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Elements of generalized ultrametric domain theory

J.J.M.M. Rutten *

CWI, P.O. Box 94079, 1090 GB Amsterdam, Netherlands

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Abstract

Generalized ultrametric spaces are a common generalization of preorders and ordinary ultrametric spaces, as was observed by Lawvere (1973). Guided by his enriched-categorical view on (ultra)metric spaces, we generalize the standard notions of Cauchy sequence and limit in an (ultra)metric space, and of adjoint pair between preorders. This leads to a solution method for recursive domain equations that combines and extends the standard order-theoretic (Smyth and Plotkin, 1982) and metric (America and Rutten, 1989) approaches.

1. Introduction

A generalized ultrametric space is a set X supplied with a distance function $X(-,-): X \times X \to [0,1]$, satisfying for all x, y, z: $X(x,x) = 0$ and $X(x,z) \leqslant \max\{X(x,y), X(y,z)\}$. This notion generalizes ordinary ultrametric spaces in that the distance need not be symmetric, and different elements may have distance 0. Generalized ultrametric spaces provide a common generalization of both preordered spaces and ordinary ultrametric spaces, as has been observed by Lawvere [14]. Therefore they are of some importance to the domain-theoretic approach to programming language semantics, in which preorders and ordinary ultrametric spaces are the most popular structures. A more direct connection with the world of semantics is provided by the observation that transition systems can be naturally endowed with a generalized ultrametric that captures their operational behaviour in terms of simulations (see Example 2.1 below).

The present paper introduces first some basic concepts such as Cauchy sequence and limit, next introduces so-called metric adjoint pairs, and then describes how these can be used to solve recursive domain equations. The latter can be seen as its main contribution. The paper concludes with some miscellaneous observations on the following topics: algebraicity; an ultrametric generalization of the category of SFP objects called SFU; the category of generalized ultrametric spaces seen as a large ultrametric space;

* E-mail: jan.rutten@cwi.nl.

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and an equivalent, purely enriched-categorical definition of metric limit, using so-called weighted colimits [6].

The present paper does *not* deal with admittedly fundamental aspects of domain theory such as completion and topology: for these, the reader is referred to [4].

Our main source of inspiration has been the aforementioned paper by Lawvere, in which he applies insights from enriched-category theory [11] to (ultra)metric spaces. One way of summarizing the relevance of this view is the fact that many properties of generalized ultrametric spaces are determined by the (categorical) structure $[0, 1]$. Notably the definition of limit of a Cauchy sequence in an arbitrary generalized ultrametric space will be phrased in terms of limits in $[0, 1]$, which are introduced first.

The above theory for generalized ultrametric spaces is developed, extending [14], along the lines of a combination of [21] and [3], which deal with the solutions of domain equations in categories of ordered and metric spaces, respectively. This it has in common with the work of Flagg and Kopperman [9] on continuity spaces, and of Wagner [22] on abstract preorders, who aim at a reconciliation of ordered and metric domain theory as well. Furthermore it has similarly been inspired by some of Smyth' results on quasimetric spaces [18]. Unlike [9, 22], we do not aim at generality. The category of generalized ultrametric spaces seems rather to be the smallest category (of sets with structure) that contains both the categories of preorders and ordinary ultrametrics. What seems to be new, amongst others, is: two fixed point theorems on generalized ultrametric spaces, generalizing the least and unique fixed point theorems of Tarski and Banach, respectively; the definition and characterizations of metric adjoint pairs; two categorical counterparts of the aforementioned fixed point theorems, based on the use of metric adjoint pairs, and generalizing the ones of [21] and [3]; the definition and characterization of the subcategory SFU of bifinite spaces; and the purely enriched-categorical definition of metric limit in terms of weighted colimits.

2. Generalized ultrametric spaces

Generalized ultrametric spaces are introduced and shown to be $[0, 1]$-categories in the sense of Lawvere. In order to see this, a brief recapitulation of Lawvere's enriched-categorical view of metric spaces is presented. For us, one of the main benefits of Lawvere's approach is the insight that many properties of generalized ultrametric spaces are determined by the unit interval of real numbers $[0, 1]$. The section concludes with a brief discussion of the category of all generalized ultrametric spaces, and a few basic definitions. (The subsection on $[0, 1]$-categories can be skipped at first reading, except for the very basic Proposition 2.2, which will be used time and again.)

A *generalized ultrametric space* (gum for short) is a set X together with a function

$$X(-,-) : X \times X \to [0, 1]$$

which satisfies, for all x, y, and z in X,

1. $X(x,x) = 0$, and

2. $X(x,z) \leqslant \max\{X(x,y), X(y,z)\}$,
where 2 is the so-called strong triangle inequality ("strong" because we have max instead of $+$). The real number $X(x,y)$ will be called the distance from x to y. A generalized ultrametric space generally does not satisfy

3. if $X(x,y) = 0$ and $X(y,x) = 0$ then $x = y$,
4. $X(x,y) = X(y,x)$,

which are the additional conditions that hold for an *ordinary* ultrametric space. Therefore it is sometimes called a *pseudo-quasi* ultrametric space. A *quasi* ultrametric space is a gum which satisfies axioms 1, 2, and 3. A gum satisfying 1, 2, and 4 is called a *pseudo* ultrametric space.

Examples 2.1. 1. Pseudo, quasi, and ordinary ultrametric spaces are generalized ultrametric spaces.

2. Any *preorder* $\langle P, \leqslant \rangle$ (where $\leqslant$ is a reflexive and transitive binary relation on P) can be viewed as a generalized ultrametric space, by defining a distance, for p and q in P,

$$P(p,q) = \begin{cases} 0 & \text{if } p \leqslant q, \\ 1 & \text{if } p \nleqslant q. \end{cases}$$

By a slight abuse of language, any gum stemming from a preorder in this way will itself be called a preorder.

3. The set A^∞ of finite and infinite words over some given set A with distance function, for v and w in A^∞,

$$A^\infty(v,w) = \begin{cases} 0 & \text{if } v \text{ is a prefix of } w, \\ 2^{-n} & \text{otherwise,} \end{cases}$$

where n is the length of the longest common prefix of v and w.

4. The set $[0,1]$ with distance, for r and s in $[0,1]$,

$$[0,1](r,s) = \begin{cases} 0 & \text{if } r \geqslant s, \\ s & \text{if } r < s. \end{cases}$$

Note that $[0,1]$ is a quasi ultrametric space.

5. The set $\bar{\omega} = \{0,1,\ldots\} \cup \{\omega\}$, with distance, for x and y in $\bar{\omega}$,

$$\bar{\omega}(x,y) = \begin{cases} 0 & \text{if } x \leqslant y, \\ 2^{-y} & \text{if } x > y. \end{cases}$$

6. A *transition system* is a pair $\langle S, \longrightarrow \rangle$ consisting of a set S of states and a transition relation $\longrightarrow \subseteq S \times S$. Let $(\leqslant_n)_n$ be a sequence of relations on S inductively defined by $\leqslant_0 = S \times S$ and

$$\leqslant_{n+1} = \{\langle s,t \rangle \in S \times S \mid \forall s' \in S \text{ s.t. } s \longrightarrow s' \; \exists t' \in S \text{ s.t. } t \longrightarrow t' \text{ and } s' \leqslant_n t'\}.$$

For s and t in S, $S(s,t) = \inf\{2^{-n} \mid s \leqslant_n t\}$ defines a generalized ultrametric on S, which measures the extent to which the transition steps from s can be simulated by steps from t.

2.1. Generalized ultrametric spaces are [0, 1]-categories

We briefly review Lawvere's [14] conception of metric spaces as $\mathscr{V}$-categories [18, 11]. Then we shall follow and further elaborate his approach for the special case of generalized ultrametric spaces, which will be shown to be [0, 1]-categories. The main point is that, in general, many properties of $\mathscr{V}$-categories derive from the structure on the underlying category $\mathscr{V}$.

The starting point is a category $\mathscr{V}$ together with a functor

$$\otimes : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$$

which is symmetric and associative, and has a unit object k (up to isomorphism). This defines a so-called symmetric monoidal structure on $\mathscr{V}$. The category $\mathscr{V}$ is required to be complete and cocomplete (i.e., all limits and colimits in $\mathscr{V}$ should exist), and its monoidal structure should be closed: that is, there exists an internal hom functor

$$Hom : \mathscr{V}^{\mathrm{op}} \times \mathscr{V} \to \mathscr{V}$$

such that for all a in $\mathscr{V}$, the functor $Hom(a, -)$ (mapping b in $\mathscr{V}$ to $Hom(a, b)$) is right adjoint to the functor $a \otimes -$ (which maps b in $\mathscr{V}$ to $a \otimes b$). A *$\mathscr{V}$-category*, or a category enriched in $\mathscr{V}$, is any set (more generally, class) X together with the assignment of an object $X(x, y)$ of $\mathscr{V}$ to every pair of elements $\langle x, y\rangle$ in X; the assignment of a $\mathscr{V}$-morphism

$$X(x, y) \otimes X(y, z) \to X(x, z)$$

to every triple $\langle x, y, z\rangle$ of elements in X; and the assignment of a $\mathscr{V}$-morphism

$$k \to X(x, x)$$

to every element x in X, satisfying a number of naturality conditions (omitted here since they are trivial in the particular case we are interested in; see [14, 5].

For instance, the category of all sets is a (complete and cocomplete) symmetric monoidal closed category (where $\otimes$ is given by the Cartesian product, and any one element set is a unit). The corresponding $\mathscr{V}$-categories are just ordinary categories: $X(x, y)$ is given by the homset of all morphisms between two objects x and y in a category X, and the $\mathscr{V}$-morphisms that are required to exist are just functions defining the composition of morphisms, and giving identity morphisms.

Generalized ultrametric spaces can now be seen to be [0, 1]-enriched categories as follows. First of all, [0, 1] is shown to be a complete and cocomplete symmetric monoidal closed category. It is a category because it is a preorder (objects are the real numbers between 0 and 1; and for r and s in [0, 1] there is a morphism from r to s if and only if $r \geqslant s$). It is complete and cocomplete: equalizers and coequalizers are trivial (because there is at most one arrow between any two elements of [0, 1]), the product $r \times s$ of two elements r and s in [0, 1] is given by $\max\{r, s\}$, and their coproduct $r + s$ by $\min\{r, s\}$. More generally, products are given by sup, and coproducts are given by inf. The monoidal structure on $[0,1]$ is given by

$$\max : [0,1] \times [0,1] \to [0,1],$$

assigning to two real numbers their maximum, which is symmetric and associative, and for which 0 is the unit element. (Note that in this particular case the monoidal product is identical to the categorical product.) Consider the following internal hom functor

$$[0,1](-,-) : [0,1]^{\mathrm{op}} \times [0,1] \to [0,1],$$

defined (as in Example 2.1) by, for r and s in $[0,1]$,

$$[0,1](r,s) = \begin{cases} 0 & \text{if } r \geqslant s, \\ s & \text{if } r < s. \end{cases}$$

The following fundamental equivalence states that $[0,1](r,-)$ is right adjoint to $\max\{r,-\}$, for any r in $[0,1]$:

Proposition 2.2. *For all r, s, and t in* $[0,1]$,

$$\max\{r,t\} \geqslant s \text{ if and only if } t \geqslant [0,1](r,s).$$

As a consequence, $[0,1]$ is a (complete and cocomplete symmetric) monoidal closed category. (In fact, since the monoidal structure is given by the categorical product on $[0,1]$, it is even Cartesian closed.)

Now $[0,1]$-categories are precisely the generalized ultrametric spaces introduced at the beginning of this section: sets X together with a function assigning to x and y in X an object, i.e., a real number $X(x,y)$ in $[0,1]$. The existence of a $[0,1]$-morphism from $X(x,y) \otimes X(y,z) = \max\{X(x,y), X(y,z)\}$ to $X(x,z)$ gives the second, and the existence of a morphism from $k = 0$ to $X(x,x)$ gives the first of the axioms for generalized ultrametric spaces.

2.2. The category of generalized ultrametric spaces

As mentioned above, many constructions and properties of generalized ultrametric spaces are determined by the category $[0,1]$. Important examples are the definitions of limit and completeness, presented in Section 3. Also the category of all gum's, which is introduced next, inherits much of the structure of $[0,1]$.

Let *Gum* be the category with generalized ultrametric spaces as objects, and *non-expansive* functions as arrows: i.e., functions $f : X \to Y$ such that for all x and x' in X,

$$Y(f(x), f(x')) \leqslant X(x,x').$$

(Non-expansive functions are precisely the $[0,1]$-functors between $[0,1]$-categories.) A function f is *isometric* if for all x and x' in X,

$$Y(f(x), f(x')) = X(x,x').$$

Two spaces X and Y are called isometric (isomorphic) if there exists an isometric bijection between them. The product $X \times Y$ of two gum's X and Y is defined as the Cartesian product of their underlying sets, together with distance, for $\langle x, y\rangle$ and $\langle x', y'\rangle$ in $X \times Y$,

$$X \times Y(\langle x,y\rangle,\ \langle x',y'\rangle) = \max\{X(x,x'),\ Y(y,y')\}.$$

Note that this definition uses the product (max) of $[0,1]$. The *exponent* of X and Y is defined by

$$Y^X = \{f : X \to Y \mid f \text{ is non-expansive }\},$$

with distance, for f and g in Y^X,

$$Y^X(f,g) = \sup\{Y(f(x),g(x)) \mid x \in X\}.$$

The fact that the category $[0,1]$ is monoidal (Cartesian) closed implies that the category *Gum* is monoidal (Cartesian) closed as well: i.e., for all gum's X, Y, and Z,

$$Z^{X\times Y} \cong (Z^Y)^X.$$

In the category *Gum*, all limits and colimits exist. Moreover, they are constructed at the level of their underlying sets. Formally:

Theorem 2.3. *Let U : Gum $\to$ Set be the functor that maps a gum to its underlying set ("forgetting" its metric structure). The functor U creates all limits and all colimits.*

Proof. Limits are easy but colimits are less trivial. They involve the use of the so-called "least-cost" [14] or "shortest-path" [20] distance. For details see [16]. □

2.3. *A few basic definitions*

This section is concluded by a number of constructions and definitions for generalized ultrametric spaces that will be used in the sequel.

The *opposite* X^{op} of a gum X is the set X with distance

$$X^{\mathrm{op}}(x,x') = X(x',x).$$

With this definition, the distance function $X(-,-)$ can be described as a function

$$X(-,-) : X^{\mathrm{op}} \times X \to [0,1].$$

Using Proposition 2.2 one can easily show that $X(-,-)$ is non-expansive, i.e., a morphism in the category *Gum*.

We saw that any preorder P induces a gum. (Note that a partial order induces a quasi ultrametric and that the non-expansive functions between preorders are precisely the monotone functions.) Conversely, any gum X gives rise to a preorder $\langle X, \leqslant_X \rangle$, where $\leqslant_X$, called the *underlying* ordering of X, is given, for x and y in X, by

$$x \leqslant_X y \text{ if and only if } X(x,y) = 0.$$

Any (pseudo or quasi) ultrametric space is a fortiori a gum. Conversely, any gum X induces a pseudo ultrametric space X^{s}, the *symmetrization* of X, with distance

$$X^{\mathrm{s}}(x,y) = \max\{X(x,y), X^{\mathrm{op}}(x,y)\}.$$

For instance, the ordering that underlies A^{∞} is the usual prefix ordering, and $(A^{\infty})^{\mathrm{s}}$ is the standard ultrametric on words. The generalized ultrametric on $[0,1]$ induces the reverse of the usual ordering: for r and s in $[0,1]$,

$$r \leqslant_{[0,1]} s \text{ if and only if } s \leqslant r;$$

and the symmetric version of $[0,1]$ is defined by

$$[0,1]^{\mathrm{s}}(r,s) = \begin{cases} 0 & \text{if } r = s, \\ \max\{r,s\} & \text{if } r \neq s. \end{cases}$$

Any gum X induces a quasi ultrametric space $[X]$ as follows. Let $\sim$ be the equivalence relation on X defined, for x and y in X, by

$$x \sim y \text{ iff } (X(x,y) = 0 \text{ and } X(y,x) = 0).$$

Let $[x]$ denote the equivalence class of x with respect to $\sim$, and $[X]$ the collection of all equivalence classes. Defining $[X]([x],[y]) = X(x,y)$ turns $[X]$ into a quasi ultrametric space. It has the following universal property: for any non-expansive function $f : X \to Y$ from X to a quasi ultrametric space Y there exists a unique non-expansive function $f' : [X] \to Y$ with $f'([x]) = f(x)$, for $x \in X$.

The above constructions give rise to various adjoint functors between the categories involved; cf. [16].

3. Cauchy sequences, limits, and completeness

The notions of forward- and backward-Cauchy sequences are introduced. It is explained what such sequences look like in $[0,1]$, and how to define in $[0,1]$ the notion of metric limit. This will give rise to a definition of metric limit for arbitrary generalized ultrametric spaces. Furthermore the notion of completeness is introduced. Two fixed point theorems for functions on complete quasi ultrametric spaces are given, generalizing those of Knaster-Tarski and of Banach.

A sequence $(x_n)_n$ in a generalized ultrametric space X is *forward-Cauchy* if

$$\forall \varepsilon > 0\ \exists N\ \forall n \geqslant N,\ \ X(x_n, x_{n+1}) \leqslant \varepsilon.$$

Note that this is equivalent to the more familiar condition:

$$\forall \varepsilon > 0\ \exists N\ \forall n \geqslant m \geqslant N,\ \ X(x_m, x_n) \leqslant \varepsilon,$$

because of the strong triangle inequality. Since our metrics need not be symmetric, the following variation exists: a sequence $(x_n)_n$ is *backward-Cauchy* if

$$\forall \varepsilon > 0 \; \exists N \; \forall n \geqslant N, \;\; X(x_{n+1}, x_n) \leqslant \varepsilon.$$

If X is an ordinary ultrametric space then forward-Cauchy and backward-Cauchy both mean Cauchy in the usual sense. If X is a preorder then forward-Cauchy sequences are eventually increasing: there exists an N such that for all $n \geqslant N$, $x_n \leqslant x_{n+1}$. (Increasing sequences in a preorder are also called *ω-chains.*) Similarly backward-Cauchy sequences are eventually decreasing.

Cauchy sequences in $[0,1]$, with the generalized ultrametric of Section 2, are particularly simple: every forward-Cauchy sequence either converges to 0 or is eventually decreasing; dually, every backward-Cauchy sequence either converges to 0 or is eventually increasing.

Proposition 3.1. *A sequence $(r_n)_n$ in $[0,1]$ is forward-Cauchy if and only if*

$$\textit{either}: \forall \varepsilon > 0 \; \exists N \; \forall n \geqslant N, \;\; r_n \leqslant \varepsilon, \;\; \textit{or}: \exists N \; \forall n \geqslant N, \;\; r_n \geqslant r_{n+1}.$$

Dually, it is backward-Cauchy if and only if

$$\textit{either}: \forall \varepsilon > 0 \; \exists N \; \forall n \geqslant N, \;\; r_n \leqslant \varepsilon, \;\; \textit{or}: \exists N \; \forall n \geqslant N, \;\; r_n \leqslant r_{n+1}.$$

Proof. We prove only the first statement, the second being dual. Sequences that converge to 0 or that are eventually decreasing are easily seen to be forward-Cauchy. Conversely, let $(r_n)_n$ be forward-Cauchy in $[0,1]$. Suppose there exists $\varepsilon > 0$ such that

$$\forall N \; \exists n \geqslant N, \;\; r_n > \varepsilon.$$

We claim that there exists an N such that *for all* $n \geqslant N$, $r_n > \varepsilon$; for suppose not:

$$\forall N \; \exists n \geqslant N, \;\; r_n \leqslant \varepsilon.$$

Because $(r_n)_n$ is forward-Cauchy, there exists M such that for all $m \geqslant M$, $[0,1](r_m, r_{m+1}) \leqslant \varepsilon$. Consider $n_1 \geqslant M$ with $r_{n_1} \leqslant \varepsilon$, and consider $n_2 \geqslant n_1$ with $r_{n_2} > \varepsilon$. Then

$$\begin{aligned} \varepsilon &< r_{n_2} \\ &= [0,1](r_{n_1}, r_{n_2}) \quad \text{[definition of the distance on [0,1]]} \\ &\leqslant \varepsilon, \end{aligned}$$

a contradiction. Therefore let N be such that for all $n \geqslant N$, $r_n > \varepsilon$. Let $M \geqslant N$ such that for all $m \geqslant M$, $[0,1](r_m, r_{m+1}) \leqslant \varepsilon$, which is equivalent to $r_{m+1} \leqslant \max\{\varepsilon, r_m\}$ by Proposition 2.2. Because $r_m > \varepsilon$, for all $m \geqslant M$, this implies $r_{m+1} \leqslant r_m$. $\square$

Because Cauchy sequences in $[0,1]$ are that simple, the following definitions are easy as well: the *forward-limit* of a forward-Cauchy sequence $(r_n)_n$ in $[0,1]$ is given by

$$\lim_{\rightarrow} r_n = \sup_n \inf_{k \geqslant n} r_k.$$

Similarly, the *backward-limit* of a backward-Cauchy sequence $(r_n)_n$ in $[0,1]$ is

$$\lim_{\leftarrow} r_n = \sup_n \inf_{k \geqslant n} r_k.$$

The following proposition shows how forward-limits and backward-limits in $[0,1]$ are related (cf. [23]).

Proposition 3.2. *For a forward-Cauchy sequence $(r_n)_n$ in $[0,1]$, and r in $[0,1]$,*

$$[0,1](\lim_{\rightarrow} r_n, r) = \lim_{\leftarrow} [0,1](r_n, r).$$

For a backward-Cauchy sequence $(r_n)_n$ in $[0,1]$, and r in $[0,1]$,

$$[0,1](r, \lim_{\leftarrow} r_n) = \lim_{\leftarrow} [0,1](r, r_n).$$

A proof follows easily from the following elementary facts:

Lemma 3.3. *For all $V \subseteq [0,1]$ and r in $[0,1]$,*
1. $[0,1](\inf V, r) = \sup_{v \in V} [0,1](v, r)$;
2. $[0,1](r, \sup V) = \sup_{v \in V} [0,1](r, v)$.

Forward-limits and backward-limits in an *arbitrary* generalized ultrametric space X can now be defined in terms of backward-limits in $[0,1]$:

Definition 3.4. Let X be a generalized ultrametric space. An element x in X is a *forward-limit* of a forward-Cauchy sequence $(x_n)_n$ in X,

$$x = \lim_{\rightarrow} x_n \text{ iff } \forall y \in X,\ X(x,y) = \lim_{\leftarrow} X(x_n, y).$$

Dually, an element x in X is a *backward-limit* of a backward-Cauchy sequence $(x_n)_n$ in X,

$$x = \lim_{\leftarrow} x_n \text{ iff } \forall y \in X,\ X(y,x) = \lim_{\leftarrow} X(y, x_n).$$

In Section 11, an alternative, equivalent definition of forward-limit and backward-limit will be given, which is, from an enriched-categorical point of view, more attractive. It will be based on the notions of weighted colimit and weighted limit.

Definition 3.4 makes use of the following.

Proposition 3.5. *Let $(x_n)_n$ be a sequence in X and y in X. If $(x_n)_n$ is forward-Cauchy in X then the sequence $(X(x_n, y))_n$ is backward-Cauchy in $[0,1]$. If $(x_n)_n$ is backward-Cauchy in X then the sequence $(X(y, x_n))_n$ is backward-Cauchy in $[0,1]$.*

Note that it follows from Proposition 3.2 that our earlier definitions of forward-limit and backward-limit in $[0, 1]$ are consistent with Definition 3.4.

For an ordinary ultrametric space X, the above definitions of forward- and backward-limit are the same and coincide with the usual notion of limit:

$$x = \underrightarrow{\lim}\, x_n = \underleftarrow{\lim}\, x_n \text{ if and only if } \forall \varepsilon > 0\ \exists N\ \forall n \geqslant N,\ \ X(x_n, x) < \varepsilon.$$

The implication from left to right is straightforward. For the converse, note that it follows from Proposition 3.1 that for y in X, the sequence $(X(x_n, y))_n$, which is both forward- and backward-Cauchy, either converges to 0 or eventually becomes constant. In both cases,

$$X(x, y) = \underleftarrow{\lim}\, X(x_n, y) = \underleftarrow{\lim}\, X(y, x_n).$$

If X is a partial order and $(x_n)_n$ is a chain in X then

$$x = \underrightarrow{\lim}\, x_n \text{ if and only if } \forall y \in X,\ \ x \leqslant_X y \Leftrightarrow \forall n \geqslant 0,\ x_n \leqslant_X y,$$

i.e., $x = \bigsqcup x_n$, the least upperbound of the chain $(x_n)_n$. Similarly, backward-limits of backward-chains correspond to greatest lowerbounds.

Since the rest of this paper mostly deals with forward-Cauchy sequences and forward-limits, we shall simply write *Cauchy* for forward-Cauchy, and

$$\lim x_n \text{ rather than } \underrightarrow{\lim}\, x_n.$$

Note that subsequences of a Cauchy sequence are Cauchy again. If a Cauchy sequence has a limit x, then all its subsequences have limit x as well. Cauchy sequences may have more than one limit. All limits have distance 0, however. As a consequence, limits are unique in quasi ultrametric spaces.

A function $f : X \to Y$ between gum's X and Y is *continuous* if it preserves limits: if $x = \lim x_n$ in X then $f(x) = \lim f(x_n)$ in Y. For ordinary ultrametric spaces, this is the usual definition. For partial orders, it means preservation of least upperbounds of ω-chains.

In dealing with generalized ultrametric spaces, one should be prepared to reconsider some basic intuitions about ordinary ultrametric spaces. For instance, any non-expansive function between ordinary ultrametric spaces is continuous. But:

Remark 3.6. The notions of "non-expansive" and "continuous" function between generalized ultrametric spaces are incomparable.

An example of a function that is continuous but not non-expansive is $f : \bar{\omega} \to \bar{\omega}$ defined, for x in $\bar{\omega}$, by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x - 1 & \text{if } 0 < x < \omega, \\ \omega & \text{if } x = \omega, \end{cases}$$

where $\bar{\omega}$ is supplied with the generalized ultrametric as defined in Example 2.1. For instance, $\bar{\omega}(f(2), f(1)) = \bar{\omega}(1,0) = 1 \not\leqslant 2^{-1} = \bar{\omega}(2,1)$. Any function between partial orders that is monotone but not continuous (i.e., least-upperbound preserving) yields an example of the converse.

A generalized ultrametric space X is *complete* if every Cauchy sequence in X has a limit. For instance, $[0,1]$ is complete. If X is a partial order completeness means that X is an ω-complete partial order, or cpo for short: all ω-chains have a least upperbound. For ordinary ultrametric spaces, the above definition of completeness is the usual one.

Limits are unique in complete *quasi* ultrametric spaces, which therefore are well suited for the construction of fixed points. There are at least two ways:

Theorem 3.7. *Let X be a complete quasi ultrametric space and $f : X \to X$ non-expansive.*

1. *If f is continuous and if there is x in X with $x \leqslant_X f(x)$ (i.e., $X(x, f(x)) = 0$), then f has a fixed point, which is the least (with respect to $\leqslant_X$) fixed point above x.*
2. *If f is continuous and* contractive:

$$\exists \varepsilon < 1 \; \forall x, y \in X, \;\; X(f(x), f(y)) \leqslant \varepsilon \cdot X(x, y),$$

and if, moreover, X is non-empty, then f has a unique *fixed point. (Note that contractiveness does not imply continuity; for an example see below.)*

Proof. 1. Suppose f is continuous and let x be such that $X(x, f(x)) = 0$. The sequence

$$(x, f(x), f^2(x), \ldots)$$

is trivially Cauchy because f is non-expansive. Since X is complete this sequence has a limit y. By continuity of f, $f(y) = \lim f(f^n(x)) = \lim f^n(x)$. In quasi ultrametric spaces, limits are unique, thus $y = f(y)$. If $x \leqslant_X z$ and $f(z) = z$, for z in X, then it follows that $y \leqslant_X z$.

2. Suppose that f is both continuous and contractive. Let x be any element in X and consider again the sequence $(x, f(x), f^2(x), \ldots)$. Because f is contractive this sequence is Cauchy: for all $n \geqslant 0$, $X(f^n(x), f^{n+1}(x)) \leqslant \varepsilon^n \cdot X(x, f(x))$. As in 1, a fixed point y is obtained by completeness of X and continuity of f. Suppose z is another one. Then $X(y,x) = X(f(y), f(z)) \leqslant \varepsilon \cdot X(y,z)$ whence $X(y,z) = 0$. Similarly $X(z,y) = 0$. Because X is a quasi ultrametric space this implies $y = z$. $\square$

Part 1 generalizes the theorem of Knaster-Tarski that continuous functions on an ω-complete partial order with a least element, have a least fixed point. Part 2 generalizes Banach's contraction theorem. Both part 1 and part 2 above are special instances of a slightly more general theorem (on quasi metric spaces) in [18].

Consider the set $\bar{\omega}$ of the natural numbers with infinity with the distance induced by the usual ordering, but for the value of $\bar{\omega}(1,0)$, which is $\frac{1}{2}$ rather than 1. Let

$f : \bar{\omega} \to \bar{\omega}$ map any $n \geqslant 0$ to 0, and ω to 1. Then f is contractive but not continuous since $\lim n = \omega$, whereas $\lim f(n) \neq f(\omega)$.

In order to prove that a function $f : P \to Q$ between partial orders is continuous (that is, preserves least upperbounds), one usually establishes first that f is monotone, from which then half of the proof follows: if $x = \bigsqcup x_n$ and f is monotone, then $x_n \leqslant_P x$ implies $f(x_n) \leqslant_Q f(x)$ whence $\bigsqcup f(x_n) \leqslant_Q f(x)$. Similarly (and more generally), non-expansiveness of a function between generalized ultrametric spaces implies "half of its continuity"; more precisely:

Proposition 3.8. *Let X and Y be generalized ultrametric spaces, $f : X \to Y$ a non-expansive function, and $(x_n)_n$ a Cauchy sequence in X with $\lim x_n = x$ in X. For all y in Y,*

$$\underleftarrow{\lim}\, Y(f(x_n), y) \leqslant Y(f(x), y).$$

Proof. Because f is non-expansive and $(x_n)_n$ is Cauchy, the sequence $(f(x_n))_n$ is again Cauchy. By Proposition 3.5, the sequence $(Y(f(x_n), y))_n$ is backward-Cauchy in $[0, 1]$, for any y in Y, and hence has a backward-limit. The inequality follows from

$$\begin{aligned}
&[0,1](Y(f(x), y), \underleftarrow{\lim}\, Y(f(x_n), y)) \\
&\quad = \underleftarrow{\lim}[0,1](Y(f(x), y),\ Y(f(x_n), y)) \quad \text{[Proposition 3.2]} \\
&\quad \leqslant \underleftarrow{\lim}\, Y(f(x_n), f(x)) \quad [Y(-, y) : Y^{\mathrm{op}} \to [0,1] \text{ is non-expansive}] \\
&\quad \leqslant \underleftarrow{\lim}\, X(x_n, x) \quad [f \text{ is non-expansive}] \\
&\quad = X(\lim x_n, x) \quad \text{[definition of limit]} \\
&\quad = X(x, x) \\
&\quad = 0,
\end{aligned}$$

and the definition of the metric on $[0, 1]$ (Example 2.1). □

Proposition 3.8 comes in handy in the following.

Proposition 3.9. *Let X and Y be generalized ultrametric spaces.*

1. *If Y is complete then Y^X is complete.*
2. *Let $[X \to Y] = \{f : X \to Y \mid f$ is both non-expansive and continuous$\}$, with distance as in Y^X. This defines a generalized ultrametric space, which is complete whenever Y is.*

Proof. The proof combines, as it were, both the proofs (of the same statements) for partial orders and ordinary ultrametric spaces, and is somewhat more complicated than both proofs individually. We list the main steps: consider a Cauchy sequence $(f_n)_n$ in X^Y. We have to show: there is f in X^Y with $\lim f_n = f$; and if all of the f_n are moreover continuous then so is f.

1. Definition: for any x in X, the sequence $(f_n(x))_n$ is Cauchy in Y. It has a limit, to be called $f(x)$, because Y is complete. This defines a function $f : X \to Y$.

2. A useful observation: $\forall \varepsilon > 0\ \exists N\ \forall n \geqslant N\ \forall x \in X,\ \ Y(f_n(x), f(x)) < \varepsilon$.

3. From 2, it follows that $f = \lim f_n$.

4. Using 3, one can prove that f is non-expansive. This proves part 1 of the theorem.

5. It remains to be shown that f is continuous if all of the f_n are. Let $\lim x_n = x$ be a converging sequence in X, and let y be in Y. By Proposition 3.8 and 4,

$$\underleftarrow{\lim}\, Y(f(x_n), y) \leqslant Y(f(x), y).$$

6. Using 2 and the fact that the functions f_n are continuous, one can also prove the converse:

$$Y(f(x), y) \leqslant \underleftarrow{\lim}\, Y(f(x_n), y).$$

From this and 5, it follows that $\lim f(x_n) = f(x)$. Thus f is continuous. □

The following fact will be useful later.

Lemma 3.10. *The composition of functions, viewed as a function* $\circ : [Y \to Z] \times [X \to Y] \to [X \to Z]$, *for generalized ultrametric spaces X, Y, and Z, is non-expansive and continuous.*

4. Distance and order

Generalized ultrametric spaces have been introduced as generalizations of ordinary ultrametric spaces. Their definition has been guided by enriched-categorical motivations. In this subsection, we shall briefly show that, alternatively, generalized ultrametric spaces can be presented as generalized preorders. A strong argument in favour of the original metric definition is the applicability of various insights from enriched category theory (see [4] for more examples). Still the presentation of a generalized ultrametric space as a generalized preorder can be useful, because it allows in certain cases a translation of familiar notions from the theory of ordered spaces into a metric variant thereof. An example will be the notion of ε-adjoint pair in Section 5.

A generalized ultrametric space X induces a family

$$\{\leqslant^{\varepsilon} \subseteq X \times X \mid \varepsilon \in [0,1]\}$$

of preorders on X defined, for $\varepsilon \in [0,1]$ and x and y in X, by

$$x \leqslant^{\varepsilon} y \iff X(x,y) \leqslant \varepsilon.$$

Note that this generalizes the definition in Section 2 of underlying ordering. The above set of relations inherits from the set of all relations on X the structure of a complete lattice. Because of the strong triangle inequality, we have for all ε and δ in $[0,1]$,

$$\leqslant^{\varepsilon} \circ \leqslant^{\delta} \subseteq \leqslant^{\max\{\varepsilon,\delta\}},$$

where on the left the composition of relations is taken.

As an illustration of a possible interest of the above representation of a generalized ultrametric space X, it is shown how both the notions of Cauchy sequence and of forward-limit can be expressed in terms of the ε-preorders:

1. A sequence $(x_n)_n$ in X is (eventually) an *ε-chain* if

$$\exists N\ \forall n \geqslant N,\ x_n \leqslant^{\varepsilon} x_{n+1}.$$

Clearly a sequence $(x_n)_n$ is Cauchy if and only if it is for every $\varepsilon > 0$ eventually an ε-chain.

2. Consider a Cauchy sequence $(x_n)_n$ and an element x in X. Define

$$x = \bigsqcup_{\varepsilon} x_n \text{ iff } (\forall y \in X,\ x \leqslant^{\varepsilon} y \iff \exists N\ \forall n \geqslant N,\ x_n \leqslant^{\varepsilon} y)$$

and call x an ε-minimal upperbound of $(x_n)_n$. Then

$$\lim x_n = x \iff \forall \varepsilon > 0,\ x = \bigsqcup_{\varepsilon} x_n.$$

5. Metric adjoint pairs

An adjoint pair between preorders is shown to be a special case of a more general metric notion of *ε-adjoint* pair. As we shall see in Sections 6 and 7, ε-adjoint pairs play a central role in the solution of recursive domain equations. Moreover, they will be used in Section 10 to turn the category of generalized ultrametric spaces itself into a large generalized ultrametric space.

A pair of non-expansive functions $f : X \to Y$ and $g : Y \to X$ between two preorders X and Y is *adjoint* (and f is *left adjoint* to g), denoted by $f \dashv g$, if

$$\forall x \in X\ \forall y \in Y,\ f(x) \leqslant_Y y \iff x \leqslant_X g(y).$$

As is well known, this is equivalent to

$$1_X \leqslant_{X^X} g \circ f \text{ and } f \circ g \leqslant_{Y^Y} 1_Y,$$

where 1_X and 1_Y are the identities on X and Y.

The approach of Section 4 gives rise, for any real number ε with $0 \leqslant \varepsilon \leqslant 1$, to the following definition. A pair of non-expansive functions $f : X \to Y$ and $g : Y \to X$ between two generalized ultrametric spaces X and Y is *ε-adjoint*, denoted by $f \dashv_{\varepsilon} g$, if

$$\forall x \in X\ \forall y \in Y,\ f(x) \leqslant_Y^{\varepsilon} y \iff x \leqslant_X^{\varepsilon} g(y),$$

or, equivalently,

$$1_X \leqslant^{\varepsilon}_{X^X} g \circ f \text{ and } f \circ g \leqslant^{\varepsilon}_{Y^Y} 1_Y.$$

By definition of the ε-preorders, the latter is equivalent to

$$\max\{X^X(1_X, g \circ f),\ Y^Y(f \circ g, 1_Y)\} \leqslant \varepsilon.$$

As a consequence, *any* pair of non-expansive functions $f : X \to Y$ and $g : Y \to X$ is ε-adjoint for ε defined as

$$\delta\langle f, g\rangle = \max\{X^X(1_X, g \circ f),\ Y^Y(f \circ g, 1_Y)\}.$$

This number can be seen as a measure for the extent to which f and g are *properly* adjoint ("the smaller the better"). It will be often used in the following sections. If f and g are 0-adjoint, then $\langle f, g\rangle$ is called a proper adjoint pair, denoted by $f \dashv g$. In that case, f and g are adjoint viewed as monotone functions between the preorders underlying X and Y. Note that for ordinary ultrametric spaces X and Y it follows that

$$f \dashv g \iff (1_X = g \circ f \text{ and } f \circ g = 1_Y),$$

i.e., f is an isomorphism with inverse g.

The next theorem, which will be used throughout the rest of the paper, characterizes ε-adjoint pairs in various ways. It uses the following definition: for a generalized ultrametric space X, x and y in X, and $\varepsilon \geqslant 0$,

$$x \approx^{\varepsilon}_X y \iff (x \leqslant^{\varepsilon}_X y \text{ and } y \leqslant^{\varepsilon}_X x).$$

Theorem 5.1. *Let $f : X \to Y$ and $g : Y \to X$ be non-expansive functions between generalized ultrametric spaces. Let ε be a real number with $0 \leqslant \varepsilon \leqslant 1$. The following are equivalent*:

1. $f \dashv_{\varepsilon} g$.
2. *For all $x \in X$ and $y \in Y$: $f(x) \leqslant^{\varepsilon}_Y y \iff x \leqslant^{\varepsilon}_X g(y)$.*
3. $\delta\langle f, g\rangle \leqslant \varepsilon$.
4. *For all $x \in X$ and $y \in Y$: $Y(f(x), y) \approx^{\varepsilon}_{[0,1]} X(x, g(y))$.*

Proof. The equivalence of 1, 2, and 3 has been discussed above. By Proposition 2.2, we have for x in X and y in Y,

$$Y(f(x), y) \leqslant \max\{X(x, g(y)),\ \varepsilon\} \iff X(x, g(y)) \leqslant^{\varepsilon}_{[0,1]} Y(f(x), y)$$

and

$$X(x, g(y)) \leqslant \max\{Y(f(x), y),\ \varepsilon\} \iff Y(f(x), y) \leqslant^{\varepsilon}_{[0,1]} X(x, g(y)).$$

As a consequence, 3 implies 4 because

$$\begin{aligned} Y(f(x), y) &\leqslant \max\{Y(f(x), f(g(y))),\ Y(f(g(y)), y)\} \\ &\leqslant \max\{X(x, g(y)),\ \varepsilon\}, \end{aligned}$$

and, similarly, $X(x, g(y)) \leqslant \max\{Y(f(x), y),\ \varepsilon\}$. Conversely, 4 implies 3 by applying the above two equivalences to $g(y)$ and y, and x and $f(x)$, respectively. □

The following lemma will be used in Section 6.

Lemma 5.2. *Let $f : X \to Y$ and $g : Y \to X$ be non-expansive functions between generalized ultrametric spaces, with $f \dashv_\varepsilon g$. Then*

$$\max\{Y^X(f, f \circ g \circ f),\ Y^X(f \circ g \circ f, f)\} \leqslant \varepsilon.$$

This section is concluded with an example of a proper adjoint pair. Consider the space A^∞ with distance as defined in Example 2.1. Let $\Delta : A^\infty \to (A^\infty \times A^\infty)$ map v in A^∞ to $\langle v, v\rangle$, and let $\wedge : (A^\infty \times A^\infty) \to A^\infty$ map $\langle v, w\rangle$ to the longest common prefix of the words v and w. Then Δ is left adjoint to $\wedge$: for all $\langle v, w\rangle$ in $A^\infty \times A^\infty$ and u in A^∞,

$$A^\infty \times A^\infty(\Delta(u),\ \langle v, w\rangle) = \max\{A^\infty(u, v),\ A^\infty(u, w)\} = A^\infty(u, v \wedge w).$$

(This defines a – $[0, 1]$-enriched – product on A^∞.)

The definition of adjoint pair between $\mathscr{V}$-categories is standard (see [14] for the case of generalized metric spaces). The definition and characterizations of ε-adjoint pairs seem to be new.

6. The category of complete quasi ultrametric spaces

Theorem 3.7 shows that complete quasi ultrametric spaces are suitable for finding fixed points of (continuous and contractive) non-expansive functions. The *category* of complete quasi ultrametric spaces turns out to be equally suitable for finding fixed points of *functors* (to be discussed in Section 7). As usual, such fixed points are obtained as colimits of certain sequences (chains) of spaces. This section gives a generalization of the standard constructions for partial orders [21] and ordinary (ultra)metric spaces [3] to complete quasi ultrametric spaces. In [22], a similar generalization is carried out using embedding–projection pairs. Interestingly, it can be carried out here using (metric) adjoint pairs rather than embedding–projection pairs. Although this is well-known for the special case of ordered spaces, it is new for ordinary (ultra)metric spaces.

As in the case of ordered spaces, the use of adjoint pairs instead of embedding–pairs will not lead to "more" fixed points of functors. Nevertheless adjoint pairs seem preferable, both because they have all properties that are needed and because their use will lead to a number of additional observations, in Section 10, on the family of all (complete) generalized ultrametric spaces, viewed itself as a large gum.

We shall consider the category $Cqum^a$, which is defined as follows: objects are complete quasi ultrametric spaces (cqum's for short); and arrows are pairs

$$\langle f, g\rangle : X \to Y$$

consisting of *non-expansive and continuous* functions $f : X \to Y$ and $g : Y \to X$. We know from Section 5 that any such pair is an ε-adjoint pair, for $\varepsilon = \delta\langle f, g\rangle$. This accounts for the superscript a in $Cqum^{a}$.

Lemma 6.1. *The composition of two arrows* $\langle f, g\rangle : X \to Y$ *and* $\langle h, i\rangle : Y \to Z$ *in* $Cqum^{a}$, *defined as* $\langle h, i\rangle \circ \langle f, g\rangle = \langle h \circ f,\ g \circ i\rangle$, *satisfies*

$$\delta(\langle h, i\rangle \circ \langle f, g\rangle) \leqslant \max\{\delta\langle f, g\rangle,\ \delta\langle h, i\rangle\}.$$

A *chain* in $Cqum^{a}$ is a sequence

$$X_0 \xrightarrow{\langle f_0, g_0\rangle} X_1 \xrightarrow{\langle f_1, g_1\rangle} \cdots$$

of cqum's and arrows between them. It will be called *Cauchy* whenever

$$\forall \varepsilon > 0\ \exists N\ \forall n \geqslant N,\ f_n \dashv_{\varepsilon} g_n$$

or, equivalently,

$$\forall \varepsilon > 0\ \exists N\ \forall n \geqslant N,\ \delta\langle f_n, g_n\rangle \leqslant \varepsilon.$$

In the special case of ω-complete partial orders, the arrows in a Cauchy chain (eventually) are (the standard) adjoint pairs.

We shall see that any Cauchy chain in $Cqum^{a}$ has a (categorical) colimit. The proof makes use of two lemmas, in which the following notation will be of help: for k and l with $0 \leqslant k < l$, define

$$f_{kl} : X_k \to X_l, \quad f_{kl} = f_{l-1} \circ \cdots \circ f_{k+1} \circ f_k,$$

$$g_{kl} : X_l \to X_k, \quad g_{kl} = g_k \circ g_{k+1} \circ \cdots \circ g_{l-1}.$$

(Note that $f_{k,k+1} = f_k$ and $g_{k,k+1} = g_k$.)

Lemma 6.2. *Consider a Cauchy chain* $(\langle f_k, g_k\rangle : X_k \to X_{k+1})_k$. *For each fixed* $k \geqslant 0$, *the sequence* $(g_{kl} \circ f_{kl})_{l>k}$ *is Cauchy in* $[X_k \to X_k]$:

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "f_{k-1}"] & X_k \arrow[l, shift left, "g_{k-1}"] \arrow[r, shift left, "f_k"] \arrow[rr, bend left, "f_{kl}"] & \cdots \arrow[l, shift left, "g_k"] \arrow[r, shift left, "f_{l-1}"] & X_l \arrow[l, shift left, "g_{l-1}"] \arrow[r, shift left, "f_l"] \arrow[ll, bend left, "g_{kl}"] & \cdots \arrow[l, shift left, "g_l"]
\end{tikzcd}$$

(*Consequently, it has a limit since* $[X_k \to X_k]$ *is complete by Theorem 3.9.*) *More generally: for every* k *and* l *with* $0 \leqslant k < l$, *the sequence* $(g_{lm} \circ f_{km})_{m>l}$ *is Cauchy in* $[X_k \to X_l]$, *and the sequence* $(g_{km} \circ f_{lm})_{m>l}$ *is Cauchy in* $[X_l \to X_k]$.

Proof. We prove only the first statement (the other ones not being more difficult). It is an immediate consequence of the Cauchy condition on the chain and the fact that, for all k and l with $0 \leqslant k < l$,

$$[X_k \to X_k](g_{kl} \circ f_{kl},\ g_{k,l+1} \circ f_{k,l+1})$$

$$= [X_k \to X_k](g_{kl} \circ f_{kl},\ g_{kl} \circ g_l \circ f_l \circ f_{kl})$$

$$\leqslant [X_k \to X_l](f_{kl},\ g_l \circ f_l \circ f_{kl}) \quad \text{[Lemma 3.10]}$$

$$= \sup_{x \in X_k} \{X_l(f_{kl}(x),\ g_l \circ f_l(f_{kl}(x)))\}$$

$$\leqslant \sup_{y \in X_l} \{X_l(y,\ g_l \circ f_l(y))\}$$

$$\leqslant \delta\langle f_l, g_l \rangle. \quad \square$$

The following lemma states that colimits of Cauchy chains are *locally determined.*

Lemma 6.3. *Consider a Cauchy chain $\Delta = (\langle f_k, g_k \rangle : X_k \to X_{k+1})_k$ and let $(\langle \alpha_k, \beta_k \rangle : X_k \to X)_k$ be a cone from Δ to X: for all $k \geqslant 0$, $\langle \alpha_k, \beta_k \rangle = \langle \alpha_{k+1}, \beta_{k+1} \rangle \circ \langle f_k, g_k \rangle$. If*
1. $\lim \alpha_k \circ \beta_k = 1_X$ *and* 2. $\forall k \geqslant 0,\ \beta_k \circ \alpha_k = \lim_{l > k} g_{kl} \circ f_{kl}$

then X is a colimiting cone.

Proof. The proof of this lemma combines the proof of the same statement for ω-complete partial orders (cf. [21, 1]) with the proof of a similar lemma (but for embedding–projection pairs) for ordinary (ultra)metric spaces in [3]. We have to show that for an arbitrary cone

$$(\langle \bar{\alpha}_k, \bar{\beta}_k \rangle : X_k \to Y)_k$$

from Δ to Y, there exists a unique arrow $\langle f, g \rangle : X \to Y$ such that $\langle f, g \rangle \circ \langle \alpha_k, \beta_k \rangle = \langle \bar{\alpha}_k, \bar{\beta}_k \rangle$, for all $k \geqslant 0$:

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "f_{k-1}"] & X_k \arrow[l, shift left, "g_{k-1}"] \arrow[rr, shift left, "f_k"] \arrow[dr, shift left, "\alpha_k"] \arrow[ddr, bend right, shift left, "\bar{\alpha}_k"] & & X_{k+1} \arrow[ll, shift left, "g_k"] \arrow[r, shift left, "f_{k+1}"] \arrow[dl, shift left, "\alpha_{k+1}"] \arrow[ddl, bend left, shift right, "\bar{\alpha}_{k+1}"'] & \cdots \arrow[l, shift left, "g_{k+1}"] \\
& & X \arrow[ul, shift left, "\beta_k"] \arrow[ur, shift left, "\beta_{k+1}"] \arrow[d, dashed, shift left, "f"] & & \\
& & Y \arrow[u, dashed, shift left, "g"] \arrow[uul, bend left, shift left, "\bar{\beta}_k"] \arrow[uur, bend right, shift right, "\bar{\beta}_{k+1}"'] & &
\end{tikzcd}$$

The Cauchy condition on Δ implies that the sequence $(\bar{\alpha}_k \circ \beta_k)_k$ is Cauchy in $[X \to Y]$, since for any $k \geqslant 0$,

$$\begin{aligned}
&[X \to Y](\bar{\alpha}_k \circ \beta_k,\ \bar{\alpha}_{k+1} \circ \beta_{k+1}) \\
&\quad = \sup_{x \in X}\{Y(\bar{\alpha}_k \circ \beta_k(x),\ \bar{\alpha}_{k+1} \circ \beta_{k+1}(x))\} \\
&\quad = \sup_{x \in X}\{Y(\bar{\alpha}_{k+1} \circ f_k \circ g_k \circ \beta_{k+1}(x),\ \bar{\alpha}_{k+1} \circ \beta_{k+1}(x))\} \\
&\quad \leqslant \sup_{x \in X_{k+1}}\{Y(\bar{\alpha}_{k+1} \circ f_k \circ g_k(x),\ \bar{\alpha}_{k+1}(x))\} \\
&\quad \leqslant \sup_{x \in X_{k+1}}\{X_{k+1}(f_k \circ g_k(x),\ x)\} \\
&\quad \leqslant \delta\langle f_k, g_k\rangle.
\end{aligned}$$

By Theorem 3.9, $[X \to Y]$ is complete so we can define $f = \lim \bar{\alpha}_k \circ \beta_k$. Similarly $g : Y \to X$ is defined as $g = \lim \alpha_k \circ \bar{\beta}_k$. Next we show, for $k \geqslant 0$, one half of $\langle f, g\rangle \circ \langle \alpha_k, \beta_k\rangle = \langle \bar{\alpha}_k, \bar{\beta}_k\rangle$:

$$\begin{aligned}
f \circ \alpha_k &= (\lim_{l} \bar{\alpha}_l \circ \beta_l) \circ \alpha_k \\
&= \lim_{l} \bar{\alpha}_l \circ \beta_l \circ \alpha_k \quad \text{[Lemma 3.10]} \\
&= \lim_{l>k} \bar{\alpha}_l \circ \beta_l \circ \alpha_l \circ f_{kl} \\
&= \lim_{l>k} \bar{\alpha}_l \circ (\lim_{m>l} g_{lm} \circ f_{lm}) \circ f_{kl} \quad \text{[by assumption 2.]} \\
&= \lim_{l>k}(\lim_{m>l} \bar{\alpha}_l \circ g_{lm} \circ f_{lm}) \circ f_{kl} \quad \text{[Lemma 3.10]} \\
&= \lim_{l>k}(\lim_{m>l} \bar{\alpha}_m \circ f_{lm} \circ g_{lm} \circ f_{lm}) \circ f_{kl}.
\end{aligned}$$

For $\varepsilon > 0$ and l and m (with $l < m$) "big enough", we have $f_{lm} \dashv_\varepsilon g_{lm}$, which implies

$$[X_l \to X_m](f_{lm} \circ g_{lm} \circ f_{lm},\ f_{lm}) \leqslant \varepsilon, \text{ and } [X_l \to X_m](f_{lm},\ f_{lm} \circ g_{lm} \circ f_{lm}) \leqslant \varepsilon,$$

by Lemma 5.2. It follows that the above sequence of equalities can be continued with

$$\begin{aligned}
&\lim_{l>k}(\lim_{m>l} \bar{\alpha}_m \circ f_{lm} \circ g_{lm} \circ f_{lm}) \circ f_{kl} \\
&\quad = \lim_{l>k}(\lim_{m>l} \bar{\alpha}_m \circ f_{lm}) \circ f_{kl} \\
&\quad = \lim_{l>k}(\lim_{m>l} \bar{\alpha}_l) \circ f_{kl} \\
&\quad = \lim_{l>k} \bar{\alpha}_l \circ f_{kl} \\
&\quad = \lim_{l>k} \bar{\alpha}_k \\
&\quad = \bar{\alpha}_k.
\end{aligned}$$

Similarly one proves $\beta_k \circ g = \bar{\beta}_k$. This shows that $\langle f, g \rangle$ is a mediating arrow. Furthermore it is unique: if $\langle p, q \rangle : X \to Y$ is another mediating arrow then

$$\begin{aligned} p &= p \circ 1_X \\ &= p \circ (\lim \alpha_k \circ \beta_k) \quad \text{[by assumption 1]} \\ &= \lim p \circ \alpha_k \circ \beta_k \quad \text{[Lemma 3.10]} \\ &= \lim \bar{\alpha}_k \circ \beta_k \\ &= f, \end{aligned}$$

and similarly $q = g$. □

Lemma 6.3 plays a crucial role in the proof of the following theorem.

Theorem 6.4. *Any Cauchy chain in* $Cqum^{\mathrm{a}}$ *has a colimit.*

Proof. Let

$$\Delta = X_0 \xrightarrow{\langle f_0, g_0 \rangle} X_1 \xrightarrow{\langle f_1, g_1 \rangle} \cdots$$

be Cauchy. The colimit we are looking for is given, as usual, by the inverse limit:

$$X = \{(x_k)_k \mid \forall k \geqslant 0,\ x_k \in X_k \text{ and } g_k(x_{k+1}) = x_k\}.$$

On X a distance is defined, for $(x_k)_k, (y_k)_k$ in X, by

$$X((x_k)_k, (y_k)_k) = \sup X_k(x_k, y_k).$$

It is a nice little exercise on generalized ultrametrics – left to the reader – to prove that X is a complete quasi ultrametric space, in which limits are determined elementwise: that is, for any Cauchy sequence $(x^k)_k$ in X, with $x^k = (x_0^k, x_1^k, \ldots)$,

$$\lim x^k = (\lim x_0^k,\ \lim x_1^k,\ \ldots).$$

Next X is turned into a cone by defining, for every $k \geqslant 0$, an arrow $\langle \alpha_k, \beta_k \rangle : X_k \to X$ as follows: for x in X_k,

$$\begin{aligned} \alpha_k(x) = (&\lim_{l>k} g_{0,l} \circ f_{kl}(x),\ \lim_{l>k} g_{1,l} \circ f_{kl}(x), \ldots, \lim_{l>k} g_{kl} \circ f_{kl}(x), \\ &\lim_{l>k+1} g_{k+1,l} \circ f_{kl}(x), \ldots), \end{aligned}$$

and for $(x_0, x_1, \ldots) \in X$,

$$\beta_k((x_0, x_1, \ldots)) = x_k.$$

The limits in the definition of α_k exist by Lemma 6.2 and α_k maps indeed into X. Also α_k and β_k are non-expansive and continuous, and $\langle \alpha_{k+1}, \beta_{k+1} \rangle \circ \langle f_k, g_k \rangle = \langle \alpha_k, \beta_k \rangle$. Furthermore,

$$\forall \varepsilon > 0\ \exists N\ \forall k \geqslant N,\ \alpha_k \dashv_\varepsilon \beta_k,$$

which is an immediate consequence of

1. $\forall \varepsilon > 0\, \exists N\, \forall k \geqslant N,\ [X \to X](\alpha_k \circ \beta_k,\ 1_X) < \varepsilon$, and
2. $\forall \varepsilon > 0\, \exists N\, \forall k \geqslant N,\ [X_k \to X_k](1_{X_k},\ \beta_k \circ \alpha_k) < \varepsilon$.

We prove only the former statement (the latter is easy): for $k \geqslant 0$ and $(x_n)_n$ in X,

$$X(\alpha_k \circ \beta_k((x_n)_n), (x_n)_n) = \sup_{m>k}\{X_m(\lim_{l>m} g_{ml} \circ f_{kl}(x_k),\ x_m)\},$$

by definition of the metric on X. Because for all $m > k$,

$$\begin{aligned} &X_m(\lim_{l>m} g_{ml} \circ f_{kl}(x_k),\ x_m) \\ &\quad = \underset{\leftarrow\, l>m}{\lim}\ X_m(g_{ml} \circ f_{kl}(x_k),\ x_m) \quad \text{[cf. Definition 3.4]} \\ &\quad = \underset{\leftarrow\, l>m}{\lim}\ X_m(g_{ml} \circ f_{kl} \circ g_{kl}(x_l),\ g_{ml}(x_l)) \quad [(x_n)_n \text{ is an element of } X] \\ &\quad \leqslant \underset{\leftarrow\, l}{\lim}\ X_l(f_{kl} \circ g_{kl}(x_l),\ x_l) \\ &\quad \leqslant \delta\langle f_{kl}, g_{kl}\rangle, \end{aligned}$$

statement 1 follows from the fact that our chain is Cauchy.

The proof of the present theorem is concluded by the verification of the conditions of Lemma 6.3, from which it follows that

$$(\langle \alpha_k, \beta_k\rangle : X_k \to X)_k$$

is a colimiting cone. Firstly, for all $k \geqslant 0$,

$$\beta_k \circ \alpha_k = \lim_{l>k} g_{kl} \circ f_{kl},$$

by definition of α_k and β_k. Secondly, we show $\lim \alpha_k \circ \beta_k = 1_X$. Because for all $\varepsilon > 0$ there exists a natural number K such that $\alpha_k \dashv_\varepsilon \beta_k$, for all $k \geqslant K$, it follows that for all x and $\bar{x}$ in X,

$$X_k(\beta_k(x), \beta_k(\bar{x})) \approx^{\varepsilon}_{[0,1]} X(\alpha_k \circ \beta_k(x), \bar{x}).$$

This implies

$$\begin{aligned} X(x, \bar{x}) &= \sup X_k(\beta_k(x), \beta_k(\bar{x})) \\ &= \underset{\leftarrow}{\lim}\, X(\alpha_k \circ \beta_k(x), \bar{x}), \end{aligned}$$

which proves $x = \lim \alpha_k \circ \beta_k(x)$. $\square$

It can be deduced from the proof above that the converse of Lemma 6.3 holds as well. Thus:

Theorem 6.5. *Consider a Cauchy chain*

$$\Delta = X_0 \xrightarrow{\langle f_0, g_0\rangle} X_1 \xrightarrow{\langle f_1, g_1\rangle} \cdots$$

and let $(\langle \alpha_k, \beta_k \rangle : X_k \to X)_k$ *be a cone from* Δ *to* X. *Then* X *is a colimit* if and only if
1. $\lim \alpha_k \circ \beta_k = 1_X$ *and* 2. $\forall k \geqslant 0,\ \beta_k \circ \alpha_k = \lim_{l>k} g_{kl} \circ f_{kl}$.

Corollary 6.6. *Let* Δ *and* $(\langle \alpha_k, \beta_k \rangle : X_k \to X)_k$ *be as in Theorem* 6.5. *If* X *is a colimit then*

$$\forall \varepsilon > 0\ \exists N\ \forall n \geqslant N,\ \alpha_n \dashv_\varepsilon \beta_n.$$

The converse does not hold: take for Δ the constant chain consisting of the ordered space $\{0, 1\}$ with $0 \leqslant 1$, and for X the space $\{1\}$.

7. Fixed points of functors

Two theorems will be formulated on the existence of fixed points of functors, which can be seen as categorical versions of parts 1 and 2 of Theorem 3.7. These theorems generalize the standard order-theoretic and (ultra)metric solutions (of [21] and [3, 17], respectively).

As usual, we shall concentrate on functors with so-called *local* properties (cf. [21]: returning for a moment to the category *Gum* of all generalized ultrametric spaces, a functor F : *Gum* $\to$ *Gum* is *locally non-expansive* if, for all gum's X and Y, the function

$$F_{XY} : Y^X \to F(Y)^{F(X)},$$

which maps $f : X \to Y$ to $F(f) : F(X) \to F(Y)$, is non-expansive. Similarly one defines the notions of *locally continuous* and *locally contractive*. (In the formulation of the latter, one should be careful with the order of the quantification: there should exist $\varepsilon < 1$ such that for all X and Y, F_{XY} is contractive "with factor ε".)

As announced in Section 6, fixed points of functors will be constructed using complete quasi ultrametric spaces. Recall that *Cqum*$^{\mathrm{a}}$ is the category of such spaces together with pairs of non-expansive and continuous functions between them. We shall concentrate on functors F^{a} : *Cqum*$^{\mathrm{a}}$ $\to$ *Cqum*$^{\mathrm{a}}$ that are "stemming from" functors F : *Cqum* $\to$ *Cqum*, where *Cqum* is the category of complete quasi ultrametric spaces with (single) non-expansive and continuous functions as arrows. More precisely, any functor F : *Cqum* $\to$ *Cqum* defines a functor F^{a} : *Cqum*$^{\mathrm{a}}$ $\to$ *Cqum*$^{\mathrm{a}}$ which acts on objects as F does, and maps an arrow $\langle f, g \rangle : X \to Y$ to $F^{\mathrm{a}}(\langle f, g \rangle) = \langle F(f), F(g) \rangle$: $F(X) \to F(Y)$. We shall use the following lemma, which can be readily verified.

Lemma 7.1. *Consider a functor* F : *Cqum* $\to$ *Cqum and an arrow* $\langle f, g \rangle : X \to Y$ *in Cqum*$^{\mathrm{a}}$.
1. *If* F *is locally non-expansive then* $\delta\langle F(f), F(g)\rangle \leqslant \delta\langle f, g\rangle$.
2. *If* F *is locally contractive with factor* ε *then* $\delta\langle F(f), F(g)\rangle \leqslant \varepsilon \cdot \delta\langle f, g\rangle$. $\square$

Note that it follows for a locally non-expansive functor F that $f \dashv_\varepsilon g$ implies $F(f) \dashv_\varepsilon F(g)$.

We are ready for the first fixed point theorem, which is the categorical version of part 1 of Theorem 3.7.

Theorem 7.2. *Let $F : Cqum \to Cqum$ be locally non-expansive. If F is locally continuous and if there exists X and $\langle f,g\rangle : X \to F(X)$ such that $f \dashv g$, then F has a fixed point.*

Proof. Consider the following chain in $Cqum^{\mathrm{a}}$,

$$\Delta = X_0 \xrightarrow{\langle f_0, g_0\rangle} X_1 \xrightarrow{\langle f_1, g_1\rangle} \cdots,$$

which is inductively defined by $X_0 = X$, $X_{n+1} = F^{\mathrm{a}}(X_n) = F(X_n)$, $\langle f_0, g_0\rangle = \langle f,g\rangle$, and

$$\langle f_{n+1}, g_{n+1}\rangle = F^{\mathrm{a}}(\langle f_n, g_n\rangle) = \langle F(f_n), F(g_n)\rangle.$$

Because F is locally non-expansive, the chain is trivially Cauchy by Lemma 7.1: for all $n \geqslant 0$, $f_n \dashv g_n$. By (the proof of) Theorem 6.4, it has a colimit

$$(\langle \alpha_n, \beta_n\rangle : X_n \to X)_n,$$

satisfying

1. $\lim \alpha_n \circ \beta_n = 1_X$, and 2. $\forall k \geqslant 0$, $\beta_k \circ \alpha_k = \lim_{l>k} g_{kl} \circ f_{kl}$.

Because F is locally continuous, this implies

$$\begin{aligned} 1.\ \lim F(\alpha_n) \circ F(\beta_n) &= \lim F(\alpha_n \circ \beta_n) \\ &= F(\lim \alpha_n \circ \beta_n) \\ &= F(1_X) \\ &= 1_{F(X)}, \end{aligned}$$

and, for all $k \geqslant 0$,

$$\begin{aligned} 2.\ F(\beta_k) \circ F(\alpha_k) &= F(\beta_k \circ \alpha_k) \\ &= F(\lim_{l>k} g_{kl} \circ f_{kl}) \\ &= \lim_{l>k} F(g_{kl} \circ f_{kl}) \\ &= \lim_{l>k} F(g_{kl}) \circ F(f_{kl}) \\ &= \lim_{l>k} g_{k+1,l+1} \circ f_{k+1,l+1}. \end{aligned}$$

By Lemma 6.3, it follows that

$$(F^{\mathrm{a}}(\langle \alpha_n, \beta_n\rangle) : F^{\mathrm{a}}(X_n) \to F^{\mathrm{a}}(X))_n,$$

which is equal to

$$(\langle F(\alpha_n), F(\beta_n)\rangle) : X_{n+1} \to F(X))_n,$$

is a colimit of

$$F^{a}(\Delta) = X_1 \xrightarrow{\langle f_1, g_1\rangle} X_2 \xrightarrow{\langle f_2, g_2\rangle} \cdots .$$

Since Δ and $F^{a}(\Delta)$ are the same but for the first element, the fact that both X and $F(X)$ are colimits implies that they are isomorphic. $\square$

A simple example is the following. Let $(-)_\perp : Cqum \to Cqum$ be defined, for any cqum X, as follows: $(X)_\perp$ is the disjoint union of $\{\perp\}$ and X, with distance, for a and b in $(X)_\perp$,

$$(X)_\perp(a,b) = \begin{cases} 0 & \text{if } a = \perp, \\ 1 & \text{if } a \in X \text{ and } b = \perp, \\ X(a,b) & \text{if } a \in X \text{ and } b \in X. \end{cases}$$

On arrows $(-)_\perp$ is defined as one would expect. This defines a functor that is both locally non-expansive and locally continuous, and applying Theorem 7.2 with $X = \{\perp\}$ yields a fixed point, which is actually a complete partial order: it is (isomorphic to) $\bar{\omega}$, the set of natural numbers plus infinity, with the usual ordering.

If X is a partial order then $(X)_\perp$ is the usual "lifting" of X. It is a special case of what could be called "ε-lifting", which is defined as follows. For ε with $0 < \varepsilon \leqslant 1$, let the set $(X)_\perp$ be as before but now with distance, for a and b in $(X)_\perp$,

$$(X)_\perp(a,b) = \begin{cases} 0 & \text{if } a = \perp, \\ 1 & \text{if } a \in X \text{ and } b = \perp, \\ \varepsilon \cdot X(a,b) & \text{if } a \in X \text{ and } b \in X. \end{cases}$$

Again Theorem 7.2 applies. For $X = \{\perp\}$ and $\varepsilon = 1/2$, the resulting fixed point is again $\bar{\omega}$ but now with metric as in Example 2.1.

The second fixed point theorem is the categorical version of part 2 of Theorem 3.7.

Theorem 7.3. *If $F : Cqum \to Cqum$ is locally contractive and locally continuous then F has a fixed point, which is unique (up to isomorphism). This fixed point is (both an initial F-algebra and) a final F-coalgebra.*

Proof. Let X_0 be an arbitrary complete quasi ultrametric space, and let $\langle f_0, g_0\rangle : X_0 \to F(X_0)$ be an arbitrary arrow. As in the proof of Theorem 7.2, we can inductively define a chain $\Delta = (\langle f_n, g_n\rangle : X_n \to X_{n+1})_n$. Part 2 of Lemma 7.1 implies that it is Cauchy. As before this leads to the existence of a fixed point. Suppose there are two such fixed points, X and Y with isomorphisms $k : X \to F(X)$ and $l : Y \to F(Y)$. It follows from the local properties of F that

$$\Phi : (X \longrightarrow Y) \to (X \longrightarrow Y),$$

defined, for h in $X \longrightarrow Y$, by $\Phi(h) = l^{-1} \circ F(h) \circ k$, is continuous and contractive. Therefore it has by Theorem 3.7 a unique fixed point $\pi : X \to Y$ with $\pi = l^{-1} \circ F(\pi) \circ k$, or, equivalently, $l \circ \pi = F(\pi) \circ k$. Similarly one can prove that there is a unique function $\rho : Y \to X$ such that $k \circ \rho = F(\rho) \circ l$; that 1_X is the unique function in $X \longrightarrow X$ such that $k \circ 1_X = F(1_X) \circ k$; and that 1_Y is the unique function in $Y \to Y$ such that $l \circ 1_Y = F(1_Y) \circ l$. Because also $k \circ (\rho \circ \pi) = F(\rho \circ \pi) \circ k$ and $l \circ (\pi \circ \rho) = F(\pi \circ \rho) \circ l$, it follows that $1_X = \rho \circ \pi$ and $1_Y = \pi \circ \rho$. Thus $X \cong Y$. Alternatively, uniqueness follows from the fact that any fixed point is a final F-coalgebra, which can be proved by a similar argument, and the fact that any two final F-coalgebras are isomorphic (cf. [17, 7]). □

An example: let 1 be a one element set and ε such that $0 < \varepsilon < 1$. Consider the functor that maps a cqum X to $1 + (\varepsilon \cdot X)$, where $\varepsilon \cdot X$ is like X but with all distances multiplied by ε. This functor is both locally continuous and locally contractive. For $\varepsilon = 1/2$, its unique fixed point is again the set $\bar{\omega}$, now with the (ordinary) ultrametric, for x and y in $\bar{\omega}$,

$$\bar{\omega}(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-\min\{x,y\}} & \text{if } x \neq y. \end{cases}$$

Note that this is the symmetric version of the distance on $\bar{\omega}$ in Example 2.1.

8. Algebraicity

We briefly discuss the notions of finiteness and algebraicity. Though they will not be used in the present paper, they are of crucial importance in the study of completion and topology of generalized ultrametric spaces [4].

An element a of a generalized ultrametric space X is *finite* if the function

$$X(a,-) : X \to [0,1], \quad x \mapsto X(a,x)$$

is continuous. The name "finite" is justified because for a preorder X, it means that for any chain $(x_n)_n$ in X,

$$X(a, \bigsqcup x_n) = \underleftarrow{\lim} X(a, x_n),$$

or, equivalently,

$$a \leqslant_X \bigsqcup x_n \text{ iff } \exists n,\ a \leqslant_X x_n,$$

which is the usual definition of finiteness for ordered spaces. If X is an ordinary ultrametric space then $X(a,-)$ is continuous for any a in X, hence all elements are finite.

The latter fact is at first sight somewhat disappointing and might suggest that the above notion of finiteness is not what it should be. In particular, one might expect that for ordinary ultrametric spaces, an element is finite if and only if it is isolated. Here we shall not argue any further in favour of the definition above. Rather we refer to [4], where the above definition of finiteness and the notion of algebraicity (introduced below) play a convincing role in the treatment of completion and topology.

A *basis* for a generalized ultrametric space X is a subset $B \subseteq X$ consisting of finite elements such that every element x in X is the limit $x = \lim a_n$ of a Cauchy sequence $(a_n)_n$ of elements in B. A gum X is *algebraic* if there exists a basis for X. For ordered spaces this is the usual definition. Note that any ordinary ultrametric space is algebraic. If X is algebraic then the collection B_X of all finite elements of X is the largest basis. Further note that algebraicity does not imply completeness. (Take any ordinary ultrametric space which is not complete.) If there exists a countable basis then X is ω*-algebraic*. For instance, the generalized ultrametric space A^∞ from Section 2 is algebraic with basis A^*, the set of all finite words over A. If A is countable then A^∞ is ω-algebraic. Note that any ordinary ultrametric space is algebraic but not necessarily ω-algebraic. Examples of the latter are compact ultrametric spaces (cf. Section 9).

The categorical structure of the collection of all algebraic generalized ultrametric spaces and, more specifically, of all algebraic complete quasi ultrametric spaces remains to be further investigated. In particular, there is the question whether the latter collection is closed under the formation of colimits of Cauchy chains.

9. SFU: sequences of finite ultrametric spaces

A complete quasi ultrametric space is called *SFU* if it is the colimit in the category *Cqum*$^{\text{a}}$ of a Cauchy "Sequence of Finite quasi Ultrametrics". (Another name could be "bifinite".) Clearly this definition is in analogy to Plotkin's definition of *SFP* objects as colimits of sequences of finite partial orders [15]. It is a little different in that (metric) adjoint pairs are used instead of embedding–projection pairs (one can show that both definitions would be equivalent). Moreover the finite quasi ultrametric spaces are not required to be pointed, which amounts to having a least element in the case of partial orders. It is straightforward to show that a complete quasi ultrametric space is SFU if and only if it is SFP (see [10] for a description of SFP objects without least element). Somewhat less trivial is the following theorem. It uses the well-known fact that an ordinary (ultra)metric space X is compact if and only if it is complete and *totally bounded*: for all $\varepsilon > 0$ there exists a finite subset $E \subseteq X$ such that

$$\forall x \in X \; \exists e \in E, \; X(e,x) \leqslant \varepsilon.$$

Note that by defining $\bar{B}_\varepsilon(e) = \{x \in X \mid X(e,x) \leqslant \varepsilon\}$, the condition on E is equivalent to

$$X = \bigcup_{e \in E} \bar{B}_\varepsilon(e).$$

Therefore such a set E is called a finite *ε-cover* for X. It is called minimal if for any e and e' in E, $X(e,e') \leqslant \varepsilon$ implies $e = e'$.

The theorem below will also make use of the following well-known property of ordinary ultrametric spaces.

Lemma 9.1. *For an ordinary ultrametric space X, x and y in X, and* $\varepsilon > 0$,

$$\bar{B}_\varepsilon(x) \cap \bar{B}_\varepsilon(y) \neq \emptyset \iff \bar{B}_\varepsilon(x) = \bar{B}_\varepsilon(y).$$

Theorem 9.2. *If X is a complete ordinary ultrametric space then*

X is SFU iff X is compact.

Proof. Let X be a complete ordinary ultrametric space, and suppose X is SFU: Consider a Cauchy chain $(\langle f_n, g_n\rangle : X_n \to X_{n+1})_n$ in $Cqum^{\mathrm{a}}$, with X_n finite for all $n \geqslant 0$, and arrows $(\langle \alpha_n, \beta_n\rangle : X_n \to X)_n$ in $Cqum^{\mathrm{a}}$ such that X is a colimit of Δ. We show that X is totally bounded. Let $\varepsilon > 0$. Let N be such that $\alpha_N \dashv_\varepsilon \beta_N$. Define $E = \{\alpha_N(a) \in X \mid a \in X_N\}$. Let x be any element of X. Then $\alpha_N \circ \beta_N(x) \in E$ and

$$X(\alpha_N \circ \beta_N(x),\ x) \leqslant \varepsilon.$$

This proves that E is a finite ε-cover for X. Thus X is totally bounded and since it is complete by assumption, it is compact.

Conversely, suppose that X is compact, and hence totally bounded. Let $(\varepsilon_n)_n$ be a decreasing sequence of real numbers with $\lim \varepsilon_n = 0$. Because X is totally bounded there are finite subsets $(X_n)_n$ of X such that, for every $n \geqslant 0$, X_n is a minimal ε_n-cover for X. Every X_n is a finite complete ultrametric space with ultrametric inherited from X. The sets X_n can be chosen such that for all $n \geqslant 0$,

$$X_n \subseteq X_{n+1},$$

because of Lemma 9.1. By ultrametricity, the collection

$$P_n = \{\bar{B}_{\varepsilon_n}(b) \mid b \in X_n\}$$

is a partitioning of X, for every $n \geqslant 0$, and P_n is refined by P_{n+1}. Let $f_n : X_n \to X_{n+1}$ be the inclusion, for $n \geqslant 0$. In the other direction, let $g_n : X_{n+1} \to X_n$ map an element b of X_{n+1} to the (uniquely determined) element a in X_n with $b \in \bar{B}_{\varepsilon_n}(a)$. The function g_n is non-expansive (and hence continuous): For any x and y in X_{n+1} with $g_n(x) \neq g_n(y)$,

$$\begin{aligned}
\varepsilon_n &< X(g_n(x), g_n(y)) \\
&\leqslant \max\{X(g_n(x),x),\ X(x,y),\ X(y,g_n(y))\} \\
&= X(x,y) \quad [\text{since } X(g_n(x),x) \leqslant \varepsilon_n \text{ and } X(y,g_n(y)) \leqslant \varepsilon_n.]
\end{aligned}$$

Moreover $f_n \dashv_{\varepsilon_n} g_n$, since $g_n \circ f_n = 1_{X_n}$ and $X_{n+1} \longrightarrow X_{n+1}(f_n \circ g_n, 1_{X_{n+1}}) \leqslant \varepsilon_n$. Thus we have defined a chain $(\langle f_n, g_n\rangle : X_n \to X_{n+1})_n$ in $Cqum^{\mathrm{a}}$. It is Cauchy because

$\lim \varepsilon_n = 0$. The space X can be turned into a colimiting cone of this chain as follows. For $n \geqslant 0$ let $\alpha_n : X_n \to X$ be the inclusion. In the other direction, let β_n map x in X to the (uniquely determined) element a in X_n with $x \in \bar{B}_{\varepsilon_n}(a)$. The function β_n can been seen to be non-expansive by the same argument used above for g_n. This defines a cone $(\langle \alpha_n, \beta_n \rangle : X_n \to X)_n$ in $Cqum^a$. It is colimiting because $\lim \alpha_n \circ \beta_n = 1_X$ and $\beta_n \circ \alpha_n = 1_{X_n}$, for every $n \geqslant 0$. This proves that X is SFU. □

The last part of the proof above refines a similar topological fact stating that any compact ordinary ultrametric space is the inverse limit of a sequence of finite discrete spaces (see, e.g., [20]).

One can show (cf. [16]) that generalized ultrametric spaces that are SFU are ω-algebraic: a countable basis is obtained by taking the union of the images of all the elements in the chain of which it is a colimit.

Although we feel that the category of SFU spaces is important from a computational point of view, its further study is left for another occasion. One of the questions to be addressed is, for instance, whether this category is closed under the formation of colimits of Cauchy chains.

10. A large generalized ultrametric space

We shall see that the *class* $\mathscr{G}$ of all generalized ultrametric spaces, which can be obtained from the *category Gum* by "forgetting" the arrows, can be turned into a large generalized ultrametric space. A number of categorical definitions and facts of the previous sections will be rephrased in terms of this ultrametric. For the special case of the class of *compact ordinary* ultrametric spaces, this will lead to a non-categorical fixed point theorem. The latter result, which has been independently obtained by F. Alessi, P. Baldan and G. Bellè, is only mentioned here. A proof can be found in [2].

A generalized ultrametric on $\mathscr{G}$ is defined, for gum's X and Y, by

$$\mathscr{G}(X,Y) = \inf\{\varepsilon \mid \exists \langle f,g \rangle : X \to Y,\ f \dashv_\varepsilon g\}.$$

(As in Section 6, $\langle f,g \rangle$ is here a pair of non-expansive and continuous functions $f : X \to Y$ and $g : Y \to X$.) The proof that this defines a generalized ultrametric is not difficult and therefore omitted. The ultrametric structure on $\mathscr{G}$ gives rise to the following observations:

1. Cauchy *chains* (as in the category $Cqum^a$) are simply Cauchy *sequences* in $\mathscr{G}$.
2. A locally non-expansive functor on the category *Gum* is a non-expansive function on $\mathscr{G}$. Similarly, a locally contractive functor is a contractive function on $\mathscr{G}$.

It remains to be seen whether the subclass $\mathscr{C}$ of $\mathscr{G}$ consisting of all complete quasi ultrametric spaces, with distance inherited from $\mathscr{G}$, is complete (in the metric sense of the word, that is). Nor do we have an answer to the following question: are locally continuous functors on the category $Cqum^a$ continuous functions on $\mathscr{C}$?

For complete *ordinary* ultrametric spaces, the answer to both questions is affirmative. Completeness follows from the observation that for any Cauchy sequence of complete ordinary ultrametric spaces, a (categorical) colimit can be constructed as in Theorem 6.4, which is then readily seen to be a (metric) limit. For the subclass $\mathscr{K}$ of compact ordinary ultrametric spaces, this leads to a non-categorical fixed point theorem: any contractive mapping – which need not be functorial – from $\mathscr{K}$ to itself has a fixed point which is unique up to isomorphism. This follows from the fact that $\mathscr{K}$ itself is a large complete pseudo ultrametric space, with the additional property: if two compact spaces have distance 0 then they are isomorphic. Hence Banach's theorem can be applied as usual. For a full proof see [2].

The idea of viewing the category of quasi metric spaces as a (large) quasi metric space is already present in [12], though the ultrametric above, based on ε-adjoint pairs, is new. The subcategory of cpo's has been described as a large cpo in [13].

11. Metric limits are weighted colimits

The definition of forward-limit and backward-limit in Section 3 is given in terms of backward-limits of backward-Cauchy sequences $(r_n)_n$ in $[0,1]$, which are defined as

$$\underset{\longleftarrow}{\lim}\, r_n = \sup_n \inf_{k \geqslant n} r_k.$$

From an (enriched-)categorical point of view, the latter definition is not entirely satisfactory because of the use of $\inf_{k \geqslant n}$, which cannot immediately be seen as a categorical construction. Below we briefly explain how forward-limits be defined, alternatively and equivalently, by means of the enriched-categorical notion of weighted colimit [6, 5]. Dually, backward-limits can be phrased in terms of weighted limits.

Consider a non-expansive function $f : D \to X$ between generalized ultrametric spaces, and a non-expansive function $g : D \to [0,1]$. An element x in X is a $[0,1]$-*limit of f weighted by g* if it satisfies, for all $y \in X$,

$$X(y,x) = [0,1]^D(g, X(y,f))$$

(where $X(y,f) : D \to [0,1]$ maps d in D to $X(y,f(d))$). In that case, we write $x = \lim_g f$. Dually, for non-expansive functions $f : D \to X$ and $g : D^{\mathrm{op}} \to [0,1]$, an element x in X is a $[0,1]$-*colimit of f weighted by g* if it satisfies, for all $y \in X$,

$$X(x,y) = [0,1]^{D^{\mathrm{op}}}(g, X(f,y))$$

(where $X(f,y) : D^{\mathrm{op}} \to [0,1]$ maps d in D to $X(f(d),y)$). In that case, we write $x = \mathrm{colim}_g f$.

The above definition of weighted limit (colimit) is a special instance of the enriched-categorical notion of the $\mathscr{V}$-limit ($\mathscr{V}$-colimit) of a functor F weighted by a functor G (see Ch. 6.6 of [5]).

It will be shown next that metric limits of forward-Cauchy sequences, as defined in Definition 3.4, are weighted colimits (leaving the case of backward-Cauchy sequences, which is dual, to the reader). Consider a generalized ultrametric space X. A sequence in X can be represented by a function $f : \mathcal{N} \to X$, where $\mathcal{N}$ is the collection of natural numbers with the discrete ultrametric. As usual, we shall write x_n for $f(n)$. Note that because of the discrete ultrametric on $\mathcal{N}$, any such function is non-expansive, and $\mathcal{N}^{\mathrm{op}} = \mathcal{N}$. One can easily verify that the sequence $(x_n)_n$ is forward-Cauchy if and only if there exists a (non-expansive) function

$$g : \mathcal{N} \to [0,1],$$

called a *weight function for* $(x_n)_n$, satisfying:
1. $\forall n \geqslant m \geqslant 0,\ g(n) \leqslant g(m)$;
2. $\inf g(n) = 0$;
3. $\forall n \geqslant m \geqslant 0,\ X(x_m, x_n) \leqslant g(m)$.

The function g gives for any $m \in \mathcal{N}$ the extent to which the sequence $(x_{m+k})_k$ is Cauchy: with the definition of ε-chain from Section 4, the last condition on g is equivalent to

$$\forall m \geqslant 0,\ \ (x_{m+k})_k \text{ is a } g(m)\text{-chain.}$$

Note that there exist many different weight functions for one and the same Cauchy sequence, and that by the definition above, an element x in X is a $[0,1]$-colimit of $(x_n)_n$ weighted by g if it satisfies, for all $y \in X$,

$$X(x,y) = \sup_n\, [0,1](g(n), X(x_n, y)).$$

Theorem 11.1. *Let X be a generalized ultrametric space and $(x_n)_n$ a Cauchy sequence in X. Let g be a weight function for $(x_n)_n$. For all x in X,*

$$x = \mathrm{colim}_g x_n \iff x = \lim x_n.$$

Proof. It is sufficient to prove, for all y in X,

$$\sup_n\, [0,1](g(n), X(x_n, y)) = \sup_n \inf_{k \geqslant n} X(x_k, y).$$

Let y be in X. We distinguish between the following two cases:

1. $\forall n \in \mathcal{N},\ g(n) \geqslant X(x_n, y)$: It follows from the definition of the distance on $[0,1]$ that

$$\sup_n\, [0,1](g(n), X(x_n, y)) = 0.$$

Furthermore,

$$\begin{aligned} \sup_n \inf_{k \geqslant n} X(x_k, y) &\leqslant \sup_n \inf_{k \geqslant n} g(k) \\ &= 0 \quad [\text{because for all } n \geqslant 0,\ \inf_{k \geqslant n} g(k) = 0.] \end{aligned}$$

2. $\exists n \in \mathcal{N}$, $g(n) < X(x_n, y)$: Let n_0 be the smallest natural number n such that $g(n) < X(x_n, y)$. For all $n \geqslant n_0$,

$$[0,1](X(x_n, y), X(x_{n_0}, y)) \leqslant X(x_{n_0}, x_n) \leqslant g(n_0),$$

which is equivalent (by Proposition 2.2) to

$$X(x_{n_0}, y) \leqslant \max\{g(n_0), X(x_n, y)\}.$$

It follows, for all $n \geqslant n_0$,

$$\begin{aligned} g(n) &\leqslant g(n_0) \\ &< X(x_{n_0}, y) \\ &\leqslant \max\{g(n_0), X(x_n, y)\} \\ &= X(x_n, y) \quad [\text{because } g(n_0) < X(x_{n_0}, y).] \end{aligned}$$

Thus $g(n) < X(x_n, y)$ and $X(x_{n_0}, y) \leqslant X(x_n, y)$. Moreover, for all $n \geqslant n_0$, it follows from $g(n) < X(x_n, y)$ by the same argument that

$$X(x_n, y) \leqslant X(x_{n+1}, y).$$

Therefore,

$$\begin{aligned} &\sup_n [0,1](g(n), X(x_n, y)) \\ &\quad = \sup_{n \geqslant n_0} [0,1](g(n), X(x_n, y)) \quad [\text{below } n_0, \text{ all these numbers are } 0] \\ &\quad = \sup_{n \geqslant n_0} X(x_n, y) \quad [\text{since } g(n) < X(x_n, y)] \\ &\quad = \sup_{n \geqslant n_0} \inf_{k \geqslant n} X(x_k, y) \quad [(X(x_{n_0+l}, y))_l \text{ is increasing}] \\ &\quad = \sup_n \inf_{k \geqslant n} X(x_k, y) \quad [(\inf_{k \geqslant n} X(x_k, y))_n \text{ is increasing.}] \end{aligned}$$

In both cases,

$$\sup_n [0,1](g(n), X(x_n, y)) = \sup_n \inf_{k \geqslant n} X(x_k, y). \quad \square$$

Note that it follows from the theorem above that if both g and h are weight functions for $(x_n)_n$, then

$$x = \mathrm{colim}_g x_n \iff x = \mathrm{colim}_h x_n.$$

The above characterization of forward-limit could have been taken as the definition to begin with. (Backward-limits could be defined similarly, using weighted limits.) Then our original Definition 3.4, as well as Proposition 3.2, would be consequences of the new definition. Also it would be interesting to prove some of the other results of this paper, such as Proposition 3.9, starting from the definition of limit in terms of weighted colimit.

But, this has to be dealt with elsewhere.

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