ON R. VON MISES' CONDITION FOR THE DOMAIN OF
ATTRACTION OF $\exp(-e^{-x})$\(^1\)

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There exist well-known necessary and sufficient conditions for a
distribution function to belong to the domain of attraction of the double
exponential distribution $\Lambda$. For practical purposes a simple sufficient con-
dition due to von Mises is very useful. It is shown that each distribution
function $F$ in the domain of attraction of $\Lambda$ is tail equivalent to some
distribution function satisfying von Mises' condition.

Suppose $X_1, X_2, X_3, \ldots$ are independent real-valued random variables with
common distribution function $F$. We say that $F$ is in the domain of attraction
of the double exponential distribution (notation $F \in D(\Lambda)$; $\Lambda(x) = \exp(-e^{-x})$)
if there exist two sequences of real constants $[b_n]$ and $[a_n]$ (with $a_n > 0$ for
$n = 1, 2, \ldots$) such that for all real $x$

$$\lim_{n \to \infty} P \left\{ \frac{\max(X_1, X_2, \ldots, X_n) - b_n}{a_n} \leq x \right\} = \exp(-e^{-x}).$$

It is convenient to use the symbol $x_v$ for the upper bound of $X_1$ defined by

$$x_v(F) = \sup \{ x \mid F(x) < 1 \}.$$

Necessary and sufficient conditions for $F \in D(\Lambda)$ are well-known (Gnedenko
(1943), de Haan (1970)) but rather intricate. The following relatively simple
criterion is due to R. von Mises ((1936) page 285):

Suppose $F(x)$ is a distribution function with a density $f(x)$ which is positive
and differentiable on a left neighborhood of $x_v$. If

$$\lim_{x \to x_v} \frac{d}{dx} \left( \frac{1 - F(x)}{f(x)} \right) = 0,$$

then $F \in D(\Lambda)$.

A distribution function $F$ satisfying (2) will be called a von Mises function.

We shall prove

**Theorem.** A distribution function $F$ lies in the domain of attraction of $\Lambda$ if and
only if there exists a von Mises function $F_*$ such that $x_v(F_*) = x_v(F) = x_v$ and

$$\lim_{x \to x_v} \frac{1 - F(x)}{1 - F_*(x)} = 1.$$

**Remark.** Relation (3) implies (see Resnick (1971) Lemma 2.5) that for the

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The convergence of the distribution functions $F^n$ and $F_n^*$ the same norming constants $a_n > 0$ and $b_n$ may be used.

**Proof.** The if statement is an immediate consequence of Gnedenko (1943), Théorème 7.

Now suppose $F \in D(\Lambda)$ with endpoint $x_0$. We define the sequence $U_0, U_1, \ldots$ by

\[
U_0(x) = 1 - F(x) \\
U_{n+1}(x) = \int_{x_0}^x U_n(t) \, dt \quad n = 0, 1, 2, \ldots
\]

By de Haan ((1970) Lemma 2.5.1 or (1971) Lemma 6 and Theorem 8) the distribution function $F_n$ defined by $F_n(x) = \max(0, 1 - U_n(x))$ belongs to $D(\Lambda)$ if $F_{n-1}$ does. In particular, the integral above converges. Then $F_n \in D(\Lambda)$ for $n = 0, 1, 2, \ldots$ and by de Haan ((1970) Theorem 2.5.2 or (1971) Theorem 10) we have

\[
\lim_{x \uparrow x_0} \frac{U_{n-1}(x)U_{n+1}(x)}{U_n(x)} = 1 \quad n = 1, 2, \ldots
\]

We now define the function $U_*$ on $(-\infty, x_0)$ by

\[
U_*(x) = \left\{ U_3(x) \right\}^4 \left\{ U_4(x) \right\}^{-1}.
\]

Then $U_*(x)$ is twice differentiable on a left neighborhood of $x_0$ and

\[
\frac{d}{dx} \log U_* = -4 \frac{U_2}{U_3} + 3 \frac{U_3}{U_4} = \frac{3 - 4U_2U_3^{-2}U_4}{U_4U_5^{-1}}.
\]

Consider

\[
\frac{U_1U_3^{-1}}{3 - 4U_2U_3^{-2}U_4} = \frac{U_*}{U_*}. \frac{d}{dx} U_*.
\]

By (4) the denominator is asymptotic to $-1$ as $x \uparrow x_0$ and both $(d/dx)U_*U_3^{-1}$ and $U_*U_3^{-1}(d/dx)(3 - 4U_2U_3^{-2}U_4)$ vanish as $x \uparrow x_0$. Hence

\[
\lim_{x \uparrow x_0} \frac{d}{dx} \left( \frac{U_*(x)}{U_*(x)} \right) = 0.
\]

Observe that

\[
U_0 = \frac{U_0U_2}{U_1^2} \cdot \left( \frac{U_1U_3}{U_2^2} \right)^2 \cdot \left( \frac{U_2U_4}{U_3^2} \right)^2 \cdot U_*.
\]

Hence by (4) we obtain

\[
\lim_{x \uparrow x_0} \frac{U_0(x)}{U_0(x)} = 1.
\]

Then $\lim_{x \uparrow x_0} U_*(x) = 0$, and since by (5) $U_*$ is decreasing on a left neighborhood of $x_0$, there exists a twice differentiable distribution function $F_*(x)$ which coincides with $1 - U_*(x)$ on a left neighborhood of $x_0$. $F_*$ is a von Mises function by (6) and satisfies (2) by (7). 

**Corollary.** A distribution function $F$ belongs to $D(\Lambda)$ if and only if there exist
a positive function \( c \) satisfying \( \lim_{x \to 0^+} c(x) = 1 \) and a positive differentiable function \( \phi \) satisfying \( \lim_{x \to 0^+} \phi'(x) = 0 \) such that

\[
1 - F(x) = c(x) \cdot \exp \left\{ - \int_{-\infty}^{x} \frac{dt}{\phi(t)} \right\} \quad \text{for } x < x_0.
\]

This improves the representation theorem (Theorem 2.5.3) in de Haan (1970).

**Proof.** Set

\[
\phi(x) = \frac{1 - F_s(x)}{F_s'(x)}
\]

in a left neighborhood of \( x_0 \).

**References**


