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# ON A CONJECTURE OF IGLEHART* †

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This paper gives an elementary proof of Iglehart's conjecture about the classical dynamic inventory model. This conjecture states that the minimal total expected cost for a planning horizon of $n$ periods minus $n$ times the minimal long-run average expected cost per period has a finite limit as $n \to \infty$ for each initial stock.

## 1. Introduction

In a fundamental paper Iglehart [4] conjectured for the dynamic inventory model with a linear purchase cost, a fixed set-up cost and convex holding and shortage costs that the minimal total expected cost for a planning horizon of $n$ periods minus $n$ times the minimal long-run average expected cost has a finite limit as $n \to \infty$ for each initial stock. In [1] this conjecture was proved amongst other results for the case of a positive discrete demand by using results in [2]. In this paper we present an elementary proof of the original conjecture offered under the assumption of a continuous demand with a density that is not concentrated on a bounded interval. The proof applies equally well to the discrete demand case.

In §2 we formulate the model and give some preliminaries. Also, we state in §2 the main theorem that will be proved in §3.

## 2. Model and Preliminaries

We consider the single-item inventory model in which the demands in successive periods form a sequence of independent random variables having a common probability distribution function $\Phi(\cdot)$ with density $\phi(\cdot)$. It is assumed that $\Phi(\xi) < 1$ for all $\xi$. Further we suppose that the demand per period has a finite expectation $\mu$. Any unfilled demand in a period is backlogged. Hence the stock level may take on any real value, where a negative value indicates the existence of a backlog. At the beginning of each period the stock on hand is reviewed. At each review an order may be placed for any positive amount of stock. An order, when placed, is delivered instantaneously. The demand in each period takes place after review and delivery (if any). The following costs are involved. The cost of ordering an amount of $z$ is $K\delta(z) + cz$, where $K \geqq 0$, $c \geqq 0$, $\delta(0) = 0$, and, $\delta(z) = 1$ for $z > 0$. Let $L(y)$ be the expected holding and shortage costs in a period when $y$ is the amount of stock at the beginning of that period just after any additions to stock. We assume that $L(y)$ is a nonnegative convex function that is continuous for all $y$. Further it is assumed that both $L(y) \to \infty$ and $cy + L(y) \to \infty$ as $|y| \to \infty$. Finally, future costs are not discounted.

We now give some known results for this model that will be needed in the sequel. For any real $x$, let $f_0(x) = 0$. It was proved by Scarf [5] (see also [3]) that there is a sequence $\{f_n(\cdot), n \geqq 1\}$ of continuous functions satisfying, for all $x$ and all $n \geqq 1$,

$$(1) \quad f_n(x) = \min_{y \geqq x}\left\{c\cdot(y-x) + K\delta(y-x) + L(y) + \int_0^\infty f_{n-1}(y-\xi)\phi(\xi)\,d\xi\right\},$$

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such that, for all $n \geqq 1$,

$$(2) \qquad \begin{aligned} f_n(x) &= -cx + K + G_n(S_n) \quad \text{for } x < s_n, \\ &= -cx + G_n(x) \qquad\qquad \text{for } x \geqq s_n, \end{aligned}$$

where $G_n(y) = cy + L(y) + \int_0^\infty f_{n-1}(y - \xi)\phi(\xi)\, d\xi$, $S_n$ is the smallest number that minimizes the function $G_n(y)$, and $s_n$ is the smallest number less than or equal to $S_n$ for which $G_n(s_n) = K + G_n(S_n)$. Hence the right side of (1) is minimal for $y = S_n$ when $x < s_n$ and for $y = x$ when $x \geqq s_n$. It was proved in [3] that the sequences $\{s_n\}$ and $\{S_n\}$ are bounded. Observe that $f_n(x)$ denotes the minimal total expected cost for a planning horizon of $n$ periods when the initial stock is $x$.

Consider now the infinite period model. Denote by $a(s, S)$ the average expected cost per period when using the $(s, S)$ policy under which the order is $S - x$ when the stock level $x < s$ and is 0 when $x \geqq s$, see [4]. Fix two finite numbers $s^*$ and $S^*$ such that $\min_{s,S} a(s, S) = a(s^*, S^*)$ and $L(s^*) + c\mu = g$ where $g = \min_{s,S} a(s, S)$. In [4] it was shown that such numbers exist and that the $(s^*, S^*)$ policy is average cost optimal among the class of all possible policies (see also [6, p. 530]). Hence the minimal average expected cost per period is independent of the initial stock and equals $g$. Next define the function $\psi(\cdot)$ by

$$(3) \qquad \begin{aligned} \psi(x) &= -c\cdot(x - s^*) \qquad\qquad\qquad\qquad\qquad \text{for } x < s^*, \\ &= L(x) - g + \int_0^\infty \psi(x - \xi)\phi(\xi)\, d\xi \quad \text{for } x \geqq s^*. \end{aligned}$$

The relation (3) constitutes for $x \geqq s^*$ a renewal equation. Using this and the relation $L(s^*) + c\mu = g$ it is easy to verify that (3) has a unique finite solution $\psi(x)$ which is continuous for all $x$. It was proved in [4] that, for all $x$,

$$(4) \quad g + \psi(x) = \min_{y \geqq x} \left\{ c\cdot(y - x) + K\delta(y - x) + L(y) + \int_0^\infty \psi(y - \xi)\phi(\xi)\, d\xi \right\},$$

where the right side of (4) is minimal for $y = S^*$ when $x < s^*$ and for $y = x$ when $x \geqq s^*$.

In the next section we prove

THEOREM 1. *There is some finite number $b$ such that*

$$\lim_{n\to\infty} \{ f_n(x) - ng \} = \psi(x) + b \quad \textit{for all } x.$$

*Moreover, the convergence of $f_n(x) - ng - \psi(x)$ for $n \to \infty$ is exponentially fast and uniform for all $x$ in any interval bounded from above.*

Iglehart [4] proved this result for the case of $K = 0$ and offered it as a conjecture for the case of $K > 0$. Actually Iglehart [4] imposed no condition on the density $\phi(\cdot)$. However, it is easy to give an example showing that $f_n(x) - ng$ may diverge as $n \to \infty$ when the demand per period is bounded, see also [1].

## 3. The Proof

To prove Theorem 1, we fix two finite numbers $L$ and $U$ such that $L < s_n \leqq S_n \leqq U$ for all $n \geqq 1$ and $L < s^* \leqq S^* \leqq U$. Let $X = \{x \mid x \leqq U\}$, and let $A(x) = \{y \mid y \geqq L, x \leqq y \leqq U\}$ for $x \in X$. By the results in §2 we have, for all $x \in X$ and

$n \geqq 1$,

(5)

$$f_n(x) = \min_{y \in A(x)} \left\{ c \cdot (y - x) + K\delta(y - x) + L(y) + \int_0^\infty f_{n-1}(y - \xi)\phi(\xi)\, d\xi \right\},$$

and, for all $x \in X$,

(6)

$$g + \psi(x) = \min_{y \in A(x)} \left\{ c \cdot (y - x) + K\delta(y - x) + L(y) + \int_0^\infty \psi(y - \xi)\phi(\xi)\, d\xi \right\}.$$

Observe that, by (2) and (3), the above integrals converge absolutely. Define $\pi$ by $\pi(x) = S^*$ for $x < s^*$ and $\pi(x) = x$ for $x \geqq s^*$ and, for $n \geqq 1$, define $\pi_n$ by $\pi_n(x) = S_n$ for $x < s_n$ and $\pi_n(x) = x$ for $x \geqq s_n$. Then, for any $x \in X$, $\pi(x)$ minimizes the right side of (6) and $\pi_n(x)$ minimizes the right side of (5).

For any $x \in X$ and $n \geqq 1$, let $e_n(x) = f_n(x) - ng - \psi(x)$. Using the definitions of $\pi$ and $\pi_n$, it follows from (5) and (6) that, for all $x \in X$ and $n \geqq 1$,

(7)

$$e_{n+1}(x) \leqq \int_0^\infty e_n(\pi(x) - \xi)\phi(\xi)\, d\xi,$$

$$e_{n+1}(x) \geqq \int_0^\infty e_n(\pi_{n+1}(x) - \xi)\phi(\xi)\, d\xi.$$

Since $U$ can be chosen arbitrarily large, Theorem 1 is an immediate consequence of the following theorem.

THEOREM 2. *The sequence $\{e_n(x), n \geqq 1\}$ has a finite limit for all $x \in X$ and this limit is independent of $x \in X$. Moreover, the convergence of $e_n(x)$ for $n \to \infty$ is exponentially fast and uniform for all $x \in X$.*

PROOF. Using the continuity of $f_1(\cdot)$ and $\psi(\cdot)$, it follows from (2) and (3) that there is a finite number $N$ such that $| e_1(x) | \leqq N$ for all $x \in X$. By induction we have from (7) that $| e_n(x) | \leqq N$ for all $x \in X$ and $n \geqq 1$, cf. [4, p. 15].

Now, define $M_n = \sup_{x \in X} e_n(x)$ and define $m_n = \inf_{x \in X} e_n(x)$ for $n \geqq 1$. It immediately follows from (7) that $M_{n+1} \leqq M_n$ and $m_{n+1} \geqq m_n$ for all $n \geqq 1$. Hence the bounded sequences $\{M_n\}$ and $\{m_n\}$ have finite limits $M$ and $m$, respectively.

Let $a = \Phi(U - L)$. Since $\Phi(\xi) < 1$ for all $\xi$, we have $0 \leqq a < 1$. By (2) and (3), $e_n(x) = \epsilon_n$ for all $x < L$ and $n \geqq 1$ where $\epsilon_n = K + G_n(S_n) - ng - cs^*$. Since $L \leqq \pi(x) \leqq U$ for all $x \in X$, we get from the first part of (7) that $e_{n+1}(x) \leqq aM_n + (1 - a)\epsilon_n$ for all $x \in X$ and $n \geqq 1$. Hence $M_{n+1} \leqq aM_n + (1 - a)\epsilon_n$ for all $n \geqq 1$. Similarly, we derive from the second part of (7) that $m_{n+1} \geqq am_n + (1 - a)\epsilon_n$ for all $n \geqq 1$. Hence $M_{n+1} - m_{n+1} \leqq a(M_n - m_n)$ for all $n \geqq 1$, so,

(8) $$0 \leqq M_n - m_n \leqq a^{n-1}(M_1 - m_1) \quad \text{for all } n \geqq 1.$$

Since $0 \leqq a < 1$ and the bounded sequences $\{M_n\}$ and $\{m_n\}$ are convergent, the theorem now follows from (8) and $m_n \leqq e_n(x) \leqq M_n$ for all $x \in X$ and $n \geqq 1$.

## References

1. HORDIJK, A. AND TIJMS, H. C., "Convergence Results and Approximations for Optimal (*s*, *S*) Policies," *Management Science*, Vol. 20 (1974), pp. 1432–1438.

2. ——, SCHWEITZER, P. J. AND ——, "The Asymptotic Behaviour of the Minimal Total Expected Cost for the Denumerable State Markov Decision Model," *Journal of Applied Probability* (to appear).
3. IGLEHART, D. L., "Optimality of $(s, S)$ Inventory Policies in the Infinite Horizon Dynamic Inventory Problem," *Management Science*, Vol. 9 (1963), pp. 259–267.
4. ——, "Dynamic Programming and Stationary Analysis of Inventory Problems," Chap. 1 in H. Scarf, D. Gilford, and M. Shelly (eds.) *Multistage Inventory Models and Techniques*, Stanford University Press, Stanford, 1963.
5. SCARF, H., "The Optimality of $(S, s)$ Policies in the Dynamic Inventory Problem," Chap. 13 in K. J. Arrow, S. Karlin, and P. Suppes (eds.) *Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, 1960.
6. VEINOTT, A. F., JR. AND WAGNER, H. M., "Computing Optimal $(s, S)$ Policies," *Management Science*, Vol. 11 (1965), pp. 525–552.