

The gaps between the sizes of large clusters in 2D critical percolation

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Abstract

Consider critical bond percolation on a large $2n \times 2n$ box on the square lattice. It is well-known that the size (i.e. number of vertices) of the largest open cluster is, with high probability, of order $n^2\pi(n)$, where $\pi(n)$ denotes the probability that there is an open path from the center to the boundary of the box. The same result holds for the second-largest cluster, the third largest cluster etcetera.

Járai showed that the differences between the sizes of these clusters is, with high probability, at least of order $\sqrt{n^2\pi(n)}$. Although this bound was enough for his applications (to incipient infinite clusters), he believed, but had no proof, that the differences are in fact of the same order as the cluster sizes themselves, i.e. $n^2\pi(n)$. Our main result is a proof that this is indeed the case.

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1 Introduction and statement of main results

For general background on percolation we refer to [10] and [3]. We consider bond percolation on the square lattice with parameter p equal to its critical value $p_c = 1/2$.

Let $\Lambda_n = [-n, n]^2 \cap \mathbb{Z}^2$ be the $2n \times 2n$ box centered at $O = (0, 0)$ and let $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$ be the (inner) boundary of the box. For each vertex $v \in \mathbb{Z}^2$, we write $\Lambda_n(v) = \Lambda_n + v$. Further, the open cluster in Λ_n of the vertex v is denoted by $\mathcal{C}_n(v)$. More precisely,

$$\mathcal{C}_n(v) := \{u \in \Lambda_n : u \leftrightarrow v \text{ inside } \Lambda_n\},$$

where ' $u \leftrightarrow v$ inside Λ_n ' means that there is an open path from u to v of which all vertices are in Λ_n . We write $\pi(n)$ for the probability $\mathbb{P}(O \leftrightarrow \partial\Lambda_n)$, the probability that there is an open path from O to $\partial\Lambda_n$. Further, we write

$$s(n) := n^2\pi(n). \tag{1.1}$$

By the *size* of a cluster we mean the number of vertices in the cluster. Let, for $i = 1, 2, \dots$, $\mathcal{C}_n^{(i)}$ denote the i -th largest open cluster in Λ_n , and let $|\mathcal{C}_n^{(i)}|$ denote its size. (If two clusters have the same size, we order them in some deterministic way).

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In [4] it was proved that $|\mathcal{C}_n^{(1)}|$ is of order $s(n)$. In the later paper [5] by the same authors it is shown that also $|\mathcal{C}_n^{(2)}|, |\mathcal{C}_n^{(3)}|$ etcetera are of order $s(n)$. They also proved an extension of this result for the case where the parameter p is not equal but close to p_c .

It was shown by Járai (see Section 1, Proposition 1, and its proof in section 3.1 in [11]) that for each i the difference $|\mathcal{C}_n^{(i)}| - |\mathcal{C}_n^{(i+1)}| \rightarrow \infty$ in probability as $n \rightarrow \infty$. In fact he showed that this difference is at least of order $\sqrt{s(n)}$. He suggested that it should be of order $s(n)$, but did not have a proof. In this paper we show that his conjecture is correct. We became interested in such problems through our investigation of frozen-percolation processes. Our main theorem is as follows.

Theorem 1.1. *For all $k \in \mathbb{N}, \delta > 0$, there exist $\varepsilon > 0, N \in \mathbb{N}$ such that for all $n \geq N$:*

$$\mathbb{P} \left(\exists i \leq k - 1 : |\mathcal{C}_n^{(i)}| - |\mathcal{C}_n^{(i+1)}| \leq \varepsilon s(n) \right) < \delta. \tag{1.2}$$

Remarks: (i) *The analog of Theorem 1.1 can be proved for site and bond percolation on other common two dimensional lattices, e.g. site percolation on the square or the triangular lattice. In this latter model (site percolation on the triangular lattice) one of the last steps of the proof can be made a little bit shorter (see the Remark below the proof of Proposition 3.2).*

(ii) *The proof, which is given in Section 3, follows the main line of Járai’s proof of the weaker bound: We divide the box Λ_n in boxes of smaller length (denoted by $2t$), and condition on the configuration outside certain open circuits in these smaller boxes. Conditioned on this information, the ‘contributions’ (to the sizes of certain open clusters) from the interiors of these circuits are independent random variables. This leads to a problem concerning the concentration function of a sum of independent random variables, to which a general (‘classical’) theorem is applied. The main difference with Járai’s arguments is that we take t proportional to n , with a proportionality factor chosen as a suitable function of the ‘parameters’ k and δ in the theorem. This makes the arguments more powerful (and also somewhat more complicated). Moreover, the theorem on concentration functions we used (see Theorem 2.6 below) is somewhat stronger than the one used in Járai’s arguments.*

Furthermore, with essentially the same argument we can show that the probability that there exists a cluster with size in a given interval of length $\varepsilon s(n)$ goes to zero as $\varepsilon \rightarrow 0$ uniformly in n :

Theorem 1.2. *For all $x, \delta > 0$, there exists an $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$:*

$$\mathbb{P} (\exists u \in \Lambda_n : xs(n) < |\mathcal{C}_n(u)| < (x + \varepsilon)s(n) < \delta). \tag{1.3}$$

This last theorem is in some sense complementary to the result in an earlier paper [1], where we proved that, for any interval (a, b) , the probability that $|\mathcal{C}_n^{(1)}|/s(n) \in (a, b)$ is bounded away from zero as $n \rightarrow \infty$.

2 Notation and Preliminaries

2.1 Preliminaries

First we need some more notation. For a cluster $\mathcal{C}_n(u)$ we define its (left-right) diameter by

$$\text{diam}(\mathcal{C}_n(u)) = \max_{v, w \in \mathcal{C}_n(u)} |v_1 - w_1|.$$

For a box Λ_n we define the spanning cluster by

$$SC_n = \{u \in \Lambda_n : u \leftrightarrow L(\Lambda_n) \text{ and } u \leftrightarrow R(\Lambda_n)\}, \tag{2.1}$$

where $L(\Lambda_n) = \{-n\} \times [-n, n] \cap \mathbb{Z}^2$ and $R(\Lambda_n) = \{n\} \times [-n, n] \cap \mathbb{Z}^2$. We use the notation $A_{m,n}$ for the annulus $\Lambda_n \setminus \Lambda_m$ and, for a vertex $v \in \mathbb{Z}^2$, the notation $A_{m,n}(v)$ for $A_{m,n} + v$.

In our proof of Theorem 1.1 and 1.2 we will use the following results from the literature, Theorems 2.1 - 2.6 below. The first one is well known, see for example [5], [2].

Theorem 2.1. ([5],[2]) *There exist constants $c_1, c_2, c_3 > 0$, such that for all $m \leq n$:*

$$c_1 \left(\frac{n}{m}\right)^{c_2} \leq \frac{\pi(m)}{\pi(n)} \leq c_3 \left(\frac{n}{m}\right)^{\frac{1}{2}}.$$

As we already mentioned in the introduction, the largest clusters in Λ_n are of order $s(n)$. This is stated in the following result.

Theorem 2.2. ([5] Thm. 3.1(i), 3.3, 3.6) *For all $i \in \mathbb{N}$,*

$$\mathbb{E}[|\mathcal{C}_n^{(i)}|] \asymp s(n), \tag{2.2}$$

and,

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left(\varepsilon < \frac{|\mathcal{C}_n^{(i)}|}{\mathbb{E}[|\mathcal{C}_n^{(i)}|]} < \frac{1}{\varepsilon} \right) \rightarrow 1 \quad \text{as } \varepsilon \rightarrow 0. \tag{2.3}$$

In an earlier paper Borgs, Chayes, Kesten and Spencer showed exponential decay for the probability that there exists a cluster with large volume, but a small diameter:

Theorem 2.3. ([4] Remark (xiii)) *There exist $C_1, C_2 > 0$ such that for all $x > 0$, $\alpha \in (0, 1]$ and $n \geq 4/\alpha$ we have*

$$\mathbb{P}(\exists u \in \Lambda_n : |\mathcal{C}_n(u)| \geq xs(n); \text{diam}(\mathcal{C}_n(u)) \leq \alpha n) \leq C_1 \alpha^{-2} \exp(-C_2 x/\alpha). \tag{2.4}$$

An easy consequence of Theorems 2.2 and 2.3 is the following.

Corollary 2.4. *Let $k \in \mathbb{N}$. For all $\delta > 0$ there exist $\alpha > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$:*

$$\mathbb{P}(\exists i \leq k : \text{diam}(\mathcal{C}_n^{(i)}) < \alpha n) < \delta. \tag{2.5}$$

In [11] a version of Theorem 2.2 for the spanning cluster is given:

Theorem 2.5. ([11] Thm. 8)

$$\mathbb{E}[|SC_n|] \asymp s(n); \tag{2.6}$$

moreover,

$$\liminf_{\varepsilon \rightarrow 0} \inf_{n \in \mathbb{N}} \mathbb{P} \left(\varepsilon < \frac{|SC_n|}{\mathbb{E}[|SC_n|]} < \frac{1}{\varepsilon} \mid SC_n \neq \emptyset \right) = 1. \tag{2.7}$$

In the proof of our main theorem we use the following inequality concerning the concentration function $Q(X, \lambda)$ of a random variable X , which is defined by

$$Q(X, \lambda) = \sup_{x \in \mathbb{R}} \mathbb{P}(x \leq X \leq x + \lambda), \tag{2.8}$$

for $\lambda > 0$.

Theorem 2.6. ([6]; [7] (B)) *Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of independent random variables, and $0 < \tilde{\lambda} \leq \lambda$. Let $a > 0$ and let $(b_k)_{k \in \mathbb{N}}$ be a sequence of real numbers such that, for all $k \in \mathbb{N}$,*

$$\mathbb{P}(X_k \leq b_k - \frac{\tilde{\lambda}}{2}) \geq a, \quad \mathbb{P}(X_k \geq b_k + \frac{\tilde{\lambda}}{2}) \geq a.$$

There exists a universal constant $C > 0$ such that, for all $m \in \mathbb{N}$

$$Q(S_m, \lambda) \leq \frac{C\lambda}{\tilde{\lambda}\sqrt{ma}},$$

where $S_m = X_1 + X_2 + \dots + X_m$.

2.2 Large clusters contain many good boxes

In the proof of our main theorem we need the following lemma, which is essentially already in [11]. First some definitions. Recall the notation $A_{m,n}$ in the beginning of Section 2.. Let $t \in 3\mathbb{N}$. (Later we will choose a suitable value for t). For any $i, j \in \mathbb{Z}$ we say that the box $\Lambda_t(2ti, 2tj)$ is ‘good’ if there is an open circuit in the annulus $A_{\frac{2}{3}t,t}(2ti, 2tj)$; in that case we denote the widest open circuit in that annulus by $\gamma_{i,j}$. (Although $\gamma_{i,j}$ depends on t , we omit that parameter from the notation). For each vertex u we denote by $G_t(\mathcal{C}_n(u))$ the set of good boxes in Λ_n of which the corresponding $\gamma_{i,j}$ is contained in the open cluster of u . More precisely,

$$G_t(\mathcal{C}_n(u)) = \{(i, j) : \Lambda_t(2ti, 2tj) \subset \Lambda_n \text{ is good ; } \gamma_{i,j} \subset \mathcal{C}_n(u)\}. \tag{2.9}$$

Lemma 2.7. *Let $\alpha > 0$. For any $\delta, \beta > 0$ there exist $\eta > 0$ and $N \in \mathbb{N}$ such that, for all $n \geq N$ and $t \in (0, \eta n) \cap 3\mathbb{N}$*

$$\mathbb{P}(\exists u \in \Lambda_n : \text{diam}(\mathcal{C}_n(u)) \geq \alpha n; |G_t(\mathcal{C}_n(u))| < \beta) < \delta. \tag{2.10}$$

Járai proved a somewhat stronger statement (see (3.15) in [11] and Proposition 3.6 in [9]), but we only need this weaker statement and give a (short) proof.

Proof. The C_i ’s in this proof denote universal constants larger than 0. Their existence is important but their precise value does not matter for the proof. First note that we can cover the box Λ_n by at most

$$\frac{C_1}{\alpha^2} \tag{2.11}$$

rectangles of width $\frac{1}{4}\alpha n$ and length $\frac{1}{2}\alpha n$, such that every cluster with diameter at least αn crosses at least one of these rectangles in the easy direction. We consider one such rectangle, namely $Q_0 := [0, \frac{1}{4}\alpha n] \times [0, \frac{1}{2}\alpha n]$. (The argument for each of the other rectangles in Λ_n is, a rotated, reflected and/or translated version of that for Q_0 .) By RSW and the BK inequality we have that the probability that there are more than C_2 disjoint horizontal open crossings of Q_0 is less than

$$\frac{\delta}{2} \frac{\alpha^2}{C_1}. \tag{2.12}$$

Let R_l denote the l -th lowest open crossing of Q_0 . We claim that there exist $\eta \in (0, 1/2)$ and $N \in \mathbb{N}$ such that, for any $n \geq N$, deterministic crossing r_0 of Q_0 , and $t \in (0, \eta n)$,

$$\mathbb{P}(|G_t(\mathcal{C}_n(r_0))| < \beta \mid R_l = r_0) < \frac{\delta \alpha^2}{2C_1 C_2}, \tag{2.13}$$

where $\mathcal{C}_n(r_0)$ denotes the open cluster which contains the crossing r_0 . From this claim we get (see (2.11) and (2.12)) that the l.h.s. of (2.10) is less than

$$\frac{C_1}{\alpha^2} \left(\frac{\delta}{2} \frac{\alpha^2}{C_1} + C_2 \frac{\delta \alpha^2}{2C_1 C_2} \right) = \delta,$$

and the lemma follows.

It remains to prove the claim concerning the inequality (2.13): The objects defined below involve a parameter i . We will always assume that i is such that the corresponding object is contained in the rectangle $[0, \frac{1}{4}\alpha n] \times [0, \alpha n]$ (Note that this rectangle is contained in Λ_n). Consider all rectangles of the form $A(i) := [2ti - t, 2ti + t] \times [0, \alpha n]$. For every i we let $j(i)$ be the smallest integer j for which the box $\Lambda_t(2ti, 2tj)$ is located above

r_0 . Let $E(i)$ be the event that $\Lambda_t(2ti, 2tj(i))$ is good and $\gamma_{i,j(i)}$ is connected with r_0 inside $A(i)$. The events $E(i)$ are conditionally independent of each other (where we condition on the event $R_l = r_0$), and, by RSW, each has probability larger than C_3 . Hence, when η is small enough (that is, n/t and thus the number of events $E(i)$ is large enough), the probability that at most β of the $E(i)$'s occur is smaller than the r.h.s. of (2.13). This proves the claim and completes the proof of Lemma 2.7. \square

Remark: *In one of the steps of J arai's proof (see the lines below our statement of Lemma 2.7), he shows that with large probability the l -th lowest crossing in Q_0 is contained in $[0, \frac{1}{4}\alpha n] \times [0, \frac{1}{2}\alpha n(1 - a)]$, for some constant $a < 1$. He used this to guarantee that the good boxes obtained are inside Q_0 . However, as the above arguments show, this (and hence the introduction of the extra constant a) is not needed in our argument.*

3 Proof of Theorems 1.1 and 1.2

3.1 Gaps between sizes of clusters with large diameter

The following lemma will be used later to show that the conditions for Theorem 2.6 are satisfied in our situation. First we define, for each circuit γ , $\text{int}(\gamma)$ as the interior of γ (that is, the bounded connected component of $\mathbb{R}^2 \setminus \gamma$, where γ is seen as subset of the plane), and

$$X_\gamma := |\{u \in \text{int}(\gamma) \cap \mathbb{Z}^2 : u \leftrightarrow \gamma\}|. \tag{3.1}$$

Lemma 3.1. *There exist universal constants $\chi, \xi > 0$ and, for all $t \in 3\mathbb{N}$ and for any circuit γ in $A_{\frac{2}{3}t,t}$, a value $c(t, \gamma) \geq 0$ such that*

$$\mathbb{P}(X_\gamma \leq c(t, \gamma)) \geq \chi; \tag{3.2}$$

$$\mathbb{P}(X_\gamma \geq c(t, \gamma) + \xi s(t)) \geq \chi. \tag{3.3}$$

Proof. Fix some $a \in (0, \frac{1}{2})$. Define the random variable $Z = |\{u \in A_{\frac{1}{3}t,t} \cap \text{int}(\gamma) : u \leftrightarrow \gamma\}|$. Let $c(t, \gamma)$ be defined by

$$c(t, \gamma) = \min\{z \in \mathbb{N} \cup \{0\} : \mathbb{P}(Z \leq z) > a\}.$$

By RSW, the probability that there is a closed dual circuit in $A_{\frac{1}{3}t, \frac{2}{3}t}$ is larger than some universal constant $C_1 > 0$. Moreover, if there is such a circuit, then $X_\gamma = Z$. Hence, $\mathbb{P}(X_\gamma \leq c(t, \gamma))$ is larger than or equal to the probability that there is such a circuit and that $Z \leq c(t, \gamma)$. By the above and FKG this is larger than $C_1 a$.

To prove (3.3) recall the notation (2.1) and define the random variable $Y = |SC_{\frac{1}{3}t}|$. Theorem 2.5 implies that there exist constants $C_2, \xi > 0$ such that, for all t , we have $\mathbb{P}(Y \geq \xi s(t)) > C_2$. Let E be the event that there is an open crossing in $\Lambda_{\frac{1}{3}t}$ from top to bottom and that this crossing is connected to γ . On E we have that $X_\gamma \geq Z + Y$, since the spanning cluster is connected to γ . By RSW, $\mathbb{P}(E)$ is larger than some universal constant C_3 . Hence

$$\mathbb{P}(X_\gamma \geq c(t, \gamma) + \xi s(t)) \geq \mathbb{P}(E; Z \geq c(t, \gamma); Y \geq \xi s(t)) \geq C_3(1 - a)C_2, \tag{3.4}$$

where the last inequality uses FKG. This proves Lemma 3.1. \square

Now we prove the following proposition, from which, as we show in the next subsection, Theorem 1.1 follows almost immediately. The set of clusters with diameter larger than αn is denoted by $\mathcal{C}_{\alpha,n}$. More precisely,

$$\mathcal{C}_{\alpha,n} = \{\mathcal{C}_n(u) : u \in \Lambda_n; \text{diam}(\mathcal{C}_n(u)) \geq \alpha n\}. \tag{3.5}$$

Proposition 3.2. *For all $\alpha, \delta > 0$ there exist $\varepsilon = \varepsilon(\alpha, \delta) > 0, N = N(\alpha, \delta) \in \mathbb{N}$ such that, for all $n \geq N$*

$$\mathbb{P}(\exists \text{ distinct } \mathcal{D}_1, \mathcal{D}_2 \in \mathbf{C}_{\alpha, n} : \|\mathcal{D}_1\| - \|\mathcal{D}_2\| < \varepsilon s(n)) < \delta. \tag{3.6}$$

Proof. Let $\alpha, \delta > 0$ be given. By a standard RSW argument, the probability that $|\mathbf{C}_{\alpha, n}| \geq 1$ is smaller than some constant < 1 which depends only on α . Hence, by the BK inequality we can choose a $\kappa = \kappa(\alpha, \delta) \in \mathbb{N}$ such that, for all n :

$$\mathbb{P}(|\mathbf{C}_{\alpha, n}| > \kappa) < \frac{\delta}{3}. \tag{3.7}$$

Let ξ and χ as in Lemma 3.1 and C as in Theorem 2.6. Take β so large that

$$\frac{\xi}{2} \leq \frac{\delta \xi \sqrt{\chi}}{6C\binom{\kappa}{2}} \cdot \sqrt{\beta}. \tag{3.8}$$

(For the time being, this property of β will play no role; it will become essential at (3.16) for a suitable choice of ε). Let η be as in Lemma 2.7 (but with $\delta/3$ instead of δ in (2.10)). It is clear from that lemma that without loss of generality we may assume that

$$\eta < \frac{\alpha}{2}. \tag{3.9}$$

For each n we take $t = t(n) = 3\lfloor \frac{1}{3}\eta n \rfloor$. Hence, by the above choice of η we have, for all sufficiently large n ,

$$\mathbb{P}(\exists \mathcal{D} \in \mathbf{C}_{\alpha, n} : |G_t(\mathcal{D})| < \beta) < \frac{\delta}{3}. \tag{3.10}$$

Denote by W the event that there are at most κ clusters in Λ_n with diameter at least αn and all these clusters have at least β good boxes. Note that the complement of W is the union of the event in the l.h.s. of (3.7) and the event in the l.h.s. of (3.10), and hence has probability smaller than $2\delta/3$. Therefore, to prove Proposition 3.2 it is sufficient to show that there exists $\varepsilon > 0$ such that for all sufficiently large n ,

$$\mathbb{P}(W \cap \{\exists \text{ distinct } \mathcal{D}_1, \mathcal{D}_2 \in \mathbf{C}_{\alpha, n} : \|\mathcal{D}_1\| - \|\mathcal{D}_2\| < \varepsilon s(n)\}) < \frac{\delta}{3}. \tag{3.11}$$

We define (compare with (2.9))

$$G_{t, n} = \{(i, j) \in \mathbb{Z}^2 : \Lambda_t(2ti, 2tj) \subset \Lambda_n \text{ is good}\}.$$

Recall that we denote the outermost open circuit in $A_{\frac{2}{3}t, t}(2ti, 2tj)$ (if it exists) by $\gamma_{i, j}$. Denote the configuration on the edges in the set

$$H := [-n, n]^2 \setminus \left(\bigcup_{(i, j) \in G_{t, n}} \text{int}(\gamma_{i, j}) \right) \tag{3.12}$$

by ω_H .

To estimate the l.h.s. of (3.11) we condition first on the $\gamma_{i, j}$'s and the configuration ω_H . Therefore, let \tilde{G} be an arbitrary set of vertices (i, j) with $\Lambda_t(2ti, 2tj) \subset \Lambda_n$, and let, for each $(i, j) \in \tilde{G}$, $\tilde{\gamma}_{i, j}$ be a (deterministic) circuit in $A_{\frac{2}{3}t, t}(2ti, 2tj)$. Let \tilde{H} be the analog of (3.12), with γ replaced by $\tilde{\gamma}$ and let $\tilde{\omega}$ be a configuration on \tilde{H} . We will consider the conditional distribution $\mathbb{P}(\cdot | G_{t, n} = \tilde{G}; \gamma_{i, j} = \tilde{\gamma}_{i, j} \forall (i, j) \in G_{t, n}; \omega_H = \tilde{\omega})$. Note that the information we condition on allows us to distinguish all the clusters in $\mathbf{C}_{\alpha, n}$ and their good boxes. (Here we used that (3.9) implies that no cluster of $\mathbf{C}_{\alpha, n}$ fits entirely in the interior of one of the above mentioned $\gamma_{i, j}$'s). We may assume that $\tilde{\omega}$ is such that W

holds. Let $\mathcal{D}_1, \mathcal{D}_2$ be two open clusters in $\mathbf{C}_{\alpha, n}$ for the configuration $\tilde{\omega}$. Their sizes can be decomposed as follows:

$$\begin{aligned}
 |\mathcal{D}_1| &= a_1 + \sum_{(i,j) \in G_t(\mathcal{D}_1)} X_{\tilde{\gamma}_{i,j}}, \\
 |\mathcal{D}_2| &= a_2 + \sum_{(i,j) \in G_t(\mathcal{D}_2)} X_{\tilde{\gamma}_{i,j}},
 \end{aligned}
 \tag{3.13}$$

where $a_1 = |\mathcal{D}_1 \cap H|$ and $a_2 = |\mathcal{D}_2 \cap H|$, and the X variables are as defined in (3.1). The terms a_1 and a_2 can be considered as ‘fixed’ (namely, determined by $\tilde{\omega}$), and the $X_{\tilde{\gamma}_{i,j}}$ ’s as independent random variables. Therefore, and because there are at most $\binom{\kappa}{2}$ choices for \mathcal{D}_1 and \mathcal{D}_2 , to prove (3.11) it is enough to show that there exists $\varepsilon > 0$, which does not depend on $a_1, a_2, G_t(\mathcal{D}_1), G_t(\mathcal{D}_2)$ and the $\tilde{\gamma}_{i,j}$ ’s, such that

$$\mathbb{P} \left(\left| \left(a_1 + \sum_{(i,j) \in G_t(\mathcal{D}_1)} X_{\tilde{\gamma}_{i,j}} \right) - \left(a_2 + \sum_{(i,j) \in G_t(\mathcal{D}_2)} X_{\tilde{\gamma}_{i,j}} \right) \right| < \varepsilon s(n) \right) < \frac{\delta}{3 \binom{\kappa}{2}}, \tag{3.14}$$

On the event W we have that $|G_t(\mathcal{D}_1)| \geq \beta$. So we can mark β of the good boxes in $G_t(\mathcal{D}_1)$, and condition (in addition to the earlier mentioned information) also on the values of $X_{\gamma_{i,j}}$ for the remaining good boxes in $G_t(\mathcal{D}_1)$ and all the good boxes in $G_t(\mathcal{D}_2)$. Hence it is enough to show that there exists an $\varepsilon > 0$ such that

$$\mathbb{P} \left(\left| \sum_{m=1}^{\beta} X_{\gamma_m} - b \right| < \varepsilon s(n) \right) < \frac{\delta}{3 \binom{\kappa}{2}}, \tag{3.15}$$

uniformly in $b \in \mathbb{N}$ and $\gamma_1, \dots, \gamma_\beta$, where the γ_m ’s are circuits in distinct annuli $A_{\frac{2}{3}t, t}(2ti, 2tj)$. We will do this by application of Theorem 2.6, where Lemma 3.1 (and our choice (3.8) for β) enables a suitable application of that theorem:

From (3.8) it follows immediately that for all n there is an $\varepsilon(n)$ such that

$$\frac{\xi}{2} \cdot \frac{s(t)}{s(n)} \leq \varepsilon(n) \leq \frac{\delta \xi \sqrt{\chi}}{6C \binom{\kappa}{2}} \cdot \sqrt{\beta} \cdot \frac{s(t)}{s(n)}. \tag{3.16}$$

By the lower bound in Theorem 2.1, the l.h.s. of (3.16) is bounded away from 0, uniformly in n . Hence, $\inf_n \varepsilon(n) > 0$. Take ε equal to this infimum. We get (with Q as in (2.8)),

$$\begin{aligned}
 \mathbb{P} \left(\left| \sum_{m=1}^{\beta} X_{\gamma_m} - b \right| < \varepsilon s(n) \right) &\leq Q \left(\sum_{m=1}^{\beta} X_{\gamma_m}, 2\varepsilon s(n) \right) \\
 &\leq Q \left(\sum_{m=1}^{\beta} X_{\gamma_m}, 2\varepsilon(n) s(n) \right) \\
 &\leq \frac{2C}{\xi \sqrt{\beta \chi}} \cdot \frac{s(n)}{s(t)} \cdot \varepsilon(n),
 \end{aligned}
 \tag{3.17}$$

where in the last inequality we used Lemma 3.1 and applied Theorem 2.6 (with $\tilde{\lambda} = \xi s(t)$, $a = \chi$, $m = \beta$ and $\lambda = 2\varepsilon(n) s(n)$). Note that the condition $\tilde{\lambda} \leq \lambda$ in that theorem is satisfied because $\xi s(t) \leq 2\varepsilon(n) s(n)$ by the first inequality in (3.16).

Now, by the second inequality of (3.16) we have that the r.h.s. of (3.17) is at most $\frac{\delta}{3 \binom{\kappa}{2}}$. This shows (3.15) and completes the proof of Proposition 3.2. \square

Remark In the case of site percolation on the triangular lattice we can, in equation (3.16) and the line above it, skip the introduction of $\varepsilon(n)$, and choose ε itself such that it is (for all sufficiently large n) between the l.h.s. and r.h.s. of (3.16). For that percolation model such ε exists because (see [8], Proposition 4.9 and the last part of the proof of Theorem 5.1 in that paper) $\pi(t)/\pi(n)$, and hence $s(t)/s(n)$, has a limit as $n \rightarrow \infty$ (with t/n fixed).

3.2 Proof of Theorem 1.1

Let δ and k be fixed. By Corollary 2.4 we can choose $\alpha = \alpha(\delta, k)$ and $N_1 = N_1(\delta, k)$ such that, for all $n \geq N_1$

$$\mathbb{P}(\exists i \leq k : \text{diam}(\mathcal{C}_n^{(i)}) < \alpha n) < \frac{\delta}{2}.$$

Further, by Proposition 3.2 there is an $\varepsilon > 0$ such that the probability that there are two clusters with diameter larger than αn of which the sizes differ less than $\varepsilon s(n)$ is smaller than $\delta/2$. Hence the l.h.s. of (1.2) is less than $\delta/2 + \delta/2$. \square

3.3 Proof of Theorem 1.2

Let x and δ be given. By Theorem 2.3 we can find an α such that

$$\mathbb{P}(\exists u \in \Lambda_n : |\mathcal{C}_n(u)| \geq xs(n); \text{diam}(\mathcal{C}_n(u)) \leq \alpha n) < \frac{\delta}{2}.$$

Let $\mathbf{C}_{\alpha,n}$ be defined as in (3.5). It is enough to show that there exist $\varepsilon = \varepsilon(\alpha, \delta) > 0$, $N = N(\alpha, \delta) \in \mathbb{N}$ such that, for all sufficiently large n ,

$$\mathbb{P}(\exists \mathcal{D} \in \mathbf{C}_{\alpha,n} : |\mathcal{D} - xs(n)| < \varepsilon s(n)) < \frac{\delta}{2}. \quad (3.18)$$

This can be proved in practically the same way as Proposition 3.2. (And, in fact, a bit easier, because now we deal with single clusters instead of pairs of clusters. In particular the factor $\binom{\kappa}{2}$ is replaced by κ in the proof.) \square

References

- [1] J. van den Berg and R. P. Conijn, *On the size of the largest cluster in 2D critical percolation*, Electron. Comm. Probab. **17** (2012). MR-3005731
- [2] J. van den Berg and H. Kesten, *Inequalities with applications to percolation and reliability*, J. Appl. Probab. **22** (1985), no. 3, 556–569. MR-799280
- [3] B. Bollobás and O. Riordan, *Percolation*, Cambridge University Press, New York, 2006. MR-2283880
- [4] C. Borgs, J. T. Chayes, H. Kesten, and J. Spencer, *Uniform boundedness of critical crossing probabilities implies hyperscaling*, Random Structures Algorithms **15** (1999), no. 3-4, 368–413. MR-1716769
- [5] C. Borgs, J. T. Chayes, H. Kesten, and J. Spencer, *The birth of the infinite cluster: finite-size scaling in percolation*, Comm. Math. Phys. **224** (2001), no. 1, 153–204, Dedicated to Joel L. Lebowitz. MR-1868996
- [6] L. Le Cam, *On the distribution of sums of independent random variables*, Proc. Internat. Res. Sem., Statist. Lab., Univ. California, Berkeley, Calif, Springer-Verlag, New York, 1965, pp. 179–202. MR-0199871
- [7] C. G. Esseen, *On the concentration function of a sum of independent random variables*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **9** (1968), 290–308. MR-0231419
- [8] C. Garban, G. Pete, and O. Schramm, *Pivotal, cluster, and interface measures for critical planar percolation*, J. Amer. Math. Soc. **26** (2013), no. 4, 939–1024. MR-3073882

- [9] C. Garban, G. Pete, and O. Schramm, *The scaling limits of the minimal spanning tree and invasion percolation in the plane*, arXiv:1309.0269 (2013), 1–50.
- [10] G. R. Grimmett, *Percolation*, second ed., Springer-Verlag, Berlin, 1999. MR-1707339
- [11] A. A. Járai, *Incipient infinite percolation clusters in 2D*, Ann. Probab. **31** (2003), no. 1, 444–485. MR-1959799
- [12] H. Kesten, *The incipient infinite cluster in two-dimensional percolation*, Probab. Theory Rel. Fields **73** (1986), no. 3, 369–394. MR-859839

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