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A singularly perturbed model problem for numerical computation

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Abstract

In this note we introduce a model problem for the numerical solution of a two-dimensional singular perturbation problem. To combine a number of typical difficulties in a relatively simple problem, we propose to solve the linear convection–diffusion problem in the domain exterior of a circle.

We describe the analytical solution of the problem and we comment on its numerical evaluation. For small values of the parameter, asymptotic approximations of the solution are given based on expansions by Friedlander (1958) and Waechter (1968). This information gives insight into the behaviour of the solution and allows the computation of a reference solution for small values of the parameter.

Keywords: Singular perturbation; Model problem; Partial differential equation; Exact solution; Numerical evaluation

AMS classification: 35J04; 65N04; 65P05

1. Introduction

Recently much renewed interest is shown in the solution of two-dimensional singular perturbation problems, see, e.g., [4, 5]. New methods have been proposed, new strategies have been introduced to form proper grids and to handle boundary and interior layers. One inconvenience is that, in order to test the algorithms only a few simple standard test problems are available. On the one hand, there are the anisotropic diffusion and the convection diffusion problem defined on a rectangular domain. On the other hand, there are complicated real-life test problems for high Reynolds-number flows in complicated domains. There seems to be a need for a relatively simple problem on a non-trivial domain, so that the problems related with a proper meshing and with the treatment of the boundary conditions can be studied more thoroughly.

In the present paper we propose a model problem on the domain exterior to a circle. The equation is the simple linear convection diffusion equation. The domain, however, is sufficiently complex to allow an exponential boundary layer as well as parabolic interior layers. To find a proper mesh is

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not trivial at all, and many different strategies are possible. It can be a challenge to see what the proper options are.

In this paper, however, we will not study discretisation methods. The problem proposed allows an explicit solution, expressed in the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$. The mathematics used for the derivation of that solution and its asymptotic expansions is classical and the approach can essentially be found in [3, 6]. We collect these results, applied to our model problem (1), (2), and present them so that they can easily be used for numerical computation. In addition, we discuss the difficulties encountered with the numerical evaluation for small values of the parameter. In particular, we show that for small values of the parameter it is necessary to use asymptotic expansions. The analysis of these expansions also enables us to get some additional insight into the behaviour of the solution.[1]

2. The problem

The model problem we propose is to find the solution of the equation

$$\varepsilon \Delta u - u_x = 0 \tag{1}$$

for some parameter $\varepsilon > 0$. The equation is defined on the exterior of the unit circle, i.e., on the region $\Omega = \{(x, y) \mid x^2 + y^2 \geqslant 1\} \subset \mathbb{R}^2$, satisfying the boundary conditions

$$u(x, y; \varepsilon) = 1 \quad \text{for } x^2 + y^2 = 1, \tag{2a}$$

$$u(x, y; \varepsilon) \to 0 \quad \text{for } x^2 + y^2 \to \infty. \tag{2b}$$

3. The analytical solution

The domain of definition and the boundary conditions suggest a treatment in polar coordinates (r, θ). The asymmetry with respect to the polar coordinates lies in the convection term in (1) only. The asymmetry in the differential operator is removed by an integrating factor. Introducing the usual polar coordinates r, θ, and the large parameter $M = 1/(2\varepsilon)$, we define

$$v(r, \theta; M) = \mathrm{e}^{-Mx} u(x, y; \varepsilon). \tag{3}$$

Passing to polar coordinates yields the equation for $v(r, \theta; M)$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = M^2 v, \tag{4}$$

[1] A set of functions, implemented in C for the evaluation of the expressions (13), (17) and (24) in this paper can be obtained from `http:// www.cwi.nl/ ~pieth/modelproblem.html`, where also additional material related to the numerical treatment can be found. Thus, the analytic solution can be used to verify approximations that are obtained by discretisation methods. The above website will be maintained to collect results from different approaches and from different authors.

with the boundary conditions

$$v(1,\theta;M) = e^{-M\cos\theta} = e^{M\cos(\pi-\theta)}, \tag{5}$$

$$v(r,\theta;M) = o(e^{-Mr\cos\theta}) \quad \text{for } r \to \infty. \tag{6}$$

3.1. *A first solution*

Using the technique of separation of variables, we obtain the solution

$$v(r,\theta;M) = \int A(\omega)\,K_\omega(Mr)e^{i\omega\theta}\,d\omega, \tag{7}$$

where the modified Bessel function $K_\omega(Mr)$ is the solution of $r^2 f_{rr} + rf_r - (\omega^2 + M^2r^2)f = 0$, that satisfies the zero boundary condition at infinity. The path of integration and the function $A(\omega)$ have to be chosen such that the remaining boundary condition is satisfied

$$e^{-M\cos\theta} = \int A(\omega)\,K_\omega(M)e^{i\omega\theta}\,d\omega. \tag{8}$$

Later, in Section 5 we find an asymptotic expansion of the solution in the form (7). First we consider the case

$$v(r,\theta;M) = \sum_{n=-\infty}^{\infty} A_n\,K_n(Mr)e^{in\theta}. \tag{9}$$

Here, A_n has to be chosen such that the boundary condition for $r = 1$ is satisfied. Therefore, we use the generating function of the Bessel function $I_n(x)$:

$$e^{x(t+1/t)/2} = \sum_{n=-\infty}^{\infty} I_n(x)\,t^n,$$

in which we substitute $t = e^{i\theta}$ and $x = -M$ in order to derive

$$e^{-M\cos\theta} = \sum_{n=-\infty}^{\infty} I_n(-M)e^{in\theta}, \tag{10}$$

which corresponds with

$$v(1,\theta;M) = \sum_{n=-\infty}^{\infty} A_n\,K_n(M)e^{in\theta}. \tag{11}$$

From (9)–(11) directly we have

$$v(r,\theta;M) = \sum_{n=-\infty}^{\infty} \frac{(-)^n I_n(M)K_n(Mr)}{K_n(M)}e^{in\theta}. \tag{12}$$

Hence, the solution of the boundary value problem (1), (2) reads

$$u(x,y;\varepsilon) = e^{Mr\cos\theta} \sum_{n=-\infty}^{\infty} \frac{I_n(M)K_n(Mr)}{K_n(M)} e^{in(\pi-\theta)}$$

$$= e^{Mr\cos\theta}\left[\frac{I_0(M)K_0(Mr)}{K_0(M)} + 2\sum_{n=1}^{\infty}\frac{I_n(M)K_n(Mr)}{K_n(M)}\cos(n(\pi-\theta))\right]. \tag{13}$$

3.2. An alternative expression

To find an alternative expression for the function $1 - u(x,y;\varepsilon)$, we use Poisson's summation formula $\sum_{k=-\infty}^{\infty} f(k) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi imx} f(x)\,dx$ to obtain

$$1 - u(x,y;\varepsilon) = e^{Mr\cos\theta} \sum_{n=-\infty}^{\infty}\left[I_n(Mr) - \frac{I_n(M)K_n(Mr)}{K_n(M)}\right] e^{in(\pi-\theta)}$$

$$= e^{Mr\cos\theta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{I_x(Mr)K_x(M) - I_x(M)K_x(Mr)}{K_x(M)}\right] e^{ix(\pi-\theta+2\pi n)}\,dx.$$

The integrand is a single-valued function of x, regarded as a complex variable. The integral can be computed by means of Watson's transformation [6]. The singular points of the integrand are found only at the zeroes of $K_x(M)$, which are simple and purely imaginary values of x, which we denote as $x = \pm i\mu_1, \pm i\mu_2, \ldots$

In the remaining part of this section we take $0 \leqslant \theta \leqslant \pi$. For $(\pi-\theta)+2\pi n > 0$ the path of integration is chosen in the upper half-plane $\Im(x) > 0$. It follows that

$$\int_{-\infty}^{\infty}\left[\frac{I_x(Mr)K_x(M) - I_x(M)K_x(Mr)}{K_x(M)}\right] e^{ix(\pi-\theta+2\pi n)}\,dx = 2\pi i \sum_{j=1,2,\ldots} \mathrm{Res}_j,$$

with

$$\mathrm{Res}_j = \lim_{x\to i\mu_j}(x - i\mu_j)\frac{I_x(Mr)K_x(M) - I_x(M)K_x(Mr)}{K_x(M)} e^{ix|\pi-\theta+2\pi n|}$$

$$= -\frac{I_{i\mu_j}(M)K_{i\mu_j}(Mr)}{\left[\frac{\partial}{\partial x}K_x(M)\right]_{x=i\mu_j}} e^{-\mu_j|\pi-\theta+2\pi n|}.$$

A similar formula is found for $(\pi-\theta)+2\pi n < 0$. Hence,

$$1 - u(x,y;\varepsilon) = -e^{Mr\cos\theta}\sum_{m=-\infty}^{\infty} -2\pi i \sum_{j=1,2,\ldots} -i\frac{I_{i\mu_j}(M)K_{i\mu_j}(Mr)}{\left[\frac{\partial}{\partial\mu}K_{i\mu}(M)\right]_{\mu=\mu_j}} e^{-\mu_j|\pi-\theta+2\pi m|}$$

or

$$u(x,y;\varepsilon) = 1 - \mathrm{e}^{Mr\cos\theta} 2\pi \sum_{j=1,2,\dots} \frac{I_{\mathrm{i}\mu_j}(M) K_{\mathrm{i}\mu_j}(Mr)}{\left[\frac{\partial}{\partial\mu} K_{\mathrm{i}\mu}(M)\right]_{\mu=\mu_j}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mu_j|\pi-\theta+2\pi m|}$$

$$= 1 - 2\pi \mathrm{e}^{Mr\cos\theta} \sum_{j=1,2,\dots} \frac{I_{\mathrm{i}\mu_j}(M)\; K_{\mathrm{i}\mu_j}(Mr)}{\left[\frac{\partial}{\partial\mu} K_{\mathrm{i}\mu}(M)\right]_{\mu=\mu_j}} \frac{\cosh(\mu_j\theta)}{\sinh(\mu_j\pi)}.$$

The expression for the Wronskian, $I_\nu(z)K_\nu'(z) - I_\nu'(z)K_\nu(z) = -z^{-1}$, in particular yields $I_{\mathrm{i}\mu_j}(M) = -1/(MK_{\mathrm{i}\mu_j}'(M))$. Thus, we obtain the analytic solution of the boundary value problem (1), (2) in terms of the modified Bessel function K with purely imaginary argument

$$u(x,y;\varepsilon) = 1 - \frac{2\pi}{M} \mathrm{e}^{Mr\cos\theta} \sum_{j=1,2,\dots} \frac{K_{\mathrm{i}\mu_j}(Mr)}{\left[\frac{\partial}{\partial\mu} K_{\mathrm{i}\mu}(M)\right]_{\mu=\mu_j} K_{\mathrm{i}\mu_j}'(M)} \frac{\cosh(\mu_j\theta)}{\sinh(\mu_j\pi)}. \tag{14}$$

4. Comment on the numerical evaluation

If we want to evaluate $u(x,y;\varepsilon)$, based on expression (13), we have to take into account the cancellation of digits in the evaluation of the sum. For not too small values of ε, expression (13) allows us to compute u with a reasonable accuracy over the whole domain of definition. However, for smaller values of ε, cancellation destroys the solution for $x > 0$. We study this in more detail. We introduce $I_n^*(M) = I_n(M)\mathrm{e}^{-M}\sqrt{2M\pi}$ and $K_n^*(M) = K_n(M)\mathrm{e}^{M}\sqrt{2M/\pi}$. Then we see that asymptotically $I_n^*(M) \sim 1$ and $K_n^*(M) \sim 1$ for $M \to \infty$. With the new notation we write (13) as

$$u(x,y;\varepsilon) = \frac{\mathrm{e}^{M(2-r+x)}}{\sqrt{2\pi Mr}} \left\{ \frac{I_0^*(M)K_0^*(Mr)}{K_0^*(M)} + 2\sum_{n=1}^{\infty} \frac{I_n^*(M)K_n^*(Mr)}{K_n^*(M)} \cos(n(\pi-\theta)) \right\}$$

$$= \mathrm{e}^{Mx}\, v(r,\pi;M) - 4\frac{\mathrm{e}^{M(2-r+x)}}{\sqrt{2\pi Mr}} \sum_{n=1}^{\infty} \frac{I_n^*(M)K_n^*(Mr)}{K_n^*(M)} \sin^2(n(\pi-\theta)/2). \tag{15}$$

Hence, all terms in the second series in (15) are positive and no cancellation takes place for $\theta = \pi$. The cancellation will be most significant for $\theta \approx 0$.

In addition, formula (15) gives an indication of the behaviour of the solution for moderate values of ε. The factor $\exp(M(2-r+x))$ shows that the behaviour is mainly characterised by the exponential curves $2 - r + x = c$, with $c \leqslant 2$, i.e., the parabolas

$$(2-c)^2 + 2(2-c)x = y^2.$$

From expression (15) it also can be seen that, for large values of M, summation fails, at least in some part of the domain Ω, no matter the number of digits used for the computation of the separate terms. This is seen by considering a point in the neighbourhood of $(x,y) = (1,0)$. At that point (15) reduces to

$$u(x,y;\varepsilon) = \frac{\mathrm{e}^{2M}}{\sqrt{2\pi M}} \left\{ I_0^*(M) + 2\sum_{n=1}^{\infty} (-1)^n I_n^*(M) \right\}. \tag{16}$$

The factor $e^{2M}/\sqrt{M}$ strongly increases for $M\to\infty$, whereas $I_n^*(M)\to 1$ for all n. However, the sum

$$I_0^*(M)+2\sum_{k=1}^{\infty}(-1)^n I_n^*(M)$$

decreases as $e^{-2M}\sqrt{M}$ for $M\to\infty$, since $u(1,0)=1$. This shows that cancellation is inevitable and the summation given in (13) is not suitable for the evaluation of u in a neighbourhood of $(1,0)$ for large values of M.

5. Asymptotic expansions

Using asymptotic expansions found in [3] for $K_{i\mu}(z)$ and $K'_{i\mu}(z)$ for large real $\mu\to\infty$ and $[\partial K_x(M)/\partial x]_{x=i\mu_j}$ we obtain from expression (14)

$$u(x,y;\varepsilon)\approx 1-2^{1/6}M^{1/3}e^{Mr\cos\theta}\sum_{j=1,2,\ldots}\frac{\zeta_j^{1/4}\,\mathrm{Ai}(\zeta_j)e^{\pi\mu_j/2}}{(M^2r^2-\mu_j^2)^{1/4}(\mathrm{Ai}'(-\alpha_j))^2}\frac{\cosh(\mu_j\theta)}{\sinh(\mu_j\pi)},\tag{17}$$

where α_j, μ_j and ζ_j are determined by

α_j is the jth zero of the Airy function $\mathrm{Ai}(-z)$,

$\mu_j=M+\alpha_j(M/2)^{1/3}+\mathcal{O}(M^{-1/3})$,

if $Mr\geqslant\mu_j$, then $\frac{2}{3}(+\zeta)_j^{3/2}=+\sqrt{M^2r^2-\mu_j^2}-\mu_j\,\arccos(\mu_j/Mr)$,

if $Mr\leqslant\mu_j$, then $\frac{2}{3}(-\zeta)_j^{3/2}=-\sqrt{\mu_j^2-M^2r^2}+\mu_j\,\mathrm{arccosh}(\mu_j/Mr)$.

Since this formula can be used to compute $u(x,y;\varepsilon)$ for values of x, y and ε, in the wake of the circle where (13) fails, it is supplementary for numerical purposes. For the analysis of the behaviour of the form (17), and in order to obtain more convenient expressions to evaluate, we distinguish the following two special cases:
(a) $r-1\approx M^{-2/3}$,
(b) $r-1\gg\mathcal{O}(M^{-2/3})$.
After these expansions for the solution inside the wake, we give an additional asymptotic expansion that holds outside the wake.

5.1. *In the wake, near the circle*

First we consider the region near the boundary, $r-1=\mathcal{O}(M^{-2/3})$. In this case we write $Mr=M+RM^{1/3}$, with $R=\mathcal{O}(1)$. Now we use the expansion [6, Eq. (2.32)] of $K_{i\mu_j}(Mr)$, for $M\to\infty$ with

$$Mr=M+RM^{1/3},$$

which reads

$$K_{i\mu_j}(Mr)\approx\pi\left(\frac{2}{M}\right)^{1/3}e^{-\pi\mu_j/2}\mathrm{Ai}(2^{1/3}R-\alpha_j),$$

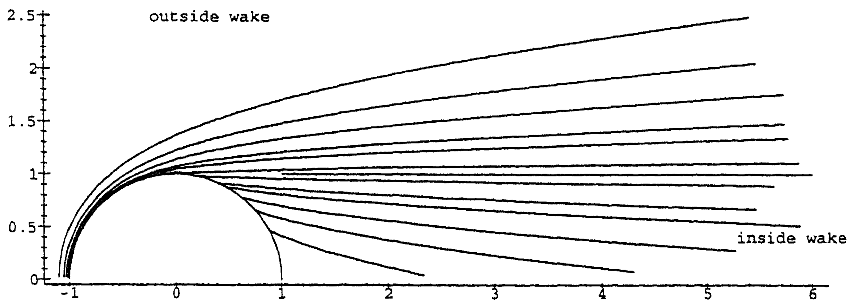

Fig. 1. Structure of the boundary and interior layer. The structure of the layers as given by the first exponential factors. In the wake $\sqrt{(r^2-1)} + r\cos\theta + |\theta| - \pi/2 - \arccos(1/r) = -\varepsilon$. Outside the wake $r\cos\beta - \cos\alpha - \cos\alpha - r\cos\theta = \sqrt{(x-\cos\alpha)^2 + (y-\sin\alpha)^2} + (\cos\alpha - x) = -\varepsilon$. The lines are lines of constant exponential factors, for the values of ε: 0.2, 0.1, 0.05, 0.02, 0.01 and 0.001.

and, together with the Wronskian relation, application to (14) yields

$$u(x,y;\varepsilon) \approx 1 - \mathrm{e}^{Mr\cos\theta} \sum_{j=1,2,\ldots} \frac{\mathrm{Ai}(2^{1/3}R - \alpha_j)\,\mathrm{e}^{\mu_j(\theta-\pi/2)}}{(\mathrm{Ai}'(-\alpha_j))^2}. \tag{18}$$

This expansion holds for $0 < \theta_0 < \theta < \theta_1 < \pi/2$.

A special case is obtained for $r-1 = \mathcal{O}(\varepsilon)$, i.e., $R = \mathcal{O}(M^{-1/3})$, when we can make a Taylor expansion of (18) in order to obtain

$$u(x,y;\varepsilon) \approx 1 - 2^{1/3}M^{2/3}(r-1)\mathrm{e}^{Mx} \sum_{j=1,2,\ldots} \frac{\mathrm{e}^{\mu_j(\theta-\pi/2)}}{\mathrm{Ai}'(-\alpha_j)}. \tag{19}$$

5.2. In the wake, away from the circle

Now we consider the case $r-1 \gg \mathcal{O}(M^{-2/3})$. Then $Mr > M + \mathcal{O}(M^{1/3})$ and so $Mr > \mu_j$. This allows us to make use of the more specific asymptotic expansion, derived from [2, 3] and [1, Eqs. (9.7.8)], of $K_{\mathrm{i}\mu}(Mr)$, given by

$$K_{\mathrm{i}\mu}(Mr) \approx \sqrt{\frac{\pi}{2M}}\,\frac{1}{(r^2-1)^{1/4}} \exp\left(-\frac{\pi\mu}{2} - \mu\left[\sqrt{r^2-1} - \arccos\left(\frac{1}{r}\right)\right]\right),$$

by which we obtain from (14)

$$u(x,y;\varepsilon) \approx 1 - \frac{\mathrm{e}^{M(r\cos\theta - \sqrt{r^2-1})}}{2^{5/6}M^{1/6}\pi^{1/2}(r^2-1)^{1/4}} \sum_{j=1,2,\ldots} \frac{\mathrm{e}^{\mu_j(\theta - \frac{\pi}{2} + \arccos(\frac{1}{r}))}}{(\mathrm{Ai}'(-\alpha_j))^2}. \tag{20}$$

This series converges for $\theta - \frac{1}{2}\pi + \arccos(1/r) < 0 \Leftrightarrow \frac{1}{2}\pi - \theta > \arccos(1/r) \Leftrightarrow \sin(\theta) < 1/r$, i.e., in the wake of the circle, cf. Fig. 1.

Thus the expansion (20) describes the behaviour of $u(x,y;\varepsilon)$ in the region

$$\{(x,y)\,|\,x^2 + y^2 > 1 + \varepsilon_0,\ x > 0,\ |y| < 1\}$$

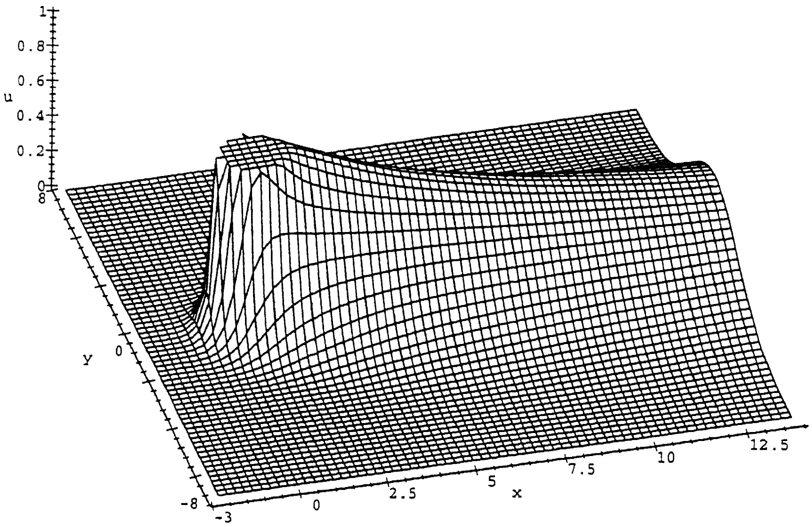

Fig. 2. Solution for $\varepsilon = 0.2$.

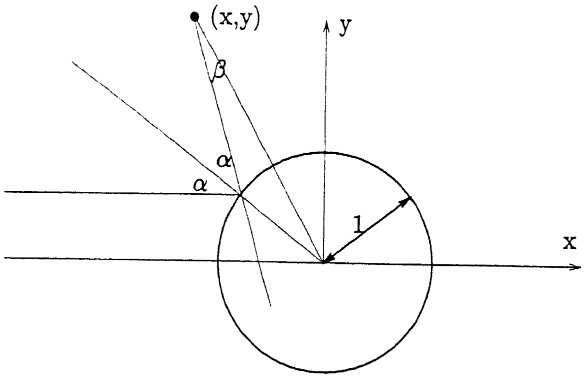

Fig. 3. The meaning of α and β in the asymptotic expression outside the wake, $\alpha < \frac{1}{2}\pi$. (Waechter).

with $\varepsilon_0 > \mathcal{O}(M^{-3/2})$, i.e., the region where for fixed (x, y) we have

$$\lim_{\varepsilon \searrow 0} u(x, y; \varepsilon) = 1.$$

5.3. *Outside the wake*

In order to obtain an asymptotic expansion that is valid in the major part of the domain not covered by the expansions in the Sections 5.1 and 5.2, we use an integral representation of the solution $u(x, y; \varepsilon)$ similar to (7). We consider the following integral, which satisfies the differential equation (1),

$$w(r,\theta;M) \equiv \int_c \frac{I_\nu(M)K_\nu(Mr)}{K_\nu(M)} e^{i\nu\theta} \, d\nu. \tag{21}$$

For the modified Bessel functions the asymptotic expansions for large orders [1, Eqs. (9.7.7)] are substituted. In these expansions we take $\nu z = M$ for $I_\nu(M)$ and $K_\nu(M)$ and we take $\nu z = Mr$ for $K_\nu(Mr)$. This yields, with μ defined by $\nu = M\mu$

$$w(r,\theta;M) \equiv \int_c f(\mu) \exp(Mg(\mu)) \, d\mu, \tag{22}$$

where $f(\mu) = M/\sqrt{2\pi M(\mu^2 + r^2)}$ and

$$g(\mu) = i\mu\theta + 2\sqrt{\mu^2+1} - \sqrt{\mu^2+r^2} - 2\mu \operatorname{arcsinh}(\mu) + \mu \operatorname{arcsinh}(\mu/r).$$

This integral is easily approximated by means of the saddle-point method. The saddle point is determined by $i\theta - 2\operatorname{arcsinh}(\mu) + \operatorname{arcsinh}(\mu/r) = 0$.

With α and β defined by $\sin\alpha = r\sin\beta$ and $2\alpha + \theta = \pi + \beta$, the saddlepoint is found to be $\mu = -i\sin\alpha$. The path of steepest decent of $g(\mu)$ is given by $\Im\mu = -\sin\alpha$. Integration (cf. [6]) yields the asymptotic expansion for large M

$$w(r,\theta;M) \approx \sqrt{\frac{\cos\alpha}{2r\cos\beta - \cos\alpha}} e^{M(2\cos\alpha - r\cos\beta)}, \tag{23}$$

which satisfies the boundary condition, $w(1,\theta;M) = \exp(M\cos\theta)$, so that

$$u(x,y;\varepsilon) \approx \sqrt{\frac{\cos\alpha}{2r\cos\beta - \cos\alpha}} e^{-M(r\cos\beta - r\cos\theta - 2\cos\alpha)}. \tag{24}$$

This expansion holds for $|\alpha| < \pi/2 - \delta$ for some $\delta > 0$, i.e., in the region outside the wake.

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