

# A $J$ -function for inhomogeneous spatio-temporal point processes

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**Abstract:** We propose a new summary statistic for inhomogeneous intensity-reweighted moment stationary spatio-temporal point processes. The statistic is defined through the  $n$ -point correlation functions of the point process and it generalises the  $J$ -function when stationarity is assumed. We show that our statistic can be represented in terms of the generating functional and that it is related to the inhomogeneous  $K$ -function. We further discuss its explicit form under some specific model assumptions and derive a ratio-unbiased estimator. We finally illustrate the use of our statistic on simulated data.

**Key words:** Generating functional, Hard core model, Inhomogeneity, Intensity-reweighted moment stationarity,  $J$ -function,  $K$ -function, Location-dependent thinning, Log-Gaussian Cox process,  $n$ -point correlation function, Papangelou conditional intensity, Poisson process, Reduced Palm measure generating functional, Second order intensity-reweighted stationarity, Spatio-temporal point process.

## 1 Introduction

A spatio-temporal point pattern can be described as a collection of pairs  $\{(x_i, t_i)\}_{i=1}^m$ ,  $m \geq 0$ , where  $x_i \in W_S \subseteq \mathbb{R}^d$ ,  $d \geq 1$ , and  $t_i \in W_T \subseteq \mathbb{R}$  describe, respectively, the spatial location and the occurrence time associated with the  $i$ th event. Examples of such point patterns include recordings of earthquakes, disease outbreaks and fires (see e.g. [12, 21, 26]).

When modelling spatio-temporal point patterns, the usual and natural approach is to assume that  $\{(x_i, t_i)\}_{i=1}^m$  constitutes a realisation of a spatio-temporal point process (STPP)  $Y$  restricted to  $W_S \times W_T$ . Then, in order to deduce what type of model could describe the observations  $\{(x_i, t_i)\}_{i=1}^m$ , one carries out an exploratory analysis of the data under some minimal set of conditions on the underlying point process  $Y$ . At this stage, one is often interested in detecting tendencies for points to cluster together, or to inhibit one another. In order to do so, one usually employs spatial or temporal summary statistics, which are able to capture and reflect such features.

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A simple and convenient working assumption for the underlying point process is stationarity. In the case of a purely spatial point pattern  $\{x_i\}_{i=1}^m \subseteq W_S$  generated by a stationary spatial point process  $X$ , a variety of summary statistics have been developed, see e.g. [7, 14, 16, 17]. One such statistic is the so-called  $J$ -function [20], given by

$$J(r) = \frac{1 - G(r)}{1 - F(r)} \quad (1)$$

for  $r \geq 0$  such that  $F(r) \neq 1$ . Here, the empty space function  $F(r)$  is the probability of having at least one point of  $X$  within distance  $r$  from the origin whereas the nearest neighbour distance distribution function  $G(r)$  is the conditional probability of some further point of  $X$  falling within distance  $r$  from a typical point of  $X$ . Hence,  $J(r) < 1$  indicates clustering,  $J(r) = 1$  indicates spatial randomness and  $J(r) > 1$  indicates regularity at inter-point distance  $r$ .

In many applications, though, stationarity is not a reasonable assumption. This observation has led to the development of summary statistics being able to compensate for inhomogeneity. For purely spatial point processes, [3] introduced the notion of second order intensity-reweighted stationarity (SIRS) and defined a summary statistic  $K_{\text{inhom}}(r)$ . It can be interpreted as an analogue of the  $K$ -function, which is proportional to the expected number of further points within distance  $r$  of a typical point of  $X$ , since it reduces to  $K(r)$  when  $X$  is stationary.

The concept of SIRS was extended to the spatio-temporal case by [12] who also defined an inhomogeneous spatio-temporal  $K$ -function  $K_{\text{inhom}}(r, t)$ ,  $r, t \geq 0$ . These ideas were further developed and studied in [22] with particular attention to the notion of space-time separability.

To take into account interactions of order higher than two, [19] introduced the concept of intensity-reweighted moment stationarity (IRMS) for purely spatial point processes and generalised (1) to IRMS point processes.

In this paper we develop a proposal given in [19] to study the spatio-temporal generalisation  $J_{\text{inhom}}(r, t)$  of (1) under suitable intensity-reweighting. In Section 2 we give the required preliminaries, which include definitions of product densities, Palm measures, generating functionals,  $n$ -point correlation functions and IRMS for spatio-temporal point processes. Then, in Section 3, we give the definition of  $J_{\text{inhom}}(r, t)$  under the assumption of IRMS and discuss its relation to the inhomogeneous spatio-temporal  $K$ -function of [12]. In Section 4 we write  $J_{\text{inhom}}(r, t)$  as a ratio of  $1 - F_{\text{inhom}}(r, t)$  and  $1 - G_{\text{inhom}}(r, t)$  in analogy with (1). As a by-product we obtain generalisations of the empty space function and the nearest neighbour distance distribution. The section also includes a representation in terms of the Papangelou conditional intensity. In Section 5 we consider three classes of spatio-temporal point processes for which the IRMS assumption holds, namely Poisson processes, location dependent thinning of stationary STPPs and log-Gaussian Cox processes. In Section 6 we derive a non-parametric estimator  $\widehat{J_{\text{inhom}}}(r, t)$  for which we show ratio-unbiasedness and in Section 7 we illustrate its use on simulated data.

## 2 Definitions and preliminaries

### 2.1 Simple spatio-temporal point process

In order to set the stage, let  $\|x\| = (\sum_{i=1}^d x_i^2)^{1/2}$  and  $d_{\mathbb{R}^d}(x, y) = \|x - y\|$ ,  $x, y \in \mathbb{R}^d$  denote respectively the Euclidean norm and metric. Since space and time must be treated differently, we endow  $\mathbb{R}^d \times \mathbb{R}$  with the supremum norm  $\|(x, t)\|_\infty = \max\{\|x\|, |t|\}$  and the supremum metric

$$d((x, t), (y, s)) = \|(x, t) - (y, s)\|_\infty = \max\{d_{\mathbb{R}^d}(x, y), d_{\mathbb{R}}(t, s)\},$$

where  $(x, t), (y, s) \in \mathbb{R}^d \times \mathbb{R}$ . Then  $(\mathbb{R}^d \times \mathbb{R}, d(\cdot, \cdot))$  is a complete separable metric space, which is topologically equivalent to the Euclidean space  $(\mathbb{R}^d \times \mathbb{R}, d_{\mathbb{R}^{d+1}}(\cdot, \cdot))$ . Note that in the supremum metric a closed ball of radius  $r \geq 0$  centred at the origin  $0 \in \mathbb{R}^d \times \mathbb{R}$  is given by the cylinder set

$$B[0, r] = \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : \max\{\|x\|, |t|\} \leq r\}.$$

Write  $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}) = \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$  for the  $d$ -induced Borel  $\sigma$ -algebra and let  $\ell$  denote Lebesgue measure on  $\mathbb{R}^d \times \mathbb{R}$ . Furthermore, given some Borel set  $A \subseteq \mathbb{R}^d \times \mathbb{R}$  and some measurable function  $f$ , we interchangeably let  $\int_A f(y)dy$  and  $\int_A f(y)\ell(dy)$  represent the integral of  $f$  over  $A$  with respect to  $\ell$ .

In this paper, a spatio-temporal point process is a simple point process on the product space  $\mathbb{R}^d \times \mathbb{R}$ . More formally, let  $N$  be the collection of all locally finite counting measures  $\varphi$  on  $\mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ , i.e.  $\varphi(A) < \infty$  for bounded  $A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ , and let  $\mathcal{N}$  be the smallest  $\sigma$ -algebra on  $N$  to make the mappings  $\varphi \mapsto \varphi(A)$  measurable for all  $A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ . Consider in addition the sub-collection  $N^* = \{\varphi \in N : \varphi(\{(x, t)\}) \in \{0, 1\} \text{ for any } (x, t) \in \mathbb{R}^d \times \mathbb{R}\}$  of simple elements of  $N$ .

**Definition 1.** *A simple spatio-temporal point process (STPP)  $Y$  on  $\mathbb{R}^d \times \mathbb{R}$  is a measurable mapping from some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  into the measurable space  $(N, \mathcal{N})$  such that  $Y$  almost surely (a.s.) takes values in  $N^*$ .*

Throughout we will denote the  $Y$ -induced probability measure on  $\mathcal{N}$  by  $P$ . To emphasise the counting measure nature of  $Y$ , we will sometimes write  $Y = \sum_{i=1}^\infty \delta_{(X_i, T_i)}$  as a sum of Dirac measures, where the  $X_i \in \mathbb{R}^d$  are the spatial components and the  $T_i \in \mathbb{R}$  are the temporal components of the points of  $Y$ . Hence, both  $Y(\{(x, t)\}) = 1$  and  $(x, t) \in Y$  will have the same meaning and both  $Y(A)$  and  $|Y \cap A|$  may be used as notation for the the number of points of  $Y$  in some set  $A$ , where  $|\cdot|$  denotes cardinality.

### 2.2 Product densities and $n$ -point correlation functions

Our definition of the inhomogeneous  $J$ -function relies on the so-called  $n$ -point correlation functions, which are closely related to the better known product densities. Here we recall their definition.

Suppose that the factorial moment measures of  $Y$  exist as locally finite measures and that they are absolutely continuous with respect to the  $n$ -fold product of  $\ell$  with itself. The

Radon–Nikodym derivatives  $\rho^{(n)}$ ,  $n \geq 1$ , referred to as *product densities*, are permutation invariant and defined by the integral equations

$$\begin{aligned} \mathbb{E} \left[ \sum_{(x_1, t_1), \dots, (x_n, t_n) \in Y}^{\neq} h((x_1, t_1), \dots, (x_n, t_n)) \right] &= \\ &= \int \cdots \int h((x_1, t_1), \dots, (x_n, t_n)) \rho^{(n)}((x_1, t_1), \dots, (x_n, t_n)) dx_1 dt_1 \cdots dx_n dt_n \end{aligned} \quad (2)$$

for non-negative measurable functions  $h : (\mathbb{R}^d \times \mathbb{R})^n \rightarrow \mathbb{R}$  under the proviso that the left hand side is infinite if and only if the right hand side is. Equation (2) is sometimes referred to as the *Campbell theorem*. The heuristic interpretation of  $\rho^{(n)}((x_1, t_1), \dots, (x_n, t_n)) dx_1 dt_1 \cdots dx_n dt_n$  is that it represents the infinitesimal probability of observing the points  $x_1, \dots, x_n \in \mathbb{R}^d$  of  $Y$  at the respective event times  $t_1, \dots, t_n \in \mathbb{R}$ .

For  $n = 1$ , we obtain the *intensity measure*  $\Lambda$  of  $Y$  as

$$\Lambda(A) = \int_{B \times C} \rho^{(1)}(x, t) dx dt$$

for any  $A = B \times C \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ . We shall also use the common notation  $\lambda(x, t) = \rho^{(1)}(x, t)$  and assume henceforth that  $\bar{\lambda} = \inf_{(x, t)} \lambda(x, t) > 0$ .

The *n-point correlation functions* [32] are defined in terms of the  $\rho^{(n)}$  by setting  $\xi_1 \equiv 1$  and recursively defining

$$\frac{\rho^{(n)}((x_1, t_1), \dots, (x_n, t_n))}{\prod_{k=1}^n \lambda(x_k, t_k)} = \sum_{k=1}^n \sum_{D_1, \dots, D_k} \prod_{j=1}^k \xi_{|D_j|}(\{(x_i, t_i) : i \in D_j\}), \quad (3)$$

where  $\sum_{D_1, \dots, D_k}$  is a sum over all possible  $k$ -sized partitions  $\{D_1, \dots, D_k\}$ ,  $D_j \neq \emptyset$ , of the set  $\{1, \dots, n\}$  and  $|D_j|$  denotes the cardinality of  $D_j$ .

For a Poisson process on  $\mathbb{R}^d \times \mathbb{R}$  with intensity function  $\lambda(x, t)$ , due to e.g. [17, Theorem 1.3], we have that  $\rho^{(n)}((x_1, t_1), \dots, (x_n, t_n)) = \prod_{k=1}^n \lambda(x_k, t_k)$ , whereby  $\xi_n \equiv 0$  for all  $n \geq 2$ . Hence, the sum on the right hand side in expression (3) is a finite series expansion of the dependence correction factor by which we multiply the product density  $\prod_{k=1}^n \lambda(x_k, t_k)$  of the Poisson process to obtain the product density  $\rho^{(n)}((x_1, t_1), \dots, (x_n, t_n))$  of  $Y$ .

A further interpretation is obtained by realising that the right hand side of the above expression is a series expansion of a higher order version of the *pair correlation function*

$$g((x_1, t_1), (x_2, t_2)) = \frac{\rho^{(2)}((x_1, t_1), (x_2, t_2))}{\lambda(x_1, t_1)\lambda(x_2, t_2)} = 1 + \xi_2((x_1, t_1), (x_2, t_2)).$$

The main definition of this section gives the class of STPPs to which we shall restrict ourselves in the sequel of this paper.

**Definition 2.** *Let  $Y$  be a spatio-temporal point process for which product densities of all orders exist. If  $\bar{\lambda} = \inf_{(x, t)} \lambda(x, t) > 0$  and for all  $n \geq 1$ ,  $\xi_n$  is translation invariant in the sense that*

$$\xi_n((x_1, t_1) + (a, b), \dots, (x_n, t_n) + (a, b)) = \xi_n((x_1, t_1), \dots, (x_n, t_n))$$

for almost all  $(x_1, t_1), \dots, (x_n, t_n) \in \mathbb{R}^d \times \mathbb{R}$  and all  $(a, b) \in \mathbb{R}^d \times \mathbb{R}$ , we say that  $Y$  is *intensity-reweighted moment stationary (IRMS)*.

By equation (3), translation invariance of all  $\xi_n$  is equivalent to translation invariance of the intensity-reweighted product densities. The property is weaker than stationarity (barring the degenerate case where  $Y$  is a.s. empty), which requires the distribution of  $Y$  to be invariant under translation, but stronger than the *second order intensity-reweighted stationary (SIRS)* of [3]. The latter property, in addition to satisfying  $\bar{\lambda} > 0$ , requires the random measure

$$\Xi = \sum_{(x,t) \in Y} \frac{\delta_{(x,t)}}{\lambda(x,t)}$$

to be second order stationary [11, p. 236].

### 2.3 Palm measures and conditional intensities

In order to define a nearest neighbour distance distribution function, we need the concept of *reduced Palm measures*. Recall that by assumption the intensity measure is locally finite. In integral terms, they can be defined by the *reduced Campbell-Mecke* formula

$$\begin{aligned} \mathbb{E} \left[ \sum_{(x,t) \in Y} g(x,t, Y \setminus \{(x,t)\}) \right] &= \int_{\mathbb{R}^d \times \mathbb{R}} \int_N g(x,t, \varphi) P^{!(x,t)}(d\varphi) \lambda(x,t) dx dt \\ &= \int_{\mathbb{R}^d \times \mathbb{R}} \mathbb{E}^{!(x,t)} [g(x,t, Y)] \lambda(x,t) dx dt \end{aligned} \quad (4)$$

for any non-negative measurable function  $g : \mathbb{R}^d \times \mathbb{R} \times N$ , with the left hand side being infinite if and only if the right hand side is infinite, see e.g. [17, Chapter 1.8]. By standard measure theoretic arguments [15], it is possible to find a regular version such that  $P^{!(x,t)}(R)$  is measurable as a function of  $(x,t)$  and a probability measure as a function of  $R$ . Thus,  $P^{!(x,t)}(R)$  may be interpreted as the conditional probability of  $Y \setminus \{(x,t)\}$  falling in  $R \in \mathcal{N}$  given  $Y(\{(x,t)\}) > 0$ .

At times we make the further assumption that  $Y$  admits a *Papangelou conditional intensity*  $\lambda(\cdot, \cdot; \varphi)$ . In effect, we may then replace expectations under the reduced Palm distribution by expectations under  $P$ . More precisely, (4) may be rewritten as

$$\mathbb{E} \left[ \sum_{(x,t) \in Y} g(x,t, Y \setminus \{(x,t)\}) \right] = \int_{\mathbb{R}^d \times \mathbb{R}} \mathbb{E} [g(x,t, Y) \lambda(x,t; Y)] dx dt \quad (5)$$

for any non-negative measurable function  $g \geq 0$  on  $\mathbb{R}^d \times \mathbb{R} \times N$ . Equation (5) is referred to as the *Georgii-Nguyen-Zessin* formula. We interpret  $\lambda(x,t; Y) dx dt$  as the conditional probability of finding a space-time point of  $Y$  in the infinitesimal region  $d(x,t) \subseteq \mathbb{R}^d \times \mathbb{R}$ , given that the configuration elsewhere coincides with  $Y$ . For further details, see e.g. [17, Chapter 1.8].

## 2.4 The generating functional

For the representation of  $J_{\text{inhom}}$  in the form (1), we will need the *generating functional*  $G(\cdot)$  of  $Y$ , which is defined as

$$G(v) = \mathbb{E} \left[ \prod_{(x,t) \in Y} v(x,t) \right] = \int_N \prod_{(x,t) \in \varphi} v(x,t) P(d\varphi)$$

for all functions  $v = 1 - u$  such that  $u : \mathbb{R}^d \times \mathbb{R} \rightarrow [0, 1]$  is measurable with bounded support on  $\mathbb{R}^d \times \mathbb{R}$ . By convention, an empty product equals 1. The generating functional uniquely determines the distribution of  $Y$  [11, Theorem 9.4.V.].

Since we assume that the product densities of all orders exist, we have that

$$\begin{aligned} G(v) &= G(1 - u) = \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int \cdots \int u(x_1, t_1) \cdots u(x_n, t_n) \rho^{(n)}((x_1, t_1), \dots, (x_n, t_n)) \prod_{i=1}^n dx_i dt_i, \end{aligned} \tag{6}$$

provided that the right hand side converges (see [7, p. 126]). The generating functional of the reduced Palm distribution  $P^{!y}$  is denoted by  $G^{!y}(v)$ .

## 3 Spatio-temporal $J$ -functions

We now turn to the definition of the inhomogeneous  $J$ -function  $J_{\text{inhom}}(r, t)$ . Before giving the definition in our general context, we define a spatio-temporal  $J$ -function  $J(r, t)$  for stationary STPPs.

### 3.1 The stationary $J$ -function

Assume for the moment that  $Y$  is stationary. Then we may set, in complete analogy to the definition in [20],

$$J(r, t) = \frac{1 - G(r, t)}{1 - F(r, t)} = \frac{\mathbb{P}^{!(0,0)}(Y \cap S_r^t = \emptyset)}{\mathbb{P}(Y \cap S_r^t = \emptyset)} \tag{7}$$

for  $r, t \geq 0$  such that  $F(r, t) \neq 1$ , where

$$S_r^t = \{(x, s) \in \mathbb{R}^d \times \mathbb{R} : \|x\| \leq r, |s| \leq t\}$$

and  $\mathbb{P}^{!(0,0)}$  is the  $P^{!(0,0)}$ -reversely induced probability measure on  $\mathcal{F}$ .

Note that the two equalities in (7) are defining ones and, clearly,  $G(r, t)$  is the spatio-temporal nearest neighbour distance distribution function whereas  $F(r, t)$  is the spatio-temporal empty space function.

### 3.2 The inhomogeneous $J$ -function

In this section, we extend the inhomogeneous  $J$ -function in [19] to the product space  $\mathbb{R}^d \times \mathbb{R}$  equipped with the supremum metric  $d(\cdot, \cdot)$ .

**Definition 3.** Let  $Y$  be an IRMS spatio-temporal point process (cf. Definition 2). For  $r, t \geq 0$ , let

$$J_n(r, t) = \int_{S_r^t} \cdots \int_{S_r^t} \xi_{n+1}((0, 0), (x_1, t_1), \dots, (x_n, t_n)) \prod_{i=1}^n dx_i dt_i$$

and set

$$J_{\text{inhom}}(r, t) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} J_n(r, t) \quad (8)$$

for all spatial ranges  $r \geq 0$  and temporal ranges  $t \geq 0$  for which the series is absolutely convergent.

Note that by Cauchy's root test absolute convergence holds for those  $r, t \geq 0$  for which  $\limsup_{n \rightarrow \infty} \left( \frac{\bar{\lambda}^n}{n!} |J_n(r, t)| \right)^{1/n} < 1$ .

Let us briefly mention a few special cases. For a Poisson process, since  $\xi_{n+1} \equiv 0$  for  $n \geq 1$ ,  $J_{\text{inhom}}(r, t) \equiv 1$ . Moreover, if  $Y$  is stationary, (8) reduces to (7).

### 3.3 Relationship to $K$ -functions

Spatio-temporal  $K$ -functions may be obtained as second order approximations of  $J_{\text{inhom}}(r, t)$ . To see this, recall that [12] defines SIRS (with isotropy) by replacing the second order stationarity of  $\Xi$  by the stronger condition that the pair correlation function  $g((x, t), (y, s)) = \bar{g}(u, v)$  depends only on the spatial distances  $u = \|x - y\|$  and the temporal distances  $v = |t - s|$ . They then introduce a spatio-temporal inhomogeneous  $K$ -function by setting

$$K_{\text{inhom}}(r, t) = \int_{S_r^t} \bar{g}(\|x_1\|, |t_1|) d(x_1, t_1) = \omega_d \int_{-t}^t \int_0^r \bar{g}(u, v) u^{d-1} du dv,$$

where  $\omega_d/d = \pi^{d/2}/\Gamma(1 + d/2) = \kappa_d$  is the volume of the unit ball in  $\mathbb{R}^d$  (see e.g. [7, p. 14]). Note that the second equality follows from a change to hyperspherical coordinates. If in addition  $Y$  is IRMS,

$$\begin{aligned} J_{\text{inhom}}(r, t) - 1 &= -\bar{\lambda} \left( \omega_d \int_{-t}^t \int_0^r \bar{g}(u, v) u^{d-1} du dv - \ell(S_r^t) \right) + \sum_{n=2}^{\infty} \frac{(-\bar{\lambda})^n}{n!} J_n(r, t) \\ &\approx -\bar{\lambda} (K_{\text{inhom}}(r, t) - \ell(S_r^t)), \end{aligned}$$

whereby  $K_{\text{inhom}}(r, t)$  may be viewed as a (scaled) second order approximation of  $J_{\text{inhom}}(r, t)$ . In relation hereto, it should be noted that even if the product densities exist only up to some finite order  $m$ , we may still obtain an approximation of  $J_{\text{inhom}}$  by truncating its series representation at  $n = m$ .

Returning now to the original definition of SIRS, where  $\bar{\lambda} > 0$  and  $\Xi$  is second order stationary, we may extend the definition of the inhomogeneous  $K$ -function in [3] to the spatio-temporal setting by defining

$$K_{\text{inhom}}^*(r, t) = \frac{1}{\ell(A)} \mathbb{E} \left[ \sum_{\substack{\neq \\ (x_1, t_1), (x_2, t_2) \in Y}} \frac{\mathbf{1}\{(x_1, t_1) \in A, \|x_1 - x_2\| \leq r, |t_1 - t_2| \leq t\}}{\lambda(x_1, t_1)\lambda(x_2, t_2)} \right] \quad (9)$$

for  $r, t \geq 0$  and some set  $A = B \times C \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$  with  $\ell(A) > 0$ . By Lemma 1 below, the definition does not depend on the choice of  $A$ .

**Lemma 1.** *For any  $A = B \times C \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$  for which  $\ell(A) > 0$ ,  $K_{\text{inhom}}^*(r, t) = \mathcal{K}_{\Xi}(S_r^t \setminus \{(0, 0)\})$ , the reduced second factorial moment measure of  $\Xi$  evaluated at  $S_r^t$  (see e.g. [11, Section 12.6]).*

*Proof.* By the Campbell formula, the intensity measure of  $\Xi$  is given by

$$\Lambda_{\Xi}(A) = \mathbb{E} \left[ \sum_{(x, t) \in Y} \frac{1}{\lambda(x, t)} \mathbf{1}_A(x, t) \right] = \ell(A),$$

so it is locally finite and has density 1. Hence, by [11, Proposition 13.1.IV.], there exist reduced Palm measures  $P_{\Xi}^{!y_1}(R)$ ,  $y_1 \in \mathbb{R}^d \times \mathbb{R}$ ,  $R \in \mathcal{N}$ , such that

$$\begin{aligned} K_{\text{inhom}}^*(r, t) &= \frac{1}{\ell(A)} \mathbb{E} \left[ \int_{\mathbb{R}^d \times \mathbb{R}} \mathbf{1}_A(y_1) \Xi((y_1 + S_r^t) \setminus \{y_1\}) \Xi(dy_1) \right] \\ &= \frac{1}{\ell(A)} \int_{\mathbb{R}^d \times \mathbb{R}} \mathbb{E}^{!y} [\mathbf{1}_A(y) \Xi(y + S_r^t)] dy = \mathbb{E}^{!(0,0)} [\Xi(S_r^t)] = \mathcal{K}_{\Xi}(S_r^t) \end{aligned}$$

and this completes the proof.  $\square$

It is not hard to see that under the stronger assumptions of [12],  $K_{\text{inhom}}^*(r, t) = K_{\text{inhom}}(r, t)$ .

## 4 Representation results

Being based on a series of integrals of  $n$ -point correlation functions, Definition 3 highlights the fact that  $J_{\text{inhom}}$  involves interactions of all orders but it is not very convenient in practice. The goal of this section is to give representations, which are easier to interpret.

### 4.1 Representation in terms of generating functionals

As for purely spatial point processes, we may express  $J_{\text{inhom}}$  in terms of the generating functionals  $G$  and  $G^!$  by making appropriate choices for the functions  $v = 1 - u$  [19]. Indeed, we may set

$$u_{r,t}^y(x, s) = \frac{\bar{\lambda} \mathbf{1}\{\|a - x\| \leq r, |b - s| \leq t\}}{\lambda(x, s)}, \quad y = (a, b) \in \mathbb{R}^d \times \mathbb{R},$$

and define the *inhomogeneous spatio-temporal nearest neighbour distance distribution function* as

$$\begin{aligned} G_{\text{inhom}}(r, t) &= 1 - G^{!y}(1 - u_{r,t}^y) \\ &= 1 - \mathbb{E}^{!(a,b)} \left[ \prod_{(x,s) \in Y} \left( 1 - \frac{\bar{\lambda} \mathbf{1}\{\|a - x\| \leq r, |b - s| \leq t\}}{\lambda(x, s)} \right) \right] \end{aligned} \quad (10)$$

and the *inhomogeneous spatio-temporal empty space function* as

$$F_{\text{inhom}}(r, t) = 1 - G(1 - u_{r,t}^y) = 1 - \mathbb{E} \left[ \prod_{(x,s) \in Y} \left( 1 - \frac{\bar{\lambda} \mathbf{1}\{\|a - x\| \leq r, |b - s| \leq t\}}{\lambda(x, s)} \right) \right]$$

for  $r, t \geq 0$ , under the convention that empty products take the value one. Then, the representation theorem below tells us that  $G_{\text{inhom}}(r, t)$  and  $F_{\text{inhom}}(r, t)$  do not depend on the choice of  $y$  and, furthermore, that  $J_{\text{inhom}}(r, t)$  may be expressed through  $G_{\text{inhom}}(r, t)$  and  $F_{\text{inhom}}(r, t)$ .

**Theorem 1.** *Let  $Y$  be an IRMS spatio-temporal point process and assume that*

$$\limsup_{n \rightarrow \infty} \left( \frac{\bar{\lambda}^n}{n!} \int_{S_r^t} \cdots \int_{S_r^t} \frac{\rho^{(n)}((x_1, t_1), \dots, (x_n, t_n))}{\lambda(x_1, t_1) \cdots \lambda(x_n, t_n)} \prod_{i=1}^n dx_i dt_i \right)^{1/n} < 1.$$

Then  $G_{\text{inhom}}(r, t)$  and  $F_{\text{inhom}}(r, t)$  are  $\ell$ -almost everywhere constant with respect to  $y = (a, b) \in \mathbb{R}^d \times \mathbb{R}$  and  $J_{\text{inhom}}(r, t)$  in expression (8) can be written as

$$J_{\text{inhom}}(r, t) = \frac{1 - G_{\text{inhom}}(r, t)}{1 - F_{\text{inhom}}(r, t)}$$

for all  $r, t \geq 0$  such that  $F_{\text{inhom}}(r, t) \neq 1$ .

*Proof.* From expression (6) it follows that

$$G(1 - u_{r,t}^y) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \int_{y+S_r^t} \cdots \int_{y+S_r^t} \frac{\rho^{(n)}((x_1, t_1), \dots, (x_n, t_n))}{\lambda(x_1, t_1) \cdots \lambda(x_n, t_n)} \prod_{i=1}^n dx_i dt_i,$$

since, by assumption, the series on the right hand side is absolutely convergent. Note that  $G(1 - u_{r,t}^y)$  is a constant for almost all  $(a, b) \in \mathbb{R}^d \times \mathbb{R}$  by the IRMS assumption. Furthermore, by an inclusion-exclusion argument,

$$G^{!y}(1 - u_{r,t}^y) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \mathbb{E}^{!(a,b)} \left[ \sum_{(x_1, t_1), \dots, (x_n, t_n) \in Y}^{\neq} \prod_{i=1}^n \frac{\mathbf{1}\{\|a - x_i\| \leq r, |b - t_i| \leq t\}}{\lambda(x_i, t_i)} \right],$$

which is well defined by the local finiteness of  $Y$  (the factor  $1/n!$  removes the implicit ordering of  $\sum^{\neq}$ ). To show the independence of the choice of  $(a, b)$ , for any bounded  $A = B \times C \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$  and any  $n \geq 1$ , consider now the non-negative measurable function

$$g_{r,t}^A((a, b), \varphi) = \frac{\mathbf{1}\{(a, b) \in A\}}{\lambda(a, b)} \sum_{(x_1, t_1), \dots, (x_n, t_n) \in \varphi}^{\neq} \prod_{i=1}^n \frac{\mathbf{1}\{\|a - x_i\| \leq r, |b - t_i| \leq t\}}{\lambda(x_i, t_i)}.$$

By rewriting the expression for  $g_{r,t}^A((a,b), Y \setminus \{(a,b)\})$ , recalling the Campbell formula (2) and taking the translation invariance of  $\xi_n$  into consideration, we obtain

$$\begin{aligned}
& \mathbb{E} \left[ \sum_{(a,b) \in Y} g_{r,t}^A((a,b), Y \setminus \{(a,b)\}) \right] = \\
&= \mathbb{E} \left[ \sum_{\substack{\neq \\ (a,b), (x_1, t_1), \dots, (x_n, t_n) \in Y}} \frac{\mathbf{1}\{(a,b) \in A\}}{\lambda(a,b)} \prod_{i=1}^n \frac{\mathbf{1}\{\|a - x_i\| \leq r, |b - t_i| \leq t\}}{\lambda(x_i, t_i)} \right] \\
&= \int_{B \times C} \left( \int_{S_r^t+y} \cdots \int_{S_r^t+y} \frac{\rho^{(n+1)}((a,b), y_1, \dots, y_n)}{\lambda(a,b) \lambda(y_1) \cdots \lambda(y_n)} dy_1 \cdots dy_n \right) dadb \\
&= \int_A \left( \int_{S_r^t} \cdots \int_{S_r^t} \frac{\rho^{(n+1)}((0,0), (x_1, t_1), \dots, (x_n, t_n))}{\lambda(0,0) \lambda(x_1, t_1) \cdots \lambda(x_n, t_n)} \prod_{i=1}^n dx_i dt_i \right) dadb.
\end{aligned}$$

On the other hand, by the reduced Campbell–Mecke formula (4),

$$\begin{aligned}
& \mathbb{E} \left[ \sum_{(a,b) \in Y} g_{r,t}^A((a,b), Y \setminus \{(a,b)\}) \right] = \\
&= \int_A \mathbb{E}^{!(a,b)} \left[ \sum_{\substack{\neq \\ (x_1, t_1), \dots, (x_n, t_n) \in Y}} \prod_{i=1}^n \frac{\mathbf{1}\{\|a - x_i\| \leq r, |b - t_i| \leq t\}}{\lambda(x_i, t_i)} \right] dx ds.
\end{aligned}$$

Hereby the two expressions above equal each other for all  $A$  and consequently the integrands are equal for  $\ell$ -almost all  $y = (a,b) \in \mathbb{R}^d \times \mathbb{R}$ . Hence, for almost all  $y = (a,b) \in \mathbb{R}^d \times \mathbb{R}$ ,

$$\begin{aligned}
& G^{!y}(1 - u_{r,t}^y) = 1 + \\
&+ \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \int_{S_r^t} \cdots \int_{S_r^t} \frac{\rho^{(n+1)}((0,0), (x_2, t_2), \dots, (x_{n+1}, t_{n+1}))}{\lambda(0,0) \lambda(x_2, t_2) \cdots \lambda(x_{n+1}, t_{n+1})} dx_2 dt_2 \cdots dx_{n+1} dt_{n+1} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \int_{S_r^t} \cdots \int_{S_r^t} \sum_{k=1}^{n+1} \sum_{D_1, \dots, D_k} \prod_{j=1}^k \xi_{|D_j|}(\{z_i : i \in D_j\}) dz_2 \cdots dz_{n+1},
\end{aligned}$$

where  $z_1 \equiv (0,0)$  and  $z_i = (x_i, t_i)$ ,  $i = 2, \dots, n+1$ . Recall that  $\sum_{D_1, \dots, D_k}$  is a sum over all possible  $k$ -sized partitions  $\{D_1, \dots, D_k\}$ ,  $\emptyset \neq D_j \in \mathcal{P}_{n+1}$ , where  $\mathcal{P}_{n+1}$  denotes the power set of  $\{1, \dots, n+1\}$ .

With the convention that  $\sum_{k=1}^0 = 1$ , we may split the above expression into terms based on whether the index sets  $D_j$  contain the index 1 (i.e. whether  $\xi_{|D_j|}$  includes  $z_1 \equiv (0,0)$ ) to obtain

$$\begin{aligned}
G^{!y}(1 - u_{r,t}^y) &= 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \sum_{D \in \mathcal{P}_n} \overbrace{\int_{S_r^t} \cdots \int_{S_r^t} \xi_{|D|+1}(0, z_1, \dots, z_{|D|}) dz_1 \cdots dz_{|D|}}^{=J_{|D|}(r,t)} \\
&\times \sum_{k=1}^{n-|D|} \sum_{\substack{D_1, \dots, D_k \neq \emptyset \text{ disjoint} \\ \cup_{j=1}^k D_j = \{1, \dots, n\} \setminus D}} \prod_{j=1}^k I_{|D_j|},
\end{aligned}$$

where  $I_n = \int_{S_r^t} \cdots \int_{S_r^t} \xi_n(z_2, \dots, z_{n+1}) dz_2 \cdots dz_{n+1}$ . The right hand side of the above expression may be written as

$$\left(1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} J_n(r, t)\right) \left(1 + \sum_{m=1}^{\infty} \frac{(-\bar{\lambda})^m}{m!} \sum_{\substack{k=1 \\ \cup_{j=1}^k D_j = \{1, \dots, m\}}} \sum_{\substack{D_1, \dots, D_k \neq \emptyset \\ \text{disjoint}}} \prod_{j=1}^k I_{|D_j|}\right),$$

which equals

$$J_{\text{inhom}}(r, t) \left(1 + \sum_{m=1}^{\infty} \frac{(-\bar{\lambda})^m}{m!} \int_{S_r^t} \cdots \int_{S_r^t} \frac{\rho^{(m)}((x_1, t_1), \dots, (x_m, t_m))}{\lambda(x_1, t_1) \cdots \lambda(x_m, t_m)} \prod_{i=1}^m dx_i dt_i, \right)$$

by Fubini's theorem and the definition of the  $n$ -point correlation functions. The absolute convergence of the individual sums in the above product imply the absolute convergence of  $G^{ly}(1 - u_{r,t}^y) = J_{\text{inhom}}(r, t)G(1 - u_{r,t}^0)$  and this in turn completes the proof.  $\square$

The intuition behind  $G_{\text{inhom}}(r, t)$  and  $F_{\text{inhom}}(r, t)$  is best seen when  $Y$  is stationary. In this case  $u_{r,t}^0(x, s) = \mathbf{1}\{(x, s) \in S_r^t\}$  and hence

$$F_{\text{inhom}}(r, t) = 1 - \mathbb{E} \left[ \prod_{(x,s) \in Y} \mathbf{1}\{(x, s) \notin S_r^t\} \right] = 1 - \mathbb{P}(Y \cap S_r^t = \emptyset) = F(r, t),$$

the empty space function in expression (7). Similarly,  $G_{\text{inhom}}(r, t)$  reduces to the distribution function of the nearest neighbour distance when  $Y$  is stationary, and the  $J$ -function is indeed a generalisation of (1).

## 4.2 Representation in terms of conditional intensities

Some families of point processes, notably Gibbsian ones [17], are defined in terms of their Papangelou conditional intensity  $\lambda(\cdot, \cdot; \cdot)$ . Below we show that for such processes,  $J_{\text{inhom}}$  may be represented in terms of  $\lambda(\cdot, \cdot; \cdot)$ .

**Theorem 2.** *Let the assumptions of Theorem 1 hold and assume, in addition, that  $Y$  admits a conditional intensity  $\lambda(\cdot, \cdot; \cdot)$ . Write  $W_{(a,b)}(Y) = \prod_{(x,s) \in Y} (1 - u_{r,t}^{(a,b)}(x, s))$ . Then  $\mathbb{E}[\lambda(a, b; Y)W_{(a,b)}(Y)/\lambda(a, b)] > 0$  implies  $E[W_{(a,b)}(Y)] > 0$  and*

$$J_{\text{inhom}}(r, t) = \mathbb{E} \left[ \frac{\lambda(a, b; Y)}{\lambda(a, b)} W_{(a,b)}(Y) \right] / \mathbb{E}[W_{(a,b)}(Y)]$$

for almost all  $(a, b) \in \mathbb{R}^d \times \mathbb{R}$ .

*Proof.* We already know that  $\mathbb{E}[W_{(a,b)}(Y)]$  is a constant for almost all  $(a, b) \in \mathbb{R}^d \times \mathbb{R}$ . Since  $0 \leq W_{(a,b)}(Y) \leq 1$ , if  $E[W_{(a,b)}(Y)] = 0$ , then  $W_{(a,b)}(Y) \stackrel{a.s.}{=} 0$  and this implies that

$\mathbb{E}\left[\frac{\lambda(a,b;Y)}{\lambda(a,b)}W_{(a,b)}(Y)\right] = 0$ . Note that by the Georgii–Nguyen–Zessin formula (5) in combination with (4), for any bounded  $A = B \times C \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ ,

$$\begin{aligned} & \int_A \mathbb{E}^{!(a,b)} \left[ \frac{1}{\lambda(a,b)} \prod_{(x,s) \in Y} \left( 1 - \frac{\bar{\lambda} \mathbf{1}\{\|a-x\| \leq r, |b-s| \leq t\}}{\lambda(x,s)} \right) \right] \lambda(a,b) dadb \\ &= \int_A \mathbb{E} \left[ \frac{\lambda(a,b;Y)}{\lambda(a,b)} \prod_{(x,s) \in Y} \left( 1 - \frac{\bar{\lambda} \mathbf{1}\{\|a-x\| \leq r, |b-s| \leq t\}}{\lambda(x,s)} \right) \right] dadb, \end{aligned}$$

whereby the integrands are equal for almost all  $(a,b) \in \mathbb{R}^d \times \mathbb{R}$  and the claim follows from Theorem 1.  $\square$

Since  $\mathbb{E}[\lambda(a,b;Y)] = \lambda(a,b)$ , we immediately see that

$$J_{\text{inhom}}(r,t) \geq 1 \iff \text{Cov}(\lambda(a,b;Y), W_{(a,b)}(Y)) \geq 0$$

and

$$J_{\text{inhom}}(r,t) \leq 1 \iff \text{Cov}(\lambda(a,b;Y), W_{(a,b)}(Y)) \leq 0.$$

In words, for clustered point processes,  $\lambda(a,b;Y)$  tends to be large if  $(a,b)$  is near to points of  $Y$  whereas  $W_{(a,b)}(Y)$  tends to be large when there are few points of  $Y$  close to  $(a,b)$ . Thus, in this case, the two random variables are negatively correlated and the  $J$ -function is smaller than one. A dual reasoning applies for regular point processes, but [5] warns against drawing too strong conclusions.

### 4.3 Scaling

In expression (8), we consider distances on the spaces  $\mathbb{R}^d$  and  $\mathbb{R}$  separately. Instead, we could have used the supremum distance on  $\mathbb{R}^d \times \mathbb{R}$  and the closed  $d$ -metric balls  $B[0,r] = S_r^r$ ,  $r \geq 0$  to define  $J_n(r) = J_n(r,r)$  and

$$J_{\text{inhom}}(r) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} J_n(r). \quad (11)$$

When the pair correlation function only depends on the spatial and temporal distances, set  $K_{\text{inhom}}^*(r) = K_{\text{inhom}}^*(r,r)$  and  $K_{\text{inhom}}(r) = K_{\text{inhom}}(r,r)$ , whence  $K_{\text{inhom}}^*(r) = K_{\text{inhom}}(r)$  and  $J_{\text{inhom}}(r) - 1 \approx -\bar{\lambda}(K_{\text{inhom}}(r) - \ell(B[0,r]))$ .

In the remainder of this subsection, we argue that (8) may be obtained from (11) by scaling. Let  $c = (c_S, c_T) \in (0, \infty)^2$  and apply the bijective transformation  $(y,s) \mapsto (c_S y, c_T s)$  to each point of the IRMS spatio-temporal point process  $Y$  to obtain

$$cY = \sum_{(y,s) \in Y} \delta_{(c_S y, c_T s)}.$$

Through a change of variables and the Campbell formula, one obtains

$$\rho_{cY}^{(n)}((x_1, t_1), \dots, (x_n, t_n)) = c_S^{-dn} c_T^{-n} \rho^{(n)}((x_1/c_S, t_1/c_T), \dots, (x_n/c_S, t_n/c_T)),$$

so that  $\lambda_{cY}(x, t) = c_S^{-d} c_T^{-1} \lambda(x/c_S, t/c_T)$  and  $\bar{\lambda}_{cY} = \inf_{(x,t)} \lambda_{cY}(x, t) = c_S^{-d} c_T^{-1} \bar{\lambda}$ . Hence,

$$\xi_n^{cY}((x_1, t_1), \dots, (x_n, t_n)) = \xi_n((x_1/c_S, t_1/c_T), \dots, (x_n/c_S, t_n/c_T))$$

whence  $cY$  is IRMS if and only if  $Y$  is and, whenever well-defined,

$$J_{\text{inhom}}^{cY}(r, t) = J_{\text{inhom}}\left(\frac{r}{c_S}, \frac{t}{c_T}\right). \quad (12)$$

In conclusion, by taking  $c_S = 1$ , and  $c_T = r/t$ , any  $J_{\text{inhom}}(r, t)$  may be obtained from  $J_{\text{inhom}}(r)$  through scaling.

## 5 Examples of spatio-temporal point processes

Below we will consider three families of models, each representing a different type of interaction.

### 5.1 Poisson processes

The inhomogeneous Poisson processes may be considered the benchmark scenario for lack of interaction between points. As we saw in Section 3.2, for a Poisson process  $J_{\text{inhom}}(r, t) \equiv 1$ . Alternative proofs may be obtained from the representation Theorems 1 and 2, by noting that the Palm distributions equal  $P$  by Slivnyak's theorem [30], or that the intensity function and the Papangelou conditional intensity coincide almost everywhere.

### 5.2 Location dependent thinning

Given a stationary STPP  $Y$  with product densities  $\rho^{(n)}$ ,  $n \geq 1$ , intensity  $\lambda > 0$  and  $J$ -function  $J(r, t)$ , consider some measurable function  $p : \mathbb{R}^d \times \mathbb{R} \rightarrow (0, 1]$  with  $\bar{p} = \inf_{(x,t)} p(x, t) > 0$ . Location dependent thinning of  $Y$  is the scenario in which a point  $(x, t) \in Y$  is retained with probability  $p(x, t)$ . Denote the resulting thinned process by  $Y_{\text{th}}$ .

The product densities of  $Y_{\text{th}}$  are

$$\rho_{\text{th}}^{(n)}((x_1, t_1), \dots, (x_n, t_n)) = \rho^{(n)}((x_1, t_1), \dots, (x_n, t_n)) \prod_{i=1}^n p(x_i, t_i)$$

by [11, Section 11.3], whereby  $\lambda_{\text{th}}(x, t) = \lambda p(x, t) > 0$  and the  $n$ -point correlation functions of  $Y_{\text{th}}$  and  $Y$  coincide. Hence,  $Y_{\text{th}}$  is IRMS with  $\bar{\lambda} = \inf_{(x,t)} \lambda_{\text{th}}(x, t) = \lambda \bar{p}$  and

$$J_{\text{inhom}}^{\text{th}}(r, t) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda \bar{p})^n}{n!} J_n(r, t)$$

for all  $r, t \geq 0$  for which the series converges. Here  $J_n(r, t)$  is the  $n$ -th coefficient in the series expansion (8) of the  $J$ -function of the original process  $Y$ .

A more informative expression for  $J_{\text{inhom}}^{\text{th}}$  can be obtained by noting that, by [7, Eq. (5.3)–(5.4)], the generating functional of  $Y_{\text{th}}$  is given by  $G_{\text{th}}(v) = G(1 - p + pv)$ , where

$G(\cdot)$  is the generating functional of  $Y$ . Hence, since applying thinning to the reduced Palm distribution of  $Y$  is equivalent to Palm conditioning in the thinned process,

$$J_{\text{inhom}}^{\text{th}}(r, t) = \frac{G_{\text{th}}^{!(0,0)}(1 - \bar{p}\mathbf{1}\{\cdot \in S_r^t\}/p)}{G_{\text{th}}(1 - \bar{p}\mathbf{1}\{\cdot \in S_r^t\}/p)} = \frac{G^{!(0,0)}(1 - \bar{p}\mathbf{1}\{\cdot \in S_r^t\})}{G(1 - \bar{p}\mathbf{1}\{\cdot \in S_r^t\})} = \frac{\mathbb{E}^{!(0,0)}[(1 - \bar{p})^{Y(S_r^t)}]}{\mathbb{E}[(1 - \bar{p})^{Y(S_r^t)}]}$$

when Theorem 1 applies.

When a Papangelou conditional intensity exists for  $Y$ , by recalling that  $\lambda = \lambda(x, t) = \mathbb{E}[\lambda(x, t; Y)]$  and applying the combination of (4) and (5) to the restriction of the function  $g(a, b, Y) = (1 - \bar{p})^{Y((a,b)+S_r^t)}$  to arbitrary bounded space-time domains, the previous expression becomes

$$J_{\text{inhom}}^{\text{th}}(r, t) = \frac{\mathbb{E}[\lambda(0, 0; Y)(1 - \bar{p})^{Y(S_r^t)}]}{\lambda \mathbb{E}[(1 - \bar{p})^{Y(S_r^t)}]}. \quad (13)$$

### 5.2.1 Thinned hard core process

The spatio-temporal hard core process is a stationary STPP defined through its Papangelou conditional intensity

$$\lambda_Y(a, b; Y) = \beta \mathbf{1}\{Y \cap ((a, b) + S_{R_S}^{R_T}) = \emptyset\} = \beta \prod_{(x,t) \in Y} \mathbf{1}\{(x, t) - (a, b) \notin S_{R_S}^{R_T}\}, \quad (14)$$

where  $(a, b) \in \mathbb{R}^d \times \mathbb{R}$ . Moreover,  $\beta > 0$  is a model parameter and  $R_S > 0$  and  $R_T > 0$  are, respectively, the spatial hard core distance and the temporal hard core distance. In words, since realisations a.s. do not contain points that violate the spatial and temporal hard core constraints, i.e.  $\mathbb{P}^{!(0,0)}(Y(S_{R_S}^{R_T}) > 0) = 0$ , there is inhibition.

By thinning  $Y$  with some suitable measurable retention function  $p : \mathbb{R}^d \times \mathbb{R} \rightarrow (0, 1]$ ,  $\bar{p} = \inf_{(x,t)} p(x, t) > 0$ , we obtain an IRMS hard core STPP  $Y_{\text{th}}$ .

**Lemma 2.** *For a hard core process  $Y$ ,  $\beta/\lambda \geq 1$ . If either  $(r, t) \in [0, R_S] \times [0, R_T]$  or  $(r, t) \in [R_S, \infty) \times [R_T, \infty)$ ,  $J(r, t)$  is increasing in  $r$  and  $t$ . Moreover, when  $(r, t) \in [0, R_S] \times [0, R_T]$  we have that  $1 \leq J(r, t) \leq \beta/\lambda$  and when  $(r, t) \in [R_S, \infty) \times [R_T, \infty)$ ,  $J(r, t) = \beta/\lambda$ . When  $R_T = R_S = R > 0$ , so that  $S_{R_S}^{R_T} = B[0, R]$ ,  $J(r) = J(r, r)$  is increasing and satisfies  $1 \leq J(r) < \beta/\lambda$  for  $r \in [0, R)$  and  $J(r) = \beta/\lambda$  for  $r \geq R$ .*

*For a thinned hard core process,  $J_{\text{inhom}}^{\text{th}}(r, t) \geq 1$  for  $r \leq R_S$  and  $t \leq R_T$ .*

*Proof.* Noting that  $\lambda = \lambda(0, 0) = \mathbb{E}[\lambda_Y(0, 0; Y)] = \beta \mathbb{P}(Y \cap S_{R_S}^{R_T} = \emptyset) \leq \beta$  we find that  $\beta/\lambda \geq 1$ . Furthermore, through Theorem 2 and expression (14) we obtain

$$J(r, t) = \frac{\mathbb{E}[\lambda_Y(0, 0; Y) \mathbf{1}\{Y \cap S_r^t = \emptyset\}]}{\lambda(0, 0) \mathbb{E}[\mathbf{1}\{Y \cap S_r^t = \emptyset\}]} = \frac{\beta \mathbb{P}(Y \cap S_{R_S}^{R_T} = \emptyset, Y \cap S_r^t = \emptyset)}{\lambda \mathbb{P}(Y \cap S_r^t = \emptyset)}.$$

Hence, when both  $r \geq R_S$  and  $t \geq R_T$  we have that  $S_{R_S}^{R_T} \subseteq S_r^t$  and consequently  $J(r, t) = \beta/\lambda$ . Moreover, when  $r \leq R_S$  and  $t \leq R_T$ , so that  $S_r^t \subseteq S_{R_S}^{R_T}$ , expression (7) gives us  $J(r, t) = 1/(1 - F(r, t))$ , which is increasing in both  $r \in [0, R_S]$  and  $t \in [0, R_T]$  and satisfies  $J(r, t) \geq 1$ . By setting  $r = R_S$  and  $t = R_T$  we confirm that  $J(r, t) = \beta/\lambda \geq 1$ .

Specialising to  $J(r) = J(r, r)$ , when  $r \leq R$  we have that  $J(r) = 1/(1 - F(r, r))$ , which is increasing to  $\beta/\lambda$ , and when  $r > R$ ,  $J(r) = \beta/\lambda$ .

When  $Y$  is thinned and  $r \leq R_S$  and  $t \leq R_T$ ,

$$J_{\text{inhom}}^{\text{th}}(r, t) = \frac{\mathbb{E}^{!(0,0)}[(1 - \bar{p})^{Y(S_r^t)}]}{\mathbb{E}[(1 - \bar{p})^{Y(S_r^t)}]} \geq \frac{\mathbb{E}^{!(0,0)}[(1 - \bar{p})^{Y(S_{R_S}^{R_T})}]}{\mathbb{E}[(1 - \bar{p})^{Y(S_r^t)}]} = \frac{1}{\mathbb{E}[(1 - \bar{p})^{Y(S_r^t)}]} \geq 1.$$

□

### 5.3 Log-Gaussian Cox processes

Our final example concerns spatio-temporal versions of log-Gaussian Cox processes (see e.g. [8, 23, 28]). In words, these models are spatio-temporal Poisson processes for which the intensity functions are given by realisations of log-Gaussian random fields [1, 2].

Recall that a Gaussian random field is completely determined by its mean function  $\mu(x, t)$  and its covariance function  $C((x, t), (y, s))$ ,  $(x, t), (y, s) \in \mathbb{R}^d \times \mathbb{R}$ , and that by Bochner's theorem  $C$  must be positive definite (see e.g. [14, Section 2.4]). Now, a spatio-temporal log-Gaussian Cox process  $Y$  has random intensity function given by

$$\exp\{\mu(x, t) + Z(x, t)\}, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R},$$

where  $Z = \{Z(x, t)\}_{(x,t) \in \mathbb{R}^d \times \mathbb{R}}$ , is a zero-mean spatio-temporal Gaussian random field. Note that the variance function of  $X$  is given by  $\sigma^2(x, t) = C((x, t), (x, t))$  and the correlation function by  $r((x, t), (y, s)) = C((x, t), (y, s))/(\sigma(x, t)\sigma(y, s))$ . By [10, Section 6.2] or [7, Section 5.2],

$$\frac{\rho^{(n)}((x_1, t_1), \dots, (x_n, t_n))}{\lambda(x_1, t_1) \cdots \lambda(x_n, t_n)} = \exp\left\{\sum_{i < j} C((x_i, t_i), (x_j, t_j))\right\}$$

and the intensity function of  $Y$  is

$$\lambda(x, t) = \exp\{\mu(x, t) + \sigma^2(x, t)/2\}. \quad (15)$$

Therefore, if  $\inf_{(x,t)} \exp\{\mu(x, t)\} > 0$  so that  $\lambda(x, t)$  is bounded away from zero, under the additional condition that  $C((x, t), (y, s)) = C(x - y, t - s)$ ,  $Y$  is IRMS. In this case,  $\sigma^2(x, t) = C(0, 0) = \sigma^2$  and  $Z$  is stationary. To exclude trivial cases, we shall assume that  $\sigma^2 > 0$ .

Before we proceed, note that we must impose conditions on  $r$  to ensure that the function  $\exp\{\mu(x, t) + Z(x, t)\}$  is integrable and defines a locally finite random measure. Further details are given in the Appendix. Henceforth, we will assume that  $\mu(x, t)$  is continuous and bounded with  $\bar{\mu} = \inf_{(x,t)} \mu(x, t) > -\infty$ , so that  $\bar{\lambda} = \exp\{\bar{\mu} + \sigma^2/2\}$ , and that  $r$  is such that  $Z$  a.s. has continuous sample paths. Combining [10, Proposition 6.2.II] with [7, (5.35)], we obtain, under the assumptions of Theorem 1,

$$J_{\text{inhom}}(r, t) = \frac{\mathbb{E}\left[e^{Z(0,0)} \exp\left\{-\int_{S_r^t} e^{\bar{\mu} + Z(x,s)} dx ds\right\}\right]}{\mathbb{E}[e^{Z(0,0)}] \mathbb{E}\left[\exp\left\{-\int_{S_r^t} e^{\bar{\mu} + Z(x,s)} dx ds\right\}\right]}$$

upon noting that the Palm distribution of the driving random measure of our log-Gaussian Cox process  $Y$  is  $\exp\{Z(x, t)\}$ -weighted. Note here that the Papangelou conditional intensity of  $Y$  exists and is given by  $\lambda(x, t; Y) = \mathbb{E}[\exp\{\mu(x, t) + Z(x, t)\} | Y]$  (see e.g. [25]).

**Lemma 3.** For a log-Gaussian Cox processes, when the above conditions are imposed on  $\mu$  and  $C$ ,  $J_{\text{inhom}}(r, t) \leq 1$  for all  $r, t \geq 0$ .

*Proof.* First, observe that  $J_{\text{inhom}}(r, t) \leq 1$  is equivalent to  $\text{Cov}(e^{Z(0,0)}, e^{-e^{\bar{\mu}} \int_{S_r^t} e^{Z(x,s)} dx ds}) \leq 0$ . Further, note that by the a.s. sample path continuity of  $Z$ ,

$$e^{-e^{\bar{\mu}} \int_{S_r^t} e^{Z(x,s)} dx ds} \stackrel{\text{a.s.}}{=} \lim_{n \rightarrow \infty} e^{-e^{\bar{\mu}} \sum_{(x_i, s_i) \in S(n)} c_{i,n} e^{Z(x_i, s_i)}},$$

where  $S(n) \subseteq S_r^t$ ,  $n \geq 1$ , are Riemann partitions. Since  $Z$  has positive correlation function, Pitt's theorem [27] tells us that  $Z$  is an associated family of random variables. Hereby  $\text{Cov}(e^{Z(0,0)}, \exp\{-e^{\bar{\mu}} \sum_{(x_i, s_i) \in S(n)} c_{i,n} e^{Z(x_i, s_i)}\}) \leq 0$  for any  $n \geq 1$  and the result follows from taking the limit in the last covariance and applying dominated convergence.  $\square$

## 6 Estimation

Assume that we observe an IRMS STPP  $Y$  within some compact spatio-temporal region  $W_S \times W_T \subseteq \mathbb{R}^d \times \mathbb{R}$  and obtain the realisation  $\{(x_i, t_i)\}_{i=1}^m$ ,  $m = Y(W_S \times W_T)$ . The goal of this section is to derive estimators for  $G_{\text{inhom}}(r, t)$ ,  $F_{\text{inhom}}(r, t)$  and  $J_{\text{inhom}}(r, t)$ . In order to deal with possible edge effects we will apply a minus sampling scheme [7, 9]. For clarity of exposition, we assume that the intensity function is known.

Denote the boundaries of  $W_S$  and  $W_T$  by  $\partial W_S$  and  $\partial W_T$ , respectively. Further, write  $W_S^{\ominus r} = \{x \in W_S : d_{\mathbb{R}^d}(x, \partial W_S) \geq r\} = \{x \in W_S : x + B_{\mathbb{R}^d}[0, r] \subseteq W_S\}$  for the eroded spatial domain and, similarly, let  $W_T^{\ominus t} = \{s \in W_T : d_{\mathbb{R}}(s, \partial W_T) \geq t\}$ . For given  $r, t \geq 0$ , we define an estimator of  $1 - G_{\text{inhom}}(r, t)$  by

$$\frac{1}{|Y \cap (W_S^{\ominus r} \times W_T^{\ominus t})|} \sum_{(x', s') \in Y \cap (W_S^{\ominus r} \times W_T^{\ominus t})} \left[ \prod_{(x, s) \in (Y \setminus \{(x', s')\}) \cap ((x', s') + S_r^t)} \left(1 - \frac{\bar{\lambda}}{\lambda(x, s)}\right) \right] \quad (16)$$

and, given a finite point grid  $L \subseteq W_S \times W_T$ , we estimate  $1 - F_{\text{inhom}}(r, t)$  by

$$\frac{1}{|L \cap (W_S^{\ominus r} \times W_T^{\ominus t})|} \sum_{l \in L \cap (W_S^{\ominus r} \times W_T^{\ominus t})} \left[ \prod_{(x, s) \in Y \cap (l + S_r^t)} \left(1 - \frac{\bar{\lambda}}{\lambda(x, s)}\right) \right]. \quad (17)$$

The ratio of (16) and (17) is an estimator of  $J_{\text{inhom}}(r, t)$ , cf. Theorem 1.

**Theorem 3.** Under the conditions of Theorem 1, the estimator (17) is unbiased and (16) is ratio-unbiased.

*Proof.* We start with (16) and note that  $\mathbb{E}[Y(W_S^{\ominus r} \times W_T^{\ominus t})] = \Lambda(W_S^{\ominus r} \times W_T^{\ominus t})$ . By the reduced Campbell-Mecke formula (4),

$$\begin{aligned} & \mathbb{E} \left[ \sum_{(x', s') \in Y \cap (W_S^{\ominus r} \times W_T^{\ominus t})} \prod_{(x, s) \in (Y \setminus \{(x', s')\}) \cap ((x', s') + S_r^t)} \left(1 - \frac{\bar{\lambda}}{\lambda(x, s)}\right) \right] = \\ & = \int_{W_S^{\ominus r} \times W_T^{\ominus t}} \mathbb{E}^{!(x', s')} \left[ \prod_{(x, s) \in Y} \left(1 - \frac{\bar{\lambda}}{\lambda(x, s)} \mathbf{1}\{(x - x', s - s') \in S_r^t\}\right) \right] \lambda(x', s') dx' ds'. \end{aligned}$$

By (10), the expectation is equal to  $G^{l_0}(1 - u_{r,t}^0)$ , from which the claimed ratio-unbiasedness follows.

Turning to (17), unbiasedness follows from the assumed translation invariance of the  $\xi_{ns}$  and equation (6) under the conditions of Theorem 1.  $\square$

In practice, the intensity function  $\lambda(x, s)$  is not known. Therefore an estimator  $\widehat{\lambda}(x, s)$  will have to be obtained and then used as a plug-in in the above estimators. E.g. [12] considers kernel estimators for  $\lambda(x, s)$  but stresses, however, that care has to be taken when  $\widehat{\lambda}(x, s)$  is close to 0, since a change of bandwidth may cause  $\widehat{\lambda}(x, s) = 0$  for some  $(x, s)$ , which would be in violation of the assumption that  $\bar{\lambda} > 0$ .

## 7 Numerical evaluations

In this section, we use the inhomogeneous  $J$ -function to quantify the interactions in a realisation of each of the three models discussed in Section 5. In order to do so, we work mostly in R and exploit functions in the package `spatstat` [4], in which versions of all summary statistics discussed in this paper have already been implemented for purely spatial point processes, both in the general and the stationary case; the spatio-temporal  $K$ -function has been implemented in `stpp` [12]. To simulate log Gaussian Cox processes we use the package `RandomFields` [29]. Realisations of spatio-temporal hard core processes can be obtained using the C++ library `MPPLIB` [31].

Throughout this section, realisations will be restricted to the observation window  $W_S \times W_T = [0, 1]^2 \times [0, 1]$ . The intensity function is either known, or known up to a constant (for the thinned hard core process). Hence, since (16)–(17) are defined in terms of the ratio  $\bar{\lambda}/\lambda(x, s)$ , there is no need to plug in intensity function estimators.

### 7.1 Poisson processes

Consider a Poisson process  $Y$  on  $\mathbb{R}^2 \times \mathbb{R}$  with intensity function  $\lambda(x, y, t) = 750 e^{-1.5(y+t)}$  as in Section 5.1. Note that  $\bar{\lambda} = 750 e^{-3} \approx 37.34$  and that the expected number of observed points of  $Y$  in  $W_S \times W_T$  is  $750(1 - e^{-1.5})^2/1.5^2$ , i.e. approximately 200. A realisation with 220 points is shown in the top-left panel of Figure 1. The temporal progress of the process is illustrated in the top-right panel of Figure 1, which shows the cumulative number of points as a function of time, i.e.  $N(t) = Y(W_S \times [0, t])$ ,  $t \in [0, 1]$ . The lower row of Figure 1 shows two spatial projections. In the left panel, we display  $Y \cap (W_S \times [0, 0.5])$ , in the right panel  $Y \cap (W_S \times [0.5, 1])$ . Here the decline in intensity, with increasing  $y$ -coordinate, is clearly visible. Furthermore, a comparison of the two spatial projections illustrates the exponential decay in the intensity function.

In Figure 2, on the left, we show a collection of 2-d plots of the estimates of  $G_{\text{inhom}}(r, t_0)$  and  $F_{\text{inhom}}(r, t_0)$  for a fixed set of values  $t = t_0$ . Similarly, in the rightmost plot, we display a collection of 2-d plots of the estimates for a fixed set of values  $r = r_0$ . In both cases the dotted lines (-◆-) represent the estimated empty space functions. We see that throughout the two estimates are approximately equal and, in addition, there are instances where each of the two is larger than the other.

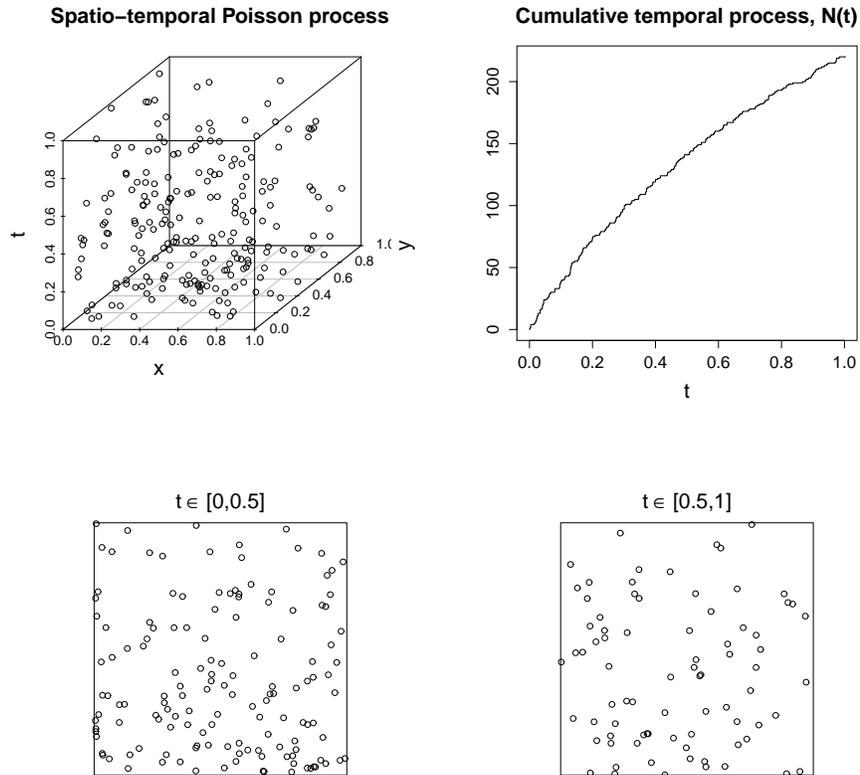


Figure 1: A realisation on  $W_S \times W_T = [0, 1]^2 \times [0, 1]$  of a Poisson process with intensity function  $\lambda((x, y), t) = 750 e^{-1.5(y+t)}$ ,  $x, y, t \in \mathbb{R}$ . Upper row: A 3-d plot (left) and a plot of the associated cumulative count process (right). Lower row: Spatial projections for the time intervals  $[0, 0.5]$  (left) and  $[0.5, 1]$  (right).

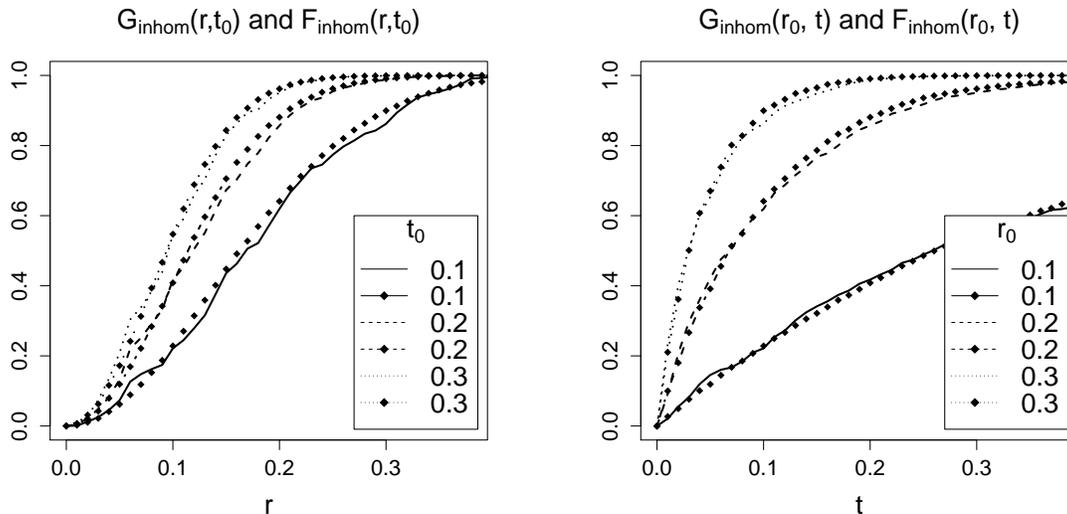


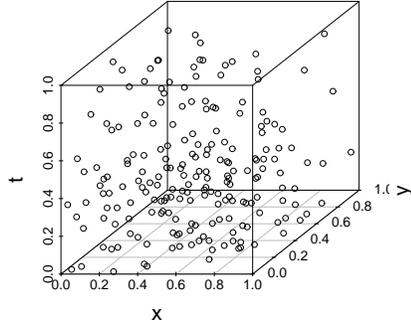
Figure 2: Plots of the estimated nearest neighbour distance distribution function and empty space function of the Poisson process sample in Figure 1. Left: As a function of spatial distance for fixed temporal distances  $t_0$ . Right: As a function of temporal distance for fixed spatial distances  $r_0$ . In both plots, the dotted lines (-◆-) represent the empty space function.

## 7.2 Thinned hard core process

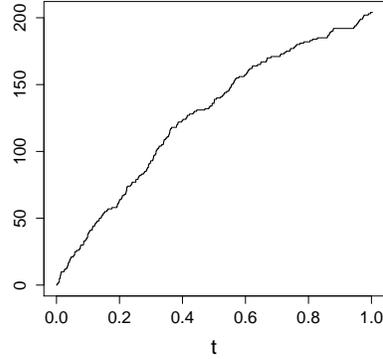
Let  $Y$  be the location dependent thinning of a stationary spatio-temporal hard core process as described in Section 5.2 with retention probability  $p(x, y, t) = e^{-1.5(y+t)}$ ,  $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}$ ,  $\beta = 1300$ ,  $R_S = 0.05$  and  $R_T = 0.05$ . The associated realisations are shown in the top panels of Figure 3. The underlying hard core process has 762 points, whereby  $\hat{\lambda} = 762$ , and the thinned process has 204 points. Note that the expected number of observed points of  $Y$  in  $W_S \times W_T$  is  $\lambda \int_{[0,1]^3} p(x, y, t) d(x, y, t) \approx \hat{\lambda}(1 - e^{-1.5})^2/1.5^2$ , i.e. approximately 200. The temporal progress of the process is illustrated in the top-right panel of Figure 3, which shows the cumulative number of points as a function of time, i.e.  $N(t) = Y(W_S \times [0, t])$ ,  $t \in [0, 1]$ . The lower row of Figure 3 shows two spatial projections. In the lower left panel we display  $Y \cap (W_S \times [0, 0.5])$  and in the lower right panel  $Y \cap (W_S \times [0.5, 1])$ . Just as in the Poisson case, the two spatial projections illustrate the decay in the intensity function, both in the  $y$ - and  $t$ -dimensions.

In order to obtain a picture of the interaction structure, in Figure 4 we have plotted the estimates of  $G_{\text{inhom}}(r, t)$  and  $F_{\text{inhom}}(r, t)$  for large  $r, t$  ranges. Again, the dotted lines (-◆-) represent the estimated empty space functions. As expected we see that the estimate of  $G_{\text{inhom}}(r, t)$  is the smaller of the two. For small values of  $r_0$  and  $t_0$ , the hardcore distances  $R_S$  and  $R_T$  are clearly identified in the lower row of Figure 4. For large values of  $r_0$  and  $t_0$  this is no longer the case due to accumulation points; see the top row of Figure 4. For instance, there are many points violating the spatial hard core constraint, which still respect the temporal hard core constraint.

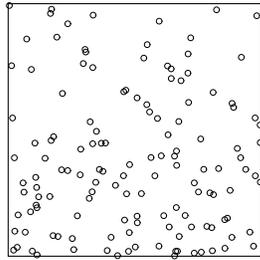
Thinned Spatio-temporal Strauss process



Cumulative temporal process, N(t)



t ∈ [0,0.5]



t ∈ [0.5,1]

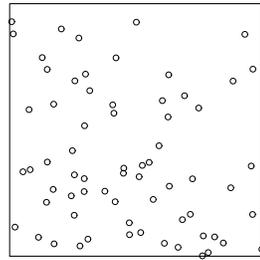


Figure 3: A realisation on  $W_S \times W_T = [0, 1]^2 \times [0, 1]$  of a location dependent thinning, with retention probability  $p(x, y, t) = e^{-1.5(y+t)}$ ,  $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}$ , of a spatio-temporal hard core process with  $\beta = 1300$ ,  $R_S = 0.05$  and  $R_T = 0.05$ . Upper row: A 3-d plot (left) and a plot of the associated cumulative count process (right). Lower row: Spatial projections for the time intervals  $[0, 0.5]$  (left) and  $[0.5, 1]$  (right).

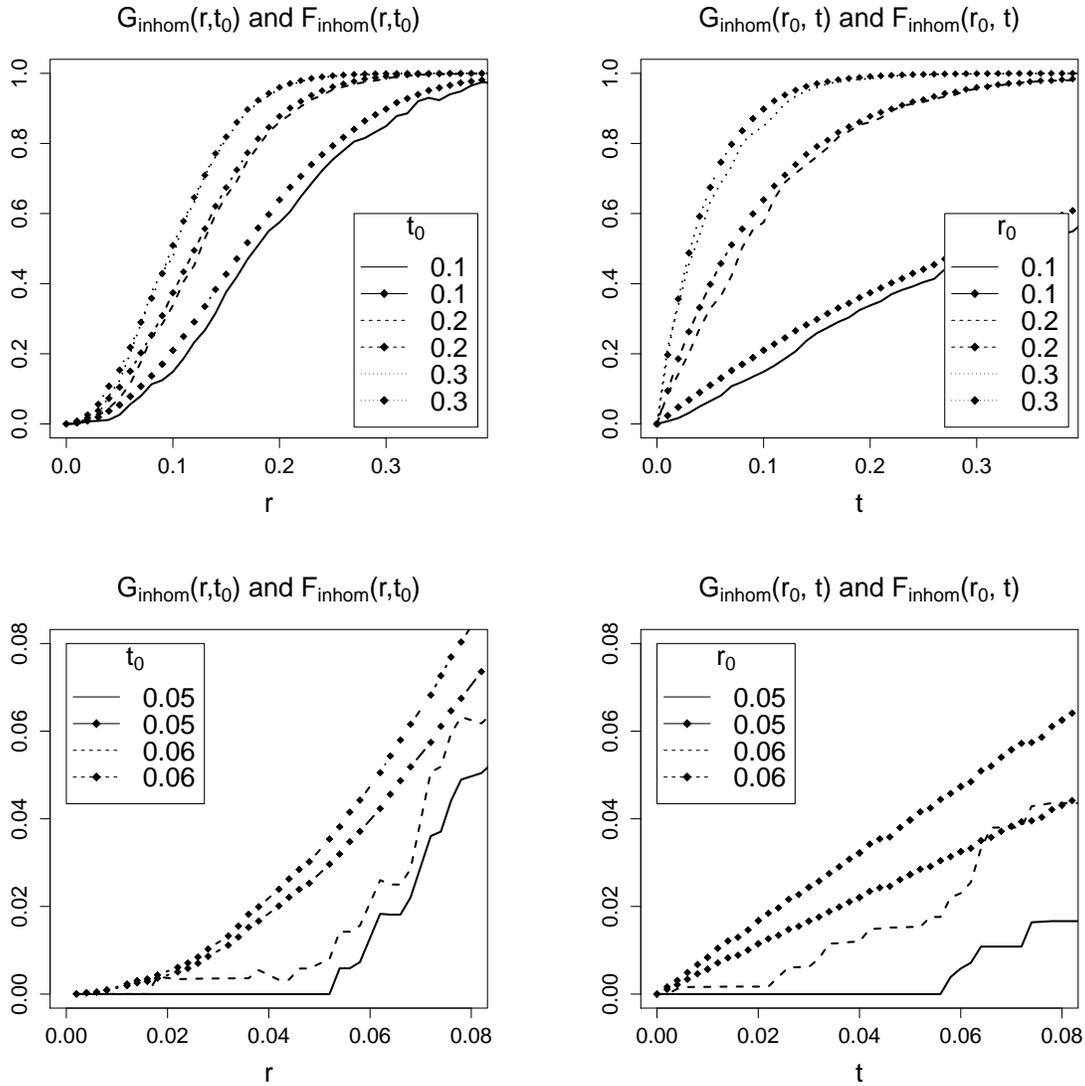


Figure 4: Plots of the estimated nearest neighbour distance distribution function and the empty space function of the thinned hard core process in Figure 3. Left: As a function of spatial distance for fixed temporal distances  $t_0$ . Right: As a function of temporal distance for fixed spatial distances  $r_0$ . In both cases the dotted lines (-◆-) represent the empty space function estimates.

### 7.3 Log-Gaussian Cox process

Recall the log-Gaussian Cox processes discussed in Section 5.3. We consider the separable covariance function (see Appendix)

$$C((x_1, y_1, t_1), (x_2, y_2, t_2)) = C_S((x_1, y_1) - (x_2, y_2))C_T(t_1 - t_2)$$

for the driving Gaussian random field of the STPP  $Y$ . Specifically, we let the component covariance functions be  $C_S(x, y) = \sigma_S^2 \exp\{-\|(x, y)\|^2\}$  (Gaussian) and  $C_T(t) = \sigma_T^2 \exp\{-|t|\}$  (exponential),  $x, y, t \in \mathbb{R}$ , where  $\sigma_S^2 = \sigma_T^2 = 1/4$ , so that  $\sigma^2 = C_S(0)C_T(0) = 1/16$  and we let the mean function be given by  $\mu(x, y, t) = \log(750) - 1.5(y + t) - \sigma^2/2$ . Figure 5 shows projections of a realisation of the driving random intensity function at time  $t = 0.5$  (left) and spatial coordinate  $x = 0.5$  (right). Note the gradient in the vertical direction in the left plot and the diagonal trend in the right one.

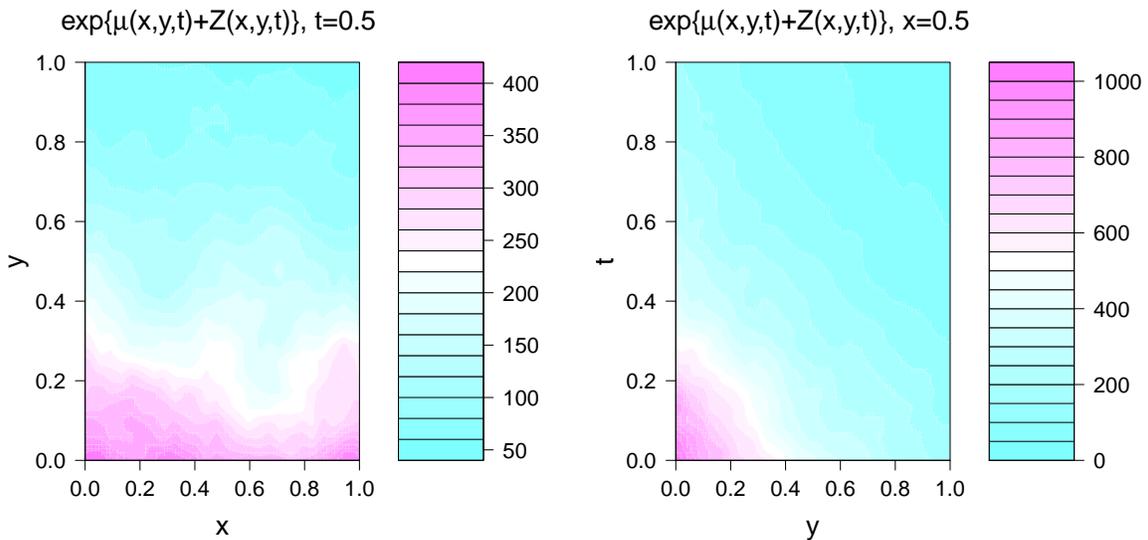
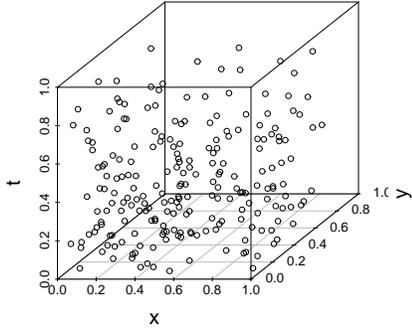


Figure 5: Projections of a realisation of the driving random intensity field at time  $t = 0.5$  (left) and at spatial coordinate  $x = 0.5$  (right).

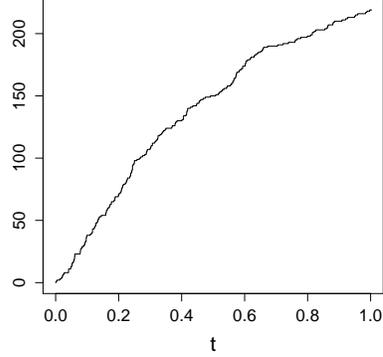
By expression (15) we obtain  $\lambda(x, y, t) = 750 e^{-1.5(y+t)}$  and consequently  $\bar{\lambda} = 750 e^{-3} \approx 37.34$ . Hereby the expected number of observed points of  $Y$  in  $W_S \times W_T$  is  $750(1 - e^{-1.5})^2 / 1.5^2$ , i.e. approximately 200. A realisation of  $Y$  with 219 points is shown in the top-left panel of Figure 6 and the cumulative number of points as a function of time, i.e.  $N(t) = Y(W_S \times [0, t])$ ,  $t \in [0, 1]$ , is illustrated in the top-right panel. The lower row of Figure 6 shows two spatial projections. In the left panel, we display  $Y \cap (W_S \times [0, 0.5])$  and in the right panel  $Y \cap (W_S \times [0.5, 1])$ . Also here the decay in the intensity function in the  $y$ - and  $t$ -dimensions is visible. In addition, by comparing the lower rows of Figures 1 and 6, the present clustering effects become evident.

In Figure 7 we have plotted the estimates of  $G_{\text{inhom}}(r, t)$  and  $F_{\text{inhom}}(r, t)$ . As before, the dotted lines ( $-\blacklozenge-$ ) represent the estimates of  $F_{\text{inhom}}(r, t)$ . Due to the structure of  $C$ , when

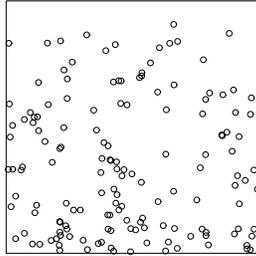
**Spatio-temporal Log-Gaussian Cox process**



**Cumulative temporal process, N(t)**



**t ∈ [0,0.5]**



**t ∈ [0.5,1]**

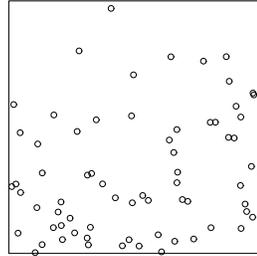


Figure 6: A realisation on  $W_S \times W_T = [0, 1]^2 \times [0, 1]$  of a log-Gaussian Cox process with  $\mu(x, y, t) = \log(750) - 1.5(y + t) - \frac{(1/4)^2}{2}$  and separable covariance function  $C_S(x, y)C_T(t) = \frac{1}{4} e^{-\|(x,y)\|^2} \frac{1}{4} e^{-|t|}$ ,  $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}$ . Upper row: A 3-d plot (left) and a plot of the associated cumulative count process (right). Lower row: Spatial projections for the time intervals  $[0, 0.5]$  (left) and  $[0.5, 1]$  (right).

e.g.  $t$  is small we find clear signs of clustering whereas for larger  $t$ , where  $C(x, y, t) \approx 0$ , as expected we have Poisson like behaviour.

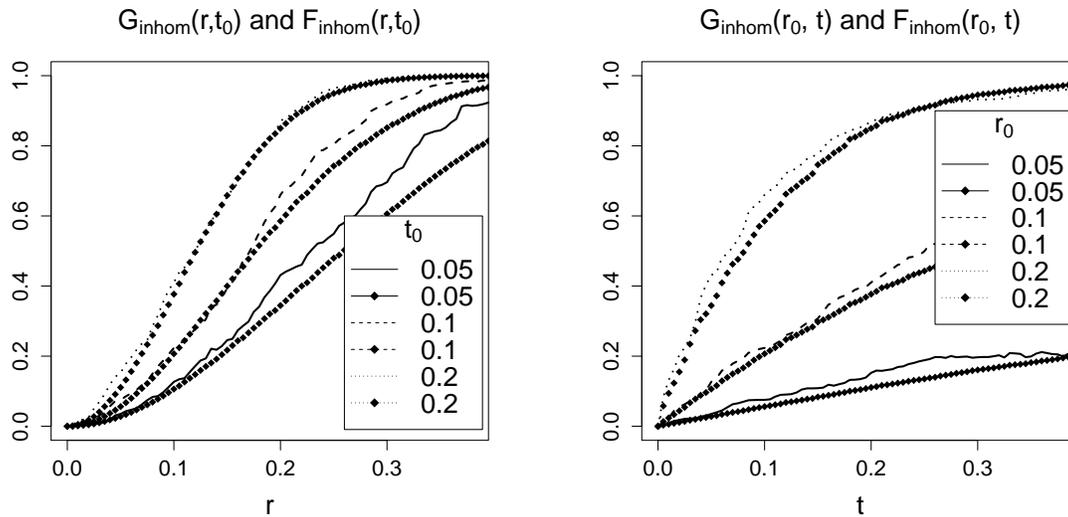


Figure 7: Plots of the estimated nearest neighbour distance distribution function and empty space function of the log-Gaussian Cox process in Figure 6. Left: As a function of spatial distance for fixed temporal distances  $t_0$ . Right: As a function of temporal distance for fixed spatial distances  $r_0$ . In both plots, the dotted lines (- $\blacklozenge$ -) represent the empty space function.

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# Appendix

## Sample path continuity of Gaussian random fields

Let  $Z$  be a stationary Gaussian random field with mean zero. We wish to impose conditions, which ensure that  $Z$  a.s. has continuous sample paths. If  $Z$  would be defined on the Euclidean space  $(\mathbb{R}^d \times \mathbb{R}, \|\cdot\|_{\mathbb{R}^{d+1}}, d_{\mathbb{R}^{d+1}}(\cdot, \cdot))$ , with  $C(x, y) = \sigma^2 r(x - y)$ ,  $\sigma^2 > 0$ , then [24, Section 5.6.1] lists sufficient conditions on the correlation function  $r(\cdot)$  as follows. There exist  $\epsilon, \delta > 0$  such that either,

1.  $1 - r(x, t) < \delta / (-\log(\|(x, t)\|_{\mathbb{R}^{d+1}}))^{1+\epsilon}$ , or
2.  $1 - r(x, t) < \delta \|(x, t)\|_{\mathbb{R}^{d+1}}^\epsilon$ ,

for all lag pairs  $(x, t) \in \mathbb{R}^d \times \mathbb{R}$  in an open Euclidean ball centred at 0. Note that the former condition, which in fact is the condition given in [1, Theorem 3.4.1], is less restrictive than the latter one but often harder to check. However, the underlying space here is  $(\mathbb{R}^d \times \mathbb{R}, \|\cdot\|_\infty, d(\cdot, \cdot))$ . Hence, one explicit way of obtaining equivalent conditions for  $r(\cdot)$  would be to consider the log-entropy related results of [2, Section 1] for Gaussian random fields on general compact spaces and exploit that  $\mathbb{R}^d \times \mathbb{R}$  is  $\sigma$ -compact. A more direct and natural approach is to note that, through the topological equivalence of  $d_{\mathbb{R}^{d+1}}(\cdot, \cdot)$  and  $d(\cdot, \cdot)$ , we have the necessary condition that, for any  $(x, t) \in \mathbb{R}^d \times \mathbb{R}$ , there exist constants  $\alpha_1, \alpha_2 > 0$  such that  $\alpha_1 d((x, t), (y, s)) \leq d_{\mathbb{R}^{d+1}}((x, t), (y, s)) \leq \alpha_2 d((x, t), (y, s))$  for all  $(y, s) \in \mathbb{R}^d \times \mathbb{R}$ . Hereby, in particular, there are  $\alpha_1, \alpha_2 > 0$  such that  $\alpha_1 \|(x, t)\|_\infty \leq \|(x, t)\|_{\mathbb{R}^{d+1}} \leq \alpha_2 \|(x, t)\|_\infty$  for all  $(x, t) \in \mathbb{R}^d \times \mathbb{R}$  and we see that the conditions above are retained in  $(\mathbb{R}^d \times \mathbb{R}, \|\cdot\|_\infty, d(\cdot, \cdot))$ , with adjusted constants  $\delta, \epsilon > 0$ . Note that the a.s. sample path continuity implies a.s. sample path boundedness on compact sets [2, Section 1].

## Covariance models

One particular family of correlation functions  $r(\cdot)$  for which the a.s. continuity conditions above are satisfied is the *power exponential family* (see [24, Section 5.6.1]),

$$r(x, t) = \exp(-\|(x, t)\|_\infty^\delta), \quad 0 \leq \delta \leq 2, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}.$$

The special case  $\delta = 1$  generates the exponential correlation,  $\delta = 2$  gives rise to the Gaussian correlation function. Note that the isotropy of  $r(\cdot)$  implies isotropy of the LGCP  $Y$  since its distribution is completely specified by  $C(\cdot)$ .

A common practical assumption when modelling spatio-temporal Gaussian random fields is to assume separability (see e.g. [14, Chapter 23]). Consider the covariance functions  $C_S(x, y) = \sigma_S^2 r_S(x - y)$  and  $C_T(t, s) = \sigma_T^2 r_T(t - s)$ ,  $x, y \in \mathbb{R}^d$ ,  $t, s \in \mathbb{R}$ , where  $\sigma_S^2, \sigma_T^2 > 0$ . We may now consider two types of separability:

1. Multiplicative separability:

$$C((x, t), (y, s)) = C_S(x, y)C_T(t, s) = \sigma_S^2 \sigma_T^2 r_S(x - y)r_T(t - s).$$

2. Additive separability:

$$C((x, t), (y, s)) = C_S(x, y) + C_T(t, s) = \sigma_S^2 r_S(x - y) + \sigma_T^2 r_T(t - s).$$

The latter is a consequence of assuming that  $Z(x, t) = Z_S(x) + Z_T(t)$ ,  $(x, t) \in \mathbb{R}^d \times \mathbb{R}$ , where  $Z_S(x)$  and  $Z_T(t)$  are independent mean zero Gaussian random fields with covariance functions  $C_S$  and  $C_T$ , respectively. In both cases a separable power exponential model can be obtained by letting  $r_S(x) = \exp(-\|x\|^{\delta_S})$  and  $r_T(t) = \exp(-|t|^{\delta_T})$  for  $\delta_S, \delta_T \in [0, 2]$ .