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Integrable Particle Systems vs Solutions to the KP and 2D Toda Equations

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Starting from the relation between integrable relativistic N-particle systems with hyperbolic interactions and elementary N-soliton solutions to the KP and 2D Toda equations, we show how fusion properties of the soliton solutions are mirrored by fusion properties of the Poisson commuting particle dynamics. We also obtain previously known relations between elliptic solutions and integrable N-particle systems with elliptic interactions, without invoking finite-gap integration theory. © 1997 Academic Press

CONTENTS

1. INTRODUCTION AND SUMMARY

This paper is concerned with relations between special solutions to the Kadomtsev–Petviashvili (KP) and 2D Toda equations on the one hand, and with integrable one-dimensional N-particle systems of Calogero–Moser type on the other hand. Our main focus consists of the well-known N-soliton solutions, which will be related to particle systems with hyperbolic interactions, but we also consider a class of solutions connected to elliptic particle systems.

Our results concerning the soliton–particle correspondence are mostly new, and may be viewed as generalizations of similar results for a class of soliton PDEs and lattices obtained in a joint paper with Schneider [1], and then extended to other

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equations in [2, 3]. By contrast, the results concerning elliptic particle systems and solutions to the KP and 2D Toda equations are not new: For the KP case the pertinent connection was made by Krichever a long time ago [4], whereas the 2D Toda case was handled quite recently by Krichever and Zabrodin [5]. Here, our contribution consists in deriving the key relations in a novel way, altogether bypassing finite-gap integration theory, the main tool in [4, 5].

We proceed by reviewing the necessary information on the solitons and equations at issue, and then summarize our main findings and the organization of this paper.

The soliton solutions we consider are of the form

$$\tau = \sum_{\mu_1, \dots, \mu_N = 0, 1} \exp\left(\sum_{1 \leqslant j < k \leqslant N} \mu_j \mu_k B_{jk} + \sum_{1 \leqslant j \leqslant N} \mu_j \xi_j \right), \tag{1.1}$$

with

$$\exp(B_{jk}) = \frac{(a_j - a_k)(b_j - b_k)}{(a_j - b_k)(b_j - a_k)}, \qquad j, k = 1, \dots, N. \tag{1.2}$$

Here, $a_1, \dots, a_N, b_1, \dots, b_N$ are distinct non-zero complex numbers, and the quantities $\xi_1, \dots, \xi_N$ are complex numbers depending linearly on an infinite sequence of complex evolution parameters—the hierarchy "times." (We shall discuss reality restrictions later on.) Accordingly, the τ-functions solve an infinite number of nonlinear Hirota type equations of motion, each of which involves a finite number of hierarchy times.

In this paper we only consider the simplest Hirota equation in the hierarchy, both for KP and for 2D Toda. For the KP case this Hirota bilinear equation can be written as ($\partial_j \equiv \partial / \partial t_j$)

$$\partial_1^4 \ln \tau + 6(\partial_1^2 \ln \tau)^2 + 4\partial_1 \partial_3 \ln \tau - 3\partial_2^2 \ln \tau = 0 \tag{1.3}$$

and ξ_j reads

$$\xi_j = \xi_j^0 + i \sum_{\kappa = 1}^{\infty} t_\kappa (a_j^\kappa - b_j^\kappa). \tag{1.4}$$

Thus, (1.3) only involves the evolution parameters t_1, t_2, t_3, and the remaining times may just as well be taken equal to 0 when studying (1.3). The original KP equation [6] results by setting

$$u \equiv -2\partial_1^2 \ln \tau \tag{1.5}$$

and reads

$$3\partial_2^2 u = \partial_1 (4\partial_3 u - 6u\partial_1 u + \partial_1^3 u). \tag{1.6}$$

The 2D Toda hierarchy involves τ-functions τ_ν, $\nu \in \mathbb{Z}$, and the simplest equation reads ($\partial_\pm \equiv \partial/\partial t_\pm$)

$$\partial_+ \partial_- \ln \tau_\nu = \frac{\tau_{\nu-1}\tau_{\nu+1}}{\tau_\nu^2} - 1, \qquad \nu \in \mathbb{Z}. \tag{1.7}$$

In terms of

$$\phi_\nu \equiv \mu^{-1} \ln(\tau_\nu/\tau_{\nu-1}) \tag{1.8}$$

this takes the more familiar form [7]

$$\partial_+ \partial_- \phi_\nu = \mu^{-1}(\exp \mu(\phi_{\nu+1} - \phi_\nu) - \exp \mu(\phi_\nu - \phi_{\nu-1})). \tag{1.9}$$

The soliton τ-functions τ_ν arise by substituting $\xi_j \to \xi_{j,\nu}$ in (1.1) with

$$\xi_{j,\nu} \equiv \xi_j^0 + \nu \ln(a_j/b_j) + i \sum_{\kappa=1}^{\infty} (t_{\kappa,+}(a_j^\kappa - b_j^\kappa) + t_{\kappa,-}(a_j^{-\kappa} - b_j^{-\kappa})). \tag{1.10}$$

Setting $t_{1,+} = t_+$, $t_{1,-} = t_-$ yields solutions to (1.7), for which the higher hierarchy times may be ignored. It is to be noted that the KP function $\tau(t)$ equals the 2D Toda function $\tau_\nu(t_+, t_-)$ with $\nu = 0$, $t_{\kappa,+} = t_\kappa$, $t_{\kappa,-} = 0$, $\kappa \in \mathbb{N}^*$.

Save for the $N=1$ case, it is not an easy matter to verify that the τ-functions just defined do satisfy the equations of motion (1.3) and (1.7). There are quite a few papers addressing this issue and presenting arguments of varying degrees of explicitness and rigor. To our knowledge, the above soliton τ-functions solving the KP and 2D Toda hierarchies were introduced by the Kyoto school—as a byproduct of the intimate connection between affine Lie algebras, Bogoliubov transformations, and soliton equations, which they discovered and elaborated on in great detail. Two comprehensive surveys of this impressive body of knowledge are [8] and [9]; in particular, the above soliton solutions can be found there. (We should add that we have an extra factor i in (1.4) and (1.10); it occurs with an eye on later reality restrictions, but it should be noted that it gives rise to sign changes at various places.)

In order to connect the above solitons to integrable particle systems, it is crucial to rewrite a_j and b_j as

$$a_j = \exp(\eta_j - ic_j), \qquad b_j = \exp(\eta_j + ic_j), \tag{1.11}$$

where η_j, c_j are complex numbers. After this reparametrization the KP and Toda quantities can be rewritten as

$$\xi_j = \xi_j^0 + 2 \sum_{\kappa=1}^{\infty} t_\kappa \sin(\kappa c_j) \exp(\kappa \eta_j) \tag{1.12}$$

and

$$\xi_{j,v}=\xi_j^0-2ivc_j+2\sum_{\kappa=1}^{\infty}\sin(\kappa c_j)(t_{\kappa,+}\exp(\kappa\eta_j)-t_{\kappa,-}\exp(-\kappa\eta_j)), \qquad (1.13)$$

resp., whereas (1.2) turns into

$$\exp(B_{jk})=\frac{\operatorname{sh}^2(\eta_j-\eta_k)/2+\sin^2(c_j-c_k)/2}{\operatorname{sh}^2(\eta_j-\eta_k)/2+\sin^2(c_j+c_k)/2}. \qquad (1.14)$$

Of particular interest—also as regards the soliton–particle correspondence—is the special case

$$\exp(2inc_j)=1, \qquad n=2,3,..., \quad j=1,...,N. \qquad (1.15)$$

Then we can and will take

$$c_j=n_j\pi/n, \qquad n_j\in\{1,...,n-1\}, \qquad (1.16)$$

and in the 2D Toda case we get periodicity mod n:

$$\tau_{v+n}=\tau_v, \qquad \phi_{v+n}=\phi_v, \qquad v\in\mathbb{Z}. \qquad (1.17)$$

From the viewpoint of Kac–Moody algebras, this amounts to a reduction of $A_\infty^{(1)}$ to $A_{n-1}^{(1)}$ [8, 9].

We are now in a position to summarize a principal result of this paper, which pertains to the above N-soliton τ-functions with parameters a_j and b_j of the form (1.11), where c_j is given by

$$c_j=n_jc, \qquad n_j\in\mathbb{N}^*, \qquad j=1,...,N, \qquad (1.18)$$

and c is required to satisfy

$$c\in(0,\pi/\max(n_1,...,n_N)). \qquad (1.19)$$

Our finding is that these N-soliton τ-functions can be tied in with integrable N-particle systems that are quite novel from one perspective, but whose definition involves integrable $(n_1+\cdots+n_N)$-particle systems that are well known by now.

To explain this statement in more detail, let us first specialize to the case

$$c_1=\cdots=c_N=c\in(0,\pi) \qquad (1.20)$$

where all n_j are equal to 1. Then the two systems just alluded to are identical: They are the integrable N-particle systems introduced in our joint paper with Schneider [1]. Henceforth, we denote these systems as $\mathrm{II}_{\mathrm{rel}}(c,N)$ systems. (The notation refers to type II (hyperbolic) and relativistic (as opposed to nonrelativistic) Calogero–Moser systems; to avoid confusion we are using a parameter c instead of the parameter τ from our papers [10–12], whose results we will need later on.)

Now for the special case (1.20) with $c=\pi/2$ the soliton–particle correspondence was already described in [1]. Indeed, the KP and 2D Toda equations then amount to the KdV and sine-Gordon equations. More generally, for the $A_{n-1}^{(1)}$ reduction (i.e., $c=\pi/n$), the connection of the $\mathrm{II}_{\mathrm{rel}}(c, N)$ system to N-soliton solutions of the reduced KP hierarchy was first pointed out in [2]—though we did not spell out the details there. Moreover, we conjectured in Section 5 of [2] that there exist integrable N-particle systems generalizing the $\mathrm{II}_{\mathrm{rel}}$ systems, in terms of which the general KP N-soliton solution should arise.

We are unable to rule out that this is true, but we do not consider this a plausible conjecture any longer. Indeed, our main finding referring to the N-soliton τ-functions with (1.18) and (1.19) in force is quite different: The above $(n_1+\cdots+n_N)$-particle system is simply the $\mathrm{II}_{\mathrm{rel}}(c, n_1+\cdots+n_N)$-particle system, and the "novel" N-particle system arises by restricting the analytic continuation of N among the commuting flows to a $2N$-dimensional invariant subspace of the complexified phase space. In this scenario a soliton with parameter $n_j=n$ (henceforth an s_n-soliton) can be viewed, roughly speaking, as a bound state of n $\mathrm{II}_{\mathrm{rel}}(c)$ particles.

We proceed with a more detailed summary of our results and the organization of this paper. Section 2 has a preparatory character. We first recall how Cauchy's identity can be exploited to write the above τ-functions as determinants of an $N\times N$ matrix. The asymptotics of the τ-function as one or more evolution parameters go to infinity can therefore be reduced to the asymptotics of the spectrum of this matrix. The pertinent spectral asymptotics was determined in a general setting in Appendix A of [10] and applies to the case at hand, too. In this way one quickly arrives at the well-known conservation of momenta and factorized position shifts associated with the above solitons.

In the second part of Section 2 we isolate a crucial fusion property of the τ-functions: The N-soliton τ-function with parameters c_j satisfying (1.18) and (1.19) can be obtained as a limit of the $(n_1+\cdots+n_N)$-soliton τ-function with all c_j equal to c. This involves a reparametrization of the ξ_j^0 that may appear quite ad hoc, but it is precisely this change of parameters that enables us to make contact with the $\mathrm{II}_{\mathrm{rel}}(c)$ systems.

This connection is worked out in Sections 3–7. Section 7 deals with the general case (1.18), (1.19), and we arrive at the general soliton–particle correspondence via several intermediate levels of generality studied in Sections 3–6. Each of these levels is of interest by itself and points the way to proceeding further.

Our starting point in Section 3 consists in specializing to s_1-solitons, i.e., we assume (1.20). (We shall refer to these solitons as *elementary* solitons—following Hollowood [13, 14], who studied the $A_{n-1}^{(1)}$ Toda case.) For this special case the correspondence can already be gleaned from [1–3]. But as it turns out, to handle the next levels of generality, we need various results from our papers [10–12], which concern action-angle transformations for systems of Calogero–Moser type.

The relevant material from [10] is reviewed in Section 3, starting from the above τ-functions. Proceeding in this reverse order, we will be able to explain more easily just what is involved in encoding soliton interactions via finite-dimensional

Hamiltonian systems. Furthermore, this approach enhances the cogency of our finding for the general case, which is not such as one might have expected.

In Section 4 we begin by recalling how an analytic continuation ("crossing substitution") of the $\mathrm{II}_{\mathrm{rel}}(c, N)$ system gives rise to a system with N_+ particles and N_- "antiparticles" ($N_+ + N_- = N$), which can form bound states whenever $N_+ N_- \neq 0$. The corresponding N-particle systems (denoted $\mathrm{II}_{\mathrm{rel}}(c, N_+, N_-)$ from now on) give rise to τ-functions with N_+ solitons and N_- antisolitons of the elementary (s_1) kind, which can form breather type bound states. Now in the special case $N = 2M, N_+ = N_- = M$, there exists a $2M$-dimensional invariant submanifold of the $2N$-dimensional phase space, which corresponds to M particle–antiparticle pairs in their ground state (i.e., with maximal binding energy). These breather type pairs do not breathe in the distant past and far future, but for finite times they are alive and interact.

The crux is now that these mortal breathers in the $\mathrm{II}_{\mathrm{rel}}(c, N, N)$ system (with $c \in (0, \pi/2)$) interact just as N particles (or N antiparticles) in the $\mathrm{II}_{\mathrm{rel}}(2c, N)$ system. This remarkable fact was pointed out and studied in detail in [11], and it yields the first stepping stone to extending the link of the $\mathrm{II}_{\mathrm{rel}}(c)$ systems to the τ-functions with more general c_j-values.

Indeed, in Section 4 we show how the N-soliton τ-functions with $c_1 = \cdots = c_N = 2c$ correspond to the "mortal breather" submanifold of the $4N$-dimensional $\mathrm{II}_{\mathrm{rel}}(c, N, N)$ phase space. Generalizing the state of affairs for mortal breathers via nth roots of unity, we then demonstrate in Section 5 how the s_n-soliton τ-functions (i.e., $c_1 = \cdots = c_N = nc \in (0, \pi)$) can be tied in with an invariant $2N$-dimensional real submanifold of the complexified $2nN$-dimensional $\mathrm{II}_{\mathrm{rel}}(c, nN)$ phase space—the commuting flows on this invariant submanifold amount to the $\mathrm{II}_{\mathrm{rel}}(nc, N)$ flows.

As a matter of fact, our account in Section 5 is not mathematically complete. Though the main points in the soliton–particle correspondence are presented there, some technical details remain to be supplied. The questions left open may be classified as "obviously true" or even "irrelevant" by a theoretical physicist, but they are crucial issues from a global analyst's viewpoint. We have shifted complete answers to later sections (and Appendix E) so as to render Section 5 more accessible to the less analytically inclined reader.

In the same vein, we have attempted to make Sections 6 and 7 more easily readable by first supplying the necessary algebraic ingredients (including coordinate changes to "harmonic oscillator variables" and renormalized dual Lax matrices), and then sketching the remaining analysis in general terms before embarking on the details. Section 6 contains a closeup of the $N = 1$ and arbitrary-n case. As will be made clear there, a mathematically complete picture of this special case can be readily obtained from the comprehensive study of a generalized Sutherland system, undertaken in our paper [12]. The latter system consists of n particles performing oscillatory motion around a freely moving center of mass, and we only need our results pertaining to the behavior near equilibrium—revealing that the s_n-soliton may be viewed as a $\mathrm{III}_{\mathrm{b}}$ n-particle molecule in its ground state.

In Section 7 we finally study the arbitrary-n_j case, making extensive use of the ingredients and outlook presented in Section 6. Though we complete the treatment of the equal-n_j case at the end of Section 7 (using Appendix E to supply a crucial algebraic ingredient), we are left with two precisely formulated assumptions that we do not prove in the most general case. We present considerable evidence supporting these technical assumptions, and we consider their general validity quite plausible, but a complete proof has not materialized thus far.

We proceed by noting that we will not have occasion to make use of the KP and Toda equations of motion (1.3) and (1.7) in all of the main text (Sections 2–7). Rather, we take the above τ-functions as a starting point, and show how they naturally arise in the context of the systems introduced in [1]. It is of considerable interest that for the Toda case the solution property can actually be proved by exploiting the connection. Indeed, in Appendix A we show how our results on action-angle maps for the $\mathrm{II}_{\mathrm{rel}}(c, N)$ systems (sketched in Section 3) can be used to reduce the equation of motion (1.7) for elementary soliton τ-functions to the functional equations (A.13). Taking (A.13) for granted, the solution property for the general case then readily follows from our fusion results in Section 2.

The functional equations (A.13) are proved in Appendix D, which is more generally devoted to various functional identities occurring in this paper. Specifically, we show how (A.13) follows from a more general set of identities (D.18). To our knowledge, these striking identities are new, just as their "quantum generalization" (D.22) is. (We actually prove (D.22), and then obtain (D.18) by taking a parameter to 0 that may be viewed as Planck's constant.)

As a consequence, we obtain a novel and complete proof of Eq. (1.7) for the above 2D Toda soliton τ-functions. In Appendix A we also show that a suitable specialization of the 2D Toda solitons gives rise to the solitons of the infinite (nonrelativistic) Toda lattice. More specifically, we introduce a parameter specialization that leads to the Toda lattice solitons in the quite non-obvious form obtained in [1]; the equation of motion for the latter is then immediate from (1.7). Appendix A is concluded with a study of the relation between the above 2D Toda solitons and the solitons of the infinite *relativistic* Toda lattice [15–17]. As far as we know, these interrelationships are obtained here for the first time.

In Appendix B we obtain an elliptic generalization of the sequence of functional equations encoding the 2D Toda equation of motion (1.9). It expresses the existence of a class of solutions to (1.9) that can be arrived at via the elliptic systems generalizing the hyperbolic ones—the systems denoted by $\mathrm{IV}_{\mathrm{rel}}$ in our survey [3] and lecture notes [18]. This surprising relation was recently established in [5] via extensive use of the theory of finite-gap integration. Our arguments are quite direct, and do not involve this theory at all.

In Appendix C we first tie in a special class of KP soliton τ-functions with the $\mathrm{I}_{\mathrm{rel}}$ and $\mathrm{II}_{\mathrm{nr}}$ systems of [10]. In the same spirit as in Appendix B, we then demonstrate that the KP Eq. (1.6) admits a more general class of elliptic solutions that can be obtained from the nonrelativistic elliptic Calogero–Moser systems—the systems denoted $\mathrm{IV}_{\mathrm{nr}}$ in [3, 18]. Again, though this result is not new by itself, our arguments are short and simple, by contrast to [4]. As it turns out, the core of our proof of the $\mathrm{IV}_{\mathrm{nr}}$/KP correspondence is a functional equation that can be tied in with the $N=2$ specialization of the sequence of elliptic functional equations (B.16) that expresses the $\mathrm{IV}_{\mathrm{rel}}$/Toda correspondence. We have relegated the proof of the general identities (B.16) to Appendix D, but Appendix C is essentially self-contained. (For the elliptic function lore used in Appendixes B–D we refer the reader to [19].)

In the final Appendix E, we reconsider the equal-n_j case as regards algebraic properties. Specifically, we show that the Lax matrix L from Section 5 can be diagonalized by a matrix $\mathscr{U}$ that transforms an auxiliary diagonal matrix A into the (dual) renormalized Lax matrix from Section 7, specialized to the equal-n_j case. Just as in our previous papers [10–12], the commutation relation between L and A is an essential tool in this enterprise. As a result, we arrive not only at spectral results already obtained in another way in Section 5, but also at explicit information that is crucial to complete our study of the equal-n_j case.

We conclude this introduction with some sketchy remarks on an eventual quantum version of the results obtained in this paper. We first recall that the Poisson commuting Hamiltonians of the $\mathrm{II}_{\mathrm{rel}}(c, N)$ system can be quantized as commuting analytic difference operators [20]. Now the eigenfunction transforms of the resulting quantum integrable particle systems are known for $N=2$ [3, 18], so the scattering of the quantum particles can be compared to the scattering of corresponding elementary quantum solitons, whenever the latter is believed to be known explicitly. This yields agreement for the special case $c=\pi/2$, where the pertinent quantum field theory is the sine-Gordon/massive Thirring model. For $c=\pi/2$, however, there is no sensible notion of fusion already at the classical level.

For $c=\pi/n$ and $n=3, 4, \ldots$, the quantum solitons are supposed to be described by the $(n-1)$-component affine Toda quantum field theory, and accordingly it is believed that there exist $n-1$ distinct elementary solitons and antisolitons. We are not aware of any analog of this picture in the context of the quantum $\mathrm{II}_{\mathrm{rel}}(c, N)$ particle systems. More specifically, from this quite different vantage point it appears unreasonable—if not nonsensical—to suppose that there is more than one type of particle, depending on the value c takes. In this connection it should be mentioned that the mod n periodicity (1.17) holds true for classical elementary solitons whenever c is a multiple of π/n. By contrast, for irrational c/π-values (which are of course dense in the c-interval $(0, \pi/2])$ the elementary solitons ϕ_v have no periodicity properties at all. Therefore, a putative quantum field version of the 2D Toda phase space subset associated with such a c-value would involve an infinity of components, and hence would give rise to infinitely many distinct elementary soliton types.

Now from a purely mathematical viewpoint there is no reason to believe that a reasonable classical field theory must have a non-perturbative quantum field version. In fact, even the existence of a bona fide quantum field theory that yields the sine-Gordon S-matrix has not been proved to date, and for the higher affine Toda quantum field theories there are unitarity problems even at a formal level [13, 14].

Even so, the S-matrix for two identical elementary solitons proposed by Hollowood (given by Eqs. (4.5), (4.6) in [14]) can be compared to the S-matrix arising in the two-particle $\mathrm{II}_{\mathrm{rel}}(\pi/n)$ system. Indeed, the two-particle scattering function $u(a_+, a_-, b; z)$ associated with the quantum $\mathrm{II}_{\mathrm{rel}}$ system is studied in considerable detail in [21], and combining Eqs. (4.39), (4.43), and (4.45) in l.c., we get a representation that can be compared to Hollowood's proposed S-matrix. Requiring equality, one needs first of all $a_+ = \pi(n\lambda)^{-1}$, $a_- = \pi$, and $z = \theta/2$ (where λ and θ are parameters from [14]). Then one would expect equality for $b = \pi/n$, but this expectation is not fulfilled. On the other hand, using Eq. (4.14) in [21], one sees that one does get equality when b equals $\pi(n^{-1} + (n\lambda)^{-1} + 1)$. (This b-value is outside the unitarity region of the eigenfunction transform.)

Finally, we note that our classical particle picture of an s_2-soliton may be translated to the quantum level by associating the quantum s_2-soliton with the ground state of the quantum $\mathrm{II}_{\mathrm{rel}}(c, 1, 1)$ system—whenever bound states exist in the first place. Now bound states do exist and are explicitly known, provided the (quantum) coupling is suitably restricted, cf. p. 189 in [3]. Unfortunately, one would need far more information on $N = 4$ eigenfunctions than is currently available in order to establish whether a quantum analog of fusion takes place for two ground state $\mathrm{II}_{\mathrm{rel}}(c, 1, 1)$ "molecules." At any rate, the customary heuristic fusion procedure for exact S-matrices associated with bound states is readily seen *not* to lead from the $\mathrm{II}_{\mathrm{rel}}(c)$ to the $\mathrm{II}_{\mathrm{rel}}(2c)$ S-matrix—in contrast to the classical scattering [11].

2. SOLITON τ-FUNCTIONS: ASYMPTOTICS AND FUSION

This section contains some results on the above N-soliton τ-functions that play a crucial role later on. It consists of two parts. In the first we recall how the solitons can be written as determinants of $N \times N$ matrices, and how this connection can be exploited to arrive at the well-known features of soliton scattering. In the second part we obtain a fusion property that will be shown to mirror the fusion properties of $\mathrm{II}_{\mathrm{rel}}(c)$ particles in later sections.

The connection between the solitons and matrices hinges on Cauchy's identity for the determinant of the matrix with elements

$$C_{jk} = \frac{a_j - b_j}{a_j - b_k}, \qquad j, k = 1, ..., N. \tag{2.1}$$

The identity reads

$$|C| = \prod_{1 \leq j < k \leq N} \exp B_{jk}. \tag{2.2}$$

Here, $\exp B_{jk}$ is given by (1.2), so it equals the 2×2 principal minor of C involving the indices j and k:

$$\exp B_{jk} = C_{jj} C_{kk} - C_{jk} C_{kj}. \tag{2.3}$$

Cauchy's identity entails that we can view τ-functions of the form (1.1), (1.2) as determinants of $N \times N$ matrices. Specifically, we deduce that τ can be written

$$\tau = \sum_{l=0}^{N} S_l(CD) = |1_N + CD|, \tag{2.4}$$

where

$$D = \mathrm{diag}(\exp \xi_1, ..., \exp \xi_N) \tag{2.5}$$

and where $S_l(M)$ denotes the lth (elementary) symmetric function of M—the sum of all lth order principal minors.

Next, we note that all principal minors of C are non-zero complex numbers, since we require $a_1, ..., b_N$ to be distinct. Indeed, this entails $\exp B_{jk} \in \mathbb{C}^*$—which is why B_{jk} is well defined (mod $2\pi i$) in the first place, cf. (1.2)—and each principal minor is the product of such factors. This already suffices to determine the asymptotics of the spectrum of the matrix CD when the quantities ξ_j are of the form

$$\xi_j = \alpha_j + t v_j, \qquad \mathrm{Re}\, v_N < \cdots < \mathrm{Re}\, v_1 \tag{2.6}$$

and t goes to ∞ or $-\infty$. Indeed, Theorem A2 in [10] entails that under these assumptions the matrix $CD(t)$ has simple spectrum for $|t|$ large enough, and eigenvalues $\lambda_1(t), ..., \lambda_N(t)$ satisfying

$$\lambda_j(t) \sim \exp(\alpha_j + t v_j) \prod_{\delta k < \delta j} \exp(B_{jk}), \qquad t \to \delta\infty, \quad \delta = +, -. \tag{2.7}$$

This formula amounts to a quite general "soliton scattering" result. Indeed, the logarithm of the τ-function satisfies

$$\ln \tau(t) \sim \sum_{j=1}^{N} \ln(1 + \exp(\alpha_j^{\pm} + t v_j)), \qquad t \to \pm\infty \tag{2.8}$$

with

$$\alpha_j^{\delta} \equiv \alpha_j + \sum_{\delta k < \delta j} B_{jk}, \qquad \delta = +, -. \tag{2.9}$$

Thus it reduces to a linear superposition of N one-soliton functions; the "velocities" $v_1, ..., v_N$ are conserved under the scattering, and the nonlinear interaction is solely visible in a factorized shift of the "positions" $\alpha_1, ..., \alpha_N$.

The asymptotics just described does not involve reality assumptions. Our primary goal is, however, to study real-valued KP τ-functions, and (more generally) 2D Toda τ-functions satisfying a generalized reality condition, viz.,

$$\tau_{-\nu} = \overline{\tau_\nu}, \qquad \nu \in \mathbb{Z}. \tag{2.10}$$

(Here and henceforth, a bar denotes complex conjugation.) Using (1.11) to reparametrize $a_1, ..., b_N$, we require from now on (1.18) and (1.19); moreover, all of the hierarchy evolution parameters in (1.12) and (1.13) will be taken real from now on. Unless explicitly indicated otherwise, we also take

$$\eta_1, ..., \eta_N \in \mathbb{R}, \qquad \eta_N < \cdots < \eta_1, \tag{2.11}$$

$$\operatorname{Im} \xi_j^0 \in \pi\mathbb{Z}, \qquad j = 1, ..., N. \tag{2.12}$$

Clearly, these choices entail that (2.10) is satisfied. It also follows that the rhs of (1.14) is positive, so we may and will take B_{jk} real-valued. Whenever the velocities corresponding to an evolution parameter t are distinct, the above asymptotics applies for $t \to \pm\infty$, yielding a shift of the positions $\operatorname{Re} \xi_j^0$, $j = 1, ..., N$, that is factorized in terms of pair shifts B_{jk}.

Next, we focus attention on a fixed N-soliton τ-function with (1.18), (1.19), (2.11) and (2.12) in force. We aim to show that this function may be viewed as a limit of M-soliton τ-functions with

$$M \equiv n_1 + \cdots + n_N, \qquad c_1 = \cdots = c_M = c \in (0, \pi/\max(n_1, ..., n_N)) \tag{2.13}$$

and M suitably chosen *complex* eta's. To prevent confusion with the N real eta's (2.11) that have been fixed, we shall work with complex quantities $\theta_1, ..., \theta_M$ (which are equal to $\eta_1, ..., \eta_N$ for $M = N$, of course). Moreover, we need a further reparametrization that is essential in the sequel.

To detail the latter, we first take $\theta_M < \cdots < \theta_1$, so that B_{jk} is real-valued and given by

$$B_{jk} = -2 \ln f(c; \theta_j - \theta_k), \tag{2.14}$$

where we have introduced the positive pair potential

$$f(c; x) \equiv (1 + \sin^2(c)/\operatorname{sh}^2(x/2))^{1/2}, \qquad c \in (0, \pi), \quad x > 0. \tag{2.15}$$

Then we define $q_1, ..., q_M \in \mathbb{C}$ by setting

$$\exp(\xi_j^0) = \exp(q_j) \prod_{k \neq j} f(c; \theta_j - \theta_k) \tag{2.16}$$

and taking

$$\operatorname{Im} q_j = \operatorname{Im} \xi_j^0 \in \pi\mathbb{Z}. \tag{2.17}$$

In terms of these new variables, the 2D Toda M-soliton τ-functions with $c_1 = \cdots = c_M = c$ can be written

$$\tau_v = \sum_{l=0}^{M} \sum_{\substack{I \subset \{1, ..., M\} \\ |I| = l}} \exp\left(\sum_{k \in I} (q_k(t_+, t_-) - 2ivc)\right) \prod_{\substack{m \in I \\ n \notin I}} f(c; \theta_m - \theta_n), \tag{2.18}$$

where

$$q_i(t_+, t_-) = q_i + 2 \sum_{\kappa=1}^{\infty} \sin(\kappa c)(t_{\kappa,+} \exp(\kappa\theta_i) - t_{\kappa,-} \exp(-\kappa\theta_i)), \tag{2.19}$$

whilst the KP τ-function is obtained by taking $v=0, t_+ = t, t_- = 0$.

With these formulas for real theta's in place, we proceed to study analytic continuations of $\theta_1, ..., \theta_M$. In view of (2.15) the only singularities that arise are poles for $\theta_m - \theta_n \in 2\pi i\mathbb{Z}$ and square-root branch points for $\theta_m - \theta_n \pm 2ic \in 2\pi i\mathbb{Z}$. Thus, the only ambiguities occurring in the analytic continuation of (2.18) to complex theta's are the *signs* of the f-products. (Note this sign ambiguity does not occur in any product involving a pair θ_m, θ_n for which $\theta_m - \theta_n = \pm 2ic$ (mod $2\pi i$)—for the simple reason that the pair potential $f(c; \theta_m - \theta_n)$ then vanishes.)

The upshot is, that for a fixed $q \in \mathbb{C}^M$ the τ-function has a finite, but multi-valued, analytic continuation to $\theta \in \mathbb{C}^M$, provided no pair of theta's differs by $2\pi ik, k \in \mathbb{Z}$. By contrast, the function $\exp B_{jk}$ diverges as $\theta_j - \theta_k \to 2ic$; this divergence reflects the pole of the matrix element

$$C_{jk} = \frac{\mathrm{sh}(-ic)\exp(\theta_j - \theta_k)/2}{\mathrm{sh}((\theta_j - \theta_k)/2 - ic)} \equiv M(-c; \theta_j - \theta_k) \tag{2.20}$$

arising for $\theta_j - \theta_k \to 2ic$, cf. (2.3). In the formula (2.18) this divergence is absent, since the reparametrization (2.16) ensures $\exp(\xi_j^0), \exp(\xi_k^0) \to 0$ for $\theta_j - \theta_k \to 2ic$ and q_j, q_k fixed.

After these preliminaries we are prepared to specify the choice of $\theta_1, ..., \theta_M$ that will lead to the fixed N-soliton τ-function, as will be demonstrated shortly. It reads

$$\theta_{n_1 + \cdots + n_{j-1} + k} = \eta_j + i(n_j + 1 - 2k)\, c, \qquad j = 1, ..., N, \quad k = 1, ..., n_j, \tag{2.21}$$

Thus the above proviso $\theta_j - \theta_k \notin 2\pi i\mathbb{Z}$ is satisfied in view of (2.11) and (1.19). Fixing an index set I, we now inquire when its contribution to τ_v (2.18), vanishes. Introducing the index sets

$$I_j \equiv \{n_1 + \cdots + n_{j-1} + 1, ..., n_1 + \cdots + n_{j-1} + n_j\}, \qquad j = 1, ..., N, \tag{2.22}$$

we assert that I yields no contribution unless each of $I_1, ..., I_N$ is either a subset of I or a subset of the complement I^c.

To prove this assertion, let us assume some indices of I_1 (say) belong to I, and the remaining one(s) to I^c. This assumption entails there is at least one pair $j, k \in I_1$ with $j \in I, k \in I^c$, and $|j-k| = 1$. But then $\theta_j - \theta_k$ equals $2ic$ or $-2ic$, so the associated pair term in the f-product (2.18) vanishes Thus the assertion readily follows.

As a result, the choice (2.21) guarantees that in (2.18) we need only sum over all index sets of the form

$$I_{k_1} \cup \cdots \cup I_{k_j} \equiv I(\{k_1, ..., k_j\}), \qquad 1 \leqslant k_1 < \cdots < k_j \leqslant N, \quad j = 0, 1, ..., N. \tag{2.23}$$

Therefore, introducing

$$Q_j \equiv \sum_{i \in I_j} q_i, \qquad j = 1, ..., N, \tag{2.24}$$

we can rewrite (2.18) with (2.21) in effect as

$$\tau_\nu = \sum_{m=0}^{N} \sum_{\substack{J \subset \{1, ..., N\} \\ |J| = m}} \exp\left(\sum_{j \in J} (Q_j(t_+, t_-) - 2i\nu c_j)\right) \prod_{\substack{i \in I(J) \\ j \notin I(J)}} f(c; \theta_i - \theta_j), \tag{2.25}$$

where we have

$$Q_j(t_+, t_-) = Q_j + 2 \sum_{\kappa=1}^{\infty} \sin(\kappa c_j)(t_{\kappa,+} \exp(\kappa\eta_j) - t_{\kappa,-} \exp(-\kappa\eta_j)), \tag{2.26}$$

cf. (2.19), (2.22).

We now invoke the "fusion identity" (D.3). It entails that the product in (2.25) equals the positive function

$$\prod_{\substack{j \in J \\ k \notin J}} \left(\frac{\mathrm{sh}^2(\eta_j - \eta_k)/2 + \sin^2(c_j + c_k)/2}{\mathrm{sh}^2(\eta_j - \eta_k)/2 + \sin^2(c_j - c_k)/2}\right)^{1/2} \tag{2.27}$$

up to a sign that depends on how we have continued from the region $\theta_M < \cdots < \theta_1$ (where the product in (2.25) is positive) to the $\theta \in \mathbb{C}^M$ defined by (2.21). We choose this continuation such that all of these signs are positive. It is not immediate that this is feasible, so we now digress to describe a continuation path with this property.

To this end we start from $\theta \in \mathbb{R}^M$ defined by (2.21) with c replaced by $-i\varepsilon$, and $\varepsilon > 0$ chosen small enough so that $\theta_M < \cdots < \theta_1$. (In view of our standing assumption (2.11) this is possible.) Thus we get N theta-clusters around $\eta_N, ..., \eta_1$. Now we rotate each of these clusters over $\pi/2$ and simultaneously scale $\varepsilon \uparrow c$, so that we wind up in the point (2.21). Along the path just detailed all of the functions $f(c; \theta_i - \theta_j)$ that occur in (2.25) are non-zero, so the product in (2.25) equals (2.27) up to a sign that is independent of the choice of η. But by taking $\eta_N \ll \cdots \ll \eta_1$ we can ensure that all of the pertinent $f(c; \theta_i - \theta_j)$ stay close to 1 (recall (2.15)), so this sign is positive, as advertised.

The upshot is, that we may replace the product in (2.25) by (2.27). Comparing the result to the given N-soliton τ-function, we need only choose

$$\exp(Q_j) = \exp(\xi_j^0) \prod_{k \neq j} \exp(B_{jk}/2) \tag{2.28}$$

(with $\exp(B_{jk})$ now given by (1.14), of course) to arrive at the desired equality. Recalling (2.12) and noting positivity of the product, it follows that $\mathrm{Re}\, Q_j$ is uniquely determined and $\mathrm{Im}\, q_i \in \pi\mathbb{Z}$ can be chosen such that (cf. (2.24))

$$\mathrm{Im}\, Q_j \equiv \sum_{i \in I_j} \mathrm{Im}\, q_i = \mathrm{Im}\, \xi_j^0 \pmod{2\pi}. \tag{2.29}$$

In summary, we have achieved our goal: We have shown that N-soliton τ-functions with (1.18), (1.19), (2.11) and (2.12) in effect may be obtained from M-soliton τ-functions satisfying (2.13). From the viewpoint of τ-functions the above reparametrization and the associated fusion relation may seem quite unnatural. But to connect the above soliton τ-functions to integrable particle systems both ingredients are indispensable, as will become clear shortly.

3. $\mathrm{II}_{\mathrm{REL}}$ PARTICLES VS ELEMENTARY SOLITONS

As we have seen in the previous section, the τ-function for the elementary soliton case (1.20) can be written as a sum of the symmetric functions S_l of the $N \times N$ matrix CD, cf. (2.4). More specifically, the Cauchy matrix is given by (2.20) in this case, and after the reparametrization (2.16) the diagonal matrix D (2.5) satisfies

$$D(\theta, q)_{jj} = \exp(q_j) \prod_{k \neq j} f(c; \theta_j - \theta_k) \tag{3.1}$$

when we take the evolution parameters and ν equal to 0.

As a result, we have actually arrived in a few steps at the N independent Hamiltonians in involution that define the hyperbolic relativistic Calogero–Moser systems of [1]. Indeed, viewing from now on $\theta_1, ..., \theta_N$ and $q_1, ..., q_N$ as canonically conjugate variables, these are simply the symmetric functions

$$S_l(\theta, q) = \sum_{\substack{I \subset \{1, ..., N\} \\ |I| = l}} \exp\left(\sum_{i \in I} q_i\right) \prod_{\substack{m \in I \\ n \notin I}} \left(1 + \frac{\sin^2(c)}{\mathrm{sh}^2(\theta_m - \theta_n)/2}\right)^{1/2}, \quad l = 1, ..., N \tag{3.2}$$

of the reparametrized matrix CD. (The Poisson commutativity of $S_1, ..., S_N$ is not supposed to be obvious; it boils down to a sequence of functional equations proved in [1].)

Now at first sight the corresponding commuting flows seem to have no relation whatsoever to the commuting flows associated to the soliton hierarchies. Indeed, the latter are encoded in the linear dependence of ξ_j on the hierarchy "times" $t_1, t_2, ...$ for the KP case, and $t_{1,\pm}, t_{2,\pm}, ...$ for the 2D Toda case, cf. (1.4) and (1.10). The generating Hamiltonians are clearly given by

$$\hat{H}_{\kappa,\pm}(\theta) = 2\kappa^{-1} \sin(\kappa c) \sum_{j=1}^{N} \exp(\pm\kappa\theta_j), \qquad \kappa = 1, 2, ..., \tag{3.3}$$

for the Toda case, and by

$$\hat{H}_\kappa = \hat{H}_{\kappa, +}, \qquad \kappa = 1, 2, ..., \tag{3.4}$$

for the KP case.

The Hamiltonians just defined generate *free* flows, which manifestly commute. By contrast, the N solitons *interact*: For asymptotic times their positions are shifted as compared to a linear superposition of N freely moving solitons. We have already recalled the character of the interaction in a quite general setting, cf. the first part of Section 2. For a special equation in the hierarchy the notions of "time" t and "position" y refer to special hierarchy evolution parameters, depending on the case at hand. For $t \to \pm\infty$ the y-dependence of the τ-function is then carried by the quantities α_j in (2.6)–(2.9); it is of the form $\alpha_j = \alpha_j^0 + y\mu_j$. Thus the quantities B_{jk}/μ_j can be interpreted as position shifts.

To model the solitons as point particles, one should therefore construct *interacting* hamiltonian dynamics, giving rise to an evolution of canonical particle positions $x_1(t), ..., x_N(t)$, whose long-time asymptotics exhibits the same factorized shifts as the soliton positions. Now for the case at issue there are obvious candidates for such particle positions. Indeed, let us write the eigenvalues of the matrix CD as $\exp x_1, ..., \exp x_N$. Then we clearly have

$$\sum_{\substack{I \subset \{1, ..., N\} \\ |I| = l}} \exp\left(\sum_{i \in I} x_i\right) = S_l(\theta, q), \qquad l = 1, ..., N, \tag{3.5}$$

with S_l given by (3.2). The crux is, that since the functions $S_1, ..., S_N$ Poisson commute, the functions $x_1(\theta, q), ..., x_N(\theta, q)$ Poisson commute, too, and hence may be viewed as canonical particle positions. By construction, they have a highly nonlinear (interacting) dependence on the hierarchy times when the quantities $q_1(t), ..., q_N(t)$ evolve linearly (freely) as specified above, and this interaction makes itself felt via a factorized shift of the asymptotic particle positions, as desired.

Taking the picture just sketched for granted, it should be emphasized that at this stage it is fully unclear that one can view the "time" dependence of the position vector $x(t_1, t_2, ...)$ for the KP case (say) as being generated by commuting Hamiltonians $H_1(x, p), H_2(x, p), ...$, where $p_1, ..., p_N$ are variables canonically conjugate to $x_1, ..., x_N$. As it happens, though, this is not only true, but the pertinent Hamiltonians can actually be written down in terms of the above matrix CD: One can take

$$H_\kappa(c; x, p) = 2\kappa^{-1} \sin(\kappa c) \operatorname{Tr}(C(x) D(x, p))^\kappa, \qquad \kappa = 1, 2, ... \tag{3.6}$$

for the KP case, whilst for the 2D Toda case one needs

$$H_{\kappa, \pm}(c; x, p) = H_\kappa(c; x, \pm p), \qquad \kappa = 1, 2, ... \tag{3.7}$$

Here, $C(x)$ is the matrix (2.20) (with $\theta \to x$) and the diagonal matrix $D(x, p)$ is given by (3.1). Since the power traces of a matrix can be written as polynomials in its symmetric functions (and vice versa), the involutivity of the functions $S_1, ..., S_N$ (3.2) amounts to that of the Hamiltonians $H_1, ..., H_N$ (3.6).

We shall presently recall how this remarkable state of affairs (*self-duality*) comes about. While doing so, we have occasion to introduce various objects that are indispensable for the particle-soliton correspondence in the general case (1.18), which we aim to establish in several steps. Before embarking on this additional machinery, however, we can already explain why and where a seemingly straightforward one-step generalization of the above ideas to an even more general correspondence of N solitons with *arbitrary* c_j-values and an integrable N-particle system with the same scattering breaks down.

This obvious generalization proceeds by first substituting (1.11) in the above formulas (2.1)–(2.9). Specializing to the KP case for convenience, we also put

$$\exp \xi_j = \exp\left(q_j + 2 \sum_{\kappa=1}^{\infty} t_\kappa \sin(\kappa c_j) \exp(\kappa \eta_j)\right) \prod_{k \neq j} \exp(-B_{jk}/2), \tag{3.8}$$

where $\exp(B_{jk})$ is given by (1.14). Then we may once again view $q_1, ..., q_N$ and $\eta_1, ..., \eta_N$ as canonically conjugate variables, and the t_κ-dependence of $q(t_1, t_2, ...)$ is now governed by free Hamiltonians

$$\hat{H}_\kappa(c_1, ..., c_N; \eta) = 2\kappa^{-1} \sum_{j=1}^{N} \sin(\kappa c_j) \exp(\kappa \eta_j), \qquad \kappa = 1, 2, ... \tag{3.9}$$

Moreover, when we write the eigenvalues of CD as $\exp x_1, ..., \exp x_N$, then their asymptotics once more models the scattering of these more general solitons.

So why does'nt this yield a starting point for setting up a soliton-particle correspondence for the case of general c_j-values? The answer to this question is quite simple: The symmetric functions $S_l(c_1, ..., c_N; \eta, q)$, $l = 1, ..., N$, of CD do not Poisson commute in this general case, so it is already inconsistent to view $x_1, ..., x_N$ as canonical particle positions, cf. (3.5). Thus it does not even make sense to ask whether the time dependence of $x(t_1, t_2, ...)$ is governed by commuting Hamiltonians $H_\kappa(c_1, ..., c_N; x, p)$, $\kappa = 1, 2, ...$

After this brief digression on the case of arbitrary c_j-values, we return to the case of equal c_j-values, and recall how the Hamiltonians (3.6), (3.7) arise. To this end we introduce the self-adjoint matrix (Lax matrix) with elements

$$L(c; x, p)_{jk} = e_j e_k \exp(-x_j) M(c; x_j - x_k). \tag{3.10}$$

Here, the function M is given by (2.20) and the vector e reads

$$e_j(c; x, p) = \exp((x_j + p_j)/2) V_j^{1/2}, \tag{3.11}$$

where we have introduced the jth potential function

$$V_j(c; x) \equiv \prod_{k \neq j} f(c; x_j - x_k), \qquad f(x) \equiv (1 + \sin^2(c)/\mathrm{sh}^2(x/2))^{1/2}. \tag{3.12}$$

Thus, L can be obtained by a similarity transformation from the matrix $C(x)^T D(x, p)$. (Here and below, T denotes transposition.) Correspondingly, (3.6) may be replaced by

$$H_\kappa = 2\kappa^{-1} \sin(\kappa c) \operatorname{Tr} L^\kappa, \qquad \kappa = 1, 2, \ldots, \tag{3.13}$$

and, more generally, the Toda Hamiltonians (3.7) can be written

$$H_{\kappa, \delta} = 2\kappa^{-1} \sin(\kappa c) \operatorname{Tr} L^{\delta\kappa}, \qquad \kappa = 1, 2, \ldots, \quad \delta = +, -. \tag{3.14}$$

(For $\delta = -$ this relation follows from the fact that $L(x, p)^{-1}$ is a diagonal similarity transform of $L(x, -p)^T$, cf. Lemma B2 in [11].)

Now the key to establish the connection between the particle variables x, p and the elementary soliton variables θ, q is a commutation relation involving the Lax matrix L and the auxiliary matrix

$$A(x) \equiv \mathrm{diag}(\exp x_1, \ldots, \exp x_N). \tag{3.15}$$

It reads

$$\coth(ic)[A, L] = 2e \otimes e - AL - LA \tag{3.16}$$

and follows readily from the definitions. It can be used to show that for all (x, p) in the phase space

$$\Omega(N) \equiv \{(x, p) \in \mathbb{R}^{2N} \mid x_N < \cdots < x_1\} \tag{3.17}$$

the spectrum of L is non-degenerate. Now L is not only self-adjoint, but actually positive on $\Omega(N)$. (Indeed, from Cauchy's identity one infers that the principal minors of L are positive.) Thus, for a fixed $(x, p) \in \Omega(N)$ the spectrum of L can be written as

$$\sigma(L) = \{\exp \theta_1, \ldots, \exp \theta_N\}, \qquad \theta_N < \cdots < \theta_1. \tag{3.18}$$

Next, the fundamental commutation relation (3.16) can be exploited to prove the existence of a unitary matrix U such that one has

$$UL(c; x, p)\, U^* = A(\theta) \tag{3.19}$$

$$UA(x)\, U^* = L(-c; \theta, q) \tag{3.20}$$

for certain real numbers $q_1, \ldots, q_N$.

It follows from straightforward linear algebra that in this way one obtains a bijection

$$\Phi\colon \Omega(N) \to \hat{\Omega}(N), \qquad (x, p) \mapsto (\theta, q), \tag{3.21}$$

where $\hat{\Omega}(N)$ is the "elementary soliton phase space"

$$\hat{\Omega}(N) \equiv \{(\theta, q) \in \mathbb{R}^{2N} \mid \theta_N < \cdots \theta_1\}. \tag{3.22}$$

Moreover, this map is involutive: It equals its own inverse. (This self-duality property can be gleaned from (3.19), (3.20); notice also the symmetric roles of L and A in the commutation relation (3.16).) It is true as well, but harder to prove, that Φ is a canonical transformation when Ω and $\hat{\Omega}$ are equipped with the symplectic forms

$$\omega \equiv \sum_{j=1}^{N} dx_j \wedge dp_j, \tag{3.23}$$

$$\hat{\omega} \equiv -\sum_{j=1}^{N} d\theta_j \wedge dq_j. \tag{3.24}$$

(Thus one has $\Phi^* \hat{\omega} = \omega$.)

In fact, Φ^{-1} is intimately related to the wave maps (Møller transformations) for the commuting $\mathrm{II}_{\mathrm{rel}}$ dynamics, and this relation is a key ingredient in our proof [10] that Φ is canonical. (To ease comparison with [10] and other previous work, we should add that we employed different orderings and variables $q, \theta, \hat{q}, \hat{\theta}$ instead of the variables x, p, q, θ used here.) The proof of [10] involves considerable technicalities; for a more leisurely account of the relation of Φ to scattering theory, and its resulting canonicity and intertwining properties, we refer to Section 5.2 of our recent lecture notes [18].

From the above features of Φ it is clear that it serves as an action-angle map: It linearizes the flows generated by Hamiltonians of the form

$$H_h = \mathrm{Tr}\, h(\ln L), \tag{3.25}$$

with h a real-valued smooth function. Indeed, one has

$$(H_h \circ \Phi^{-1})(\theta, q) = \sum_{j=1}^{N} h(\theta_j) \equiv \hat{H}_h(\theta), \tag{3.26}$$

so the nonlinear flow

$$(x, p) \mapsto (x(t), p(t)) \tag{3.27}$$

generated by H_h maps to the linear flow

$$(\theta, q) \mapsto (\theta, q(t)), \qquad q_j(t) = q_j + th'(\theta_j), \quad j = 1, ..., N \tag{3.28}$$

generated by $\hat{H}_h$. Hence, this holds true in particular for the choices

$$h_{\kappa,\pm}(u)=2\kappa^{-1}\sin(\kappa c)\exp(\pm\kappa u), \qquad \kappa=1,2,... \tag{3.29}$$

which yield the above interacting Hamiltonians $H_{\kappa,\pm}(x,p)$ (3.14) and free Hamiltonians $\hat{H}_{\kappa,\pm}(\theta)$ (3.3).

We are now prepared to tie in the above with τ-functions of the form (2.18), (2.19), taking $M\to N$ and

$$\theta_N<\cdots<\theta_1, \qquad q\in\mathbb{R}^N. \tag{3.30}$$

Indeed, we deduce from the above developments that we may write

$$\tau_\nu=|1_N+e^{-2i\nu c}L(-c;\theta,q(t_+,t_-))|. \tag{3.31}$$

Therefore, recalling (3.20) and (3.15), it follows that $\ln\tau_\nu$ may be viewed as a linear superposition of N "single-particle" terms:

$$\ln\tau_\nu=\sum_{j=1}^{N}\ln(1+\exp[x_j(t_+,t_-)-2i\nu c]), \tag{3.32}$$

the $t_{\kappa,\delta}$ evolution being governed by the particle Hamiltonian $H_{\kappa,\delta}(x,p)$ (3.14).

Having described various mathematical aspects of the $\mathrm{II}_{\mathrm{rel}}(c,N)$ particle system and its relation to the N-soliton τ-functions with (1.20) and (3.30) in effect, we conclude this section with a brief overview of physical features. First, let us note that the particles are governed by repulsive forces. Indeed, since the Hamiltonian

$$H(c;x,p)\equiv H_{1,+}+H_{1,-}=4\sin(c)\sum_{j=1}^{N}\mathrm{ch}(p_j)\prod_{k\neq j}\left(1+\frac{\sin^2(c)}{\mathrm{sh}^2(x_j-x_k)/2}\right)^{1/2} \tag{3.33}$$

is conserved, the particle ordering $x_N<\cdots<x_1$ is invariant under the commuting flows. The long-time behavior of the flows generated by the hierarchy Hamiltonians $H_{\kappa,\delta}$ (and, more generally, an extensive class of commuting Hamiltonians) exhibits the conservation of asymptotic momenta and factorized asymptotic position shifts characteristic of soliton collisions. For instance, for the Hamiltonian H (3.33) one obtains in the same way as already sketched for solitons in Section 2

$$p_{\substack{j\\N-j+1}}(t)\sim\theta_j, \qquad t\to\pm\infty, \tag{3.34}$$

$$x_{\substack{j\\N-j+1}}(t)\sim q_j+4t\sin(c)\,\mathrm{sh}(\theta_j)\pm\frac{1}{2}\left(\sum_{k>j}-\sum_{k<j}\right)\ln\left(1+\frac{\sin^2(c)}{\mathrm{sh}^2(\theta_j-\theta_k)/2}\right),$$
$$t\to\pm\infty. \tag{3.35}$$

Here, (θ,q) is the image of $(x(0),p(0))$ under the action-angle map Φ (3.21).

The physical correspondence can be further enhanced by introducing a notion of soliton space-time trajectories. This renders it possible to ascribe a position to a soliton even in the regions where the interaction takes place (revealing in particular that solitons repel each other), and the trajectories coincide with the soliton positions for asymptotic times—where the notion of "soliton position" is unambiguous. (As mentioned before, "space" and "time" refer to special equations, cf. [1, 2, 3], but much of the analysis involved in a detailed study of the trajectories applies very generally, cf. Section 7A in [11].) Thus one is led to a picture of solitons as deformations of an elastic medium, which hides an underlying point particle motion.

4. BOUND PARTICLE-ANTIPARTICLE PAIRS VS s_2-SOLITONS

In order to handle the s_2-soliton case, it is expedient to begin this section by detailing how $\mathrm{II}_{\mathrm{rel}}(c)$ particles, antiparticles, and bound particle-antiparticle pairs (with $c \in (0, \pi/2)$) correspond to elementary solitons, antisolitons, and breathers. To this end, we analytically continue some of the particle positions, $x_j \to x_j + i\pi$, and note that this has the effect of turning the repulsive pair potential $(1+\sin^2(c)/\mathrm{sh}^2(x_j - x_k)/2)^{1/2}$ into the attractive potential $(1-\sin^2(c)/\mathrm{ch}^2(x_j - x_k)/2)^{1/2}$ whenever only one of x_j, x_k is continued. Accordingly, the above Hamiltonians $H_{\kappa,\delta}(x, p)$ (3.14) turn into real-valued commuting particle-antiparticle Hamiltonians.

Taking first $N=2$, an inspection of the Hamiltonian

$$H = 4 \sin c \sum_{j=1}^{2} \mathrm{ch}(p_j)(1-\sin^2(c)/\mathrm{ch}^2(x_1 - x_2)/2)^{1/2} \tag{4.1}$$

(obtained from H (3.33) by taking $x_2 \to x_2 + i\pi$) reveals the presence of a bound state subset of the phase space $\Omega(1, 1) = \mathbb{R}^4$: It is characterized by $H(x, p) < 4 \sin c$. (Indeed, since H is conserved, this condition yields an upper bound on the distance $|x_1 - x_2|$.) Moreover, employing sum and difference variables, one easily sees that the 2-dimensional submanifold $\{x_1 = x_2, p_1 = p_2\}$ is left invariant by the H flow (and by all of the $H_{\kappa,\delta}$ flows). This subspace corresponds to bound states with maximal binding energy, and no oscillation takes place in that case.

The general case where N_+ particles and N_- antiparticles are present can be described as follows [11]. The phase space

$$\Omega(N_+, N_-) \equiv \{(x, p) \in \mathbb{R}^{2N} \mid x_N < \cdots < x_{N_+ + 1}, x_{N_+} < \cdots < x_1\}, \qquad N = N_+ + N_- \tag{4.2}$$

splits up into invariant subsets corresponding to various scattering channels, and an exceptional set of codimension one (separatrix). The channels correspond to a long-time behavior of freely moving particles and antiparticles, together with l bound pairs; here, one has $l \in \{0, ..., \min(N_+, N_-)\}$, and all possibilities occur. Once again, scattering theory gives rise to asymptotic positions and momenta

$$(q_1^s, ..., q_{k_+}^s, \theta_1^s, ..., \theta_{k_+}^s), (q_1^{\bar{s}}, ..., q_{k_-}^{\bar{s}}, \theta_1^{\bar{s}}, ..., \theta_{k_-}^{\bar{s}}), \quad k_+ \equiv N_+ - l, \quad k_- \equiv N_- - l \tag{4.3}$$

describing the unbound particles and antiparticles. Furthermore, the l bound pairs can be described with complex momenta

$$(\theta_j^b - i\delta_j)/2, \qquad \theta_j^b \in \mathbb{R}, \quad \delta_j \in [-2c, 0), \quad j = 1, ..., l. \tag{4.4}$$

Here, the minimum value $\delta_j = -2c$ gives rise to maximal binding energy; in that case the two positions x_j, x_k involved become equal for asymptotic times. To specify the center of mass positions and oscillation phases one also needs variables $q_j^b \in \mathbb{R}$ and $\gamma_j \in (-\pi, \pi]$, $j = 1, ..., l$; the latter phases are not needed for ground state pairs—a feature to which we return at the end of this section.

Just as for the pure soliton situation, a key tool for obtaining these results is the commutation relation (3.16), or, more precisely, its analytic continuation to the subset of $\mathbb{C}^N \times \mathbb{R}^N$ that corresponds to $\Omega(N_+, N_-)$ (4.2). Then the matrix A turns into

$$A(N_+, N_-; x) = \mathrm{diag}(\exp x_1, ..., \exp x_{N_+}, -\exp x_{N_+ + 1}, ..., -\exp x_N) \tag{4.5}$$

and the new Lax matrix $\tilde{L}(x, p)$ is no longer self-adjoint. But it is still pseudo-self-adjoint with respect to the indefinite metric $\mathrm{diag}(1_{N_+}, -1_{N_-})$, and this key feature can be exploited to obtain spectral information reflecting the above decomposition of $\Omega(N_+, N_-)$.

Specifically, $\tilde{L}(x, p)$ is not diagonalizable precisely on the separatrix, whereas on the l-pair component $\tilde{L}(x, p)$ has simple spectrum and one has

$$\begin{aligned}\tilde{L} &\sim \mathrm{diag}(\exp \theta_1^s, ..., \exp \theta_{k_+}^s, \exp(\theta_1^b - i\delta_1)/2, ..., \exp(\theta_l^b + i\delta_l)/2, \exp \theta_1^{\bar{s}}, ..., \exp \theta_{k_-}^{\bar{s}})\\ &\equiv \mathrm{diag}(\exp \theta_1, ..., \exp \theta_N).\end{aligned} \tag{4.6}$$

(Recall $\sim$ denotes similarity.) Provided $\delta_1, ..., \delta_l > -2c$, the similarity can be effected by an invertible matrix $\mathscr{P}$ such that the corresponding transform of A (4.5) becomes

$$(\mathscr{P}A\mathscr{P}^{-1})_{jk} = \exp(-\theta_j/2) \frac{\mathrm{sh}(-ic)}{\mathrm{sh}((\theta_j - \theta_k)/2 - ic)} \exp(\theta_k/2) \exp(q_k)\, V_k(\theta). \tag{4.7}$$

Here, $\theta \in \mathbb{C}^N$ is defined by (4.6), and $q \in \mathbb{C}^N$ satisfies

$$(q_1, ..., q_N) \equiv (q_1^s, ..., q_{k_+}^s, q_1^b + i\gamma_1, ..., q_l^b - i\gamma_l, q_1^{\bar{s}}, ..., q_{k_-}^{\bar{s}}) \qquad (\mathrm{mod}\ i\pi). \tag{4.8}$$

(The $i\pi$-multiples are uniquely determined mod $2i\pi$, cf. [11], p. 904. For brevity we neither specify the multiples nor the square-root conventions in $V_k(\theta)$, but we shall return to the phase of $\exp(q_k)\, V_k(\theta)$ shortly.) The map

$$\tilde{\Phi}: (x, p) \mapsto (\theta_1^s, ..., \theta_{k_+}^s, \theta_1^b, ..., \delta_l, \theta_1^{\bar{s}}, ..., \theta_{k_-}^{\bar{s}}, q_1^s, ..., q_{k_+}^s, q_1^b, ..., \gamma_l, q_1^{\bar{s}}, ..., q_{k_-}^{\bar{s}}) \tag{4.9}$$

from the l-pair component to the asymptotic variables may be viewed as the analytic continuation of the action-angle map Φ (3.21) from the "pure particle" case. Accordingly, one has

$$\tilde{\Phi}^*\left(-\sum_{j=1}^{N} d\theta_j \wedge dq_j\right)=\sum_{j=1}^{N} dx_j \wedge dp_j, \tag{4.10}$$

where the complex coordinates $\theta_1, ..., q_N$ are defined by (4.6) and (4.8). (Note that in terms of the asymptotic (action-angle) variables the symplectic form is real and canonical.) Moreover, one deduces

$$H_{\kappa,\pm}(x, p)=2\kappa^{-1}\sin(\kappa c)\operatorname{Tr}\tilde{L}(x, p)^{\pm\kappa}=2\kappa^{-1}\sin(\kappa c)\sum_{j=1}^{N}\exp(\pm\kappa\theta_j). \tag{4.11}$$

In view of (4.6), the rhs can be rewritten

$$\begin{aligned}\hat{H}_{\kappa,\pm}=2\kappa^{-1}\sin(\kappa c)\Bigg(&\sum_{j=1}^{k_+}\exp(\pm\kappa\theta_j^s)\\ &+2\sum_{j=1}^{l}\cos(\kappa\delta_j/2)\exp(\pm\kappa\theta_j^b/2)+\sum_{j=1}^{k_-}\exp(\pm\kappa\theta_j^s)\Bigg),\end{aligned} \tag{4.12}$$

so it is real-valued, as announced in the first paragraph of this section.

With these results in hand, we are prepared to return to the above N-soliton τ-functions. To each point (x, p) in the l-pair component of $\Omega(N_+, N_-)$, $N_+ + N_- = N$, we can associate a τ-function by substituting

$$a_j=\exp(\theta_j - ic), \qquad b_j=\exp(\theta_j + ic), \tag{4.13}$$

$$\exp(\xi_j^0)=\exp(q_j)\prod_{k\neq j}\left(1+\frac{\sin^2 c}{\operatorname{sh}^2(\theta_j-\theta_k)/2}\right)^{1/2}, \tag{4.14}$$

with $(\theta, q)\in\mathbb{C}^{2N}$ now defined via the asymptotic variables $\tilde{\Phi}(x, p)$, cf. (4.6), (4.8). The unspecified $i\pi$-multiples in (4.8) then combine with the unspecified sign conventions for the square roots in (4.14) to yield phases satisfying

$$\begin{aligned}\exp(\xi_j^0)&\in(0,\infty), && j=1, ..., k_+,\\ \exp(\xi_{k_+ + j}^0)&\in\mathbb{C}^*, && j=1, ..., l \quad (\delta_j > -2c),\\ \exp(\xi_{N-k_- + j}^0)&\in(-\infty, 0), && j=1, ..., k_-,\end{aligned} \tag{4.-5}$$

cf. Eq. (2.42) in [11].

In this way we obtain τ-functions that may be transformed to the (x, p)-variables via $\tilde{\Phi}^{-1}$, yielding

$$\ln \tau_\nu = \sum_{j=1}^{N_+} \ln(1 + \exp[-2i\nu c + x_j(t_+, t_-)]) + \sum_{j=N_++1}^{N} \ln(1 - \exp[-2i\nu c + x_j(t_+, t_-)]) \tag{4.16}$$

as a generalization of (3.32). Here, the 2D Toda evolutions are governed by the real-valued Hamiltonians (4.11), and the τ-functions thus associated to a point in the l-pair component of $\Omega(N_+, N_-)$ consist of k_+ solitons, k_- antisolitons, and l breathers (bound soliton-antisoliton pairs). Upon specializing to concrete equations, with corresponding notions of "space" and "time," the long-time asymptotics of such τ-functions leads to a picture of widely separated, freely evolving, one-soliton, one-antisoliton, and one-breather solutions—the interaction being solely visible in factorized position and phase shifts.

We have filled in the analytical details of this picture for the special case of the sine-Gordon and modified KdV equations in Section 7B of [11]. By contrast, in this paper we are not presenting a mathematically and physically complete discussion of τ-functions obeying (2.10) and their connections to (real) finite-dimensional symplectic manifolds and commuting hamiltonian flows. (Note, for instance, that (4.16) gives rise to τ-functions satisfying (2.10) for *all* points of $\Omega(N_+, N_-)$, in particular on the separatrix, where the representation (1.1) is no longer valid.)

At this point we should also repeat that we take $c \in (0, \pi/2)$ throughout this section, whereas for the sine-Gordon and mKdV equations one needs $c = \pi/2$. Observe in this connection that the repulsive $\mathrm{II}_{\mathrm{rel}}(c)$ system of the previous section is nonsingular for $c = \pi/2$, and invariant under $c \to \pi - c$. By contrast, for $c = \pi/2$ the particle-antiparticle interaction becomes singular for coinciding positions, cf. e.g. (4.1).

We now turn to establishing the connection between the fusion procedure for the s_2-case (cf. Section 2) and special points in the phase space $\Omega(N, N)$ of the $\mathrm{II}_{\mathrm{rel}}(c, N, N)$ system. Thus, we choose $c_1 = \cdots = c_N = 2c \in (0, \pi)$. Of course, it is immediate from the previous section that the N-soliton τ-functions can then be tied in with the $\mathrm{II}_{\mathrm{rel}}(2c, N)$ particles. It is not at all immediate, though, that the latter can be associated with a $2N$-dimensional invariant submanifold of the $\mathrm{II}_{\mathrm{rel}}(c, N, N)$ system—and this is the picture that generalizes to the τ-functions with (1.18) in effect.

Therefore, we now detail this relation between the $\mathrm{II}_{\mathrm{rel}}(2c, N)$ and $\mathrm{II}_{\mathrm{rel}}(c, N, N)$ systems. The invariant submanifold of $\Omega(N, N)$ reads

$$\{(x, p) \in \Omega(N, N) \mid x_j = x_{j+N},\ p_j = p_{j+N},\ j = 1, ..., N\}. \tag{4.17}$$

It consists of N particle-antiparticle pairs in their ground state, which do not oscilllate and whose ordering $x_N < \cdots < x_1$ remains fixed under all of the commuting flows. On this manifold—which can obviously be identified with $\Omega(N)$ (3.17)—the Lax matrix $\tilde{L}(x, p)$ has spectrum

$$\sigma(\tilde{L}) = \{\exp(\theta_1^b + 2ic)/2, ..., \exp(\theta_N^b - 2ic)/2\}, \qquad \theta_N^b < \cdots < \theta_1^b. \tag{4.18}$$

Clearly, the matrix at the rhs of (4.7) is ill defined in this case: One obtains poles for $\theta_{2j} - \theta_{2j-1} = 2ic$. But it is possible to get rid of these singularities. Indeed, there exists a renormalized invertible matrix $\mathscr{P}_r$ effecting the similarity (4.6) such that the "dual Lax matrix" $\mathscr{P}_r A \mathscr{P}_r^{-1}$ takes a different and non-singular form (cf. [11], Chapter 5).

In Section 7 we shall have occasion to define such a non-singular dual Lax matrix in a far snore general setting. Here, we observe that (4.7) and Cauchy's identity entail

$$S_l(\mathscr{P}A\mathscr{P}^{-1}) = \sum_{\substack{I \subset \{1, ..., 2N\} \\ |I| = l}} \exp\left(\sum_{i \in I} q_i\right) \prod_{\substack{i \in I \\ j \notin I}} f(c; \theta_i - \theta_j), \tag{4.19}$$

where the signs of the terms are governed by the $i\pi$-multiples in (4.8) and square-root sign conventions that we did not specify. The point to be made, however, is that the rhs has a finite limit when one lets $\theta_{2j} - \theta_{2j-1} \to 2ic$ (which already indicates that the above poles can be transformed away). Indeed, performing these limits is tantamount to the fusion procedure from Section 2, with $n_1 = \cdots = n_N = 2$ and the η_j in (2.21) equal to $\theta_j^b/2$, $j = 1, ..., N$.

To be quite specific, we recall the pertinent limit yields $S_m \to 0$ for m odd, whereas

$$S_{2l}(\mathscr{P}a\mathscr{P}^{-1}) \to \sum_{\substack{J \subset \{1, ..., N\} \\ |J| = l}} \exp\left(\sum_{j \in J} Q_j\right) \prod_{\substack{j \in J \\ k \notin J}} f(2c; (\theta_j^b - \theta_k^b)/2), \qquad l = 1, ..., N. \tag{4.20}$$

Here, we have $\operatorname{Im} Q_j = i\pi (\operatorname{mod} 2\pi i)$, $j = 1, ..., N$, which yields a sign $(-)^l$ for each term in the sum. This should be the case, as this agrees with the signs obtained from the definition (4.5) of the matrix $A = A(N, N; x)$: From

$$|A + \lambda 1_{2N}| = \sum_{j=1}^{N} (\lambda^2 - \exp(2x_j)) \tag{4.21}$$

one deduces

$$S_m(A) = \begin{cases} 0, & m \text{ odd}, \\ (-)^l \displaystyle\sum_{\substack{J \subset \{1, ..., N\} \\ |J| = l}} \exp\left(\sum_{j \in J} x_j\right), & m = 2l, \quad l = 1, ..., N. \end{cases} \tag{4.22}$$

As a further preparation for the following sections, we review one more crucial feature of the l-pair component Ω_l of $\Omega(N, N)$ with $l > 0$. When one tries to extend the action-angle map $\tilde{\Phi}: (x, p) \mapsto (\theta, q)$ to the subset of Ω_l for which one or more pairs are in their ground state, one finds that the variables (θ, q) are no longer appropriate. In order to obtain a real-analytic extension, one needs to trade the internal pair coordinates δ, γ for harmonic oscillator variables u, v. Then δ is a function of $r^2 \equiv u^2 + v^2$, and γ becomes the angle that describes the location on the circle $r > 0$, with $r \downarrow 0$ corresponding to $\delta \downarrow -2c$.

In terms of the new variables, the map $\tilde{\Phi}$ extends to a canonical diffeomorphism. Moreover, the Hamiltonians $\hat{H}_{\kappa, \delta}$ (4.12) become functions of the external (center of mass) momenta θ_j^b and the r_j^2, and by inspection one sees that the dimension of the span of their gradients reduces by 1 for each r_j becoming 0. Since $\tilde{\Phi}$ is a symplectomorphism, this holds for the Hamiltonians $H_{\kappa, \delta}$ on $\Omega(N, N)$ as well; in particular, on the subspace (4.17) this dimension equals N.

Later on, we shall reobtain the latter result from [11] in the course of handling the s_n-case; it arises by simply taking $n = 2$. We shall however employ harmonic oscillator variables inspired by the study of Sutherland type systems that we undertook in [12]; the resulting subset of $\mathbb{C}^{4N}$ intersects $\Omega(N, N)$ solely in (4.17). This is one reason why we are not going into further details (in particular as regards the above phase factors). These details turn out to be rather intricate and cannot be avoided when one seeks a comprehensive and rigorous view of the physical state of affairs on all of $\Omega(N, N)$.

In the next sections, however, our aim is rather to obtain generalizations of the subspace (4.17), clarify their local structure, and tie them in with the N-soliton τ-functions with (1.18), (1.19) in force. In this general setting we do not know whether these subspaces are embedded in symplectic manifolds with $2(n_1 + \cdots + n_N)$ real dimensions, equipped with $(n_1 + \cdots + n_N)$ independent real-valued commuting Hamiltonians—which would be the analog of the picture in [11] concerning (4.17) vs $\Omega(N, N)$.

5. THE s_N-SOLITON CASE

Our account in this section clarifies to a large extent how solitons of type s_n (i.e., with $c_1, ..., c_N$ equal to nc, $c \in (0, \pi/n)$) arise in the complexification of the $\mathrm{II}_{\mathrm{rel}}(c, nN)$ system. However, it leaves open some technical questions, whose detailed answers are postponed to Section 6 for $N = 1$, and to Section 7 for $N > 1$ (using also Appendix E in the latter case). This approach enables us to bring out the main points unobscured by technicalities, and moreover avoids a repetition of arguments.

A key role in the following is played by a subspace Ω_N of the complexified phase space $\mathbb{C}^{2nN}$, which may be viewed as a copy of $\Omega(N)$ (3.17). Specifically, choosing $(X, P) \in \Omega(N)$, so that

$$X_N < \cdots < X_1, \qquad P \in \mathbb{R}^N, \tag{5.1}$$

we define $(x, p) \in \Omega_N \subset \mathbb{C}^{nN} \times \mathbb{R}^{nN}$ by setting

$$x_{(j-1)n+k} = [X_j + 2\pi i(n-k)]/n, \qquad j = 1, ..., N, \quad k = 1, ..., n, \tag{5.2}$$

$$p_{(j-1)n+k} = P_j, \qquad j = 1, ..., N, \quad k = 1, ..., n. \tag{5.3}$$

Due to the branch points in the elements of the Lax matrix $L(c; x, p)$ (given by (3.10)–(3.12)), there is no unique continuation of L and the associated Hamiltonians from the (real) $\mathrm{II}_{\mathrm{rel}}(c, nN)$ phase space to the submanifold Ω_N. However, we may and will fix the ambiguities by requiring that the functions $V_1^{1/2}(c; x), ..., V_{nN}^{1/2}(c; x)$ be positive in the point (5.2). To be more specific, we are claiming that there exists a path from a point in $\Omega(nN)$ to the point (5.1)–(5.3) in Ω_N for which the analytic continuations of the latter yield positive numbers.

To prove this claim, we exhibit such a path: We choose

$$x_{(j-1)n+k}(\phi) = [X_j + 2\pi(n-k)\, a(\phi)\, e^{i\phi}]/n, \qquad p_{(j-1)n+k}(\phi) = P_j, \tag{5.4}$$

where $\phi \in [0, \pi/2]$, and $a(\phi)$ increases from $\varepsilon > 0$ to 1 as ϕ goes from 0 to $\pi/2$. In view of (5.1), we may and will choose ε such that $(x(0), p(0)) \in \Omega(nN)$. Now it is clear that $V_l^2(c; x(\phi))$ is positive for $\phi = 0, \pi/2$, and we also have $V_l^{1/2}(c; x(0)) > 0$ for all $l \in \{1, ..., nN\}$. Moreover, $V_l^2(c; x(\pi/2))$ is non-zero for $c \in (0, \pi/n)$, so the sign of $V_l^{1/2}(c; x(\pi/2))$ is constant on this c-interval. Since all of the pair potentials $f(c; x_j - x_k)$ stay close to 1 along the path (5.4) for c very small, it now follows that the continuations of $V_1^{1/2}, ..., V_{nN}^{1/2}$ are positive in (5.2), as claimed.

Accordingly, we now obtain a well-defined Lax matrix on Ω_N, and Hamiltonians $H_{\kappa,\delta}(c; x, p)$ on Ω_N given by (3.14). At this point it is far from clear that all of these Hamiltonians are real-valued and that their flows leave Ω_N invariant, but we are going to prove that this is true. Let us begin by observing that the invariance property amounts to the associated hamiltonian vector fields (symplectic gradients) being tangent to Ω_N. Now from the definition (5.2), (5.3) of Ω_N one sees that a vector field V is tangent to Ω_N if and only if

$$V_l \in \mathbb{R}, \qquad l = 1, ..., 2nN, \tag{5.5}$$

$$V_{(j-1)n+k} = V_{(j-1)+1}, \quad V_{nN+(j-1)n+k} = V_{nN+(j-1)n+1}, \qquad j = 1, ..., N, \quad k = 2, ..., n. \tag{5.6}$$

Thus we have to show that on Ω_N the Hamiltonians $H_{\kappa,\delta}$ and their gradients are real-valued, and that the gradients have the symmetry properties (5.6).

In fact, we will prove a far stronger result: On Ω_N the gradients not only satisfy (5.6), but one also has

$$H_{\kappa,\delta}(c, nN; x, p) = H_{\kappa,\delta}(nc, N; X, P), \qquad \kappa = 1, 2, ..., \quad \delta = +, -. \tag{5.7}$$

(This entails in particular real-valuedness of the lhs, as announced.) Combining these two properties with (5.2), (5.3) and the chain rule, one deduces

$$\partial H_{\kappa,\delta}(nc, N; X, P)/\partial P_j = n\,\partial H_{\kappa,\delta}(c, nN; x, p)/\partial p_{(j-1)n+k}), \qquad k=1, \ldots, n, \tag{5.8}$$

$$\partial H_{\kappa,\delta}(nc, N; X, P)/\partial X_j = \partial H_{\kappa,\delta}(c, nN; x, p)/\partial x_{(j-1)n+k}), \qquad k=1, \ldots, n. \tag{5.9}$$

Thus the hamiltonian vector field associated to $H_{\kappa,\delta}(c, nN)$ is not only tangent to Ω_N: It is related to the hamiltonian vector field associated to $H_{\kappa,\delta}(nc, N)$ in such a way that the flows on Ω_N and $\Omega(N)$ coincide.

In this section we present a quite direct proof of (5.7) and the gradient symmetries (5.6) for the special case $\kappa=1, \delta=+$. Combining this result with the known asymptotics of the $H_{1,+}(nc, N)$ flow, we obtain the identities (5.7) for the general case. The relations (5.8) and (5.9) are then highly plausible, but they do not rigorously follow, unless one has previously proved (5.6), or, equivalently, invariance of Ω_N under the $H_{\kappa,\delta}(c, nN)$ flow.

Unfortunately, our direct proof of (5.6) for $\kappa=1, \delta=+$ does not appear to admit an easy generalization. We prove invariance of Ω_N by other means in Section 6 for $N=1$, and for $N>1$ in Section 7—alongside with the arbitrary-n_j case. On the other hand, the considerable analysis involved in completing the proof is not necessary for understanding how the fusion property of the τ-functions obtained in Section 2 fits in, and how the variables Q_j in (2.26) should be chosen. At the end of this section we shall therefore detail these kinematic features, before completing the dynamical picture in later sections.

Turning now to the proof of (5.7)–(5.9) for $\kappa=1, \delta=+$, we collect some identities that readily follow from Lemma D.2. To start with, (D.5) entails

$$\prod_{k=0}^{n-1}\left(1+\frac{\sin^2 c}{\mathrm{sh}^2(a+i\pi k/n)}\right)^{1/2}=\left(1+\frac{\sin^2 nc}{\mathrm{sh}^2 na}\right)^{1/2}, \qquad a>0. \tag{5.10}$$

From this we obtain

$$\prod_{k=1}^{n-1}\left(1+\frac{\sin^2 c}{\mathrm{sh}^2(a+i\pi k/n)}\right)^{1/2}=\left(\frac{\mathrm{sh}^2 a+\sin^2(nc)\,\mathrm{sh}^2 a/\mathrm{sh}^2 na}{\mathrm{sh}^2 a+\sin^2 c}\right)^{1/2}, \qquad a>0, \tag{5.11}$$

so taking $a\downarrow 0$ yields

$$\prod_{k=1}^{n-1}\left(1-\frac{\sin^2 c}{\sin^2(\pi k/n)}\right)^{1/2}=\frac{\sin nc}{n\sin c}, \qquad c\in(0, \pi/n). \tag{5.12}$$

Now from (3.14) we have

$$H_{1,+}(c, nN; x, p)=2\sin(c)\sum_{l=1}^{nN}\exp(p_l)\prod_{m\neq l}\left(1+\frac{\sin^2 c}{\mathrm{sh}^2(x_l-x_m)/2}\right)^{1/2}, \tag{5.13}$$

$$H_{1,+}(nc, N; X, P)=2\sin(nc)\sum_{i=1}^{N}\exp(P_i)\prod_{k\neq i}\left(1+\frac{\sin^2 nc}{\mathrm{sh}^2(X_i-X_k)/2}\right)^{1/2}. \tag{5.14}$$

Thus from (5.10) and (5.12) we obtain (5.7) and (5.8) for $\kappa = 1, \delta = +$. Moreover, using the a-derivative of (5.10) and noting that the a-derivative of (5.11) converges to 0 for $a \downarrow 0$ (indeed, the rhs of (5.11) is even in a), we also deduce (5.9) for $\kappa = 1$, $\delta = +$.

Next, we aim to explicitly determine the spectrum of the Lax matrix on Ω_N. To this end we begin by recalling that the $H_{1,+}(c, nN)$ flow is isospectral, i.e., the spectrum of

$$L_t \equiv L \cdot \exp(tH_{1,+}(c, nN)) \tag{5.15}$$

does not depend on t. Therefore, we can combine the known $t \to \infty$ asymptotics of the $H_{1,+}(nc, N)$ flow on $\Omega(N)$ and the equality of this flow and the $H_{1,+}(c, nN)$ flow on Ω_N to obtain $\sigma(L_\infty)$ and hence $\sigma(L_0)$.

The $t \to \infty$ asymptotics of the $H_{1,+}(nc, N)$ flow can be read off from Theorem 3.4 in [10] (with $\beta = \mu = 1$, $z = inc$, $q, \theta \to X, P$ and an extra scale factor $2 \sin(nc)$). It reads

$$P_j(t) \sim \eta_j, \qquad t \to \infty, \tag{5.16}$$

$$X_j(t) \sim y_j + 2t \sin(nc) \exp \eta_j + \frac{1}{2}\left(\sum_{l>j} - \sum_{l<j}\right) \ln\left(1 + \frac{\sin^2(nc)}{\operatorname{sh}^2(\eta_j - \eta_l)/2}\right), \quad t \to \infty, \tag{5.17}$$

where $j = 1, ..., N$, and where the asymptotic positions and momenta satisfy

$$\eta_N < \cdots < \eta_1, \qquad y \in \mathbb{R}^N. \tag{5.18}$$

Combining now the above definition of L on Ω_N with (5.15)–(5.18), we obtain

$$L_\infty \equiv \lim_{t \to \infty} L_t = \operatorname{diag}(\exp(\eta_1) E, ..., \exp(\eta_N) E), \tag{5.19}$$

where E is the $n \times n$ matrix with elements

$$E_{kl} = \frac{\sin nc}{n \sin(c - \pi(k-l)/n)}, \qquad k, l = 1, ..., n, \quad c \in (0, \pi/n). \tag{5.20}$$

Thus, to determine $\sigma(L_\infty)$ we need only obtain $\sigma(E)$.

Now E is just the equilibrium matrix of the $\mathrm{III_b}$ n-particle system studied in [12]. As we proved in Lemma A.6 of l.c., its spectrum reads

$$\sigma(E) = \{\exp i(n-1)c, \exp i(n-3)c, ..., \exp i(-n+1)c\}. \tag{5.21}$$

Thus we deduce

$$\sigma(L(c; x, p)) = \sigma(L_\infty) = \{\exp(\eta_j + i(n+1-2k)c) \mid j = 1, ..., N, k = 1, ..., n\}, \tag{5.22}$$

so we have achieved our aim.

We now exploit the spectral characteristics just obtained to prove the identities (5.7). First, we note that (5.18) and our standing assumption $c \in (0, \pi/n)$ entail that $\sigma(L)$ is non-degenerate. Thus L is diagonalizable:

$$L(c; x, p) \sim \mathrm{diag}(\exp(\eta_1 + i(n-1)\, c), ..., \exp(\eta_N + i(-n+1)\, c)). \qquad (5.23)$$

Now we also have (using again the above $t \to \infty$ asymptotics)

$$L(nc; X, P) \sim \mathrm{diag}(\exp \eta_1, ..., \exp \eta_N). \qquad (5.24)$$

Combining the definition (3.14) with the similarities (5.23) and (5.24), we deduce

$$\begin{aligned} H_{\kappa,\delta}(c; x, p) &= 2\kappa^{-1} \sin(\kappa c) \sum_{j=1}^{N} \exp(\delta\kappa\eta_j) \sum_{k=1}^{n} \exp(i\delta\kappa(n+1-2k)\, c) \\ &= 2\kappa^{-1} \sin(n\kappa c) \sum_{j=1}^{N} \exp(\delta\kappa\eta_j), \end{aligned} \qquad (5.25)$$

and

$$H_{\kappa,\delta}(nc; X, P) = 2\kappa^{-1} \sin(\kappa nc) \sum_{j=1}^{N} \exp(\delta\kappa\eta_j), \qquad (5.26)$$

resp. Thus the identities (5.7) follow.

As announced, we relegate the proof of invariance of Ω_N under all of the $H_{\kappa,\delta}$ flows to later sections, and conclude this section by tying in the above with the fusion picture established in Section 2. Since this only involves kinematics, we may as well consider the time-zero case, taking also $v = 0$ to ease the notation.

Obviously, the nN theta's in (2.21) correspond to the eigenvalues of the Lax matrix $L(c; x, p)$, with the N eta's equal to the eta's of this section. To understand the connection in more detail, however, we should reconsider the analytic continuations of various relevant quantities along the path (5.4). First, we recall from Section 3 that in the initial point $\phi = 0$ (which belongs to the $\mathrm{II}_{\mathrm{rel}}(c, nN)$ phase space) the (time-zero, $v = 0$) τ-function can be written both as

$$\tau = |1_{nN} + A(x(0))| \qquad (5.27)$$

and as

$$\tau = |1_{nN} + L(-c; \theta(0), q(0))|, \qquad (\theta(0), q(0)) = \Phi(c; x(0), p(0)). \qquad (5.28)$$

Moreover, as we have seen in this section, the spectrum of the Lax matrix $L(c; x(\phi), p(\phi))$ changes from $\exp(\theta_{nN}(0)) < \cdots < \exp(\theta_1(0))$ to (5.22) as ϕ goes from 0 to $\pi/2$, with η determined by

$$(\eta, y) = \Phi(nc; X, P). \qquad (5.29)$$

It is therefore quite plausible that the path (5.4) amounts to a "fusion path" as detailed in Section 2. Indeed, in later sections we will present the somewhat involved arguments from which this expectation rigorously follows. In this section, however, we take the path correspondence for granted, and observe that it is quite easy to continue the "particle representation" (5.27) along (5.4): In the endpoint we get, successively,

$$\begin{aligned}\tau &= \prod_{l=1}^{nN} (1+\exp x_l(\pi/2)) = \prod_{j=1}^{N} (1+(-)^{n-1} \exp X_j) \\ &= |1_N + (-)^{n-1} A(X)| = |1_N + (-)^{n-1} L(-nc; \eta, y)|. \end{aligned} \tag{5.30}$$

Thus the parameter Q_j from Section 2 (cf. (2.24)–(2.26)) can be chosen

$$Q_j = y_j + (n-1)\pi i, \qquad j = 1, ..., N, \tag{5.31}$$

so as to obtain equality of the fused τ-function from Section 2 and the τ-function obtained from (5.27) by analytic continuation.

The choice of the imaginary part in (5.31) is in accordance with the analytic continuation of the identity

$$\sum_{l=1}^{nN} x_l(0) = \sum_{l=1}^{nN} q_l(0). \tag{5.32}$$

(This identity follows from $|A(x)| = |L(-c; \theta, q)|$, cf. (3.20).) Indeed, the lhs continues to

$$\sum_{j=1}^{N} X_j + N(n-1)\pi i = \sum_{j=1}^{N} y_j + N(n-1)\pi i = \sum_{j=1}^{N} Q_j \tag{5.33}$$

and, in view of (2.24), this is what one gets from continuing the rhs. Notice that the imaginary part (or, equivalently, the "extra" factor $(-)^{n-1}$ in (5.30)) suggests that one should view the above n-particle bound states as antiparticles for n even and as particles for n odd. In this connection we point out that a shift of one or more of the X_j in (5.2) by πi has the effect of flipping the charge—this can be read off from our account in Section 4.

6. INTERLUDE: THE III_b MOLECULE

This section is concerned with an n-particle system that is of interest in itself, but which plays only an auxiliary role in this paper. Indeed, as mentioned and used in the previous section, E (5.20) is the equilibrium Lax matrix of the n-particle system referred to as the III_b system in [12]. Our results on this integrable system will be invoked in particular to deduce invariance of Ω_N for $N=1$ under all of the $H_{\kappa,\delta}$

flows. More importantly, however, the insights gained from this special case can be exploited to get a handle on the arbitrary-n_j setting of the next section.

Let us observe first of all that the pertinent formulas from the previous section are very simple for $N=1$. Indeed, $H_{1,+}(nc, 1; X, P)$ (5.14) becomes a free Hamiltonian $2\sin(nc)\exp(P)$, and (5.16), (5.17) become

$$P=\eta, \qquad X(t)=y+2t\sin(nc)\exp(P). \tag{6.1}$$

Thus the flow generated by $H_{1,+}(c, n; x, p)$ (5.13) turns into

$$x_k(t)=[X(t)+2\pi i(n-k)]/n, \qquad p_k(t)=P, \quad k=1, ..., n. \tag{6.2}$$

Hence, L_t is in fact time-independent, and reads

$$L(c; x, p)=\exp(P)\,E. \tag{6.3}$$

Now in [12] the $\mathrm{II}_{\mathrm{rel}}$ and $\mathrm{III}_{\mathrm{b}}$ systems were connected via analytic continuation in the scale parameters μ and β multiplying the x_k's and p_k's, resp., cf. the proof of Theorem 3.5 in l.c. In this paper we take $\mu=\beta=1$, but the continuation along the path (5.4) has substantially the same effect as the μ, β continuation, since it yields purely imaginary x and p differences (namely, $x_k-x_l=2\pi i(l-k)/n$, $p_k-p_l=0$), giving rise to the $\mathrm{III}_{\mathrm{b}}$ equilibrium Lax matrix E.

There is however a slight difference resulting from the two different continuation procedures. This concerns the center of mass coordinates X and P, which appear as multiplicative factors $\exp(X)$ and $\exp(P)$ when the matrices $A(x)$ (3.15) and $L(c; x, p)$ (3.10) are continued along the path (5.4). By contrast, the μ, β continuation in [12] gives rise to factors $\exp(iX)$ and $\exp(iP)$ (taking $|\mu|=|\beta|=1$).

Now this difference does have consequences (to be discussed shortly). Even so, most of the results from [12] apply here, too. This is because the center of mass motion and the internal motion decouple when the $\mathrm{III}_{\mathrm{b}}$ system is defined on a suitable phase space, viz., the space Ω^c given by (1.43) in [12]. In that setting the center of mass coordinates vary over $\mathbb{R}$ (cf. (1.44) in l.c.), and the action-angle map and its harmonic oscillator extension factorize in a trivial center of mass part and an internal map encoding the physical behavior of the system.

We can therefore use various features associated with the internal variables as a starting point for tackling the invariance problem for arbitrary n_j. Now this problem is of a local nature, so we need only generalize those results from [12] that have a bearing on the situation close to the $\mathrm{III}_{\mathrm{b}}$ equilibrium. In particular, there is no need for the somewhat involved coordinatizations we employed in l.c. to prove that the $\mathrm{III}_{\mathrm{b}}$ n-particle system (or, more precisely, a mathematically convenient version) wants to live on a phase space $\mathbb{R}^2\times\mathbb{P}^{n-1}$ in which Ω^c is dense. Instead, we are going to introduce local oscillator variables that are simpler to work with, and in terms of which we can study Ω_N and its generalization in the next section.

We begin by embedding Ω_1 in a subspace $\Omega_{\mathrm{loc}} \subset \mathbb{C}^{2n}$ of real dimension $2n$, by allowing $x_1, ..., p_n$ to take values

$$x_k = [X + 2\pi i(n-k)]/n + i\alpha_k, \qquad X \in \mathbb{R}, \quad \alpha_k \in (-\varepsilon_1, \varepsilon_1), \quad k = 1, ..., n, \quad (6.4)$$

$$p_k = P + i\beta_k, \qquad P \in \mathbb{R}, \quad \beta_k \in (-\varepsilon_2, \varepsilon_2), \quad k = 1, ..., n, \quad (6.5)$$

where $\varepsilon_1 \in (0, \pi/n - c)$ and $\varepsilon_2 \in (0, \pi/2)$ are at our disposal, and where

$$\sum_{k=1}^{n} \alpha_k = 0, \qquad \sum_{k=1}^{n} \beta_k = 0. \quad (6.6)$$

Since we take $\varepsilon_1 \in (0, \pi/n - c)$, the Lax matrix $L(c; x, p)$ is well defined and holomorphic on Ω_{loc}. Moreover, from Section 2.3 in [12] it follows that its spectrum is simple and may be written

$$\sigma(L) = \{\exp(\theta_1), ..., \exp(\theta_n)\}, \qquad \mathrm{Re}\, \theta_k = P, \quad k = 1, ..., n, \qquad \sum_{k=1}^{n} \mathrm{Im}\, \theta_k = 0, \quad (6.7)$$

with the difference variables

$$\delta_k \equiv -i(\theta_k - \theta_{k+1})/2, \qquad k = 1, ..., n-1 \quad (6.8)$$

satisfying

$$\delta_k \geqslant c, \qquad k = 1, ..., n-1. \quad (6.9)$$

Now on Ω_1 we have $\delta_k = c < \pi/n$, so from now on we may and will choose $\varepsilon_1, \varepsilon_2$ sufficiently small so that

$$\delta_k \in [c, \pi/n], \qquad k = 1, ..., n-1 \quad (6.10)$$

on all of Ω_{loc}.

We proceed by introducing variables

$$(X, P; v_1, ..., v_{n-1}, u_1, ..., u_{n-1}) \in \mathbb{R}^{2n} \quad (6.11)$$

on Ω_{loc} that are not only canonical,

$$\sum_{k=1}^{n} dx_k \wedge dp_k = dX \wedge dP + \sum_{k=1}^{n-1} dv_k \wedge du_k, \quad (6.12)$$

but also satisfy

$$u_k^2 + v_k^2 = 4(\delta_k - c), \qquad k = 1, ..., n-1. \quad (6.13)$$

Thus, the subspace Ω_1 of Ω_{loc} is characterized by all (u_k, v_k) being equal to (0,0). To detail these "harmonic oscillator variables" $u_1, ..., v_{n-1}$, we continue with some preparations.

To begin with, the map Φ (3.21) defined on the $\mathrm{II}_{\mathrm{rel}}$ phase space $\Omega(n)$ (3.17) can be analytically continued to

$$\Omega_r \equiv \Omega_{\mathrm{loc}} \backslash \Omega_e. \tag{6.14}$$

By definition, the exceptional variety Ω_e consists of all points where one or more inequalities (6.9) are equalities (in particular, $\Omega_1 \subset \Omega_e$). The continuation of $\theta(x, p)$ can and will be chosen such that on the restricted space Ω_r it satisfies (6.7) and $\delta_k > c$, $k = 1, ..., n-1$. The corresponding continuation of $q(x, p)$, however, is multi-valued: Going around noncontractible loops in Ω_r yields additive multiples of $2\pi i$. The sum of these multiples vanishes, since one has

$$Q \equiv \sum_{k=1}^{n} q_k = \sum_{k=1}^{n} x_k = X + (n-1)\,\pi i \tag{6.15}$$

in view of (6.4) and (6.6). (These assertions readily follow from Chapter 3 in [11]; cf. also below.)

Next, we introduce the variables

$$\gamma_k \equiv i[(q_1 + \cdots + q_k) - k(q_1 + \cdots + q_n)/n], \qquad k = 1, ..., n-1, \tag{6.16}$$

so that we have

$$\begin{aligned} q_1 &= Q/n - i\gamma_1 \\ q_2 &= Q/n - i(\gamma_2 - \gamma_1) \\ &\vdots \\ q_n &= Q/n + i\gamma_{n-1}. \end{aligned} \tag{6.17}$$

It is not obvious, but true, that the function $\gamma_k(x, p)$ is real-valued on Ω_r, and since its values are determined mod 2π, it may and will be viewed as an angle coordinate. Moreover, using the definition (6.8) of δ_k one easily verifies the equality

$$d\theta \wedge dq = dP \wedge dX + 2\, d\delta \wedge d\gamma. \tag{6.18}$$

We are now prepared to define the announced oscillator variables: They read

$$u_k \equiv -2(\delta_k - c)^{1/2} \cos \gamma_k, \qquad v_k \equiv 2(\delta_k - c)^{1/2} \sin \gamma_k, \quad k = 1, ..., n-1. \tag{6.19}$$

Obviously, this entails (6.13), and one readily checks

$$2\, d\delta \wedge d\gamma = du \wedge dv. \tag{6.20}$$

Composing the multi-valued map

$$\Phi\colon \Omega_r \to \mathbb{C}^{2n}, \qquad (x, p) \mapsto (\theta, q) \tag{6.21}$$

with the two coordinate changes just detailed, one now obtains a one-valued bijective map

$$\Psi: \Omega_r \to \mathbb{R}^2 \times \mathscr{U}, \qquad (x, p) \mapsto (P, X; u, v). \tag{6.22}$$

(The map Ψ is surjective by definition. Its injectivity is not immediate, but this property follows from the definition of Ω_{loc} and [12].) Since canonicity of Φ is preserved under analytic continuation, one now deduces (6.12) from (6.20) and (6.18).

By construction, the set $\mathscr{U}$ in (6.22) is an open neighborhood of the origin in $\mathbb{R}^{2(n-1)}$ on which $(u_k, v_k) \neq (0, 0)$, $k = 1, ..., n-1$. But one can now extend the map Ψ to a canonical bijective map $\Psi^\#$ on all of Ω_{loc}—this results in all origins being included in the image, and then one obtains (6.12) and

$$\Omega_1 = \{(x, p) \in \Omega_{\text{loc}} \mid (u_k, v_k) = (0, 0), k = 1, ..., n-1\}, \tag{6.23}$$

as announced above. (Note that Φ does *not* extend: the angles do not make sense at the origins.)

We have now detailed how the harmonic oscillator coordinates u, v are defined in terms of the analytically continued quantities $q(x, p)$, $\theta(x, p)$, using the action-angle variables δ (6.8) and γ (6.16) as an intermediate step. Before assembling further tools from [11] and [12] (to be generalized in the next section), it is expedient to add three remarks.

First of all, let us point out that invariance of Ω_1 under the $H_{\kappa,\delta}(c, n)$ flow is a simple corollary of the above developments. Indeed, by virtue of real-analyticity the functions $(x, p) \mapsto P, X, u, v$ extend to holomorphic canonical coordinates on a complex neighborhood of $\Omega_1 \subset \mathbb{C}^{2n}$, and since $u_1 = \cdots = v_{n-1} = 0$ on Ω_1, invariance of Ω_1 can be read off from the dependence of $H_{\kappa,\delta}$ on these variables. Specifically, we recall

$$H_{\kappa,\delta} = 2\kappa^{-1} \sin(\kappa c) \sum_{k=1}^{n} \exp(\delta\kappa\theta_k) \tag{6.24}$$

and note that (6.7) and (6.8) entail

$$\begin{aligned} \theta_1 &= P + 2i[(n-1)\delta_1 + (n-2)\,\delta_2 + \cdots + \delta_{n-1}]/n \\ \theta_2 &= P + 2i[-\delta_1 + (n-2)\,\delta_2 + \cdots + \delta_{n-1}]/n \\ &\vdots \\ \theta_n &= P + 2i[-\delta_1 - 2\delta_2 - \cdots - (n-1)\,\delta_{n-1}]/n. \end{aligned} \tag{6.25}$$

Since δ_k depends linearly on $u_k^2 + v_k^2$ (recall (6.13)), it follows that the gradients $\nabla_u H_{\kappa,\delta}$ and $\nabla_v H_{\kappa,\delta}$ vanish on Ω_1, implying the announced invariance.

Secondly, it should be emphasized that complex neighborhoods and holomorphicity are essential in the previous paragraph. Indeed, on Ω_r the gradients

$\nabla_u H_{\kappa,\delta}$ and $\nabla_t H_{\kappa,\delta}$ are not real-valued, so the flows $\exp(t_{\kappa,\delta} H_{\kappa,\delta})$ do not leave Ω_r invariant. (In the holomorphic context the expression "flow" often refers to a complex flow parameter. However, here as elsewhere in this paper, we think of flow parameters $t_{\kappa,\delta} \in \mathbb{R}$. Furthermore, we are referring to the arbitrary-n case; For $n = 2$ one does get real gradients and the flows stay in Ω_{loc} for non-zero time intervals depending on the initial point.)

Our third remark concerns a comparison of the III_b system studied in [12] and the slightly different version arising in this paper. As explained below (6.3), the difference consists in the dependence on the center of mass coordinates X and P. In the context of [12] these need not necessarily vary over $\mathbb{R}$, since A and L have dependence $\exp(iX)$ and $\exp(iP)$ on X and P, resp. Thus it becomes physically natural and mathematically feasible to study the system on quotient phase spaces, modeling n particles on a ring with angular positions $x_k \in (-\pi, \pi]$, all of whose momenta p_k vary over the "first Brillouin zone" $(-\pi, \pi]$. Furthermore, one obtains real-valued commuting Hamiltonians by taking for instance $\mathrm{Tr}(L^\kappa + L^{-\kappa})$, $\kappa = 1, 2, \ldots$.

By contrast, the matrices A and L arising in the present setting have dependence $\exp(X)$ and $\exp(P)$. Thus one must let X and P vary over $\mathbb{R}$—which is of course in accord with the s_n-soliton picture we are modeling. A second consequence is, that the Hamiltonians $\mathrm{Tr}(L^\kappa + L^{-\kappa})$ are now not real-valued on Ω_r (which may be viewed as a subset of the phase space Ω^c (1.44) in [12]). Indeed, this can be read off from (6.24) and (6.25). Correspondingly, the associated flows are not even locally defined on Ω_r.

As announced above, we now turn to further ingredients from [12] and [11] that we need to generalize in the next section. Specifically, we need a renormalization of the dual Lax matrix $L(-c; \theta, q)$ (in terms of which the τ-functions can be defined) that enables us to continue the new matrix from $\hat{\Omega}(M)$ to points (θ, q) with θ of the form (2.21). Moreover, in the next section we are going to define the generalization of the space Ω_N to the arbitrary-n_j setting indirectly, namely as the image of an explicit space $\hat{\Omega}_N$ under a suitable branch of the analytically continued, reparametrized and extended inverse action-angle map Φ^{-1}, defined at first on $\hat{\Omega}(M)$. Roughly speaking, we are going to reduce the branch fixing to the $N = 1$ case by letting $c \downarrow 0$. Thus, we must first fix the branch for $N = 1$, ensuring in particular that the image of $\hat{\Omega}_1$ equals the space Ω_1 as defined directly already in Section 5. (Here, we are switching again to a picture of maps relating distinct spaces—as opposed to viewing the maps as different coordinatizations of the same space, which we did in the first part of this section. This is notationally and conceptually more convenient, as long as we have not yet proved what we intend to prove.)

In keeping with this program, we begin by defining a similarity transform L_r of the dual Lax matrix. For $N = 1$ (2.21) reads

$$\eta + i(n + 1 - 2k)c, \qquad \eta \in \mathbb{R}, \quad k = 1, \ldots, n, \tag{6.26}$$

and when we try to continue $L(-c;\theta,q)$ or its diagonal similarity transform $C(\theta)\,D(\theta,q)$ (with C, D given by (2.20), (3.1), resp.) to the points (6.26), then we obtain diverging matrix elements due to the poles in the elements $C_{k,k+1}$, $k=1,\ldots,n-1$, of the Cauchy matrix. We now define a renormalized matrix L_r that does not diverge as θ converges to (6.26), and which has further properties to be revealed shortly. It reads

$$L_{r,kl}\equiv \exp\left(-kQ/n+\sum_{m\leqslant k} q_m\right)\prod_{m\neq k}\left(\frac{\operatorname{sh}((\theta_k-\theta_m)/2-ic)}{\operatorname{sh}(\theta_k-\theta_m)/2}\right)^{1/2}\cdot\frac{\operatorname{sh}(-ic)}{\operatorname{sh}((\theta_k-\theta_l)/2-ic)}$$
$$\cdot\exp\left(lQ/n-\sum_{m\leqslant l-1} q_m\right)\prod_{m\neq l}\left(\frac{\operatorname{sh}((\theta_l-\theta_m)/2+ic)}{\operatorname{sh}(\theta_l-\theta_m)/2}\right)^{1/2},\qquad k,l=1,\ldots,n. \quad (6.27)$$

(For $(\theta,q)\in\hat{\Omega}(n)$ the square-root sign ambiguity in each product term is fixed by requiring positivity for $c=0$, and then continuing to $c\in(0,\pi/n)$.) Equivalently, we may set

$$L_r\equiv D_r CDD_r^{-1}, \quad (6.28)$$

where D_r is the diagonal matrix with elements

$$D_{r,kk}=\exp\left(-\theta_k/2-kQ/n+\sum_{l\leqslant k} q_l\right)\prod_{l\neq k}\left(\frac{\operatorname{sh}((\theta_k-\theta_l)/2-ic)}{\operatorname{sh}(\theta_k-\theta_l)/2}\right)^{1/2},\qquad k=1,\ldots,n. \quad (6.29)$$

Thus L_r is indeed a similarity transform of CD.

We proceed by showing that L_r does have a finite limit as θ converges to (6.26). To render the limit unique, we must specify the path along which L_r is to be continued from $\hat{\Omega}(n)$ to (6.26). (Of course, no ambiguities arise from continuing q to $\mathbb{C}^n$.) To this end we start from the point (6.26) with c replaced by $-ic^+$, $c^+\in(c,\pi/n]$. Thus we obtain a theta-cluster around η ordered in the required fashion. Now we rotate this cluster over $\pi/2$ (around η). Then we wind up with positive radicands in (6.27) (for instance, $\sin((l-j)\,c^+-c)/\sin(l-j)\,c^+>0$), and taking c very small one sees that all square roots are in fact positive in this point. Letting now $c^+\downarrow c$, one readily verifies that all matrix elements of L_r converge to 0, save for the elements 12, 23, ..., $n1$. To be more specific, a straightforward calculation yields

$$\lim_{c^+\downarrow c} L_{r,n1}=\exp(Q/n), \quad (6.30)$$

$$\lim_{c^+\downarrow c} L_{r,k,k+1}=-\exp(Q/n),\qquad k=1,\ldots,n-1. \quad (6.31)$$

Next, we consider L_r on the image of the above space Ω_r (6.14) under the map Φ. It can be written

$$L_r=\exp([X+(n-1)\,\pi i]/n)\operatorname{diag}(\exp(-i\gamma_1),\ldots,\exp(-i\gamma_{n-1}),1)$$
$$\cdot\, U(\delta_1,\ldots,\delta_{n-1})\operatorname{diag}(1,\exp(i\gamma_1),\ldots,\exp(i\gamma_{n-1})) \quad (6.32)$$

in terms of the variables γ (6.16) and δ (6.8). (Here, we also used (6.15).) It is far from evident, but true, that the matrix $U(\delta)$ is *unitary*—this follows from Appendix A in [12]. Thus, the matrix $\exp(-X/n)\, L_r$ is unitary on the image of Ω_r. Transforming to the harmonic oscillator coordinates (6.19), a second key property of L_r is that it extends to a real-analytic function of $u_1, ..., v_{n-1}$ on $\Psi^{\#}(\Omega_{\text{loc}})$, as we now explain.

Of course, real-analyticity of L_r on $\Psi(\Omega_r)$ is evident. The crux is, that allowing origins $(u_k, v_k) = (0, 0)$ does not give rise to singularities: The harmonic oscillator coordinates serve to uniformize the square-root branch varieties occurring in the dual Lax matrix when the variables θ, q are used. This real-analytic extension property of L_r is not immediate, and we continue by showing why it holds true. (As will be seen, the key point applies to the general setting of the next section.) First we observe that by virtue of (6.8) the dependence of $U(\delta)$ in (6.32) on $\delta_1, ..., \delta_{n-1}$ occurs via functions

$$\sin\left(\sum_{k=l}^{m} \delta_k + \alpha\right)^{1/2}, \qquad 1 \leqslant l \leqslant m \leqslant n-1, \quad \alpha \in \{-c, 0, c\} \tag{6.33}$$

that are positive on Ω_r and that stay away from 0 on Ω_{loc} unless $\alpha = -c$ and $m = l$. In view of (6.13), all of these functions except $\sin(\delta_k - c)^{1/2}$ are real-analytic in u, v on Ω_{loc}. Now in $U(\delta)_{kl}$ the exceptional functions $\sin(\delta_k - c)^{1/2}$ and $\sin(\delta_{l-1} - c)^{1/2}$ occur in the products for $k = 1, ..., n-1$ and $l = 2, ..., n$, but they are canceled by the Cauchy matrix denominator $\sin(\delta_k - c)$ when l equals $k+1$, cf. (6.27). Since $L_{r,12}, ..., L_{r,n-1,n}$ are γ-independent (cf. (6.32)), it follows that these matrix elements have a real-analytic extension. Taking now $l \neq k+1$, the Cauchy matrix element is manifestly real-analytic, and we need only observe that we may write

$$\begin{aligned} &\exp(-i\gamma_k) \sin(\delta_k - c)^{1/2} \sin(\delta_l - c)^{1/2} \exp(i\gamma_l) \\ &\quad = (u_k + iv_k)(u_l - iv_l) \left(\frac{\sin((u_k^2 + v_k^2)/4)}{u_k^2 + v_k^2}\right)^{1/2} \left(\frac{\sin((u_l^2 + v_l^2)/4)}{u_l^2 + v_l^2}\right)^{1/2} \end{aligned} \tag{6.34}$$

to deduce real-analyticity of the remaining factors in $L_{r,kl}$. (Indeed, the function $x \mapsto (\sin(x)/x)^{1/2}$ is holomorphic at $x = 0$.)

The remainder of this section deals with the analytic continuation and branch fixing of the inverse action-angle map

$$\mathscr{E}: \hat{\Omega}(n) \to \Omega(n), \qquad (\theta, q) \mapsto (x, p). \tag{6.35}$$

We studied the continuation properties of the functions $x(\theta, q)$, $p(\theta, q)$ in considerable detail in Chapter 3 of [11], and we begin by summarizing the pertinent features obtained there. First of all, we required $|\text{Im}\, \theta_k| < c$, so as to block multi-valuedness coming from the pair potentials $f(c; \theta_k - \theta_l)$ (2.15). Then we proved that the map $\mathscr{E}$ extends to a multi-valued holomorphic map R in a domain $\mathscr{H}_s^{\flat}$ obtained from this polystrip by discarding three analytic subvarieties of complex codimension one.

The first variety is the pole variety of the potentials—the subset where $\theta_k = \theta_l$ (mod $2\pi i$) for some $k \neq l$. The second variety consists of all points where the dual Lax matrix (chosen equal to $C(\theta)\,D(\theta, q)$ in [11]) has degenerate spectrum. The third variety consists of all points where the dual Lax matrix has a pair of eigenvalues α_k, α_l such that $\alpha_k = \exp(2ic)\,\alpha_l$. (Such points give rise to square-root branch points for the functions $V_k(c; x)$, cf. the similarities (3.19), (3.20).)

Now in this paper we need to continue R beyond the strips $|\mathrm{Im}\,\theta_k| < c$, as is clear from (2.21) and (6.26). Inspection of the arguments in Chapter 3 of [12] shows that this can be done: One need only delete the branch varieties of the potentials (the subset where $\theta_k - \theta_l = 2ic$ (mod $2\pi i$) for some pair k, l), and discard once more the above three varieties, taking however into account that the two spectral varieties now depend on the sheet for the potentials $V_k(c; \theta)$ (3.12). (Observe that the spectrum of the dual Lax matrix is not invariant under flipping the signs of one or more pair potentials.)

The latter source of multi-valuedness of R is a new feature, not handled in [11], since this was not needed. We can avoid it in the present context, too, as will be made clear shortly. On the other hand, we do need to know about the multivaluedness of R restricted to the domain $\mathscr{H}_s^\flat$: When (x^0, p^0) is one value of the restricted map lying over some $(\theta^0, q^0) \in \mathscr{H}_s^\flat$, then all other values are obtained by taking products of permutations $(\sigma(x^0), \sigma(p^0))$, and adding multiples of $2\pi i$ to x_k^0 and multiples of πi to p_k^0. For all of these branches the numbers $\exp(x_k^0)$ are the eigenvalues of $L(-c; \theta^0, q^0)$ and the numbers $\exp(\theta_k^0)$ are the eigenvalues of $L(c; x^0, p^0)$ [11].

The sums of the multiples just mentioned vanish, as is clear from the sum rules

$$\sum x_k = \sum q_k, \qquad \sum p_k = \sum \theta_k. \tag{6.36}$$

These hold true on $\hat{\Omega}$ (as follows by comparing determinants in the similarities (3.20), (3.19)), and since $\hat{\Omega}$ is a subset of $\mathscr{H}_s^\flat$, they are preserved under analytic continuation. Of course, the equalities (6.36) continue to hold when R is continued beyond the strips $|\mathrm{Im}\,\theta_k| < c$. Another key feature is preserved as well: Since $\mathscr{E}$ (6.35) is a canonical map, R is canonical, too. This entails in particular that R has a local holomorphic inverse.

Finally, letting c vary over $(0, \pi)$, the map $\mathscr{E}$ (6.35) is analytic in c, too, cf. Appendix B in [10]. More generally, combining the arguments in l.c. with the analysis in Chapter 3 of [11], it readily follows that when $R(c_0; \theta_0, q_0)$, $c_0 \in (0, \pi)$, is defined and holomorphic in a (θ, q)-polydisc around (θ_0, q_0), then it is holomorphic in a (c, θ, q)-polydisc around (c_0, θ_0, q_0). (Actually, one does not need Hartogs' theorem in the proof of Theorem B1 of [10], but only Osgood's lemma, whose proof is quite easy [22].) This fact will be exploited to fix a branch of R in the next section.

After this summary we return to the case at hand, and consider the path

$$\theta_k(\phi) = \eta + c^+(n+1-2k)\exp(i\phi), \qquad c^+ \in (c, \pi/n], \quad k = 1, ..., n, \tag{6.37}$$

$$q_k(\phi) = [y + 2\pi(n-k)\exp(i\phi)]/n, \qquad k = 1, ..., n, \tag{6.38}$$

where $\phi \in [0, \pi/2]$. This path does not meet the pole and branch varieties of the pair potentials $f(c; \theta_k - \theta_l)$, and so the matrix $L_r(-c; \theta, q)$ is well defined and holomorphic along the path. Now for $c^+ \downarrow c$ the matrix L_r evaluated in the endpoint converges to a multiple of the antiperiodic shift, cf. (6.31), (6.30). Thus we deduce

$$\lim_{c^+ \downarrow c} \sigma(L_r(-c; \theta(\pi/2), q(\pi/2))) = \{\exp([y + 2\pi i(n-k)]/n) \mid k = 1, ..., n\}. \tag{6.39}$$

From this spectrum one reads off that the endpoint does not belong to the above spectral varieties for c^+ close to c.

As a consequence, the map R can be continued to the endpoint for any c^+ in the interval $(c, c+\varepsilon)$ with $\varepsilon > 0$ small enough. Indeed, though we do not know whether the above path (6.37), (6.38) is disjoint from the spectral varieties, we can always avoid the latter by deforming the path slightly, since the varieties have real codimension two. Furthermore, we can select a suitable branch of R by changing the part of the path that lies in $\mathscr{H}_s^\flat$, without changing L_r in the endpoint. (This is because the potentials $V_k(c; \theta)$ are one-valued in $\mathscr{H}_s^\flat$.)

Now we fix the branch of R as follows. First we change to oscillator variables u, v (6.19) via (6.8) and (6.16). Then the endpoint belongs to $\Psi(\Omega_r)$ for ε small enough, cf. (6.22), so we may and will choose the branch to be Ψ^{-1}. (Note that our definition of Ω_{loc} is precisely such as to fix the ambiguities concerning ordering and addition of πi-multiples.) Denoting this reparametrized branch again by $\mathscr{E}$, it extends real-analytically to the oscillator origins $\delta_k = c$. Then the extended map $\mathscr{E}^\# = \Psi^{\# -1}$ satisfies by construction

$$\mathscr{E}^\#(\hat{\Omega}_1) = \Omega_1, \tag{6.40}$$

where

$$\hat{\Omega}_1 \equiv \{\eta, y; u, v) \in \mathbb{R}^{2n} \mid (u_k, v_k) = (0, 0), k = 1, ..., n-1\}, \tag{6.41}$$

and, as advertised,

$$\Omega_1 = \{(x, p) \in \mathbb{C}^n \times \mathbb{R}^n \mid x_k = [y + 2\pi i(n-k)]/n, p_k = \eta, k = 1, ..., n\}. \tag{6.42}$$

(Indeed, (6.42) coincides with the space Ω_1 from Section 5, cf. (5.2), (5.3).)

7. THE GENERAL CASE

We are finally prepared to study the case of arbitrary $n_1, ..., n_N \in \mathbb{N}^*$. Thus, putting

$$M \equiv n_1 + \cdots + n_N \tag{7.1}$$

and requiring

$$c \in (0, \pi/\max(n_1, ..., n_N)) \tag{7.2}$$

as in Section 2 (cf. (2.13)), our objective is once more to arrive at particle dynamics (3.14), with L an $M \times M$ Lax matrix $L(c; x, p)$ continued from $\Omega(M)$ (3.17) to a subspace Ω_N of $\mathbb{C}^{2M}$ with $2N$ real dimensions, modeling the fused τ-functions from Section 2. For unequal n_j our results are less explicit and complete than before. In particular, by contrast to Section 5, we are only able to define Ω_N indirectly. Moreover, in the process we need some assumptions that hold true in various special cases (notably the equal-n_j case), but that we have not proved in full generality.

Our starting point is a renormalized dual Lax matrix $L_r(-c; \theta, q)$, where we take (θ, q) in $\hat{\Omega}(M)$ (3.22) at first. This matrix is defined by (6.28), with C, D given by (2.20), (3.1), resp., and with the renormalizing matrix D_r given by

$$D_r \equiv \mathrm{diag}(D_r^{(1)}, ..., D_r^{(N)}). \tag{7.3}$$

Here, $D_r^{(j)}$ is the diagonal $n_j \times n_j$ matrix defined by

$$D_{r,kk}^{(j)} \equiv \exp\left(-\theta_{\rho_j+k}/2 - kQ_j/n_j + \sum_{i=1}^{k} q_{\rho_j+1}\right) \cdot \prod_{\substack{l=\rho_j+1 \\ l \neq \rho_j+k}}^{\rho_j+1} \left(\frac{\mathrm{sh}((\theta_{\rho_j+k} - \theta_l)/2 - ic)}{\mathrm{sh}(\theta_{\rho_j+k} - \theta_l)/2}\right)^{1/2}, \qquad k = 1, ..., n_j, \tag{7.4}$$

where

$$\rho_1 \equiv 0, \qquad \rho_j \equiv n_1 + \cdots + n_{j-1}, \qquad j = 2, ..., N, \tag{7.5}$$

$$Q_j \equiv \sum_{i=1}^{n_j} q_{\rho_j+i}, \qquad j = 1, ..., N. \tag{7.6}$$

Just as in the special case $N = 1$, the matrix L_r thus defined on $\hat{\Omega}(M)$ has a finite limit as θ converges to points of the form (2.21). The continuation path is chosen once more in such a fashion that the radicands in (7.4) are positive in the points

$$\theta^+_{\rho_j+k} \equiv \eta_j + i(n_j + 1 - 2k)\, c^+, \qquad c^+ \in (c, \pi/\max(n_1, ..., n_N)]. \tag{7.7}$$

(Since η satisfies (2.1), this can be done e.g. by increasing first the distance between successive eta's so that the theta clusters described below (6.29) yield the required ordering $\theta_M < \cdots < \theta_1$. Rotating the clusters over $\pi/2$ and letting the eta's reconverge to their fixed values then yields the desired positivity.) With this convention one obtains in the diagonal blocks

$$\lim_{c^+ \downarrow c} L_{r, p_j+n_j, p_j+1} = \exp(Q_j/n_j) \prod_{m \in \{1, ..., p_j\} \cup \{p_{j+1}+1, ..., M\}} f(c; \theta_{p_j+1} - \theta_m) \quad (7.8)$$

$$\lim_{c^+ \downarrow c} L_{r, p_j+k, p_j+k+1} = -\exp(Q_j/n_j) \prod_{m \in \{1, ..., p_j\} \cup \{p_{j+1}+1, ..., M\}} f(c; \theta_{p_j+k+1} - \theta_m) \quad (7.9)$$

(where $k = 1, ..., n_j - 1$), whereas all other elements have limit 0. (It is to be noted that (7.8) applies to the special case $n_j = 1$ as well.) In the off-diagonal blocks all elements converge to 0 save for the element in the left lower corner: It yields

$$\lim_{c^+ \downarrow c} L_{r, p_j+n_j, p_l+1} = \exp(Q_l/n_l) \frac{[\sin(n_j c)\sin(n_l c)]^{1/2}}{i\,\mathrm{sh}((\theta_{p_j+n_j} - \theta_{p_l+1})/2 - ic)} \cdot \prod_{m \in \{1, ..., p_l\} \cup \{p_{l+1}+1, ..., M\}} f(c; \theta_{p_l+1} - \theta_m). \quad (7.10)$$

(Note this amounts to (7.8) for $j = l$.) For later use we also point out that the analytic continuation described above entails that all of the pair potentials $f(\cdots)$ continued to the points (7.7) and occurring in (7.9)–(7.10) converge to 1 when the distances between the eta's go to ∞. (Of course, in this limit the elements (7.10) converge to 0 for $j \neq l$.)

To proceed, we define local oscillator variables generalizing the $N = 1$ variables (6.19). Thus we begin by introducing

$$\delta_k^{(j)} \equiv -i(\theta_{p_j+k} - \theta_{p_j+k+1})/2, \qquad j = 1, ..., N, \quad k = 1, ..., n_j - 1 \quad (7.11)$$

$$\gamma_k^{(j)} \equiv i[(q_{p_j+1} + \cdots + q_{p_j+k}) - kQ_j/n_j], \qquad j = 1, ..., N, \quad k = 1, ..., n_j - 1 \quad (7.12)$$

as a generalization of (6.8) and (6.16), resp. Next, we first restrict attention to (θ, q) such that

$$\eta_N < \cdots < \eta_1, \quad y_j \in \mathbb{R}, \quad \delta_k^{(j)} \in (c, \pi/n_j], \quad \gamma_k^{(j)} \in \mathbb{R}, \quad j = 1, ..., N, \quad k = 1, ..., n_j - 1, \quad (7.13)$$

where we have

$$\eta_j \equiv \sum_{k=p_j+1}^{p_{j+1}} \theta_k/n_j, \qquad j = 1, ..., N, \quad (7.14)$$

$$y_j \equiv Q_j - (n_j - 1)\pi i, \qquad j = 1, ..., N. \quad (7.15)$$

Then we can unambiguously define

$$u_k^{(j)} \equiv -2(\delta_k^{(j)} - c)^{1/2} \cos \gamma_k^{(j)}, \qquad v_k^{(j)} \equiv 2(\delta_k^{(j)} - c)^{1/2} \sin \gamma_k^{(j)}, \quad (7.16)$$

by insisting that the square roots be positive. The crux is now, that when $L_r(-c;\theta,q)$ is rewritten in terms of the variables η, y and $u^{(1)},\ldots,v^{(N)}$, it is holomorphic not only in the points (7.13), but also in all points obtained by allowing in addition $\delta_k^{(j)}=c$. Indeed, this follows by the same arguments as in the $N=1$ case, cf. the paragraph containing the key identity (6.34).

We continue by introducing a $2N$-dimensional space

$$\hat{\Omega}_N \equiv \{(\eta, y; u^{(1)},\ldots,v^{(N)})\in\mathbb{R}^{2M} \mid \eta_N<\cdots<\eta_1,\ u^{(1)},\ldots,v^{(N)}=0\}. \tag{7.17}$$

As we have just seen, the matrix $L_r(-c;\eta,y;u^{(1)},\ldots,v^{(N)})$ is well defined and holomorphic on a complex neighborhood $\hat{\mathcal{N}}\subset\mathbb{C}^{2M}$ of $\hat{\Omega}_N$. By construction, its restriction to $\hat{\Omega}_N$ is such that the determinant $|1_M+\exp(-2ivc)\,L_r|$ equals the τ-function obtained via the fusion procedure described in Section 2, cf. in particular (2.25)–(2.27). Therefore, it remains to connect the "soliton picture" associated with $\hat{\Omega}_N$ to a "particle picture" associated with a particle phase space Ω_N generalizing the spaces thus denoted in Sections 5 and 6.

In order to do so, we need certain assumptions already alluded to above. More specifically, we need a spectral assumption A(sp) and a topological assumption A(top). We are now in the position to detail A(sp). It concerns the spectrum of L_r restricted to $\hat{\Omega}_N$. We denote this $M\times M$ matrix by

$$\hat{A}(n_1,\ldots,n_N;\eta,y), \qquad (\eta,y)\in\hat{\Omega}(N). \tag{7.18}$$

Its non-zero elements are given explicitly by (7.9)–(7.10), with θ related to η via (2.21), and Q related to y via (7.15). Our spectral assumption A(sp) now reads: For all $(\eta,y)\in\hat{\Omega}(N)$ the matrix $\hat{A}$ has simple spectrum, with no pair of eigenvalues α_j, α_k satisfying $\alpha_j=\exp(2ic)\,\alpha_k$.

We shall discuss this assumption in more detail shortly. Taking it for granted, we first wish to sketch its consequences. In the process, the topological assumption A(top) will be detailed. To begin with, because L_r is holomorphic on a complex neighborhood $\hat{\mathcal{N}}$ of $\hat{\Omega}_N$, we may as well assume (by eventually shrinking $\hat{\mathcal{N}}$) that L_r has the above spectral features on all of $\hat{\mathcal{N}}$. In particular, this is then the case for points of the form (7.13), provided c^+ is chosen sufficiently close to c (how close may depend on the other variables). The crux is now, that the inverse action-angle map $\mathscr{E}$ (6.35) (taking $n\to M$ (7.1), of course) continues to a multi-valued holomorphic map R (cf. Section 6), which is defined in particular in the latter points (7.13), and when we compose R with the parameter change (7.16) on (7.13), it admits an analytic continuation $R^{\#}$ to all of $\hat{\mathcal{N}}$.

Now the space $\hat{\Omega}_N$ (7.17) is manifestly convex, so we may and will choose $\hat{\mathcal{N}}$ simply-connected. Then the monodromy theorem ensures that no multi-valuedness can arise from loops in $\hat{\mathcal{N}}$. We now select a suitable branch of $R^{\#}$ on $\hat{\mathcal{N}}$ (our choice will be detailed later on), and denote this branch of the reparametrized and extended map by $\mathscr{E}^{\#}$. The particle phase space Ω_N is then defined by

$$\Omega_N\equiv\mathscr{E}^{\#}(\hat{\Omega}_N). \tag{7.19}$$

Since canonicity is preserved under analytic continuation, and since the coordinate change to oscillator variables is manifestly canonical, the holomorphic map

$$\mathcal{E}^{\#}: \hat{\mathcal{V}} \to \mathcal{V} \subset \mathbb{C}^{2M}, \qquad (\eta, y; u^{(1)}, \ldots, v^{(N)}) \mapsto (x, p) \tag{7.20}$$

satisfies

$$\mathcal{E}^{\#*}(dx \wedge dp) = dy \wedge d\eta + \sum_{j=1}^{N} dv^{(j)} \wedge du^{(j)}. \tag{7.21}$$

This canonicity property entails that $\mathcal{E}^{\#}$ admits a *local* holomorphic inverse on $\hat{\mathcal{V}}$. Unfortunately, we do not know whether $\mathcal{E}^{\#}$ is *globally* invertible on a suitably restricted complex neighborhood $\hat{\mathcal{V}}_r \subset \hat{\mathcal{V}}$ of $\hat{\Omega}_N$ in the general case at issue here. This injectivity question is intimately related to the problem of obtaining more information on Ω_N.

To obviate this snag, we assume from now on that Ω_N (7.19) is simply-connected. Indeed, this topological assumption A(top), which we conjecture to be true in all cases, readily entails that such a neighborhood $\hat{\mathcal{V}}_r$ does exist. To explain this, let $\Phi^{\#}$ denote a local inverse of $\mathcal{E}^{\#}$ around a point in Ω_N. Then it is easy to see that $\Phi^{\#}$ continues to a one-valued global inverse on Ω_N. Next, continuing $\Phi^{\#}$ to a simply-connected complex neighborhood $\mathcal{V}_r \subset \mathcal{V}$ of Ω_N, one deduces that the map $\mathcal{E}^{\#}$ is injective on

$$\hat{\mathcal{V}}_r \equiv \Phi^{\#}(\mathcal{V}_r), \tag{7.22}$$

with inverse $\Phi^{\#}$, as advertised.

We are now in the position to define pullback "particle Hamiltonians"

$$H_{\kappa,\delta} \equiv \hat{H}_{\kappa,\delta} \circ \Phi^{\#}, \qquad \Phi^{\#} \equiv \mathcal{E}^{\# -1} \tag{7.23}$$

on $\mathcal{V}_r$, where

$$\hat{H}_{\kappa,\delta} \equiv 2\kappa^{-1} \sin(\kappa c) \sum_{m=1}^{M} \exp(\delta \kappa \theta_m), \qquad \kappa = 1, 2, \ldots, \quad \delta = +, - \tag{7.24}$$

are the usual "soliton Hamiltonians" on $\hat{\mathcal{V}}_r$. The latter are real-valued on $\hat{\Omega}_N$, and now the invariance of Ω_N under all of the $H_{\kappa,\delta}$ flows follows in the same way as in Section 6 (cf. the paragraph containing (6.24)).

The definition of the maps is such that the particle Hamiltonians (7.23) coincide with (3.14), where $L = L(c; x, p)$ denotes the Lax matrix, suitably continued to $(x, p) \in \mathcal{V}_r$. More specifically, the definition of the position vector $x \in \mathbb{C}^M$ is such that one has

$$\sigma(\hat{A}) = \{\exp(x_1), \ldots, \exp(x_M)\}. \tag{7.25}$$

Therefore, the spectral assumption A(sp) guarantees that the (meromorphic) radicands in the potentials $V_j(c; x)$ (3.12) have no poles and zeros on $\mathcal{V}_r$, so that

$L(c; x, p)$ is well defined on $\mathcal{M}_r$. (The branch choice for L is fixed by the branch choice for $R^{\#}$, cf. below.)

The above sketch contains the gist of the particle-soliton correspondence in the general case. We continue by filling in various details, beginning with the spectral assumption A(sp). It is satisfied in several cases, which we now consider. First, taking $n_j \in \{1, 2\}$, $j = 1, \ldots, N$, its validity follows from [11]. Indeed, in that case the space $\hat{\Omega}_N$ amounts to a subspace of the space $\hat{\Omega}^{\#}$ (5.20) in [11], and the assumption is satisfied on all of $\hat{\Omega}^{\#}$, cf. also Section 4. (The renormalized dual Lax matrix employed in Section 5B of l.c. differs from, but is similar to the above matrix $\hat{A}$ (7.18) on $\hat{\Omega}_N$.) Choosing $N = 2$, $n_1 = 1$, $n_2 = 2$, the first remark on p. 935 of [11] reveals that in the unequal-n_j case (of which this choice is the simplest instance) one should not *expect* a simple explicit description of the space Ω_N.

In the special case $n_j = n$, $j = 1, \ldots, N$, studied in Section 5, the spectral assumption is satisfied as well. Indeed, from the fusion argument in Section 2 (cf. (2.21)–(2.27) with $\nu = 0$, $t_{\kappa,\delta} = 0$) we obtain

$$\begin{aligned}
&|\lambda 1_{nN} + \hat{A}(n, \ldots, n; \eta, y)| \\
&= \sum_{m=0}^{N} (\lambda^n)^{N-m} \sum_{\substack{J \subset \{1, \ldots, N\} \\ |J| = m}} \exp\Big(\sum_{j \in J} Q_j\Big) \prod_{\substack{j \in J \\ l \notin J}} f(nc; \eta_j - \eta_l) \\
&= |\lambda^n 1_N + (-)^{n-1} L(-nc; \eta, y)| \\
&= \prod_{j=1}^{N} (\lambda^n + (-)^{n-1} \exp(X_j)), \qquad (7.26)
\end{aligned}$$

where we also used (7.15) and (5.29). Therefore the spectrum of $\hat{A}$ is given by (7.25) and (5.2), and so the validity of A(sp) can be read off.

Turning now to the general case, we assert that the two spectral properties involved in A(sp) hold true on an open dense full measure subset of $\hat{\Omega}_N$. To prove this assertion we exploit perturbation theory. First, letting

$$\eta_j - \eta_{j+1} \geqslant d, \qquad j = 1, \ldots, N-1, \quad d > 0, \qquad (7.27)$$

we deduce from (7.9)–(7.10) the limit

$$\lim_{d \to \infty} \hat{A}(n_1, \ldots, n_N; \eta, y) = \hat{A}(y), \qquad (7.28)$$

where

$$\hat{A}(y) \equiv \operatorname{diag}(\hat{A}^{(1)}(y_1), \ldots, \hat{A}^{(N)}(y_N)), \qquad (7.29)$$

with $\hat{A}^{(j)}$ the $n_j \times n_j$ matrix whose non-zero elements read

$$\hat{A}^{(j)}_{n_j 1} = -\exp[(y_j - \pi i)/n_j], \qquad \hat{A}^{(j)}_{k,k+1} = \exp[(y_j - \pi i)/n_j], \quad k = 1, \ldots, n_j - 1. \quad (7.30)$$

Thus $\hat{A}^{(j)}$ has spectrum

$$\sigma(\hat{A}^{(j)}) = \{\exp[(y_j + 2\pi i(k-1))/n_j] \mid k = 1, ..., n_j\}. \tag{7.31}$$

Therefore, after fixing $y \in \mathbb{R}^N$ such that

$$y_j/n_j - y_{j+1}/n_{j+1} > 0, \qquad j = 1, ..., N-1, \tag{7.32}$$

the spectral properties hold true when the minimal distance d between the eta's is large enough. (Since the limit matrix $\hat{A}(y)$ (7.29) is normal, the spectrum of $\hat{A}(n_1, ..., n_N; \eta, y)$ can be handled via the second resolvent formula.) From this one deduces they can break down only on a real-analytic subvariety of codimension at least one, and so the assertion follows.

We conjecture that the spectral properties hold true on *all* of $\hat{\Omega}_N$, or, equivalently, that the assumption A(sp) is valid. However, we do not need any assumption to prove that the multi-valued holomorphic map R can be continued to points $(\theta, q) \in \mathbb{C}^{2M}$ of the following form: q is given by

$$q_{\rho_j+k} = [y_j + 2\pi i(n_j - k)]/n_j, \tag{7.33}$$

where y is fixed and obeys (7.32), and θ is given by (7.7) where η obeys (7.27) with d large enough and where c^+ varies over $(c, c+\varepsilon]$ with ε small enough. Indeed, consider the path

$$\theta_{\rho_j+k}(\phi) = \eta_j + c^+(n_j + 1 - 2k)\exp(i\phi), \qquad j = 1, ..., N, \quad k = 1, ..., n_j, \tag{7.34}$$

$$q_{\rho_j+k}(\phi) = [y_j + 2\pi(n_j - k)\exp(i\phi)]/n_j, \qquad j = 1, ..., N, \quad k = 1, ..., n_j, \tag{7.35}$$

where $\phi \in [0, \pi/2]$. We first choose d large enough so that $(\theta(0), q(0))$ belongs to $\hat{\Omega}(M)$. Then the path does not meet the pole and branch varieties of the pair potentials $f(c; \theta_l - \theta_m)$, so $L_r(-c; \theta, q)$ is holomorphic along the path. Now for $d \to \infty$ and $\varepsilon \downarrow 0$ the matrix $L_r(-c; \theta(\pi/2), q(\phi/2))$ converges to $\hat{A}(y)$ (7.29), so it readily follows that the endpoint does not belong to the two spectral varieties described below (6.35), provided d is sufficiently large and ε sufficiently small. Therefore, R can be continued to these points, as asserted.

Changing next to oscillator variables (7.16) via (7.11) and (7.12), the reparametrized map R extends real-analytically to the oscillator origins, since the matrix L_r has this property. As a consequence, the extended map $R^{\#}$ is holomorphic on a subset $\hat{\Omega}_s$ of $\hat{\Omega}_N$ consisting of points for which y is a fixed vector satisfying (7.32), and η satisfies (7.27) with d large enough (the choice of d depends on y). Now $\hat{\Omega}_s$ is simply-connected, so $R^{\#}$ is holomorphic in a simply-connected complex neighborhood $\hat{\mathcal{N}}_s \subset \mathbb{C}^{2M}$ of $\hat{\Omega}_s \subset \hat{\Omega}_N$. Since no multi-valuedness arises from loops in $\hat{\mathcal{N}}_s$, we can fix a branch $\mathcal{S}^{\#}$ of $R^{\#}$ by fixing a branch on $\hat{\Omega}_s$.

To this end we recall that by construction the vectors x, p satisfy

$$\mathcal{U}^{-1}\hat{A}(n_1, \ldots, n_N; \eta, y)\,\mathcal{U} = \mathrm{diag}(\exp x_1, \ldots, \exp x_M), \tag{7.36}$$

$$(\mathcal{U}^{-1}\hat{L}(n_1, \ldots, n_N; \eta)\,\mathcal{U})_{ll} = \exp(p_l)\, V_l(c; x), \qquad l = 1, \ldots, M, \tag{7.37}$$

where

$$\hat{L} \equiv \mathrm{diag}(\exp[\eta_1 + ic(n_1 - 1)], \ldots, \exp[\eta_N + ic(-n_N + 1)]), \tag{7.38}$$

and where $\mathcal{U} \in GL(M, \mathbb{C})$ is uniquely determined up to right multiplication by a diagonal invertible matrix. Now the definition of $\hat{A}$ (7.18) entails

$$\lim_{c \downarrow 0} \hat{A}(n_1, \ldots, n_N; \eta, y) = \hat{A}(y), \tag{7.39}$$

with $\hat{A}(y)$ given by (7.29), cf. (7.9)–(7.10). Thus we may and will fix the branch of x by requiring

$$x_{\rho_j + k} \sim [y_j + 2\pi i(n_j - k)]/n_j, \qquad y_N/n_N < \cdots < y_1/n_1, \quad c \downarrow 0. \tag{7.40}$$

(Indeed, this fixes the ordering and $2\pi i$-multiples.)

Next, we readily deduce from (7.36)–(7.38) that we have

$$\exp(p_{\rho_j + k})\, V_{\rho_j + k}(c; x) \sim \exp(\eta_k), \qquad y_N/u_N < \cdots < y_1/u_1, \quad c \downarrow 0. \tag{7.41}$$

(Note that (7.39) and (7.40) entail that $\mathcal{U}$ may be chosen to converge to a block diagonal matrix for $c \downarrow 0$.) Thus we may and will fix the branch of p by requiring

$$p_{\rho_j + k} \sim \eta_j, \qquad y_N/n_N < \cdots < y_1/n_1, \quad c \downarrow 0. \tag{7.42}$$

(This fixes also the branch for the potentials $V_l(c; x)$: they all converge to 1 for $c \downarrow 0$.)

We repeat that we have reached these conclusions without making any assumptions. To proceed further, however, we have to invoke the above spectral assumption A(sp). It entails that the map $\mathcal{E}^{\#}$ already defined on the subset $\hat{\Omega}_s$ of $\hat{\Omega}_N$ has an analytic continuation to all of $\hat{\Omega}_N$. Since $\hat{\Omega}_N$ is convex, no continuation ambiguities arise in the process, and so the space Ω_N (7.19) is well defined. More generally, we are now in a position to invoke the topological assumption A(top) to finish our account of the arbitrary-n_j case as described above, cf. the paragraphs containing (7.20)–(7.25).

We conclude this section by discussing A(top) and by completing our account of the equal-n_j case. First, we should point out that A(top) is equivalent to the assumption that $\mathcal{E}^{\#}$ is injective on $\hat{\Omega}_N$. Indeed, assuming injectivity, it follows that $\mathcal{E}^{\#}$ is a homeomorphism from $\hat{\Omega}_N$ onto Ω_N, so that Ω_N must be simply-connected. (Recall $\hat{\Omega}_N$ is convex, hence simply-connected.) The converse implication was already used above: Simply-connectedness of Ω_N entails that a local inverse extends to a one-valued global inverse.

As a result, the problem of proving the assumptions A(sp) and A(top) boils down to the algebraic problem of obtaining more information on the spectrum of the explicitly known matrix $\hat{A}(n_1, ..., n_N; \eta, y)$ (7.18) on the explicit space $\hat{\Omega}_N$ (7.17), and on the suitably normalized diagonalizing matrix $\mathscr{U}$, cf. (7.36)–(7.38). As we have already seen above, in the case where all n_j equal 1 or 2, the needed information can be found in [11], whereas [12] can be invoked to handle the $N=1$ and arbitrary-n case. We supply the missing information for the equal-n_j case in Appendix E, but we have not found a synthesis of our algebraic arguments (which heavily lean on the fundamental commutation relation) that would dispose of the arbitrary-n_j case.

Turning finally to the case of equal n_j's, let us take stock of what remains to be proved: We have defined a space Ω_N via (7.19), and now it suffices to show that this space coincides with the space thus denoted in Section 5. Indeed, we recall that the latter is given by the equations (5.2) and (5.3), with (X, P) varying over $\Omega(N)$, cf. (5.1). Hence it is manifestly convex, so that equality of the two spaces entails A(top). Moreover, we recall that the argument in the paragraph containing (7.26) implies A(sp). More specifically, setting $n_1 = \cdots = n_N = n$ in (7.36)–(7.42), it says that the spectrum of $\hat{A}(n, ..., n; \eta, y)$ reads

$$\sigma(\hat{A}) = \{\exp[X_j + 2\pi i(n-k)]/n \mid j = 1, ..., N, k = 1, ..., n\}, \tag{7.43}$$

where X is determined by

$$(X, P) = \mathscr{E}(nc; \eta, y), \qquad \mathscr{E} \equiv \Phi^{-1}, \tag{7.44}$$

cf. (5.29).

In order to prove equality of the two spaces, we now begin by noting that when one fixes y with $y_N < \cdots < y_1$, one obtains

$$\mathscr{E}(nc; \eta, y) \sim (y, \eta), \qquad c \downarrow 0. \tag{7.45}$$

Therefore, the branch choice (7.40) ensures that the vector $x(\eta, y)$ is indeed given by (5.2). As a consequence, we may choose $V_l(c; x)$ in (7.37) positive. Doing so, it remains to prove that the vector $p(\eta, y)$ determined by (7.37) and (7.42) obeys

$$\exp(p_{(j-1)n+k}) = \exp(P_j), \qquad j = 1, ..., N, \quad k = 1, ..., n, \tag{7.46}$$

with P given by (7.44). Indeed, in view of (7.45), the branch choice (7.42) combined with (7.46) entails (5.3).

To prove the remaining equality (7.46), we invoke Appendix E. We show there that the Lax matrix $L(c; x(X), p(P))$ on the space Ω_N defined in Section 5 can be diagonalized by a matrix $\mathscr{U} \in GL(nN, C)$ such that

$$\mathscr{U}^{-1} L(c; x, p)\, \mathscr{U} = \hat{L}(n, ..., n; \eta), \tag{7.47}$$

$$\mathscr{U}^{-1} A(x)\, \mathscr{U} = \hat{A}(n, ..., n; \eta, y), \tag{7.48}$$

where (η, y) is given by (5.29). Consequently, we need only choose $\mathscr{U} = \mathscr{U}^{-1}$ in (7.36) and (7.37) to deduce (7.46). We have therefore proved equality of the space (7.19) and the space Ω_N from Section 5, and so our account of the equal-n_j case is now complete.

APPENDIX A: TODA SOLITONS REVISITED

In this appendix we reconsider the 2D Toda τ-functions specified in Section 1. Specifically, we present a novel proof that these functions satisfy the equation of motion (1.7), and we study the relation of these 2D Toda solitons to the soliton solutions of the infinite nonrelativistic and relativistic Toda lattices.

We begin by proving the solution property for the elementary soliton τ-functions considered in Section 3. As we have shown in that section (cf. (3.32)), these τ-functions admit a representation

$$\tau_N(t_+, t_-) = \sum_{j=1}^{N} F((x_j(t_+, t_-)), \tag{A.1}$$

where the t_+, t_- dependence of x_j is governed by Hamiltonians of the form

$$H_\delta = \lambda \sum_{j=1}^{N} \exp(\delta p_j)\, V_j(x), \qquad \delta = +, -, \tag{A.2}$$

$$V_j = \prod_{k \neq j} f(x_j - x_k). \tag{A.3}$$

Specifically, we have

$$F(u) = 1 + \exp(u - 2ivc) \tag{A.4}$$

$$\lambda = 2\sin(c) \tag{A.5}$$

$$f(x) = (1 + \sin^2(c)/\mathrm{sh}^2(x/2))^{1/2} \tag{A.6}$$

in the case at hand.

In the next lemma, however, we only assume (A.1)–(A.3). Thus we will be able to use the lemma in Appendix B, where (A.4)–(A.6) are generalized to an elliptic setting.

LEMMA A.1. *Assume that* (A.1)–(A.3) *hold true, with $F(u)$ entire and $f(x)$ even. Then one has*

$$\partial_+ \partial_- \ln \tau_N = -\lambda^2 \sum_{j=1}^{N} \partial_{x_j}\left(\frac{F'(x_j)}{F(x_j)}\, V_j^2(x)\right). \tag{A.7}$$

Proof. The lhs can be written

$$\sum_{j=1}^{N}\left(\frac{F'(x_j)}{F(x_j)}\partial_+\partial_- x_j+\partial_+ x_j\,\partial_- x_j\,\partial_{x_j}\left(\frac{F'(x_j)}{F(x_j)}\right)\right). \tag{A.8}$$

Now from (A.2) we calculate

$$\partial_\delta x_j=\{x_j, H_\delta\}=\delta\lambda\exp(\delta p_j)\,V_j(x). \tag{A.9}$$

In order to simplify the second derivative, we note that evenness of $f(x)$ entails

$$V_k\,\partial V_j/\partial x_k+V_j\,\partial V_k/\partial x_j=0,\qquad k\neq j. \tag{A.10}$$

Thus we get

$$\begin{aligned}\partial_+\partial_- x_j&=\{\lambda\exp(p_j)\,V_j, H_-\}\\&=\lambda^2\sum_k\left(\exp(p_j)\,\partial_k V_j(-\exp(-p_k))\,V_k-\exp(p_j)\,V_j\exp(-p_k)\,\partial_j V_k\right)\\&=-\lambda^2\,\partial_j(V_j^2).\end{aligned} \tag{A.11}$$

Substituting (A.9) and (A.11) in (A.8), we obtain the identity (A.7). ∎

Let us now study the choices (A.4)–(A.6). Then one has

$$\frac{\tau_{\nu-1}\tau_{\nu+1}}{\tau_\nu^2}=\sum_{j=1}^{N}\frac{\operatorname{ch}(x_j/2-i(\nu-1)c)\operatorname{ch}(x_j/2-i(\nu+1)c)}{\operatorname{ch}^2(x_j/2-i\nu c)}. \tag{A.12}$$

Combining this with (A.7), we deduce that the above elementary soliton τ-functions solve the 2D Toda equation (1.7) if and only if the functional equations

$$\begin{aligned}&-4\sin^2(c)\sum_{j=1}^{N}\partial_j\left(\frac{\exp(x_j)}{1+\exp(x_j)}\prod_{k\neq j}\left(1+\frac{\sin^2(c)}{\operatorname{sh}^2((x_j-x_k)/2)}\right)\right)\\&\quad=\prod_{j=1}^{N}\frac{\operatorname{ch}(x_j/2+ic)\operatorname{ch}(x_j/2-ic)}{\operatorname{ch}^2(x_j/2)}-1\end{aligned} \tag{A.13}$$

hold true. Now it is easy to verify (A.13) for $N=1$, and for the case $N=2$ a direct verification is still feasible, too. For $N>2$, however, this is no longer tractable.

We shall prove (A.13) for general N in Appendix D. In fact, we shall obtain a more general sequence of functional identities, which also entail the formula

$$\sum_{j=1}^{N}\partial_j(V_j^2)=0, \tag{A.14}$$

when V_j (A.3) is given by (A.6). This equation encodes the relativistic invariance of the $\mathrm{II}_{\mathrm{rel}}(c, N)$ system, and can be proved in various other ways, cf. [1, 20].

We continue by proving that the solution property of the elementary soliton τ-functions established via the functional equations (A.13) entails the solution property for the more general τ-functions given by (1.1), (1.2) and (1.10). To this end we first transform (A.1) back to the original representation (3.31), which amounts to (2.18), (2.19), with $q \in \mathbb{R}^M$, $\theta_M < \cdots < \theta_1$. Now it is clear that (1.7) still holds when we continue q to $\mathbb{C}^M$. Similarly, we can continue θ to $\mathbb{C}^M$ with pole varieties deleted.

Next, we invoke the fusion property established in Section 2. Combined with the previous paragraph, it entails that any N-soliton τ-function with (1.18), (1.19), (2.11) and (2.12) in effect satisfies (1.7). By analyticity, this still holds for $\xi_j^0, \eta_j \in \mathbb{C}$.

The upshot is, that (1.7) holds true for the τ-functions (1.1), (1.2), with $a_j, b_j, \xi_{j,\nu}$ given by (1.11), (1.13), where $\xi^0, \eta \in \mathbb{C}^N$ and $c_1, ..., c_N$ are restricted by (1.18), (1.19). Now the points $(c_1, ..., c_N)$ thus obtained are easily seen to be dense in $(0, \pi)^N$. Therefore, invoking analyticity in $c_1, ..., c_N$, it follows that the 2D Toda equation (1.7) is satisfied for arbitrary $\xi^0, a, b \in \mathbb{C}^N$, as announced.

We continue by detailing how the solitons of the infinite nonrelativistic Toda lattice can be obtained from the 2D Toda solitons by a specialization. To this end we substitute

$$a_j = i\,\mathrm{th}(\theta_j/2), \qquad b_j = i\coth(\theta_j/2) \tag{A.15}$$

in (1.1), (1.2), which yields

$$\exp(B_{jk}) = \mathrm{th}^2(\theta_j - \theta_k)/2. \tag{A.16}$$

Thus, when we also substitute

$$\exp(\xi_j^0) = \exp(q_j) \prod_{k \neq j} |\coth(\theta_j - \theta_k)/2| = \exp(q_j)\, V_j(\pi/2; \theta), \tag{A.17}$$

we obtain the Toda lattice solitons as parametrized in [1]. More specifically, (1.10) turns into

$$\xi_{j,\nu} = \xi_j^0 + \nu \ln(\mathrm{th}^2(\theta_j/2)) + 2t/\mathrm{sh}(\theta_j), \qquad t \equiv t_{1,+} + t_{1,-} \tag{A.18}$$

when we take $t_{\kappa,\delta} = 0$, $\kappa > 1$, and this amounts to the Toda lattice τ-functions given by Eqs. (6.5), (6.6), (6.18) in [1].

It should be noted that the 2D Toda equation (1.7) (which we have proved above) entails that the τ-functions $\tau_\nu(t)$ just defined satisfy

$$\partial_t^2 \ln \tau_\nu = \frac{\tau_{\nu-1}\tau_{\nu+1}}{\tau_\nu^2} - 1. \tag{A.19}$$

Of course, when we now set

$$x_\nu \equiv \ln(\tau_\nu/\tau_{\nu-1}), \tag{A.20}$$

we obtain the infinite Toda lattice equation

$$\ddot{x}_\nu = \exp(x_{\nu+1} - x_\nu) - \exp(x_\nu - x_{\nu-1}), \qquad \nu \in \mathbb{Z}. \tag{A.21}$$

But the equation (A.19) is not implied by (A.21), cf. also the next appendix.

Consider now the more general substitution

$$a_j = i\,\frac{\mathrm{sh}(\theta_j - \alpha)/2}{\mathrm{ch}(\theta_j + \alpha)/2}, \qquad b_j = i\,\frac{\mathrm{ch}(\theta_j - \alpha)/2}{\mathrm{sh}(\theta_j + \alpha)/2}, \qquad \alpha \in [0, \infty). \tag{A.22}$$

Amazingly, this *still* yields (A.16), and now (1.10) becomes

$$\xi_{j,\nu} = \xi_j^0 + \nu \ln\left(\mathrm{th}\,\frac{1}{2}(\theta_j - \alpha)\,\mathrm{th}\,\frac{1}{2}(\theta_j + \alpha)\right) + 2\,\mathrm{ch}\,\alpha\left(\frac{t_+}{\mathrm{sh}(\theta_j + \alpha)} + \frac{t_-}{\mathrm{sh}(\theta_j - \alpha)}\right), \tag{A.23}$$

when we take $t_{\kappa,\delta} = 0$, $\kappa > 1$, and $t_{1,\delta} \to t_\delta$. Parametrizing $\exp(\xi_j^0)$ via (A.17), we obtain once more real-valued positions $\hat{x}_\nu(t_+, t_-)$ via (A.20), provided we require

$$(\theta, q) \in \hat{\Omega}(N), \qquad \pm\theta_j \in (\alpha, \infty), \quad j = 1, ..., N. \tag{A.24}$$

(The restriction ensures posivity of the argument of ln in (A.23), and hence positivity of τ_ν.) We conjecture that the functions $\hat{x}_\nu(t_+, t_-)$ yield N-soliton solutions of the infinite *relativistic* Toda lattice, in a sense that we shall now explain.

The light cone Hamiltonians of this lattice may be taken to be [15]

$$\begin{aligned} S_\pm = \frac{1}{\beta^2} \sum_{\nu \in \mathbb{Z}} (&\exp(\pm\beta p_\nu)(1 + \beta^2 \exp(x_{\nu+1} - x_\nu))^{1/2} \\ &\times (1 + \beta^2 \exp(x_\nu - x_{\nu-1}))^{1/2} - (1 + \beta^2)). \end{aligned} \tag{A.25}$$

Here we think of boundary conditions

$$p_\nu \to 0, \qquad x_\nu - x_{\nu-1} \to 0, \qquad |\nu| \to \infty, \tag{A.26}$$

with sufficiently fast decrease so that the series converge. Also, the parameter $c \equiv 1/\beta$ may be viewed as the speed of light, but it should be observed that the boundary conditions are not Lorentz invariant. (In fact, the boost generator $\sum_{\nu \in \mathbb{Z}} x_\nu$ is readily seen to diverge for the one-soliton solutions occurring below.)

Consider now the position part $x_\nu(t_+, t_-)$ of the 2-parameter flow $\exp(t_+ S_+ + t_- S_-)$. (This makes sense, since S_+ and S_- Poisson commute, as is easily verified.) It satisfies

$$\begin{aligned} \partial_\delta x_\nu = \{x_\nu, S_\delta\} = \delta\beta^{-1} &\exp(\delta\beta p_\nu)(1 + \beta^2 \exp(x_{\nu+1} - x_\nu))^{1/2} \\ &\times (1 + \beta^2 \exp(x_\nu - x_{\nu-1}))^{1/2}. \end{aligned} \tag{A.27}$$

From this one readily obtains

$$\partial_+\partial_- x_\nu = \{\partial_- x_\nu, S_+\} = \exp(x_{\nu+1} - x_\nu) - \exp(x_\nu - x_{\nu-1}). \tag{A.28}$$

Consequently, one obtains a solution to the 2D Toda equation (1.9) with $\mu = 1$, $\phi \to x$.

Of course, the solutions to (1.9) thus obtained from infinite relativistic Toda lattice solutions are very special: They also satisfy two further equations

$$\partial_\delta^2 x_\nu = \{\partial_\delta x_\nu, S_\delta\} = \frac{\beta^2 \exp(x_{\nu+1} - x_\nu)}{1 + \beta^2 \exp(x_{\nu+1} - x_\nu)} \times (\partial_\delta x_{\nu+1})(\partial_\delta x_\nu) - (\nu \to \nu - 1), \qquad \delta = +, -. \tag{A.29}$$

Now in the nonrelativistic limit $\beta \to 0$ these two equations become trivial. But if we slightly alter x_ν by setting

$$\tilde{x}_\nu(t_+, t_-) \equiv x_\nu(t_+, t_-) - t_+ r(\beta)/\beta + t_- r(\beta)/\beta, \tag{A.30}$$

then we obtain

$$\partial_\delta^2 \tilde{x}_\nu = \frac{\exp(x_{\nu+1} - x_\nu)}{1 + \beta^2 \exp(x_{\nu+1} - x_\nu)} (r(\beta) + \delta\beta\, \partial_\delta \tilde{x}_{\nu+1})(r(\beta) + \delta\beta\, \partial_\delta \tilde{x}_\nu) - (\nu \to \nu - 1). \tag{A.31}$$

Requiring now

$$r(\beta) = 1 + O(\beta), \qquad \beta \to 0, \tag{A.32}$$

the ensuing nonrelativistic limit equations clearly equal (A.21).

To connect these simple observations to the above functions $\tilde{x}_\nu(t_+, t_-)$, obtained from the 2D Toda solitons (1.1), (1.2) via the substitutions (A.22), (A.23), (A.17) and (A.20), it is crucial to note first of all that the change (A.30) may be viewed as a consequence of changing the functions $S_\pm$ (A.25), namely, employing

$$\tilde{S}_\pm \equiv S_\pm \mp \frac{r(\beta)}{\beta} \sum_{\nu \in \mathbb{Z}} p_\nu. \tag{A.33}$$

These renormalized Hamiltonians still Poisson commute, and in view of (A.32) both converge to the nonrelativistic Toda lattice Hamiltonian

$$H_{\mathrm{nr}} = \sum_{\nu \in \mathbb{Z}} (p_\nu^2/2 + \exp(x_{\nu+1} - x_\nu) - 1) \tag{A.34}$$

as $\beta \to 0$, which explains why (A.31) turns into (A.21).

We are now in the position to render our conjecture precise: The above functions $\tilde{x}_\nu(t_+, t_-)$ satisfy (A.31) and hence may be viewed as position parts of the two-parameter flow $\exp(t_+ \tilde{S}_+ + t_- \tilde{S}_-)$, with the renormalizing function $r(\beta)$ given by

$$r = 1 + \beta^2 \tag{A.35}$$

and with the parameter α in (A.22) related to β via

$$\beta = \operatorname{sh} \alpha. \tag{A.36}$$

Let us turn to the evidence for this guess. First of all, our expectation is fulfilled in the one-soliton case $N = 1$. This may seem fortuitous at first sight, but a far stronger result holds true: If x_ν is defined by (A.20) with τ_ν of the form

$$\tau_\nu = 1 + C^\nu \exp(At)\, d, \tag{A.37}$$

then x_ν satisfies the equation of motion

$$\ddot{x}_\nu = \frac{\exp(x_{\nu+1} - x_\nu)}{1 + \beta^2 \exp(x_{\nu+1} - x_\nu)} (1 + \beta \dot{x}_{\nu+1})(1 + \beta \dot{x}_\nu) - (\nu \to \nu - 1) \tag{A.38}$$

for arbitrary $d \in \mathbb{C}$ if and only if β, C and A are related by

$$(1 + \beta^2)^2 A^2 - \beta(1 + \beta^2)\, A(C - C^{-1}) - C - C^{-1} + 2 = 0. \tag{A.39}$$

(This follows from quite elementary, but rather grueling computations, which we will spare the reader.) Parametrizing β by (A.36) and C by taking

$$C = \operatorname{th} \tfrac{1}{2}(\theta + \alpha) \operatorname{th} \tfrac{1}{2}(\theta - \alpha), \tag{A.40}$$

one now obtains solutions

$$A_\pm = \pm \frac{2}{\operatorname{ch}(\alpha) \operatorname{sh}(\theta \pm \alpha)} \tag{A.41}$$

to the dispersion relation (A.39). Via a rescaling of time one can then transform (A.38) to (A.31), and verify that (A.23) is of the desired form for solving (A.31).

A second argument of a general character is now that for $|t_\delta| \to \infty$ the above N-soliton functions $\tilde{x}_\nu(t_+, t_-)$ become linear superpositions of N exponentially separated one-soliton functions, so that (A.31) holds true asymptotically.

Third, by construction the functions $\tilde{x}_\nu$ satisfy the 2D Toda equation (A.28), and their $\beta \to 0$ limits yield the nonrelativistic Toda lattice solitons (in agreement with the $\beta \to 0$ limit of (A.31)).

Fourth, the τ-functions $\tau_\nu(t_+, t_-)$ obtained via the substitution (A.22) are associated to the $\mathrm{II}_{\mathrm{rel}}(\pi/2)$ N-particle system in the same way as a host of other well-known soliton τ-functions [1, 2, 3].

Further circumstantial evidence can be found in papers by Cosentino [16] and Ohta *et al.* [17]. Let us first discuss Cosentino's results. He constructs an inverse scattering transform for the infinite relativistic Toda lattice, taking $\beta = 1$ and concentrating on S_+ (A.25). Unfortunately, the solitons he finds by insisting on vanishing reflection are not in a form that admits an easy comparison with the above functions. For one thing, he only considers differences $x_{\nu+1} - x_\nu$, for which the above renormalization (A.30) is irrelevant. Therefore, even if these differences amount to our $\tilde{x}_{\nu+1} - \tilde{x}_\nu$ (something we do consider plausible), it would not follow from his results that (A.31) is indeed satisfied by our functions $\tilde{x}_\nu(t_+, t_-)$. Secondly, even for $N = 1$ the expected equality of our $\tilde{x}_{\nu+1}(t_+) - \tilde{x}_\nu(t_+)$ with $\beta = 1$ to Cosentino's one-soliton solution (to be found on pp. 550–551 of [16]) is not at all apparent.

On the positive side, however, we point out that the substitution

$$\alpha_{\text{Cosentino}} = -\text{th}\,\tfrac{1}{2}(\theta + \alpha), \qquad \alpha = \text{arsh}(1) \tag{A.42}$$

yields the expected plane wave in Eq. (144) of l.c.; that is, it yields the second term at the rhs of (A.37) with C and A given by (A.40) and A_+ (A.41), resp. Moreover, the restriction $|\theta_j| > \alpha$ in (A.24) then agrees with Cosentino's result on the poles of the transmission, cf. Lemma 4 on p. 547 of l.c.

Next, we turn to a comparison with the relativistic Toda lattice solitons that can be found in the paper by Ohta *et al.* [17]. They obtain τ-functions giving rise to solutions of (A.38) when (A.20) is used to get x_ν from τ_ν. (Actually, their f_n corresponds to our τ_{-n}.) Since their τ-functions are written in terms of Casorati determinants, they look quite different from our functions $\tilde{x}_\nu(t_+)$ and the Cosentino solitons. (In fact, the authors of [17] suggest their solitons differ from Cosentino's solitons.) Even so, it can be seen from pp. 5192–5193 in [17] that their one-soliton solution does amount to our function $\tilde{x}_\nu(t_+)$: One need only substitute

$$p_{\text{Ohta}} = \text{th}\,\tfrac{1}{2}(\theta + \alpha), \qquad a_{\text{Ohta}} = \text{th}(\alpha), \qquad x_{\text{Ohta}} = t_+ \text{ch}(\alpha) \tag{A.43}$$

to obtain agreement up to a constant. (The t_--dependence becomes quite awkward when the variables p and a are used.)

We conclude this appendix with some further remarks pertaining to our functions $\tilde{x}_\nu(t_+, t_-)$. Since C (A.40) is a number in (0,1) for $|\theta| > \alpha$, the above one-soliton solution to (A.31) obeys

$$\lim_{\nu \to \infty} \tilde{x}_\nu(t_+, t_-) = 0, \tag{A.44}$$

$$\lim_{\nu \to -\infty} \tilde{x}_\nu(t_+, t_-) = C. \tag{A.45}$$

More generally, it can be seen that the N-soliton functions satisfy (A.44) and

$$\lim_{\nu \to -\infty} \tilde{x}_\nu(t_+, t_-) = \sum_{j=1}^{N} \text{th}\,\tfrac{1}{2}(\theta_j + \alpha)\,\text{th}\,\tfrac{1}{2}(\theta_j - \alpha). \tag{A.46}$$

Thus the "total elongation" of the lattice is well defined for these functions. Notice that in view of (A.30), the renormalizing function $r(\beta)$ (A.35) is the unique function for which (A.44) holds true.

Assuming that Cosentino's solitons amount to our functions $\tilde{x}_{l+1}(t_+) - \tilde{x}_l(t_+)$, it follows that the scattering map of the solitons equals the S-map of the $\mathrm{II}_{\mathrm{rel}}(\pi/2)$ particles. Thus it is manifestly canonical in terms of the variables $\theta_1, \ldots, \theta_N$, $q_1, \ldots, q_N$. But the "obvious" symplectic structure employed by Cosentino yields action-angle variables for the solitons that are not symplectically related to our θ, q, and the S-map is *not* canonical w.r.t. these variables. We have discussed the same dilemma for various other soliton equations (including the nonrelativistic Toda lattice) on pp. 195–198 of our survey [3]. The observations to be found there may well have a translation to the present case (contingent, of course, on the above assumption). In particular, Oevel *et al.* [23] have shown that the relativistic Toda lattice admits two additional local symplectic structures. A further elaboration on these matters is beyond the scope of the present paper, however.

APPENDIX B: $\mathrm{IV}_{\mathrm{rel}}$ PARTICLES VS 2D TODA SOLUTIONS

We begin this appendix by generalizing the elementary soliton τ-functions (A.1)–(A.6) to an elliptic setting. To this end we recall that the hyperbolic $\mathrm{II}_{\mathrm{rel}}(c)$ systems admit such a generalization, denoted $\mathrm{IV}_{\mathrm{rel}}(c)$ from now on. These elliptic systems are characterized by a pair potential

$$f(c; x) = \lambda(c)(\wp(x; \omega, i\pi) - \wp(2ic; \omega, i\pi))^{1/2}, \qquad c \in (0, \pi), \tag{B.1}$$

where $x \in (0, 2\omega)$ and the normalization constant $\lambda(c) > 0$ is at our disposal. We have chosen the imaginary period of the Weierstrass $\wp$-function equal to $2i\pi$, so that the hyperbolic pair potential (A.6) arises by taking $\omega \to \infty$ (up to a constant determined by the $\omega \to \infty$ limit of $\lambda(c)$).

The $\wp$-function difference in (B.1) can be factored in terms of the Weierstrass σ-function. However, we find it convenient to work with the slightly altered function

$$\tilde{\sigma}(x; \omega, i\pi) \equiv \exp(i\eta' x^2/2\pi)\, \sigma(x; \omega, i\pi), \tag{B.2}$$

which is $2i\pi$-antiperiodic and satisfies

$$\lim_{\omega \to \infty} \tilde{\sigma}(x; \omega, i\pi) = 2\mathrm{sh}(x/2). \tag{B.3}$$

Choosing

$$\lambda(c) \equiv -i\tilde{\sigma}(2ic; \omega, i\pi) > 0, \qquad c \in (0, \pi), \tag{B.4}$$

we can rewrite (B.1) as

$$f(c;x)=\left(\frac{\tilde{\sigma}(x-2ic;\omega,i\pi)\,\tilde{\sigma}(x+2ic;\omega,i\pi)}{\tilde{\sigma}(x;\omega,i\pi)^2}\right)^{1/2}. \tag{B.5}$$

With this normalization the elliptic pair potential (B.5) reduces to the hyperbolic pair potential (A.6) for $\omega\to\infty$, and $\lambda(c)$ (B.4) reduces to (A.5), cf. (B.3).

Our object of study is now the τ-function (A.1)–(A.3), with λ and f given by (B.4) and (B.5), and with F given by

$$F(u)\equiv -i\exp(u/2-ivc)\,\tilde{\sigma}(u+i\pi-2ivc). \tag{B.6}$$

Thus $F(u)$ is $2i\pi$-periodic, and (B.3) entails

$$\lim_{\omega\to\infty} F(u)=1+\exp(u-2ivc), \tag{B.7}$$

so (B.6) amounts to (A.4) for $\omega=\infty$. Moreover, we have with the above definitions

$$c=\pi/n\Rightarrow \tau_{v+n}=\tau_v,\qquad \phi_{v+n}=\phi_v,\quad v\in\mathbb{Z} \tag{B.8}$$

(which generalizes (1.17) for elementary solitons with $c=\pi/n$), and

$$\begin{aligned}\frac{\tau_{v-1}\tau_{v+1}}{\tau_v^2}&=\prod_{j=1}^{N}\frac{\tilde{\sigma}(x_j+i\pi-2i(v-1)\,c)\,\tilde{\sigma}(x_j+i\pi-2i(v+1)\,c)}{\tilde{\sigma}(x_j+i\pi-2ivc)^2}\\ &=\lambda(c)^{2N}\sum_{j=1}^{N}(\wp(x_j+i\pi-2ivc)-\wp(2ic)),\end{aligned} \tag{B.9}$$

where $x_j=x_j(t_+,t_-)$. Thus, this τ-function quotient is elliptic in $x_1,\ldots,x_N$.

Next, we invoke Lemma A.1. For the case at hand it yields

$$\partial_+\partial_-\ln\tau_v=-\lambda(c)^{2N}\sum_{j=1}^{N}\partial_j\left(\frac{1}{2}+\frac{\tilde{\sigma}'}{\tilde{\sigma}}(x_j+i\pi-2ivc)\right)\prod_{k\neq j}(\wp(x_j-x_k)-\wp(2ic)). \tag{B.10}$$

Now from (B.2) we have

$$\tilde{\sigma}'(x)/\tilde{\sigma}(x)=i\eta' x/\pi+\zeta(x), \tag{B.11}$$

where ζ is the Weierstrass ζ-function, so if we let ∂_j act on $\tilde{\sigma}'/\tilde{\sigma}$, we obtain a function that is elliptic in x_j. But $\tilde{\sigma}'/\tilde{\sigma}$ is not elliptic in x_j, so if ∂_j acts on the product, we obtain *non-elliptic* terms. Thus the equation of motion (1.7) for τ_v is *not* satisfied by the τ-function just defined—the rhs of (1.7) is elliptic in x_j, whereas its lhs is not.

However, let us now define ϕ_ν by (1.8), taking $\mu = 1$ to ease the notation. Then the lhs of the equation of motion (1.9) for ϕ_ν reads, using (B.10) and (B.11),

$$- \lambda(c)^{2N} \sum_{j=1}^{N} \partial_j (2\eta' c/\pi + \zeta(x_j + i\pi - 2i\nu c) - \zeta(x_j + i\pi - 2i(\nu - 1)c))$$

$$\times \prod_{k \neq j} (\wp(x_j - x_k) - \wp(2ic)). \tag{B.12}$$

The key difference with (B.10) is, that this expression *is* elliptic in x_j. Indeed, this assertion easily follows from the quasi-periodicity relations

$$\zeta(x + 2\omega) = \zeta(x) + \eta, \qquad \zeta(x + 2i\pi) = \zeta(x) + \eta'. \tag{B.13}$$

The upshot is that the non-ellipticity obstruction for (1.7) is not present for (1.9): The function ϕ_ν, $\nu \in \mathbb{Z}$, obeys (1.9) if and only if the functional equation

$$\sum_{j=1}^{N} \partial_j (- 2\eta' c/\pi + \zeta(y_j - ic) - \zeta(y_j + ic)) \prod_{k \neq j} (\wp(y_j - y_k) - \wp(2ic))$$

$$= \prod_{j=1}^{N} (\wp(y_j + ic) + \wp(2ic)) - \prod_{j=1}^{N} (\wp(y_j - ic) - \wp(2ic)) \tag{B.14}$$

holds true.

At this point we recall from [1, 20] the functional equation

$$\sum_{j=1}^{N} \partial_j \prod_{k \neq j} (\wp(x_j - x_k) - \wp(\mu)) = 0, \qquad \mu \in \mathbb{C}, \tag{B.15}$$

which encodes the relativistic invariance of the $\mathrm{IV}_{\mathrm{rel}}(c, N)$ system. It entails that we may drop the constant $-2\eta' c/\pi$ at the lhs of (B.14). Thus we deduce that (B.14) follows from the more general functional equation

$$\sum_{j=1}^{N} \partial_j (\zeta(y_j + \alpha) - \zeta(y_j - \alpha)) \prod_{k \neq j} (\wp(y_j - y_k) + \lambda)$$

$$+ \sum_{j=1}^{N} (\wp(y_j + \alpha) + \lambda) - \sum_{j=1}^{N} (\wp(y_j - \alpha) + \lambda) = 0, \tag{B.16}$$

where $\alpha, \lambda \in \mathbb{C}$.

In Appendix D we prove that (B.16) holds true. Thus, summarizing the above result, the function

$$\phi_\nu = \mu^{-1} \sum_{j=1}^{N} \ln(e^{-ic} \tilde{\sigma}(x_j + i\pi - 2i\nu c; \omega, i\pi) / \tilde{\sigma}(x_j + i\pi - 2i(\nu - 1)c; \omega, i\pi)) \tag{B.17}$$

satisfies the equation of motion (1.9) when the t_+, t_- dependence of $x_j = x_j(t_+, t_-)$ is governed by the $\mathrm{IV}_{\mathrm{rel}}(c)$ particle Hamiltonians

$$H_\pm = (-i\tilde{\sigma}(2ic;\omega, i\pi))^N \sum_{j=1}^{N} \exp(\pm p_j) \times \prod_{k\neq j} (\wp(x_j - x_k;\omega, i\pi) - \wp(2ic;\omega, i\pi))^{1/2}, \qquad c \in (0,\pi). \tag{B.18}$$

We conclude this appendix with some remarks. First, let us note that by construction the elliptic result just stated has a hyperbolic specialization, cf. (B.3), (B.7). But in Appendix A we obtain a stronger result, namely the equation of motion (1.7) for τ_ν. As we have seen above, (1.7) does not hold for (A.1)–(A.3) with (B.6) and (B.18) in effect, and there appear to be no simple changes that repair this. Thus the elementary soliton result in Appendix A cannot be derived from the elliptic result in this Appendix, and neither have we found a way to derive from the elliptic functional equations (B.16) the hyperbolic identities (A.13).

Secondly, we comment on the special case $c = \pi/2$. Then (B.8) yields $\phi_{\nu+2} = \phi_\nu$, $\nu \in \mathbb{Z}$, so (1.9) entails

$$\partial_+ \partial_- \phi = -4\mu^{-1} \operatorname{sh} \mu\phi, \qquad \phi \equiv \phi_0 - \phi_1 \quad (c = \pi/2). \tag{B.19}$$

In particular, taking $\mu = i$, this amounts to the sine-Gordon equation. Now it should be noticed that the above solution properties still hold true when the function $F(u)$ is replaced by $F(u + C)$, where $C \in \mathbb{C}$ is arbitrary. This follows by inspection from the functional equations equivalent to the solution property; the crux is, that $V_j(x)$ depends only on differences. We have chosen the constant such that we obtain real-valued KP solutions and 2D Toda solutions obeying (2.10), but this convention does not yield a real-valued $\phi = \phi_0 - \phi_1$ for $c = \pi/2$. Rather, one should choose $C = -i\pi/2$, so that the terms $x_j + i\pi$ in (B.17) become $x_j + i\pi/2$. Then one winds up with real-valued, real-analytic sine-Gordon solutions associated to the $\mathrm{IV}_{\mathrm{rel}}(\pi/2)$ particle systems, which reduce to the well-known soliton solutions in the hyperbolic limit $\omega \to \infty$.

APPENDIX C: $\mathrm{II}_{\mathrm{nr}}$ AND $\mathrm{IV}_{\mathrm{nr}}$ PARTICLES VS KP SOLUTIONS

The t_1-dependence of the KP solitons (1.1), (1.2), (1.4) is governed by factors $\exp(it_1(a_j - b_j))$, $j = 1, ..., N$, so in general they are not periodic in t_1. This appendix is concerned with solutions to (1.3) and (1.6) that *are* periodic in t_1 with period $2i\pi$. The first class of solutions consists of *special* KP solitons, and we tie them in with the rational relativistic $\mathrm{I}_{\mathrm{rel}}$ and hyperbolic nonrelativistic $\mathrm{II}_{\mathrm{nr}}$ Calogero–Moser systems. The second class consists of solutions to (1.6) that are elliptic in t_1 with periods $2\omega > 0$ and $2i\pi$. They can be obtained via the elliptic nonrelativistic $\mathrm{IV}_{\mathrm{nr}}$ systems, and reduce to the first class for $\omega \to \infty$.

The pertinent KP solitons arise by substituting

$$a_j = \theta_j - i/2, \quad b_j = \theta_j + i/2, \qquad j = 1, ..., N, \tag{C.1}$$

in (1.2) and (1.4), with $\theta_N < \cdots < \theta_1$. Then (1.4) turns into

$$\xi_j = \xi_j^0 + t_1 + 2\theta_j t_2 + (3\theta_j^2 - 1/4)\, t_3 + \cdots \tag{C.2}$$

and so we do obtain solutions to (1.3) that are $2i\pi$-periodic in t_1. Moreover, (1.2) becomes

$$\exp B_{jk} = (1 + (\theta_j - \theta_k)^{-2})^{-1}, \tag{C.3}$$

and the Cauchy matrix (2.1) reads

$$C_{jk} = \frac{-i}{\theta_j - \theta_k - i}. \tag{C.4}$$

Next, we choose $\xi_1^0, ..., \xi_N^0$ real, introduce the positive pair potential

$$f_r(x) \equiv (1 + 1/x^2)^{1/2}, \qquad x > 0 \tag{C.5}$$

and real numbers $q_1, ..., q_N$ by requiring

$$\exp(\xi_j^0) = \exp(q_j) \prod_{k \neq j} f_r(\theta_j - \theta_k). \tag{C.6}$$

Since we clearly have

$$B_{jk} = -2 \ln f_r(\theta_j - \theta_k), \tag{C.7}$$

cf. (C.3), it now follows from Cauchy's identity (2.2) that the special KP solitons at issue can be written

$$\tau = \sum_{l=0}^{N} \sum_{\substack{I \subset \{1, ..., N\} \\ |I| = l}} \exp\left(\sum_{k \in I} q_k(t)\right) \prod_{\substack{m \in I \\ n \notin I}} f_r(\theta_m - \theta_n), \tag{C.8}$$

where

$$q_i(t) = q_i + t_1 + 2\theta_i t_2 + (3\theta_i^2 - 1/4)\, t_3 + \cdots. \tag{C.9}$$

(This should be compared to (2.18) with $\nu = 0$, $t_+ = t$, $t_- = 0$.)

After the above reparametrizations, the matrix $C(\theta)\, D(\theta, q)$ amounts to the $\mathrm{I}_{\mathrm{rel}}$ Lax matrix from [10] (up to a diagonal similarity). Thus, invoking the action-angle map

$$\Phi\colon \mathrm{I}_{\mathrm{rel}} \to \mathrm{II}_{\mathrm{nr}}, \qquad (\theta, q) \mapsto (x, p), \tag{C.10}$$

from [10], we can rewrite (C.8) as

$$\tau = \prod_{j=1}^{N} (1 + \exp(x(t)), \tag{C.11}$$

where the t-dependence is governed by power traces of the dual ($\mathrm{II}_{\mathrm{nr}}$) Lax matrix

$$L_{jk} = p_j \delta_{jk} + i(1 - \delta_{jk}) \frac{1}{2\mathrm{sh}(x_j - x_k)/2}, \qquad j, k = 1, ..., N. \tag{C.12}$$

Specifically, since L has similarity transform $\mathrm{diag}(\theta_1, ..., \theta_N)$, the t_1, t_2, t_3 generators read

$$H_1 = \mathrm{Tr}\, L = \sum_{j=1}^{N} p_j, \tag{C.13}$$

$$H_2 = \mathrm{Tr}\, L^2 = \sum_{j=1}^{N} p_j^2 + 2 \sum_{j<k} V(x_j - x_k), \tag{C.14}$$

$$H_3 = \mathrm{Tr}\, L^3 + \alpha\, \mathrm{Tr}\, L = \sum_{j=1}^{N} p_j^3 + 3 \sum_{j<k} V(x_j - x_k)(p_j + p_k) + \alpha \sum_{j=1}^{N} p_j, \tag{C.15}$$

where

$$V(x) = 1/4\, \mathrm{sh}^2(x/2), \qquad \alpha = -1/4 \tag{C.16}$$

for the case at hand.

We continue by proving that (1.3) indeed holds true when τ is *defined* by (C.11), with the t_1, t_2, t_3 dependence governed by the particle Hamiltonians (C.13)–(C.16), so that we have

$$x_j(t_1, t_2, t_3) = t_1 + x_j(t_2, t_3). \tag{C.17}$$

In fact, we shall study first (with an eye on the elliptic generalization) a τ-function of the form

$$\tau(t) = \prod_{j=1}^{N} F(t_1 + x_j(t_2, t_3)), \tag{C.18}$$

where the t_2, t_3 dependence is governed by Hamiltonians of the form (C.14), (C.15). Thus we leave F, V and α unspecified for the moment, and view (C.18) as an Ansatz for (1.3).

Plugging (C.18) into (1.3), we obtain

$$\sum_{j=1}^{N} G'''(x_j)+6\left(\sum_{j=1}^{N} G'(x_j)\right)^2+4\sum_{j=1}^{N}(\partial_3 x_j)\,G'(x_j)$$
$$-3\sum_{j=1}^{N}((\partial_2 x_j)^2\,G'(x_j)+(\partial_2^2 x_j)\,G(x_j))=0, \qquad G(u)\equiv F'(u)/F(u). \quad (C.19)$$

Now we have from (C.14) and (C.15)

$$\partial_2 x_j=\{x_j, H_2\}=2p_j, \quad (C.20)$$

$$\partial_2^2 x_j=\{2p_j, H_2\}=-4\sum_{k\neq j} V'(x_j-x_k), \quad (C.21)$$

$$\partial_3 x_j=3p_j^2+3\sum_{k\neq j} V(x_j-x_k)+\alpha, \quad (C.22)$$

so (C.19) reduces to a functional equation that can be written

$$\sum_{j=1}^{N}\left(G'''(x_j)+4\alpha G'(x_j)+6G'(x_j)^2+12\partial_j(G(x_j)\sum_{k\neq j} V(x_j-x_k))\right)$$
$$+12\sum_{j<k} G'(x_j)\,G'(x_k)=0. \quad (C.23)$$

Specializing to

$$F(u)=1+e^u\Rightarrow G(u)=e^u/(1+e^u), \qquad G'(u)=1/4\mathrm{ch}^2(u/2) \quad (C.24)$$

and $\alpha=-1/4$, we obtain

$$G'''(u)-G'(u)+6G'(u)^2=0. \quad (C.25)$$

Thus (C.23) holds true for $N=1$, and for $N>1$ we are left with

$$\sum_{j<k} J(x_j, x_k)=0, \quad (C.26)$$

where

$$J(x, y)=G'(x)\,G'(y)+(G'(x)+G'(y))\,V(x-y)+(G(x)-G(y))\,V'(x-y). \quad (C.27)$$

It is straightforward to verify that $J(x, y)$ indeed vanishes for $G(x)=e^x/(1+e^x)$ and $V(x)=1/4\,\mathrm{sh}^2(x/2)$. (Note this also follows from (A.13) with $N=2$ by taking $\mathrm{Im}\, c\to\infty$.) Thus we have now proved that the above special type of KP soliton τ-function indeed satisfies the equation of motion (1.3).

Next, we assume that τ is of the form (C.18) with the t_j-dependence governed by the elliptic $\mathrm{IV}_{\mathrm{nr}}$ generalizations of the above Hamiltonians. Thus we take in (C.14), (C.15)

$$V(x) = \wp(x; \omega, i\pi) \tag{C.28}$$

and try to find $F(u)$ and α such that (C.18) solves (1.3) and is elliptic in t_1 with periods $2\omega, 2i\pi$. Proceeding as before, we infer that we should find $G = F'/F$ and α such that the functional equation (C.23) holds true when V is given by (C.28). For $N=1$ this yields the ODE

$$H'' + 4\alpha H + 6H^2 = 0, \qquad H = G'. \tag{C.29}$$

Now since the $\wp$-function satisfies

$$-\wp'' + 6\wp^2 = g_2/2, \tag{C.30}$$

we can solve (C.29) by taking

$$\alpha = g_2^{1/2}/2, \qquad H(u) = -\wp(u + C; \omega, i\pi) + g_2^{1/2}/6, \tag{C.31}$$

where the constant C is at our disposal. But then $G(u)$ is of the form $\zeta(u + C) + du + e$ and so (C.23) cannot hold for $N = 2$: the term $(\zeta(x_1) - \zeta(x_2))\, \wp'(x_1 - x_2)$ in (C.23) is not elliptic in x_1 and x_2, whereas all other terms are elliptic.

In summary, the above Ansatz appears incompatible with (1.3). However, we shall now prove that (1.6) *can* be solved with an Ansatz of this type. (Just as in the 2D Toda case, the presence of additional derivatives in (1.5) obviates the non-ellipticity obstruction.) More specifically, we start from the Ansatz

$$u(t_1, t_2, t_3) = 2 \sum_{j=1}^{N} (\wp(t_1 + x_j(t_2, t_3) + C; \omega, i\pi) + A) \tag{C.32}$$

and require (1.6). Using the second derivative of (C.30) and the relations (C.20)–(C.22) with V given by (C.28), we deduce that (1.6) amounts to the functional equation

$$\left(\frac{\alpha}{3} - A\right) \sum_{j=1}^{N} \wp''(x_j) = \sum_{j<k} M(x_j, x_k), \tag{C.33}$$

where

$$\begin{aligned} M(x, y) \equiv{} & 2\wp'(x)\, \wp'(y) + \wp''(x)\, \wp(y) + \wp(x)\, \wp''(y) \\ & - \wp(x - y)(\wp''(x) + \wp''(y)) - \wp'(x - y)(\wp'(x) - \wp'(y)). \end{aligned} \tag{C.34}$$

We claim that $M(x, y)$ vanishes identically. Taking this for granted, we need only choose

$$\alpha = 3A \tag{C.35}$$

to deduce that $u(t)$ (C.32) solves (1.6) when the t_2, t_3 dependence is governed by the Hamiltonians H_2 (C.14), H_3 (C.15) with V given by (C.28).

It remains to prove the claim. To this end we note that we can write

$$M(x, y) = \partial_a L(x+a, y+a)|_{a=0}, \tag{C.36}$$

where

$$\begin{aligned} L(x, y) \equiv{}& \wp'(x)\,\wp(y) + \wp(x)\,\wp'(y) - \wp(x-y)(\wp'(x) + \wp'(y)) \\ &- \wp'(x-y)(\wp(x) - \wp(y)). \end{aligned} \tag{C.37}$$

We assert that $L(x, y)$ vanishes identically. Accepting this, the claim follows from (C.36), so it suffices to prove the assertion.

To show $L(x, y) = 0$, we observe that we may write

$$L(x, y) = \partial_a K(a, x, y)|_{a=0}, \tag{C.38}$$

where

$$\begin{aligned} K(a, x, y) \equiv{}& \wp(x+a)\,\wp(y+a) - \wp(x-y)(\wp(x+a) + \wp(y+a)) \\ &+ \wp'(x-y)(\zeta(x+a) - \zeta(y+a)). \end{aligned} \tag{C.39}$$

This function is not identically zero, since the term $\zeta(x+a)\wp'(x-y)$ is not elliptic in x, whereas all other terms are. It is however *constant* in a, so that the assertion $L = 0$ follows from (C.38).

To see that K is indeed a-independent, one need only note that it is elliptic in a, and inspect the poles at $a = -x$, $a = -y$. Putting $a + x = \varepsilon$, the singular terms as $\varepsilon \to 0$ read

$$\begin{aligned} &\wp(\varepsilon)\,\wp(y+\varepsilon-x) - \wp(x-y)\,\wp(\varepsilon) + \wp'(x-y)\,\zeta(\varepsilon) \\ &= \varepsilon^{-2}(\wp(y-x) + \varepsilon\wp'(y-x)) - \wp(x-y)\,\varepsilon^{-2} + \wp'(x-y)\,\varepsilon^{-1} + O(1) \\ &= O(1), \qquad \varepsilon \to 0, \end{aligned} \tag{C.40}$$

so K is non-singular for $a \to -x$, and, similarly, for $a \to -y$. Thus constancy in a follows from Liouville's theorem, and the proof is complete.

To conclude this appendix, we point out that the KP solution (C.32) is a real-valued real-analytic function of t_1, t_2, t_3 provided we choose the undetermined constant C equal to $i\pi$ and A real. Letting A converge to $-1/12$ for $\omega \to \infty$, we then obtain the hyperbolic limit

$$\lim_{\omega \to \infty} u(t) = -\sum_{j=1}^{N} 1/2\,\mathrm{ch}^2((t_1 + x_j(t_2, t_3))/2), \tag{C.41}$$

with the $\mathrm{II}_{\mathrm{nr}}$ Hamiltonians (C.14)–(C.16) governing the t_2, t_3 dependence. Thus we obtain the special KP solitons studied in the first part of this appendix.

APPENDIX D: SOME FUNCTIONAL IDENTITIES

In this appendix we collect and prove a number of identities that are used in the main text and in Appendixes A, B and E. In Lemma D.1 we prove a general fusion identity, which is used in Section 2 and Appendix E. Lemmas D.2 and D.3 concern elementary functional equations that will be exploited in Section 5 and Appendix E. The remainder of the appendix deals with the hyperbolic and elliptic identities (A.13) and (B.16).

LEMMA D.1. *Let $M(z)$ be a meromorphic function of the form*

$$M(z)=\frac{s(z-2d)\, s(z+2d)}{s^2(z)}, \qquad d\in\mathbb{C}^*, \tag{D.1}$$

where $s(z)$ is entire. Putting

$$w_m=w+(a+1-2m)\,d, \quad z_n=z+(b+1-2d)\,d, \quad m=1,\ldots,a, \quad n=1,\ldots,b, \tag{D.2}$$

one has

$$\prod_{m=1}^{a}\prod_{n=1}^{b} M(w_m-z_n)=\frac{s(u+d(a+b))\, s(u-d(a+b))}{s(u+d(a-b))\, s(u-d(a-b))}, \qquad u\equiv w-z. \tag{D.3}$$

Proof. This follows by writing

$$\begin{aligned}
\text{LHS}&=\prod_{n=1}^{b}\frac{(\prod_{j=2}^{a+1} s(u+d(a-b-2(j-n))))(\prod_{j=0}^{a-1} s(u+d(a-b-2(j-n))))}{(\prod_{j=1}^{a} s(u+d(a-b-2(j-n))))^2}\\
&=\prod_{n=1}^{b}\frac{s(u+d(-a-b+2(n-1)))\, s(u+d(a-b+2n))}{s(u+d(a-b+2(n-1)))\, s(u+d(-a-b+2n))}\\
&=\frac{s(u+d(-a-b))\, s(u+d(a+b))}{s(u+d(a-b))\, s(u+d(-a+b))}=\text{RHS},
\end{aligned} \tag{D.4}$$

where we canceled terms appropriately. ∎

LEMMA D.2. *Letting $n\in\mathbb{N}^*$, one has*

$$\prod_{k=0}^{n-1}\sin(z+\pi k/n)=\sin(nz)/2^{n-1}, \qquad \forall z\in\mathbb{C}. \tag{D.5}$$

Proof. Consider the function

$$Q(z)=\frac{1}{\sin(nz)}\prod_{k=0}^{n-1}\sin(z+\pi k/n). \tag{D.6}$$

It has period π/n and is regular for $z \to 0$; in fact,

$$\lim_{z\to 0} Q(z) = \frac{1}{n} \prod_{k=1}^{n-1} \sin(\pi k/n). \tag{D.7}$$

Thus $Q(z)$ is entire, and since we have

$$\lim_{|\mathrm{Im}\, z| \to \infty} Q(z) = \frac{1}{2^{n-1}}, \tag{D.8}$$

it follows from Liouville's theorem that $Q(z)$ is constant. Hence, (D.5) follows. ∎

LEMMA D.3. *Letting $n \in \mathbb{N}^*$, one has*

$$\sum_{k=0}^{n-1} \frac{e^{i\pi k(2l-1)/n}}{\sin(z+\pi k/n)} = \frac{n e^{iz(n+1-2l)}}{\sin(nz)}, \qquad \forall l \in \{1, 2, ..., n\}, \qquad \forall z \in \mathbb{C}. \tag{D.9}$$

Proof. The functions at the lhs and at the rhs are both meromorphic and π-antiperiodic in z. Moreover, both have limit 0 for $|\mathrm{Im}\, z| \to \infty$. Finally, both have simple poles iff z equals $\pi m/n$, $m \in \mathbb{Z}$. Comparing residues, one sees they are equal. Therefore, (D.9) follows from Liouville's theorem. ∎

We remark that we arrived at the identities (D.9) before, as a corollary of Lemmas A.5 and A.6 in our paper [12]. Indeed, (A.38) in l.c. amounts to (D.9).

We continue by deriving functional equations ((D.17) below) that are equivalent to (A.13), and then prove a more general sequence of identities (D.18) (already mentioned in the introduction). To begin with, we note that (A.13) can be rewritten

$$-4\lambda \sum_{j=1}^{N} \partial_j \left(\frac{\exp(x_j)}{1+\exp(x_j)} \prod_{k \neq j} \left(1 + \frac{\lambda}{\mathrm{sh}^2(x_j - x_k)/2}\right)\right) = \prod_{j=1}^{N} \left(1 - \frac{\lambda}{\mathrm{ch}^2(x_j/2)}\right) - 1, \tag{D.10}$$

where $\lambda = \sin^2 c$. Since c is an arbitrary number in $(0, \pi)$, it follows by expanding (D.10) in powers of λ that the sequence (A.13) for $N \in \mathbb{N}^*$ is equivalent to the sequence of identities

$$4 \sum_{j=1}^{M} \partial_j \left(\frac{\exp(x_j)}{1+\exp(x_j)} \prod_{k \neq j} \frac{1}{\mathrm{sh}^2(x_j - x_k)/2}\right) = (-)^{M-1} \prod_{j=1}^{M} \frac{1}{\mathrm{ch}^2(x_j/2)} \tag{D.11}$$

for $M \in \mathbb{N}^*$.

Next, we turn (D.11) into a sequence of rational identities via the substitution

$$y_j = \mathrm{th}(x_j/2). \tag{D.12}$$

Indeed, this entails

$$\frac{1}{\mathrm{ch}^2(x_j/2)}=1-y_j^2, \qquad \frac{\partial}{\partial x_j}=\frac{1}{2}(1-y_j^2)\frac{\partial}{\partial y_j}. \tag{D.13}$$

Moreover, one readily verifies

$$\frac{1}{\mathrm{sh}^2(x_j-x_k)/2}=\frac{(1-y_j^2)(1-y_k^2)}{(y_j-y_k)^2}. \tag{D.14}$$

Finally, we may write

$$\frac{e^{x_j}}{1+e^{x_j}}=\frac{1}{2}-\frac{1}{2}\mathrm{th}(x_j/2)=\frac{1}{2}(1-y_j), \tag{D.15}$$

so (D.11) is equivalent to

$$\sum_{j=1}^{M}(1-y_j^2)\,\partial_{y_j}\left((1-y_j)\prod_{k\neq j}\frac{(1-y_j^2)(1-y_k^2)}{(y_j-y_k)^2}\right)=(-)^{M-1}\prod_{j=1}^{M}(1-y_j^2). \tag{D.16}$$

Comparing lhs and rhs of (D.16), we infer that this sequence can be simplified as

$$\sum_{j=1}^{M}\partial_j\left((1-y_j)(y_j^2-1)^{M-1}\prod_{k\neq j}\frac{1}{(y_j-y_k)^2}\right)=1. \tag{D.17}$$

Clearly, (D.17) (and hence (A.13)) follows from the more general identities

$$\sum_{j=1}^{M}\partial_j\left(y_j^l\prod_{k\neq j}\frac{1}{(y_j-y_k)^2}\right)=\begin{cases}0, & l=0,1,\ldots,2M-2\\ 1, & l=2M-1\end{cases} \tag{D.18}$$

that we will prove shortly. Before doing so, it is of interest to note that the latter identities contain the functional equations encoding relativistic invariance of the $\mathrm{I}_{\mathrm{rel}}$ and $\mathrm{II}_{\mathrm{rel}}$ systems. Indeed, these can be written

$$\sum_{j=1}^{M}\partial_{y_j}\prod_{k\neq j}\frac{1}{(y_j-y_k)^2}=0, \tag{D.19}$$

$$\sum_{j=1}^{M}\partial_{x_j}\prod_{k\neq j}\frac{1}{\mathrm{sh}^2(x_j-x_k)/2}=0, \tag{D.20}$$

resp. Now (D.19) equals (D.18) for $l=0$, whereas (D.20) turns into (D.18) with $l=M-1$ when one substitutes $e^{x_j}=y_j$.

Turning to (D.18), we first sketch a quite direct, but rather arduous proof. To begin with, note it suffices to show that the rational function at the lhs is actually a polynomial in $y_1, \ldots, y_M$. (Indeed, accepting this, the pertinent polynomial is clearly symmetric in $y_1, \ldots, y_M$, so one need only take $y_1 \to \infty$ and check that this yields the limits at the rhs.) By symmetry, one therefore need only show that the limit $\varepsilon \to 0$ is finite when one puts $y_2 = y$, $y_1 = y + \varepsilon$ after the differentiations. The singular terms arise by taking $j = 1, 2$, and a Taylor expansion then shows that the coefficients of ε^{-3}, ε^{-2} and ε^{-1} vanish.

We continue with a simpler proof of a yet more general polynomial property. To this end we introduce

$$E_{l,N}(\lambda, \mu, x) \equiv \sum_{j=1}^{M} \left((x_j - \mu)^l (x_j + \mu + \lambda)^l \prod_{k \neq j} \frac{1}{(x_j - x_k)(x_j - x_k + \lambda)} - (x \to -x) \right). \tag{D.21}$$

LEMMA D.4. *For all* $l \in \mathbb{N}$, $N \in \mathbb{N}^*$ *the function* $E_{l,N}$ *is a polynomial in* $\lambda, \mu, x_1, ..., x_N$. *Moreover, one has*

$$E_{l,N}(\lambda, \mu, x) = 0, \qquad l = 0, 1, ..., N-1. \tag{D.22}$$

Proof. From (D.21) one reads off that $E_{l,N}$ is odd in x and symmetric in $x_1, ..., x_N$, and that all terms have simple poles for generic x and λ. Thus we need only show that the residues at the simple poles (1) $x_1 = x_2$ and (2) $x_1 = x_2 - \lambda$ vanish. Now in case (1) there are two terms in the "left" product that have a non-zero residue; these residues are manifestly equal up to sign. The same holds true for the "right" product, so no pole occurs for $x_1 = x_2$.

To handle case (2), we note that only one term in the left and one term in the right product yield a non-zero residue for $x_1 = x_2 - \lambda$. The sum of these two residues reads

$$(x_2 - \lambda - \mu)^l (x_2 + \mu)^l \left(\frac{1}{-\lambda}\right) \prod_{k>2} \frac{1}{(x_2 - \lambda - x_k)(x_2 - x_k)}$$
$$+ (x_2 + \mu)^l (x_2 - \mu - \lambda)^l \left(\frac{1}{\lambda}\right) \prod_{k>2} \frac{1}{(x_2 - x_k)(x_2 - x_k - \lambda)} = 0. \tag{D.23}$$

Thus $E_{l,N}$ is pole-free, and hence is a polynomial in $\lambda, \mu, x_1, ..., x_N$, as asserted.

Now from (D.21) it is clear that one has

$$\lim_{x_1 \to \infty} E_{l,N}(\lambda, \mu, x) = 0, \qquad l \leq N-1. \tag{D.24}$$

Since $E_{l,N}$ is symmetric in $x_1, ..., x_N$, it follows that (D.22) holds true. ∎

To obtain (D.18) from this lemma, note first that $E_{l,N}$ vanishes for $\lambda = 0$, so that $E_{l,N}/\lambda$ is still a polynomial in $\lambda, \mu, x_1, ..., x_N$. Therefore, the limit function

$$E_{l,N}(\mu, x) \equiv \lim_{\lambda \to 0} \frac{E_{l,N}(\lambda, \mu, x)}{\lambda} = \sum_{j=1}^{N} \partial_j \left((x_j - \mu)^l (x_j + \mu)^l \prod_{k \neq j} \frac{1}{(x_j - x_k)^2} \right) \tag{D.25}$$

is a polynomial in $\mu, x_1, ..., x_N$. Setting $x_j = y_j + \mu$ and expanding in powers of μ, it then follows that the functions at the lhs of (D.18) are polynomials in $y_1, ..., y_M$ for all $l \in \mathbb{N}$. Taking e.g. $\lambda_1 \to \infty$, one now deduces (D.18).

The introduction of the function $E_{l,N}$ (D.21) and the corresponding proof of (D.18) were inspired by results in Appendix A from our paper [20], cf. especially l.c. (A17)–(A23). In this connection we observe that (D.22) can be reformulated as the commutativity of the two analytic difference operators

$$A_{l,\pm} \equiv \sum_{j=1}^{N} (x_j - \mu)^{l/2} \prod_{k \neq j} \frac{1}{(x_j - x_k)} \exp(\pm \lambda\, \partial_j)(x_j + \mu)^{l/2} \prod_{k \neq j} \frac{1}{(x_j - x_k)}, \qquad \text{(D.26)}$$

for $l = 1, ..., N-1$.

We conclude this appendix by proving the identities (B.16). As will be shown shortly, they easily follow from the next lemma.

LEMMA D.5. *For all $M \in \mathbb{N}^*$ the function*

$$E_M(a) \equiv \sum_{j=1}^{M} \partial_{x_j} \left(\zeta(x_j + a) \prod_{k \neq j} \wp(x_j - x_k) \right) + \prod_{j=1}^{M} \wp(x_j + a) \qquad \text{(D.27)}$$

does not depend on a.

Proof. By virtue of the functional equation

$$\sum_{j=1}^{M} \partial_j \prod_{k \neq j} \wp(x_j - x_k) = 0, \qquad \text{(D.28)}$$

proved in [1] and [20], the function $E_M(a)$ is elliptic in a. Since it is symmetric in $x_1, ..., x_M$, we need only show that it has a finite limit for $a \to -x_1$. (Indeed, this entails $E_M(a)$ is pole-free and hence constant.) Now the singular terms as $\varepsilon = x_1 + a \to 0$ read

$$\begin{aligned}
&-\wp(\varepsilon) \prod_{k>1} \wp(x_1 - x_k) + \zeta(\varepsilon)\, \partial_{x_1} \prod_{k>1} \wp(x_1 - x_k) + \wp(\varepsilon) \prod_{j>1} \wp(x_j - x_1 + \varepsilon) \\
&= -\varepsilon^{-2} \prod_{k>1} \wp(x_1 - x_k) + \varepsilon^{-1} \prod_{k>1} \wp(x_1 - x_k) \sum_{m>1} \wp'(x_1 - x_m)/\wp(x_1 - x_m) \\
&\quad + \varepsilon^{-2} \left(\prod_{j>1} \wp(x_j - x_1) + \varepsilon \prod_{j>1} \wp(x_j - x_1) \sum_{m>1} \wp'(x_m - x_1)/\wp(x_m - x_1) \right) + O(1) \\
&= O(1), \qquad \varepsilon \to 0, \qquad \text{(D.29)}
\end{aligned}$$

so there is no pole for $a + x_1 = 0$. ∎

To derive (B.16) from the lemma, one need only expand its lhs in powers of λ. The crux is then that the coefficient of λ^{N-M} can be written as a sum of terms that are of the form $E_M(\alpha) - E_M(-\alpha) = 0$.

Finally, we point out that $E_M(a)$ does not vanish identically for $M > 1$ (since it is not elliptic in x_1), and that the function K (C.39) amounts to $E_2(a)$. Thus the functional equation encoding the KP solution property may be viewed as a special case of the functional equations encoding the 2D Toda solution property.

APPENDIX E: THE EQUAL-n_j TRANSFORM

Our starting point in this appendix is the Lax matrix $L(c; x, p)$ on the space $\Omega_N \subset \mathbb{C}^{nN} \times \mathbb{R}^{nN}$ defined by (5.1)–(5.3). In view of the identities (5.10) and (5.12) its elements may be written

$$L_{(l-1)n+m,(j-1)n+k} = \frac{d_l \operatorname{sh}(inc)\, d_j}{n \operatorname{sh}([X_l - X_j - 2\pi i(m-k)]/2n + ic)}, \tag{E.1}$$

with

$$d_j = \exp(P_j/2) \prod_{\substack{p=1 \\ p \neq j}}^{N} f(nc; X_j - X_p). \tag{E.2}$$

(In this appendix the indices j, l, p take values $1, ..., N$ and the indices k, m take values $1, ..., n$, unless explicitly indicated otherwise.)

We recall that the spectrum of L on Ω_N is given by (5.22), where $\eta(X, P)$ is the action part of the $\mathrm{II}_{\mathrm{rel}}(nc, N)$ action-angle transform (5.29). Since one has $\eta_N < \cdots < \eta_1$, the spectrum (5.22) is non-degenerate. Thus there exists a matrix $\mathscr{V} \in GL(nN, \mathbb{C})$ such that

$$\mathscr{V}^{-1} L \mathscr{V} = \operatorname{diag}(\exp[\eta_1 + i(n-1)c], ..., \exp[\eta_N + i(-n+1)c]) \equiv \hat{L}. \tag{E.3}$$

Clearly, $\mathscr{V}$ is uniquely determined by (E.3) up to right multiplication by a diagonal invertible matrix $\mathscr{D}$.

Next, we introduce the matrix

$$B \equiv \mathscr{V}^{-1} A \mathscr{V}, \tag{E.4}$$

where

$$A \equiv \operatorname{diag}(\exp x_1, ..., \exp x_{nN}) = \operatorname{diag}(\exp[(X_1 + 2\pi i(n-1))/n], ..., \exp[X_N/n]). \tag{E.5}$$

Note that non-zero off-diagonal elements of B are not gauge-invariant. (That is, invariant under taking $\mathscr{V} \to \mathscr{V}\mathscr{D}$ in (E.4).)

We are now prepared to state the main result of this appendix: we claim there exists a matrix $\mathscr{U} \in GL(nN, \mathbb{C})$ satisfying

$$\mathscr{U}^{-1} L \mathscr{U} = \hat{L} \tag{E.6}$$

and

$$\mathscr{U}^{-1} A \mathscr{U} = \hat{A}, \tag{E.7}$$

where $\hat{A}$ is the matrix whose non-zero elements read

$$\hat{A}_{ln,(j-1)n+1} = -\exp[(y_j - \pi i)/n] \frac{\sin(nc)}{i\,\mathrm{sh}((\eta_l - \eta_j)/2 - inc)} \cdot \prod_{\substack{p=1 \\ p \neq j}}^{N} \prod_{m=1}^{n} f(c; \eta_j - \eta_p - 2i(1-m)c), \tag{E.8}$$

$$\hat{A}_{(j-1)n+k,(j-1)n+k+1} = \exp[(y_j - \pi i)/n] \cdot \prod_{\substack{p=1 \\ p \neq j}}^{N} \prod_{m=1}^{n} f(c; \eta_j - \eta_p - 2i(k+1-m)c), \tag{E.9}$$

where $k = 1, ..., n-1$. Thus, $\hat{A}$ equals the matrix $\hat{A}(n, ..., n; \eta, y)$ (7.18), cf. (7.9)–(7.10), and (E.6), (E.7) amount to (7.47), (7.48).

To prove our claim, we begin by exploiting the commutation relation (3.16) between L and A. To this end we choose a matrix $\mathscr{V}$ satisfying (E.3), and transform (3.16) to obtain

$$B_{ab}[\mathrm{ch}(ic)(-\lambda_a + \lambda_b) + \mathrm{sh}(ic)(\lambda_a + \lambda_b)] = 2\,\mathrm{sh}(ic) f_a g_b, \qquad a, b = 1, ..., nN. \tag{E.10}$$

Here, $\lambda_1, ..., \lambda_{nN}$ are the ordered eigenvalues of L,

$$\lambda_{(j-1)n+k} = \exp[\eta_j + i(n+1-2k)c], \tag{E.11}$$

and we have introduced the vectors

$$f \equiv \mathscr{V}^{-1} e, \qquad g \equiv \mathscr{V}^{T} e. \tag{E.12}$$

(As before, the superscript T stands for transpose.)

Now from (E.11) we deduce that the square-bracket term in (E.10) is non-zero when a equals jn or b equals $(j-1)n+1$. Thus we must have

$$f_{jn} \neq 0, \qquad g_{(j-1)n+1} \neq 0, \tag{E.13}$$

and so

$$B_{ln,(j-1)n+1} \neq 0. \tag{E.14}$$

(Indeed, if one of the above components of f or g were zero, then it would follow from (E.10) that B has a row or column consisting of zeros, contradicting $|B| = |A| \neq 0$.)

On the other hand, (E.11) entails that the square-bracket term vanishes when $a = (j-1)n+k$ and $b = (j-1)n+k+1$, where $k = 1, ..., n-1$, so that

$$f_{(j-1)n+k}\, g_{(j-1)n+k+1} = 0, \qquad k = 1, ..., n-1. \tag{E.15}$$

From this we readily deduce

$$B_{(j-1)n+k,\,(j-1)n+k+1} \neq 0, \qquad k = 1, ..., n-1. \tag{E.16}$$

(Indeed, assume one of these elements were 0. Since (E.15) entails that at least one of $f_{(j-1)n+k}$, $g_{(j-1)n+k+1}$ vanishes, it would then follow from (E.10) that B has a row or column consisting of zeros, a contradiction.)

We now exploit (E.13) and (E.16) to trade $\mathscr{V}$ for a uniquely determined gauge-transformed matrix

$$\mathscr{U} = \mathscr{V}\mathscr{D} \tag{E.17}$$

by insisting on

$$(\mathscr{U}^{-1}e)_{jn} = \exp[\eta_j/2], \tag{E.18}$$

$$(\mathscr{U}^{-1}A\mathscr{U})_{(j-1)n+k,\,(j-1)n+k+1} = \hat{A}_{(j-1)n+k,\,(j-1)n+k+1}, \qquad k = 1, ..., n-1. \tag{E.19}$$

(Indeed, requiring (E.18) fixes the elements $\mathscr{D}_{jn,\,jn}$, and then (E.19) fixes the remaining elements.) Therefore, it remains to prove that the matrix

$$A' \equiv \mathscr{U}^{-1}A\mathscr{U} \tag{E.20}$$

is equal to $\hat{A}$.

Before embarking on a second line of argument that will lead to this conclusion, it is illuminating to explain why a quite different reasoning appears inevitable. Possibly, a more refined analysis of the transformed commutation relation (E.10) may lead directly to the conclusion

$$f_{(j-1)n+k} = 0, \qquad g_{(j-1)n+k+1} = 0, \qquad k = 1, ..., n-1. \tag{E.21}$$

(Note these relations are compatible with, but not implied by (E.15).) We have not found such a proof however, and we are only able to establish the validity of (E.21) via a detour. The point to be made is, that even when one takes (E.21) for granted, it does not yet follow that A' (E.20) equals $\hat{A}$. (To be more specific, it is unclear why the elements $A'_{ln,\,(j-\ )n+1}$ are equal to (E.8); (E.19) and (E.21) only entail equality of the remaining elements.)

Our new argument (from which we will derive not only (E.21) but also (E.7)) starts "from scratch": we transform L to diagonal form with a matrix $\mathscr{W}$ that differs from $\mathscr{U}$, but that we are able to construct explicitly. In the process, we will reobtain the spectrum of L on Ω_N in a novel way—we need not appeal to the reasoning embodied in (5.15)–(5.22).

Turning now to the details, we begin by noting that we can view L as an $N \times N$ block matrix with blocks that are $n \times n$ matrices. In view of (E.1), the (lj)-block can be written

$$d_l d_j \sum_{k=0}^{n-1} \frac{\mathrm{sh}(inc)}{n\,\mathrm{sh}((X_l - X_j)/2n + ic + i\pi k/n)} S^k, \tag{E.22}$$

where S is the antiperiodic $n \times n$ shift:

$$S \equiv \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -1 & 0 & \cdots & 0 \end{pmatrix}. \tag{E.23}$$

Introducing the $nN \times nN$ matrix

$$\mathscr{S} \equiv \mathrm{diag}(S, ..., S), \tag{E.24}$$

we deduce from (E.22)

$$\mathscr{S} L = L \mathscr{S}. \tag{E.25}$$

Now S has eigenvalues

$$s_k = e^{i\pi(2k-1)/n} \tag{E.26}$$

and corresponding orthonormal eigenvectors

$$u_m^{(k)} = n^{-1/2} e^{i\pi(2mk - m - k)/n}. \tag{E.27}$$

Accordingly, $\mathscr{S}$ has N-fold degenerate eigenvalues $s_1, ..., s_n$, so in view of (E.25) we can diagonalize L in each eigenspace. Specifically, we may and will look for eigenvectors of the form

$$(a_1 u^{(k)}, ..., a_N u^{(k)}) \equiv a \otimes u^{(k)}, \qquad a_1, ..., a_N \in \mathbb{C}, \tag{E.28}$$

where the tensor notation will be clear.

The crux is now that when the block structure (E.22) is combined with the identity (D.9), then the eigenvalue equation

$$L a \otimes u^{(k)} = \lambda a \otimes u^{(k)} \tag{E.29}$$

is readily seen to reduce to

$$L^{(k)} a = \lambda e^{-i(n+1-2k)c} a. \tag{E.30}$$

Here, $L^{(k)}$ is the $N \times N$ matrix with elements

$$L^{(k)}_{lj} = \exp[(X_l - X_j)(n+1-2k)/2n] \, L(nc; X, P)_{lj}. \tag{E.31}$$

In words, $L^{(k)}$ is a similarity transform of the $\mathrm{II}_{\mathrm{rel}}(nc, N)$ Lax matrix from Section 3. Now the latter has eigenvalues $\exp \eta_1, ..., \exp \eta_N$, with η the action part of $\Phi(nc; X, P)$; moreover, the associated eigenvectors $v^{(1)}, ..., v^{(N)}$ may be chosen orthonormal and such that

$$\overline{v^{(l)}} \cdot A(X) \, v^{(j)} = L(-nc; \eta, y)_{lj}, \tag{E.32}$$

cf. (3.19), (3.20).

Let us now introduce the similarity-transformed bases $v^{(1,k)}, ..., v^{(N,k)}$ of $\mathbb{C}^N$ with components

$$v^{(j,k)}_l \equiv \exp[X_l(n+1-2k)/2n] \, v^{(j)}_l. \tag{E.33}$$

Then we deduce that the matrix $L(c; x, p)$ has eigenvalues (E.11) with eigenvectors $v^{(j,k)} \otimes u^{(k)}$. (This is the spectral result announced above (E.22).) Setting

$$\mathscr{W} \equiv \mathrm{Col}(v^{(1,1)} \otimes u^{(1)}, ..., v^{(1,n)} \otimes u^{(n)}, ..., v^{(N,n)} \otimes u^{(n)}), \tag{E.34}$$

we now infer using (E.33) that

$$\mathscr{W}^{-1} = \mathrm{Row}(\tilde{v}^{(1,1)} \otimes \overline{u^{(1)}}, ..., \tilde{v}^{(N,n)} \otimes \overline{u^{(n)}}), \tag{E.35}$$

where

$$\tilde{v}^{(j,k)}_l \equiv \exp[-X_l(n+1-2k)/2n] \, \overline{v^{(j)}_l}. \tag{E.36}$$

(Recall the bases $v^{(1)}, ..., v^{(N)}$ of $\mathbb{C}^N$ and $u^{(1)}, ..., u^{(n)}$ of $\mathbb{C}^n$ are orthonormal.) Moreover, by construction we have

$$\mathscr{W}^{-1} L \mathscr{W} = \hat{L}, \tag{E.37}$$

with $\hat{L}$ given by (E.3).

We continue by deriving explicit information on the vectors

$$\hat{e} \equiv \mathscr{W}^{-1} e, \qquad \tilde{e} \equiv \mathscr{W}^T e \tag{E.38}$$

and the matrix

$$\tilde{A} \equiv \mathscr{W}^{-1} A \mathscr{W}. \tag{E.39}$$

To this purpose we first note that we may write

$$e = \tilde{d} \otimes u^{(n)}, \tag{E.40}$$

where $\tilde{d}$ is defined by

$$\tilde{d}_j \equiv \frac{\sin(nc)^{1/2}}{\sin(c)^{1/2}} \exp[X_j/2n]\, d_j, \tag{E.41}$$

with d_j given by (E.2). Now we have $\overline{u^{(k)}} \cdot u^{(n)} = 0$ for $k = 1, ..., n-1$, so from (E.35) and (E.40) we deduce that the vector $\hat{e}$ (E.38) satisfies

$$\hat{e}_{(j-1)n+k} = 0, \qquad k = 1, ..., n-1. \tag{E.42}$$

Moreover, since $u^{(n)}$ is proportional to $\overline{u^{(1)}}$ (cf. (E.27)), we derive similarly from (E.34) and (E.40) that we have

$$\tilde{e}_{(j-1)n+k} = 0, \qquad k = 2, ..., n. \tag{E.43}$$

Next, we calculate, using (E.27), (E.33), and (E.36),

$$\begin{aligned}
\tilde{A}_{(j-1)n+k,(j-1)n+k+1} &= (\tilde{v}^{(j,k)} \otimes \overline{u^{(k)}}) \cdot A(v^{(j,k+1)} \otimes u^{(k+1)}) \\
&= \sum_{l=1}^{N} \sum_{m=1}^{n} \tilde{v}_l^{(j,k)}\, \overline{u_m^{(k)}} \exp[(X_l + 2\pi i(n-m))/n]\, v_l^{(j,k+1)} u_m^{(k+1)} \\
&= e^{-i\pi/n} \sum_{l=1}^{N} \exp[-X_l(n+1-2k)/2n]\, \overline{v_l^{(j)}} \\
&\quad \cdot \exp[X_l/n] \exp[X_l(n+1-2(k+1))/2n]\, v_l^{(j)} \\
&= e^{-i\pi/n}, \qquad k = 1, ..., n-1,
\end{aligned} \tag{E.44}$$

and similarly,

$$\begin{aligned}
\tilde{A}_{(j-1)n+n,(j-1)n+1} &= \sum_{l,m} \tilde{v}_l^{(j,n)}\, \overline{u_m^{(n)}} \exp[(X_l - 2\pi i m)/n]\, v_l^{(j,1)} u_m^{(1)} \\
&= -e^{-i\pi/n} \sum_l \overline{v_l^{(j)}} \exp(X_l)\, v_l^{(j)} \\
&= -e^{-i\pi/n} L(-nc; \eta, y)_{jj} \\
&= -e^{-i\pi/n} \exp(y_j) \prod_{\substack{p=1 \\ p \neq j}}^{N} f(nc; \eta_j - \eta_p),
\end{aligned} \tag{E.45}$$

where we used (E.32).

This is the only information on $\hat{e}$, $\tilde{e}$ and $\tilde{A}$ we need to complete the proof of our claim. Indeed, in view of (E.37), the matrix $\mathscr{W}$ is a gauge transform of $\mathscr{V}$ (i.e., there exists a diagonal invertible matrix $\tilde{\mathscr{D}}$ such that $\mathscr{V}$ equals $\mathscr{U}\tilde{\mathscr{D}}$). Consequently, we can now derive the announced equalities (E.21) from (E.42) and (E.43). Then the transformed commutation relation (E.10) entails that all matrix elements of B vanish, but for (E.14) and (E.16). This result is evidently gauge-invariant, so A' and $\tilde{A}$ have the same property.

We proceed by noting that all principal minors of B are gauge-invariant, too. In particular, the determinant Δ_j of the (jj)-block is gauge-invariant, so we may calculate it by taking $B=\tilde{A}$. On account of (E.44) and (E.45) this yields

$$\Delta_j=(-)^{n-1}\exp(y_j)\prod_{\substack{p=1\\p\neq j}}^{N} f(nc;\eta_j-\eta_p). \tag{E.46}$$

We assert that this entails

$$A'_{(j-1)n+n,(j-1)n+1}=-\exp[(y_j-\pi i)/n]\prod_{\substack{p=1\\p\neq j}}^{N}\prod_{m=1}^{n} f(c;\eta_j-\eta_p-2i(1-m)c). \tag{E.47}$$

To prove this assertion, we need only point out that this matrix element is uniquely determined by the determinant (E.46) and the remaining non-zero matrix elements in the (jj)-block (given by (E.19) and (E.9)). Using the fusion identity (D.3) it is now straightforward to establish that one must have (E.47) for the determinant of the (jj)-block of A' to equal (E.46).

Of course, the rhs of (E.47) equals the rhs of (E.8) for $l=j$, so it remains to show equality for $l\neq j$. Now when the commutation relation is transformed with $\mathscr{U}$, then we obtain (cf. (E.10), (E.11))

$$-A'_{ln,(j-1)n+1}\exp[(\eta_l+\eta_j)/2]\,\mathrm{sh}((\eta_l-\eta_j)/2-inc)=\mathrm{sh}(ic)(\mathscr{U}^{-1}e)_{ln}(\mathscr{U}^{T}e)_{(j-1)n+1}. \tag{E.48}$$

Taking first $l=j$, we may substitute (E.47) and (E.18) to deduce

$$(\mathscr{U}^{T}e)_{(j-1)n+1}=-\frac{\sin(nc)}{\sin(c)}\exp[\eta_j/2]\exp[(y_j-\pi i)/n]\cdot\prod_{\substack{p=1\\p\neq j}}^{N}\prod_{m=1}^{n} f(c;\eta_j-\eta_p-2i(1-m)c). \tag{E.49}$$

Finally, substituting (E.49) and (E.18) in (E.48) for $l\neq j$, we deduce equality of the remaining matrix elements $A'_{ln,(j-1)n+1}$ and (E.8). Thus, the proof of our claim is now complete.

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