and percolation processes. *Ann. Prob.* **19**, 984–1009.

Daley, D. J. and Vere-Jones, D. (1988) *An Introduction to the Theory of Point Processes.* Springer, New York.

Grimmett, G. R. (1989) *Percolation.* Springer, New York.

MEESTER, R. AND ROY, R. (1994) Uniqueness of unbounded occupied and vacant components in Boolean models. *Ann. Appl. Prob.* **4**, 933–951.

MENSHIKOV, M. V. AND SIDORENKO, A. F. (1987) Coincidence of critical points for Poisson percolation models. *Theory Prob. Appl.* **32**, 603–606 (in Russian) (547–550 in translation).

PENROSE, M. D. (1991) On a continuum percolation model. *Adv. Appl. Prob.* **23**, 536–556.

REIMER, D. (1994) *Butterflies.* Preprint.

ROY, R. (1990) The RSW theorem and the equality of critical densities and the 'dual' critical densities for continuum percolation on $\mathbb{R}^2$. *Ann. Prob.* **18**, 1563–1575.

ROY, R. AND SARKAR, A. (1992) *On some questions of Hartigan in cluster analysis: an application of BK inequality for continuum percolation.* Preprint.

SARKAR, A. (1994) *Some Problems of Continuum Percolation.* PhD thesis, Indian Statistical Institute, New Delhi, India.