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ROBUST PRICING OF EUROPEAN OPTIONS WITH WAVELETS AND THE CHARACTERISTIC FUNCTION*

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Abstract. We present a novel method for pricing European options based on the wavelet approximation method and the characteristic function. We focus on the discounted expected payoff pricing formula and compute it by means of wavelets. We approximate the density function associated to the underlying asset price process by a finite combination of jth order B-splines, and recover the coefficients of the approximation from the characteristic function. Two variants for wavelet approximation will be presented, where the second variant adaptively determines the range of integration. The compact support of a B-splines basis enables us to price options in a robust way, even in cases where Fourier-based pricing methods may show weaknesses. The method appears to be particularly robust for pricing long-maturity options, fat-tailed distributions, as well as staircase-like density functions encountered in portfolio loss computations.

Key words. option pricing, European options, Haar wavelets, B-spline wavelets

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1. Introduction.

The robust and efficient valuation of financial products, like options, by numerical techniques is an interesting recent field in applied mathematics and scientific computing. The best known option pricing partial differential equation (PDE) is without any doubt the Black–Scholes equation, pricing a European (plain vanilla) option under geometric Brownian motion (GBM) asset price dynamics. When considering more realistic asset dynamics, like stochastic volatility or Lévy jump asset models, other option pricing PDEs, or even partial integro-differential equations, will be encountered.

Option pricing is often done by the discounted expected payoff approach, which is nothing but a Green's function integral formulation for the PDEs mentioned above. The connection between the solution of the option pricing PDEs and the discounted expected payoff lies in the Feynman–Kac theorem. In many cases in option pricing, we do not have the Green's function (read, the conditional probability density function for the asset prices) available, but we do have its Fourier transform. This Fourier transform is the characteristic function, and the theory in [Duf00] shows that for affine PDEs the characteristic function can easily be determined. This derivation technique is related to classical theory regarding the use of the Laplace transform for linear PDEs.

Highly efficient numerical pricing techniques in this context are found in the class of Fourier-based numerical integration [Car99, Cho05, Lor07]. An exponent of this

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class is the COS method [Fan08], which is based on Fourier cosine expansions and on the availability of the characteristic function of the asset price dynamics. However, the COS method, although highly efficient, may lack robustness for a number of cases. For call options, for example, it is often recommended to employ the put-call parity relation, and price calls via puts. The integration interval must be defined a priori, and the method also shows sensitivity in the choice of the interval for certain underlying processes like Carr–Geman–Madan–Yor (CGMY) model [Car02]. Furthermore, with the COS method it is nontrivial to handle accurately long maturity contracts as roundoff errors are produced in some cases.

Other techniques that have been employed for pricing options in this context include the Hilbert transform [Fen08] and the Gauss transform [Bro03].

In the present paper we will focus on the application of *wavelet techniques* in the context of the discounted expected payoff approach. Wavelets have been used before in option pricing [Mat04, Mat05], but then directly as part of the numerical discretization for the option pricing PDEs. In the framework we consider here, the wavelet approximation (WA) approach is novel.

We consider the use of Haar wavelets as well as low-order B-spline wavelets. Compared to the use of cosines in the COS method, the computation of wavelet coefficients may be somewhat more involved. However, we will see that the use of wavelets is in particular robust. In terms of the sensitivity with respect to the size of the integration interval, we propose an adaptive wavelet method, avoiding an a priori choice of the interval. The fact that these wavelets are functions with compact support enables us to deal properly with roundoff errors that we may encounter when pricing long maturity options. In the context of portfolio loss computations, the use of this wavelet-based method is also of benefit, since we deal with step-like density functions and a Haar basis fits better to the shape of the probability density function than a cosine basis.

Haar wavelets have been used before [Mas11, Ort12a] in the context of portfolio losses, but there the derivations were based on the Laplace transform while we focus on the use of the Fourier transform and the characteristic function in the present paper.

The outline of the paper is as follows: In section 2 we present the option pricing problem and discuss wavelets and their properties. In section 3 we explain two different wavelet-based approximations for the option pricing problem. The first method is related to a fixed integration range, whereas the second method adaptively determines the suitable range of integration. It is in particular this second method which is free of open parameters and highly robust. Error analysis is subsequently presented in section 4, and numerical experiments focusing on the method's robustness are found in section 5. Finally, section 6 is devoted to conclusions.

2. Option pricing with wavelets. We consider the risk-neutral valuation formula,

$$v(x,t) = e^{-r_{\text{int}}(T-t)}\mathbb{E}^{\mathbb{Q}}\left(v(y,T)|x\right) = e^{-r_{\text{int}}(T-t)}\int_{\mathbb{R}} v(y,T)f(y|x)dy, \tag{1}$$

where v denotes the option value, T is the maturity, t is the initial date, $\mathbb{E}^{\mathbb{Q}}$ is the expectation operator under the risk-neutral measure $\mathbb{Q}$, x and y are state variables at time t and T, respectively, $f(y|x)$ is the probability density of y given x, and r_{int} is the risk-neutral interest rate.

Whereas f is typically not known, the characteristic function of the log-asset price is often known, that is, the Fourier transform of f. We represent the payoff as

a function of the log-asset price and denote the log-asset prices by

$$x = \log(S_0/K) \quad \text{and} \quad y = \log(S_T/K)$$

with S_t the underlying price at time t and K the strike price. The payoff $v(y,T)$ for European options in log-asset price then reads

$$v(y,T) = [\alpha \cdot K\,(e^y - 1)]^+ \quad \text{with} \quad \alpha = \begin{cases} 1 & \text{for a call,} \\ -1 & \text{for a put.} \end{cases}$$

We derive the WA method for pricing European options by replacing the density function by an approximation with jth order B-splines at a fixed scale level m.

A natural and convenient way to introduce wavelets is following the notion of multiresolution analysis (MRA). Here we provide the basic definitions and properties regarding MRA and B-spline wavelets; for further information see [Chu92, Chu97].

2.1. Wavelets and dual wavelets. We start with the definition of a Riesz basis, as follows.

Definition 1. *A countable set $\{f_n\}$ of a Hilbert space is a Riesz basis if every element f of the space can be uniquely written as $f = \sum_n c_n f_n$, and there exist positive constants A and B such that*

$$A\|f\|^2 \le \sum_n |c_n|^2 \le B\|f\|^2.$$

Definition 2. *A function $\psi \in L^2(\mathbb{R})$ is called an $\mathcal{R}$-function if $\{\psi_{j,k}\}$ defined as $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), j,k \in \mathbb{Z}$, is a Riesz basis of $L^2(\mathbb{R})$.*

If we assume that ψ is an $\mathcal{R}$-function, then there exists a unique Riesz basis $\{\psi^{l,m}\}$ of $L^2(\mathbb{R})$, which is dual to $\{\psi_{j,k}\}$ in the sense that $\langle \psi_{j,k}, \psi^{l,m} \rangle = \delta_{j,l}\delta_{k,m}$ for all $j,k,l,m \in \mathbb{Z}$, and $\delta_{p,q}$ is the Kronecker delta defined in the usual way as

$$\delta_{p,q} = \begin{cases} 1, & p = q, \\ 0 & \text{otherwise.} \end{cases}$$

With the above definitions, we can give the definition of wavelets.

Definition 3. *An $\mathcal{R}$-function $\psi \in L^2(\mathbb{R})$ is called an $\mathcal{R}$-wavelet, or simply a wavelet, if there exists a function $\widetilde{\psi} \in L^2(\mathbb{R})$, such that $\{\psi_{j,k}\}$ and $\{\widetilde{\psi}_{j,k}\}$ defined as $\widetilde{\psi}_{j,k}(x) = 2^{j/2}\widetilde{\psi}(2^j x - k), j,k \in \mathbb{Z}$, are dual bases of $L^2(\mathbb{R})$. If ψ is a wavelet, then $\widetilde{\psi}$ is called a dual wavelet corresponding to ψ.*

Some definitions relevant to our present work are given below.

Definition 4. *Let ψ be the wavelet function in Definition* 3.

(i) *A wavelet ψ is said to have a vanishing moment of order s if $\int_{\mathbb{R}} x^p \psi(x)dx = 0, p = 0, \ldots, s-1$. All wavelets must satisfy this condition for $p = 0$.*

(ii) *A wavelet ψ is an orthogonal wavelet if the family $\{\psi_{j,k}\}$ forms an orthonormal basis of $L^2(\mathbb{R})$, that is, $\langle \psi_{s,t}, \psi_{u,v} \rangle = \delta_{s,u}\delta_{t,v}$ for all $s,u,t,v \in \mathbb{Z}$.*

(iii) *A wavelet ψ is called a semiorthogonal (SO) wavelet if the family $\{\psi_{j,k}\}$ satisfies, $\langle \psi_{s,t}, \psi_{u,v} \rangle = 0, s \ne u$, for all $s,u,t,v \in \mathbb{Z}$.*

A dual wavelet $\widetilde{\psi}$ is unique and is itself a wavelet. The pair $(\psi, \widetilde{\psi})$ is symmetric in the sense that ψ is also the dual wavelet of $\widetilde{\psi}$. If ψ is an orthogonal wavelet, then

it is self-dual in the sense of $\widetilde{\psi} \equiv \psi$. Moreover, it is important to emphasize that if $f \in L^2(\mathbb{R})$, then

$$f = \sum_{j,k=-\infty}^{+\infty} \langle f, \psi_{j,k} \rangle \, \widetilde{\psi}_{j,k} = \sum_{j,k=-\infty}^{+\infty} \left\langle f, \widetilde{\psi}_{j,k} \right\rangle \psi_{j,k}.$$

For each $j \in \mathbb{Z}$, let W_j denote the closure of the linear span of $\{\psi_{j,k} : k \in \mathbb{Z}\}$, namely, $W_j := clos_{L^2(\mathbb{R})} \langle \psi_{j,k} : k \in \mathbb{Z} \rangle$. Then, $L^2(\mathbb{R})$ can be decomposed as a direct sum of the spaces W_j,

$$L^2(\mathbb{R}) = \sum_{j \in \mathbb{Z}}^{\bullet} W_j := \cdots \cup W_{-1} \cup W_0 \cup W_1 \cup \cdots, \tag{2}$$

in the sense that every function $f \in L^2(\mathbb{R})$ has a unique decomposition,

$$f(x) = \cdots + g_{-1}(x) + g_0(x) + g_1(x) + \cdots,$$

where $g_j \in W_j, j \in \mathbb{Z}$.

If ψ is an orthogonal wavelet, then the subspaces W_j of $L^2(\mathbb{R})$ are mutually orthogonal, and consequently the direct sum in (2) becomes an orthogonal sum,

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j := \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \cdots. \tag{3}$$

Obviously, every SO wavelet generates an orthogonal decomposition (3) of $L^2(\mathbb{R})$, and every orthogonal wavelet is also an SO wavelet.

2.2. MRA, scaling functions, and B-splines. Any wavelet, SO or not, generates a direct sum decomposition (2) of $L^2(\mathbb{R})$. For each $j \in \mathbb{Z}$, let us consider the closed subspaces,

$$V_j = \cdots \cup W_{j-2} \cup W_{j-1}, \quad j \in \mathbb{Z},$$

of $L^2(\mathbb{R})$. These subspaces have the following properties:

(i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$,
(ii) $clos_{L^2} \left(\bigcup_{j \in \mathbb{Z}} V_j \right) = L^2(\mathbb{R})$,
(iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,
(iv) $V_{j+1} = V_j \cup W_j, j \in \mathbb{Z}$,
(v) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}, j \in \mathbb{Z}$.

Observe that every function $f \in L^2(\mathbb{R})$ can be approximated as closely as desirable by its projections $P_j f$ in V_j as described by (ii).

If the reference subspace V_0, say, is generated by a single function $\phi \in L^2(\mathbb{R})$ in the sense that

$$V_0 := clos_{L^2(\mathbb{R})} \langle \phi_{0,k} : k \in \mathbb{Z} \rangle,$$

where $\phi_{j,k}(x) := 2^{j/2} \phi(2^j x - k)$, and then all subspaces V_j are also generated by the same ϕ, namely,

$$V_j := clos_{L^2(\mathbb{R})} \langle \phi_{j,k} : k \in \mathbb{Z} \rangle, \quad j \in \mathbb{Z}. \tag{4}$$

Definition 5. *A function $\phi \in L^2(\mathbb{R})$ is said to generate an MRA if it generates a nested sequence of closed subspaces V_j that satisfy* (i), (ii), (iii), *and* (v) *in the sense of* (4), *such that $\{\phi_{0,k}\}$ forms a Riesz basis of V_0. If ϕ generates a MRA, then ϕ is called a scaling function.*

Typical examples of scaling functions ϕ are the jth order cardinal B-splines, $N_j(x)$, defined recursively by a convolution,

$$N_j(x) = \int_{-\infty}^{\infty} N_{j-1}(x-t)N_0(t)dt = \int_0^1 N_{j-1}(x-t)dt, \quad j \geq 1,$$

where,

$$N_0(x) = \chi_{[0,1)}(x) = \begin{cases} 1 & \text{if } x \in [0,1), \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively,

$$N_j(x) = \frac{x}{j}N_{j-1}(x) + \frac{j+1-x}{j}N_{j-1}(x-1), \quad j \geq 1.$$

We note that cardinal B-spline functions are compactly supported, since the support of the j^{th} order B-spline function N_j is $[0, j+1]$, and they have as the Fourier transform

$$\widehat{N_j}(w) = \left(\frac{1-e^{-iw}}{iw}\right)^{j+1}.$$

To describe the space V_0 that N_j generates, we define π_n as the collection of all polynomials of degree at most n, and C^n denotes the collection of all functions f such that $f, f^{(1)}, \ldots, f^{(n)}$ are continuous everywhere.

The subspace V_0 generated by N_j consists of all functions $f \in C^{j-1} \cap L^2(\mathbb{R})$ such that the restriction of each function f to any interval $[k, k+1), k \in \mathbb{Z}$, is in π_j. From property (v) of an MRA, we can now identify all other subspaces V_j, namely,

$$V_j = \left\{ f \in C^{s-1} \cap L^2(\mathbb{R}) : f|_{[\frac{k}{2^j}, \frac{k+1}{2^j})} \in \pi_s, k \in \mathbb{Z} \right\}.$$

Since splines are only piecewise polynomial functions, they are very easy to implement in a computer code.

From the nested sequence of splines subspaces V_j, we have the orthogonal complementary subspaces W_j, such that $V_{j+1} = V_j \oplus W_j, j \in \mathbb{Z}$. Just as the B-spline of order s is the minimally supported generator of $\{V_j\}$, we can find the minimally supported $\psi^s \in W_0$ that generates the mutually orthogonal subspaces W_j. These compactly supported functions will be called B-wavelets of order s. In Chapter 6 in [Chu92], explicit formulas for all ψ^s and their duals are derived.

In this paper we consider $\phi^j = N_j$ as the scaling function which generates a MRA (see Figure 1), and we restrict ourselves to the orders $j = 0, 1$. Clearly, for $j = 0$ we have the scaling function of the Haar wavelet system. We also remark that from the previous discussions, for every function $f_m \in V_m$, there exists a unique sequence $\{c^j_{m,k}\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$, such that

$$f_{m,j}(x) = \sum_{k\in\mathbb{Z}} c^j_{m,k}\phi^j_{m,k}(x). \tag{5}$$

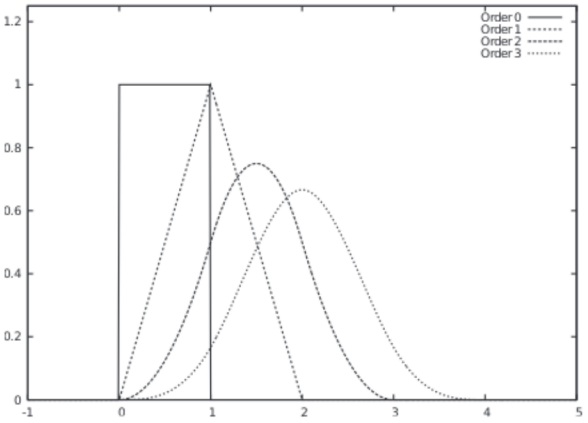

Fig. 1. *Cardinal B-splines of orders* $j = 0, 1, 2, 3$.

In this paper we present two different methods for approximating the density function in (1). In the first method we restrict ourselves to a *bounded interval* $[a, b]$,

$$f^c_{m,j}(x) = \sum_{k=0}^{(j+1)\cdot(2^m-1)} c^j_{m,k}\phi^j_{m,k}\left((j+1)\cdot\frac{x-a}{b-a}\right), \quad j \geq 0,$$

while in the second method we carry out the approximation on the *entire real line* as in (5). For this reason, it is convenient to give a short review about B-splines on a bounded interval. We refer the reader to [Chu97] for a detailed description of scaling functions on a bounded interval.

Let us assume that $[0, n], n \in \mathbb{N}$, is the working interval. We must distinguish between *interior* B-splines and *boundary* B-splines. We have

$$N_j(x-k), \quad k = 0, \ldots, n-j-1.$$

These are the interior B-splines for the bounded interval $[0, n]$. The remaining B-splines,

$$\begin{cases} N_j(x-k), & k = -j, \ldots, -1, \text{and} \\ N_j(x-k), & k = n-j, \ldots, n-1, \end{cases}$$

are the boundary B-splines for the interval $[0, n]$. Here, the first group is for the boundary $x = 0$, while the second group is for the boundary $x = n$.

3. The WA method. We present two different methods to recover a probability density function f from its characteristic function, i.e., from its Fourier transform. The first method relies on fixing a priori the interval for the approximation and then using B-splines wavelets on this bounded interval, while the second method performs the approximation in an adaptive way, computing the domain of the approximation such that a predefined tolerance error is met. Later on, we will apply both methods to the option pricing problem and we will discuss the pros and cons of each of them.

3.1. Approximation in a fixed interval $[a, b]$: the WA$^{[a;b]}$ method. Let us consider a probability density function $f \in L^2(\mathbb{R})$ associated to a certain continuous

random variable X and its Fourier transform,

$$\widehat{f}(w) = \int_{-\infty}^{+\infty} e^{-iwx} f(x)dx. \tag{6}$$

We can expect that f decays to zero at infinity, so it can be approximated in a finite interval $[a, b]$ by

$$f^c(x) = \begin{cases} f(x) & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

To determine the interval of integration $[a, b]$, we can consider the approximation

$$[a, b] := \left[c_1 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + L\sqrt{c_2 + \sqrt{c_4}} \right], \tag{7}$$

where c_n denotes the nth cumulant[1] of X, as in [Fan08], where the authors recommend $L \geq 7.5$. This has been confirmed accurate for a variety of asset processes and options with different maturities.

Following the theory of MRA in a bounded interval, we can approximate $f^c(x) \simeq f^c_{m,j}(x)$ for all $x \in [a, b]$, where

$$f^c_{m,j}(x) = \sum_{k=0}^{(j+1)\cdot(2^m-1)} c^j_{m,k} \phi^j_{m,k} \left((j+1) \cdot \frac{x-a}{b-a} \right), \quad j \geq 0,$$

with convergence in L^2-norm. Note that we are not considering the left and right boundary scaling functions. For the sake of simplicity, we set the coefficients corresponding to the boundary basis functions equal to zero. Observe that in the case that $j = 0$ (Haar wavelets) there are no boundary functions.

The main idea behind the WA method is to approximate $\widehat{f}$ by $\widehat{f}^c_{m,j}$ and then to compute the coefficients $c^j_{m,k}$ by inverting the Fourier transform. Proceeding this way, we have

$$\begin{aligned} \widehat{f}(w) &= \int_{-\infty}^{+\infty} e^{-iwx} f(x)dx \simeq \int_{-\infty}^{+\infty} e^{-iwx} f^c_{m,j}(x)dx \\ &= \sum_{k=0}^{(j+1)\cdot(2^m-1)} c^j_{m,k} \left(\int_{-\infty}^{+\infty} e^{-iwx} \phi^j_{m,k} \left((j+1) \cdot \frac{x-a}{b-a} \right) dx \right). \end{aligned}$$

Introducing a change of variables, $y = (j+1) \cdot \frac{x-a}{b-a}$, gives us

$$\begin{aligned} \widehat{f}(w) &\simeq \frac{b-a}{j+1} \cdot e^{-iaw} \sum_{k=0}^{(j+1)\cdot(2^m-1)} c^j_{m,k} \int_{-\infty}^{+\infty} e^{-iw\frac{b-a}{j+1}y} \phi^j_{m,k}(y)dy \\ &= \frac{b-a}{j+1} \cdot e^{-iaw} \sum_{k=0}^{(j+1)\cdot(2^m-1)} c^j_{m,k} \widehat{\phi}^j_{m,k} \left(\frac{b-a}{j+1} \cdot w \right). \end{aligned}$$

[1]The cumulants are the power series coefficients of the cumulant generating function $c(s) = \log \mathbb{E}(e^{sX})$.

Finally, taking into account that $\widehat{\phi^j}_{m,k}(\xi) = 2^{-\frac{m}{2}}\widehat{\phi^j}(\frac{\xi}{2^m})e^{-i\frac{k}{2^m}\xi}$ and performing a change of variables, $z = e^{-i\frac{b\ -a}{2^m\ (j+1)}w}$, we find

$$\widehat{f}\left(\frac{2^m(j+1)}{b-a}i\cdot\log(z)\right) \simeq 2^{-\frac{m}{2}}\frac{b-a}{j+1}\cdot z^{\frac{2^m\ (j+1)a}{b\ -a}}\widehat{\phi^j}\left(i\cdot\log(z)\right)\sum_{k=0}^{(j+1)\cdot(2^m-1)} c^j_{m,k}z^k. \tag{8}$$

If we define

$$P_{m,j}(z) := \sum_{k=0}^{(j+1)\cdot(2^m-1)} c^j_{m,k}z^k \quad \text{and}$$

$$Q_{m,j}(z) := \frac{2^{\frac{m}{2}}(j+1)z^{-\frac{2^m\ (j+1)a}{b\ -a}}\widehat{f}\left(\frac{2^m\ (j+1)}{b-a}i\cdot\log(z)\right)}{(b-a)\widehat{\phi^j}\left(i\cdot\log(z)\right)},$$

then, according to the previous formula (8), we have

$$P_{m,j}(z) \simeq Q_{m,j}(z). \tag{9}$$

Since $P_{m,j}(z)$ is a polynomial, it is (in particular) analytic inside a disc of the complex plane $\{z \in \mathbb{C} : |z| < r\}$ for $r > 0$. We can obtain expressions for the coefficients $c^j_{m,k}$ by means of Cauchy's integral formula. This is

$$c^j_{m,k} = \frac{1}{2\pi i}\int_\gamma \frac{P_{m,j}(z)}{z^{k+1}}dz, \quad k = 0,\dots,(j+1)\cdot(2^m-1),$$

where γ denotes a circle of radius r, $r > 0$, about the origin.

Considering now the change of variables $z = re^{iu}$, $r > 0$, gives us

$$c^j_{m,k} = \frac{1}{2\pi r^k}\int_0^{2\pi} \frac{P_{m,j}(re^{iu})}{e^{iku}}du, \tag{10}$$

where $k = 0,\dots,(j+1)\cdot(2^m-1)$.

Then, we can further expand expression (10) by (see Appendix A for a detailed proof)

$$c^j_{m,0} = \frac{1}{\pi}\int_0^{\pi} \Re(P_{m,j}(re^{iu}))du \tag{11}$$

and

$$c^j_{m,k} = \frac{2}{\pi r^k}\int_0^{\pi} \Re(P_{m,j}(re^{iu}))\cos(ku)du, \quad k = 1,\dots,(j+1)\cdot(2^m-1). \tag{12}$$

On the other side, since $\widehat{\phi^j}(i\cdot\log(z)) = (\frac{z-1}{\log(z)})^{j+1}$, we have

$$Q_{m,j}(z) = \frac{2^{\frac{m}{2}}(j+1)z^{-\frac{2^m\ (j+1)a}{b\ -a}}\widehat{f}\left(\frac{2^m\ (j+1)}{b-a}i\cdot\log(z)\right)(\log(z))^{j+1}}{(b-a)(z-1)^{j+1}},$$

and it has a pole at $z = 1$. Finally, making use of (9) and taking into account

the former observation, we can exchange $P_{m,j}$ by $Q_{m,j}$ in (11) and (12) to obtain, respectively,

$$c^j_{m,0} \simeq \frac{1}{\pi}\int_0^\pi \Re(Q_{m,j}(re^{iu}))du \tag{13}$$

and

$$c^j_{m,k} \simeq \frac{2}{\pi r^k}\int_0^\pi \Re(Q_{m,j}(re^{iu}))\cos(ku)du, \quad k = 1,\dots,(j+1)\cdot(2^m-1), \tag{14}$$

where $r \neq 1$ is a positive real number.

In practice, both integrals in (13) and (14) are computed by means of the trapezoidal rule, and we can define

$$I(k) = \int_0^\pi \Re(Q_{m,j}(re^{iu}))\cos(ku)du \tag{15}$$

and

$$I(k;h) = \frac{h}{2}\left(Q_{m,j}(r) + (-1)^k Q_{m,j}(-r) + 2\sum_{s=1}^{M-1}\Re(Q_{m,j}(re^{ih_s}))\cos(kh_s)\right), \tag{16}$$

where $h = \frac{\pi}{M}$ and $h_s = sh$ for all $s = 0,\dots,M$. Proceeding this way we find

$$\begin{aligned} c^j_{m,k} &\simeq \frac{2}{\pi r^k} I(k) \simeq \frac{2}{\pi r^k} I(k;h) \\ &= \frac{1}{Mr^k}\left(Q_{m,j}(r) + (-1)^k Q_{m,j}(-r) + 2\sum_{s=1}^{M-1}\Re(Q_{m,j}(re^{ih_s}))\cos(kh_s)\right), \end{aligned} \tag{17}$$

where $k = 1,\dots,(j+1)\cdot(2^m-1)$.

3.2. Approximation in $\mathbb{R}$: The WA$^{\mathbb{R}}$ method. Let f be a probability density function as in section 3.1. The main drawback of the WA$^{[a,b]}$ method is that we do not have an estimate about the density when truncating the interval. Here, we propose an adaptive method that allows us to control the mass of the density recovered.

Following the theory of MRA in $L^2(\mathbb{R})$, we can approximate $f(x)$ by $f_{m,j}(x)$ in (5) for all $x \in \mathbb{R}$, where

$$f_{m,j}(x) = \sum_{k\in\mathbb{Z}} c^j_{m,k}\phi^j_{m,k}(x), \quad j \geq 0,$$

with convergence in the L^2-norm. Note that the coefficients $c^j_{m,k}$ are different from those in section 3.1. For the sake of clarity, however, we keep the same notation as in the previous section.

Let us consider the finite sum,

$$f^{tr}_{m,j}(z) = \sum_{k=k_1}^{k_2} c^j_{m,k}\phi^j_{m,k}(x), \tag{18}$$

where $k_1 < k_2, k_1, k_2 \in \mathbb{Z}$.

Substituting (18) into the Fourier transform expression (6), making the change of variables $z = e^{-i\frac{w}{2^m}}$, and following the steps as in the previous section gives us

$$\widehat{f}\left(2^m i \cdot \log(z)\right) \simeq 2^{-\frac{m}{2}} \cdot \widehat{\phi^j}\left(i \cdot \log(z)\right) \sum_{k=k_1}^{k_2} c_{m,k}^j z^k.$$

If we define

$$P_{m,j}^*(z) := \sum_{k=k_1}^{k_2} c_{m,k}^j z^k \quad \text{and} \quad Q_{m,j}^*(z) := \frac{2^{\frac{m}{2}} \widehat{f}\left(2^m i \cdot \log(z)\right)}{\widehat{\phi^j}\left(i \cdot \log(z)\right)},$$

then, according to the previous formula, we have

$$P_{m,j}^*(z) \simeq Q_{m,j}^*(z).$$

Now, if we consider the polynomial of degree $k_2 - k_1$,

$$P_{m,j}(z) := z^{-k_1} \cdot P_{m,j}^*(z) = \sum_{k=0}^{k_2-k_1} c_{m,k_1+k} z^k,$$

we can approximate it by $Q_{m,j}(z)$ as

$$P_{m,j}(z) \simeq Q_{m,j}(z),$$

where $Q_{m,j}(z) = z^{-k_1} \cdot Q_{m,j}^*(z)$.

We can derive the coefficients of the approximation following the same steps as in the $\text{WA}^{[a,b]}$ method to obtain

$$c_{m,k_1}^j \simeq \frac{1}{\pi} \int_0^\pi \Re(Q_{m,j}(re^{iu}))du$$

and

$$c_{m,k_1+k}^j \simeq \frac{2}{\pi r^k} \int_0^\pi \Re(Q_{m,j}(re^{iu})) \cos(ku)du, \quad k = 1, \ldots, k_2 - k_1. \tag{19}$$

Finally, we must choose the integers k_1 and k_2 to recover the density function without loosing significant accuracy. For that purpose, let us assume first that the density is unimodal. (This is observed in practice for the densities considered in the numerical examples section.) In this case, k_1 and k_2 can be chosen such that

$$f_{m,j}\left(\frac{k}{2^m}\right) \leq f_{m,j}\left(\frac{k_1}{2^m}\right) < \epsilon_{\text{tol}} \quad \text{for all} \quad k \leq k_1$$

and

$$f_{m,j}\left(\frac{k}{2^m}\right) \leq f_{m,j}\left(\frac{k_2}{2^m}\right) < \epsilon_{\text{tol}} \quad \text{for all} \quad k \geq k_2,$$

where ϵ_{tol} is a predefined tolerance error such that $f_{m,j}(x) < \epsilon_{\text{tol}}$ for all $x \in \mathbb{R}\backslash[k_1, k_2]$.

We can start the algorithm by considering an initial seed $k_1 = \lfloor 2^m \cdot a \rfloor$, where a is defined in expression (7) and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Note that, although we rely on the cumulants to facilitate the work, we could

choose a random value as the initial seed. We update the value k_1 until the condition $f_{m,j}(\frac{k_1}{2^{\mathrm{m}}}) < \epsilon_{\mathrm{tol}}$ is satisfied. We compute the coefficients c^j_{m,k_1+k} until the requirement $f_{m,j}(\frac{k_2}{2^{\mathrm{m}}}) < \epsilon_{\mathrm{tol}}$ is met. Furthermore, we immediately calculate the area below the computed density (as a byproduct). In case we cannot assume unimodality, we may finish the algorithm when that area equals (approximately) one, recomputing k_1 if necessary.

3.3. Coefficients $\mathsf{V}^{\mathsf{j};\ \mathsf{l}}_{\mathsf{m};\mathsf{k}}$ for plain vanilla options. Now, we return to the option pricing problem and we consider option valuation formula (1). From now on, we assume that the density functions are $L^2(\mathbb{R})$ functions. Since we can expect that the density function $f(y|x)$ rapidly decays to zero, we can apply the methods explained in section 3.1 and section 3.2.

We must distinguish between the two methods presented previously:

- **WA$^{[\mathsf{a};\mathsf{b}]}$ method.** We first truncate the infinity integration range to $[a,b] \subset \mathbb{R}$ as in section 3.1 and then approximate the density function,

$$\begin{aligned} v^c(x,t) &= e^{-r_{\mathrm{int}}(T-t)} \int_a^b v(y,T) f(y|x)\,dy, \\ v^c_{m,j}(x,t) &= e^{-r_{\mathrm{int}}(T-t)} \int_a^b v(y,T) f^c_{m,j}(y|x)\,dy, \end{aligned} \tag{20}$$

  where,

$$f^c_{m,j}(y|x) = \sum_{k=0}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c^j_{m,k}(x)\phi^j_{m,k}\left((j+1)\cdot\frac{y-a}{b-a}\right).$$

  We then expand the expression (20) to obtain

$$v(x,t) \simeq v^c_{m,j}(x,t) = e^{-r_{\mathrm{int}}(T-t)} \sum_{k=0}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c^j_{m,k}(x)\cdot V^{j,\alpha}_{m,k}, \tag{21}$$

  where

$$V^{j,\alpha}_{m,k} := \int_a^b v(y,T)\phi^j_{m,k}\left((j+1)\cdot\frac{y-a}{b-a}\right)dy.$$

- **WA$^{\mathbb{R}}$ method.** Here, we first approximate the density function by a series of scaling functions in $L^2(\mathbb{R})$ and then truncate the series expansion as in section 3.2, i.e.,

$$\begin{aligned} v_{m,j}(x,t) &= e^{-r_{\mathrm{int}}(T-t)} \int_{\mathbb{R}} v(y,T) f_{m,j}(y|x)\,dy, \\ v^{tr}_{m,j}(x,t) &= e^{-r_{\mathrm{int}}(T-t)} \int_{\mathbb{R}} v(y,T) f^{tr}_{m,j}(y|x)\,dy, \end{aligned}$$

  where

$$f^{tr}_{m,j}(y|x) := \sum_{k=k_1}^{k_2} c^j_{m,k}(x)\phi^j_{m,k}(y).$$

Then,

$$v(x,t) \simeq v_{m,j}^{tr}(x,t) = e^{-r_{\text{int}}(T-t)} \sum_{k=k_1}^{k_2} c_{m,k}^{j}(x) \cdot V_{m,k}^{j,\alpha},$$

where

$$V_{m,k}^{j,\alpha} := \int_{\mathbb{R}} v(y,T)\phi_{m,k}^{j}(y)dy.$$

Remark 1. Once we have an expression for the scaling coefficients $c_{m,k}^{j}(x)$ and the payoff coeffcients $V_{m,k}^{j,\alpha}$, the Greeks can be easily derived by differentiation.

3.3.1. Coefficients with Haar wavelets. We first consider $j = 0$. Then, the following proposition holds.

Proposition 1. *Let us define* $\Delta_m = \frac{b-a}{2^m}, \beta_k = a + k\Delta_m, \gamma_k = \beta_k + \Delta_m, \delta_k = \max(0,\beta_k)$, *and* $\zeta_k = \min(0,\gamma_k)$. *Assuming that* $a < 0 < b$, *the coefficients computed by the* $WA^{[a,b]}$ *method are as follows:*

$$V_{m,k}^{0,1} = \begin{cases} 2^{m/2}K\left(e^{\gamma_k} - e^{\delta_k} + \delta_k - \gamma_k\right), & \gamma_k > 0, \\ 0 & \textit{otherwise}, \end{cases}$$
$$V_{m,k}^{0,-1} = \begin{cases} 2^{m/2}K\left(e^{\beta_k} - e^{\zeta_k} + \zeta_k - \beta_k\right), & \beta_k < 0, \\ 0 & \textit{otherwise} \end{cases} \tag{22}$$

for $k = 0, \ldots, 2^m - 1$.

If $a < b < 0$, *then* $V_{m,k}^{0,1} = 0, V_{m,k}^{0,-1} = 2^{m/2}K(e^{\beta_k} - e^{\gamma_k} + \gamma_k - \beta_k)$, *for* $k = 0, \ldots, 2^m - 1$. *And conversely, if* $0 < a < b$, *then* $V_{m,k}^{0,-1} = 0, V_{m,k}^{0,1} = 2^{m/2}K(e^{\gamma_k} - e^{\beta_k} + \beta_k - \gamma_k)$, *for* $k = 0, \ldots, 2^m - 1$.

Proof. We consider three cases separately:

- Case $a < 0 < b$. We first consider call options, i.e., the case that $\alpha = 1$. By definition,

$$V_{m,k}^{0,1} := \int_a^b \left[K\left(e^y - 1\right)\right]^+ \phi_{m,k}^{0}\left(\frac{y-a}{b-a}\right) dy = \int_0^b K\left(e^y - 1\right) \phi_{m,k}^{0}\left(\frac{y-a}{b-a}\right) dy.$$

Taking into account that

$$\phi_{m,k}^{0}\left(\frac{y-a}{b-a}\right) = \begin{cases} 2^{m/2}, & \frac{k}{2^m} \le \frac{y-a}{b-a} < \frac{k+1}{2^m}, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$V_{m,k}^{0,1} = \int_{[0,b]\cap[\beta_k,\gamma_k]} 2^{m/2}K\left(e^y - 1\right) dy = \begin{cases} \int_{\delta_k}^{\gamma_k} 2^{m/2}K\left(e^y - 1\right) dy, & \gamma_k > 0, \\ 0 & \text{otherwise}. \end{cases} \tag{23}$$

For put options, i.e., $\alpha = -1$, we find

$$V_{m,k}^{0,-1} = \int_a^0 -K\left(e^y - 1\right) \phi_{m,k}^{0}\left(\frac{y-a}{b-a}\right) dy = \int_{[a,0]\cap[\beta_k,\gamma_k]} -2^{m/2}K\left(e^y - 1\right) dy$$
$$= \begin{cases} \int_{\beta_k}^{\zeta_k} -2^{m/2}K\left(e^y - 1\right) dy, & \beta_k < 0, \\ 0 & \text{otherwise}. \end{cases}$$

- Case $a < b < 0$. In this case, supp $\phi^0_{m,k} \subset (-\infty, 0)$ and for call options $v(y,T) = 0, y \leq 0$. Then, $V^{0,1}_{m,k} = 0, k = 0, \ldots, 2^m - 1$. For put options $v(y,T) > 0, y \leq 0$, and therefore $V^{0,-1}_{m,k} = \int_{[a,b]\cap[\beta_k,\gamma_k]} -2^{m/2}K(e^y - 1)dy = \int_{\beta_k}^{\gamma_k} -2^{m/2}K(e^y - 1)dy, k = 0, \ldots, 2^m - 1$.
- Case $0 < a < b$. In this case, supp $\phi^0_{m,k} \subset (0, +\infty)$ and for call options $v(y,T) > 0, y > 0$. Then, $V^{0,1}_{m,k} = \int_{[a,b]\cap[\beta_k,\gamma_k]} 2^{m/2}K(e^y - 1)dy = \int_{\beta_k}^{\gamma_k} 2^{m/2} K(e^y - 1)dy, k = 0, \ldots, 2^m - 1$. For put options $v(y,T) = 0, y \geq 0$, and therefore $V^{0,-1}_{m,k} = 0, k = 0, \ldots, 2^m - 1$. ∎

Corollary 1. *The prices for a put and for a call can be obtained simultaneously with a WA$^{[a,b]}$ method at resolution level m, by computing 2^m coefficients $c^0_{m,k}$ and 2^m coefficients $V^{0,\alpha}_{m,k}$.*

Proof. For the cases $a < b < 0$ and $0 < a < b$ the statement holds immediately. We consider the nontrivial case $a < 0 < b$. By Proposition 1 we just need to compute the coefficients such that $\gamma_k > 0$ and $\beta_k < 0$. Following the same notation as before,

$$\gamma_k > 0 \Leftrightarrow \beta_k + \Delta_m > 0 \Leftrightarrow a + k\Delta_m + \Delta_m > 0 \Leftrightarrow k + 1 > -\frac{a}{\Delta_m}$$

and

$$\beta_k < 0 \Leftrightarrow a + k\Delta_m < 0 \Leftrightarrow k < -\frac{a}{\Delta_m}.$$

Since $-\frac{a}{\Delta_m} > 0$,

$$\gamma_k > 0 \Leftrightarrow \left\lfloor -\frac{a}{\Delta_m} \right\rfloor \leq k \leq 2^m - 1 \quad \text{and} \quad \beta_k < 0 \Leftrightarrow 0 \leq k \leq \left\lfloor -\frac{a}{\Delta_m} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and this completes the proof. ∎

Remark 2. Digital options are popular in the financial markets for hedging and speculation. They are also important to financial engineers as building blocks for constructing more complex option products. Here, we consider the payoff of a cash-or-nothing call option as an example, which has value 0 if $S_T \leq K$ and K if $S_T > K$. For this contract, the $V^{0,1}_{m,k}$ coefficients, like (23), can also be obtained analytically with the WA$^{[a,b]}$ method as follows (we assume that $a < 0 < b$):

$$V^{0,1}_{m,k} = \begin{cases} 2^{m/2}K(\delta_k - \gamma_k), & \gamma_k > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Remark 3. If we define $a := \frac{k_1}{2^m}$ and $\Delta_m := \frac{1}{2^m}$, then the formulae (22) and (24) also hold to compute the coefficients $V^{0,\alpha}_{m,k}$ by the WA$^{\mathbb{R}}$ method. Moreover, we can price a put and a call simultaneously by computing $k_2 - k_1 + 1$ coefficients $c^0_{m,k}$ and $k_2 - k_1 + 1$ coefficients $V^{0,\alpha}_{m,k}$.

3.3.2. Coefficients with linear B-splines. We consider $j = 1$. Then, the following proposition holds.

Proposition 2. *Let us define $\bar{\Delta}_m = \frac{b-a}{2^{m+1}}, \bar{\beta}_k = a + k\bar{\Delta}_m, \bar{\gamma}_k = \bar{\beta}_k + 2\bar{\Delta}_m, \kappa = \frac{\bar{\beta}_k + \bar{\gamma}_k}{2}, \iota_k = \max(0, \kappa), \lambda_k = \min(0, \kappa), \bar{\delta}_k = \max(0, \bar{\beta}_k)$, $\bar{\zeta}_k = \min(0, \bar{\gamma}_k)$, $I_1(a,b) :=$*

$e^b - e^a$, $I_2(a,b) := be^b - ae^a - I_1(a,b)$, *and* $I_3(a,b) := \frac{b^2-a^2}{2}$. *Assuming that* $a < 0 < b$, *the coefficients computed with the* $WA^{[a,b]}$ *method are as follows:*

$$V_{m,k}^{1,1} = \begin{cases} 2^{m/2}K \begin{bmatrix} \left(k + \frac{a}{\bar{\Delta}_m}\right)\left(I_1(\iota_k, \bar{\gamma}_k) - I_1(\bar{\delta}_k, \iota_k)\right) + 2I_1(\iota_k, \bar{\gamma}_k) \\ + \frac{1}{\bar{\Delta}_m}\left(I_2(\bar{\delta}_k, \iota_k) - I_2(\iota_k, \bar{\gamma}_k) - I_3(\bar{\delta}_k, \iota_k) + I_3(\iota_k, \bar{\gamma}_k)\right) \\ + \left(k + \frac{a}{\bar{\Delta}_m}\right)(\iota_k - \bar{\delta}_k) - \left(k + \frac{a}{\bar{\Delta}_m} + 2\right)(\bar{\gamma}_k - \iota_k) \end{bmatrix}, & \bar{\gamma}_k > 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$V_{m,k}^{1,-1} = \begin{cases} -2^{m/2}K \begin{bmatrix} \left(k + \frac{a}{\bar{\Delta}_m}\right)\left(I_1(\lambda_k, \bar{\zeta}_k) - I_1(\bar{\beta}_k, \lambda_k)\right) + 2I_1(\lambda_k, \bar{\zeta}_k) \\ + \frac{1}{\bar{\Delta}_m}\left(I_2(\bar{\beta}_k, \lambda_k) - I_2(\lambda_k, \bar{\zeta}_k) - I_3(\bar{\beta}_k, \lambda_k) + I_3(\lambda_k, \bar{\zeta}_k)\right) \\ + \left(k + \frac{a}{\bar{\Delta}_m}\right)(\lambda_k - \bar{\beta}_k) - \left(k + \frac{a}{\bar{\Delta}_m} + 2\right)(\bar{\zeta}_k - \lambda_k) \end{bmatrix}, & \bar{\beta}_k < 0, \\ 0 & \text{otherwise} \end{cases}$$

for $k = 0, \ldots, 2^{m+1} - 2$.

If $a < b < 0$, *then* $V_{m,k}^{1,1} = 0$ *and*

$$V_{m,k}^{1,-1} = -2^{m/2}K \begin{bmatrix} \left(k + \frac{a}{\bar{\Delta}_m}\right)\left(I_1(\kappa_k, \bar{\gamma}_k) - I_1(\bar{\beta}_k, \kappa_k)\right) + 2I_1(\kappa_k, \bar{\gamma}_k) \\ + \frac{1}{\bar{\Delta}_m}\left(I_2(\bar{\beta}_k, \kappa_k) - I_2(\kappa_k, \bar{\gamma}_k) - I_3(\bar{\beta}_k, \kappa_k) + I_3(\kappa_k, \bar{\gamma}_k)\right) \\ + \left(k + \frac{a}{\bar{\Delta}_m}\right)(\kappa_k - \bar{\beta}_k) - \left(k + \frac{a}{\bar{\Delta}_m} + 2\right)(\bar{\gamma}_k - \kappa_k) \end{bmatrix}$$

for $k = 0, \ldots, 2^{m+1} - 2$. *And conversely, if* $0 < a < b$, *then* $V_{m,k}^{1,-1} = 0$ *and*

$$V_{m,k}^{1,1} = 2^{m/2}K \begin{bmatrix} \left(k + \frac{a}{\bar{\Delta}_m}\right)\left(I_1(\kappa_k, \bar{\gamma}_k) - I_1(\bar{\beta}_k, \kappa_k)\right) + 2I_1(\kappa_k, \bar{\gamma}_k) \\ + \frac{1}{\bar{\Delta}_m}\left(I_2(\bar{\beta}_k, \kappa_k) - I_2(\kappa_k, \bar{\gamma}_k) - I_3(\bar{\beta}_k, \kappa_k) + I_3(\kappa_k, \bar{\gamma}_k)\right) \\ + \left(k + \frac{a}{\bar{\Delta}_m}\right)(\kappa_k - \bar{\beta}_k) - \left(k + \frac{a}{\bar{\Delta}_m} + 2\right)(\bar{\gamma}_k - \kappa_k) \end{bmatrix}$$

for $k = 0, \ldots, 2^{m+1} - 2$.

Proof. See Appendix B for a detailed proof. ∎

We can also price simultaneously calls and puts, due to the compact support of B-splines scaling functions. The coefficients $V_{m,k}^{1,\alpha}$ for digital options with linear B-splines can easily be derived following a similar procedure as in the case of Haar wavelets. Moreover, also the adaptive $WA^{\mathbb{R}}$ method applies in the case of B-splines basis functions, again due to the compact support property of these functions.

4. Error analysis. Let us distinguish four sources of error in our computation of price $v(x,t)$ in (1) by means of the $WA^{[a,b]}$ method:

(A) Truncation of the integration range,

$$\mathcal{E}_1(x) := v(x,t) - v^c(x,t) = e^{-r_{\text{int}}(T-t)} \int_{\mathbb{R}\setminus[a,b]} v(y,t) f(y|x) dy.$$

(B) The approximation error at scale m,

$$\mathcal{E}_2(x) := v^c(x,t) - v^c_{m,j}(x,t) = e^{-r_{\text{int}}(T-t)} \int_a^b v(y,t) \left[f^c(y|x) - f^c_{m,j}(y|x) \right] dy.$$

(C) The discretization error, which results when approximating the integral $I(k)$ in (15) by $I(k;h)$ in (16) using the trapezoidal rule. We can apply here the formula for the error of the composite trapezoidal rule, considering

$$q^j_{m,k}(u) = \Re(Q_{m,j}(re^{iu})) \cos(ku), \quad \mathcal{E}_3 := I(k) - I(k;h),$$

and assuming that $q^j_{m,k} \in C^2([0,\pi])$. Then,

$$|\mathcal{E}_3| = \frac{\pi^3}{12M^2} \left| (q^j_{m,k}(\mu))'' \right|, \quad \mu \in (0,\pi). \tag{25}$$

(D) The roundoff error. If we can calculate the sum in expression (17) with a precision of $10^{-\eta}$, then the roundoff error after multiplying by a factor $\frac{1}{Mr^k}$ is approximately $\mathcal{E}_4 := \frac{1}{Mr^k} \cdot 10^{-\eta}$. This roundoff error increases when r in (13), (14), (15) approaches 0.

Here, we focus on the study of the error of types (A) and (B), and we refer the reader to [Ort12b] for a detailed analysis of the error of types (C) and (D).

If we define $\mathcal{E} := v(x,t) - v^c_{m,j}(x,t)$, then obviously $|\mathcal{E}| \le |\mathcal{E}_1| + |\mathcal{E}_2|$. If we consider an appropriate interval $[a,b]$ for the approximation, then the overall error $\mathcal{E}$ is dominated by $\mathcal{E}_2$.

Looking at

$$|\mathcal{E}_2| = e^{-r_{\text{int}}(T-t)} \left| \int_a^b v(y,t) \left[f^c(y|x) - f^c_{m,j}(y|x) \right] dy \right|,$$

by the Cauchy–Schwarz inequality, we have

$$|\mathcal{E}_2| \le e^{-r_{\text{int}}(T-t)} \parallel v(\cdot,T) \parallel_2 \cdot \parallel f^c(\cdot|x) - f^c_{m,j}(\cdot|x) \parallel_2, \tag{26}$$

where $\| \cdot \|_2$ indicates the norm in the $L^2([a,b])$ space.

Following the theory of MRA in section 2.2, we can write

$$\begin{aligned} f^c(y|x) = & \sum_{k=0}^{(j+1)\cdot(2^{\text{m}}-1)} c^j_{m,k} \phi^j_{m,k} \left((j+1) \cdot \frac{y-a}{b-a} \right) \\ & + \sum_{l \ge m} \sum_{k=0}^{(j+1)\cdot(2^{\text{l}}-1)} d^j_{l,k} \psi^j_{l,k} \left((j+1) \cdot \frac{y-a}{b-a} \right), \end{aligned} \tag{27}$$

where $d^j_{l,k}$ are the wavelet coefficients defined as

$$d^j_{l,k} := \int_a^b f^c(y|x) \cdot \tilde{\psi}^j_{l,k} \left((j+1) \cdot \frac{y-a}{b-a} \right) dy \tag{28}$$

with $\tilde{\psi}^j$ the jth order dual wavelet. (Note that in the case of the Haar system, the jth order wavelet is the same as the jth order dual wavelet, that is, $\psi^j = \tilde{\psi}^j$.)

The following proposition and theorem give us a bound for $\mathcal{E}_2$ in the case of B-splines of order zero ($j = 0$), i.e., for the Haar system.

Proposition 3. *Let us assume that $f^c \in C^2([a,b])$, and let $j = 0$. If $\mathcal{M}$ is a constant such that $(f^c)'(y|x) \leq \mathcal{M}$ for all $y \in [a,b]$, then*

$$|d^j_{l,k}| \leq \frac{1}{4}(b-a)^2 \mathcal{M} \cdot 2^{-\frac{3}{2}l} + \mathcal{O}\left(2^{-\frac{5}{2}l}\right).$$

Proof. Let $I_{l,k} := [a + k \cdot \frac{b-a}{2^l}, a + (k+1) \cdot \frac{b-a}{2^l}] \subset [a,b]$, $y_{l,k}$ be the midpoint of the interval $I_{l,k}$, $I^w_{l,k} := [a + k \cdot \frac{b-a}{2^l}, y_{l,k}]$, and $I^e_{l,k} := [y_{l,k}, a + (k+1) \cdot \frac{b-a}{2^l}]$.

We have that f^c is twice differentiable with continuity in the interval $I_{l,k}$, and then we can expand f^c about $y_{l,k}$ by means of Taylor's formula. It follows that for all $y \in I_{l,k}$,

$$f^c(y|x) = f^c(y_{l,k}|x) + (f^c)'(y_{l,k}|x)(y - y_{l,k}) + \frac{1}{2}(f^c)''(\xi_{l,k}(y)|x)(y - y_{l,k})^2,$$

where $\xi_{l,k} \in I_{l,k}$.

If we substitute f^c in (28) by its Taylor expansion and we take into account that ψ has a vanishing moment of order 1, then

$$\begin{aligned} d^j_{l,k} &= (f^c)'(y_{l,k}|x) \int_{I_{l,k}} (y - y_{l,k}) \psi^j_{l,k}\left(\frac{y-a}{b-a}\right) dy \\ &\quad + \frac{1}{2} \int_{I_{l,k}} (f^c)''(\xi_{l,k}(y)|x)(y - y_{l,k})^2 \psi^j_{l,k}\left(\frac{y-a}{b-a}\right) dy. \end{aligned}$$

Finally, we have

$$\begin{aligned} \int_{I_{l,k}} (y - y_{l,k}) \psi^j_{l,k}\left(\frac{y-a}{b-a}\right) dy &= 2^{\frac{1}{2}} \int_{I^w_{l,k}} (y - y_{l,k}) dy - 2^{\frac{1}{2}} \int_{I^e_{l,k}} (y - y_{l,k}) dy \\ &= -\frac{1}{4}(b-a)^2 \cdot 2^{-\frac{3}{2}l} \end{aligned}$$

and

$$\begin{aligned} &\left| \int_{I_{l,k}} (f^c)''(\xi_{l,k}(y)|x)(y - y_{l,k})^2 \psi^j_{l,k}\left(\frac{y-a}{b-a}\right) dy \right| \\ &\quad \leq 2^{\frac{1}{2}} \max_{y \in I_{l,k}} \left|(f^c)''(y|x)\right| \int_{I_{l,k}} (y - y_{l,k})^2 dy \\ &\quad = \frac{(b-a)^3}{12} \max_{y \in I_{l,k}} \left|(f^c)''(y|x)\right| \cdot 2^{-\frac{5}{2}l}, \end{aligned}$$

and this completes the proof. ∎

Proposition 4. *Under the same assumptions as Proposition 3, we have*

$$|\mathcal{E}_2| \leq \mathcal{C} \sqrt{(b-a)^5} \cdot e^{-r_{\text{int}}(T-t)} \cdot 2^{-m}.$$

Proof. If we consider expression (27) and we take into account that $\| \psi^j_{l,k}(\frac{y-a}{b-a}) \|_2^2 = b - a$, then

$$\| f^c(\cdot|x) - f^c_{m,j}(\cdot|x) \|_2^2 = (b-a) \sum_{l \geq m} \sum_{k=0}^{2^l - 1} |d_{l,k}|^2,$$

since the Haar wavelets are orthogonal. By Proposition 3,

$$\| f^c(\cdot|x) - f^c_{m,j}(\cdot|x) \|_2^2 \leq \frac{1}{16}(b-a)^5\mathcal{M}^2 \sum_{l\geq m} 2^{-2l} = \frac{1}{12}(b-a)^5\mathcal{M}^2 2^{-2m}. \tag{29}$$

Finally, by (26) and (29), we have

$$|\mathcal{E}_2| \leq \mathcal{C}\sqrt{(b-a)^5}\cdot e^{-r_{\text{int}}(T-t)}\cdot 2^{-m},$$

and this completes the proof. ∎

Let us consider the WA$^{\mathbb{R}}$ method. Errors of type (C) and (D) also apply in this case. Furthermore, we define

$$\mathcal{E}(x) := v(x,t) - v^{tr}_{m,j}(x,t) = e^{-r_{\text{int}}(T-t)} \int_{\mathbb{R}} v(y,t)\left[f(y|x) - f^{tr}_{m,j}(y|x)\right] dy.$$

Then, we have

$$\mathcal{E} = e^{-r_{\text{int}}(T-t)}\left[\int_{\mathbb{R}\setminus\left[\frac{k_1}{2^m},\frac{k_2+1}{2^m}\right]} v(y,t)f(y|x)dy + \int_{\frac{k_1}{2^m}}^{\frac{k_2+1}{2^m}} v(y,T)\left[f(y|x) - f^{tr}_{m,j}(y|x)\right] dy\right], \tag{30}$$

since supp $f^{tr}_{m,j} = [\frac{k_1}{2^m}, \frac{k_2+1}{2^m}]$.

If we consider appropriate values k_1 and k_2, the error in (30) is mainly dominated by the second integral. Taking into account the Cauchy–Schwarz inequality, this integral can be bounded by

$$\begin{aligned}
&\left|\int_{\frac{k_1}{2^m}}^{\frac{k_2+1}{2^m}} v(y,T)\left[f(y|x) - f^{tr}_{m,j}(y|x)\right] dy\right| \\
&\quad\leq \left(\int_{\frac{k_1}{2^m}}^{\frac{k_2+1}{2^m}} |v(y,T)|^2\right)^2 \cdot \left(\int_{\frac{k_1}{2^m}}^{\frac{k_2+1}{2^m}} |f(y|x) - f^{tr}_{m,j}(y|x)|^2\right)^2 \\
&\quad\leq \left(\int_{\frac{k_1}{2^m}}^{\frac{k_2+1}{2^m}} |v(y,T)|^2\right)^2 \cdot \| f(\cdot|x) - f^{tr}_{m,j}(\cdot|x) \|_2,
\end{aligned}$$

where $\|\cdot\|_2$ indicates the norm in the $L^2(\mathbb{R})$ space.

Finally, the Strang–Fix theory [Str73, Str89] states that for smooth functions,

$$\| f(\cdot|x) - f^{tr}_{m,j}(\cdot|x) \|_2 \leq \mathcal{D}\cdot 2^{-m(j+1)} \| (f^c)^{(j+1)}(\cdot|x) \|_2, \quad j \geq 0, \tag{31}$$

where $\mathcal{D}$ is a constant.

Notice that in this case we obtain an estimation similar to Proposition 4, since we can consider $\mathcal{M} = \| (f^c)'(\cdot|x) \|_\infty$ in Proposition 3, where $\| (f^c)'(\cdot|x) \|_\infty := \max_{y\in[a,b]} |(f^c)'(y|x)|$.

Remark 4. B-splines of order 1 can achieve a higher rate of convergence than B-splines of order 0. Following a similar procedure as in Proposition 3 and taking into account that B-spline wavelets have a vanishing moment of order 2, it can be shown that the wavelet coefficients $|d^j_{l,k}|$ for WA$^{[a,b]}$ method are of order 2^{-3l}. Using this estimate, it is straightforward to prove that $|\mathcal{E}_2|$ is of order 2^{-2m}. Finally, for the WA$^{\mathbb{R}}$ method, we can also apply (31) with $j = 1$.

5. Numerical examples. In this section, we carry out several numerical experiments to evaluate the accuracy and robustness of the WA method. We put our attention to plain vanilla European options with GBM, the Heston stochastic volatility model, and the CGMY model as the associated process for the underlying asset. We will consider several maturity times, ranging from very short maturities to very long maturities, to stress the robustness of the WA method. Apart from the pricing of European options, we also perform a test to approximate a loss distribution function in a credit portfolio. This problem was treated in [Mas11] by inverting the Laplace transform of the distribution function. Here, we consider the characteristic function instead of the Laplace transform.

The recovery process of the density function is carried out by means of the Fourier inversion method presented in section 3. We will apply two different wavelet-based approximation methods, the $\text{WA}^{[a,b]}$ method, and the $\text{WA}^{\mathbb{R}}$ method, and we will compare the results with the COS method [Fan08]. The $\text{WA}^{[a,b]}$ method, as in the case of the COS method, relies in the a priori computation of the approximation interval based on the cumulants for the underlying models. On the contrary, the $\text{WA}^{\mathbb{R}}$ method computes *adaptively* the approximation interval to meet a predefined tolerance error in the density approximation.

To determine the interval of integration $[a, b]$ within the COS method and with the $\text{WA}^{[a,b]}$ method, we consider

(32)
$$[a,b] := \left[x + c_1 - L\sqrt{c_2 + \sqrt{c_4}}, x + c_1 + L\sqrt{c_2 + \sqrt{c_4}}\right] \quad \text{with} \quad x = \log(S_0/K).$$

Here, c_n denotes the nth cumulant of $\log(S_T/K)$. The cumulants for the models employed are presented for example in [Fan08].

We set the parameters to compute the coefficients for the $\text{WA}^{[a,b]}$ method at the scale of approximation m as follows:

$$r = 0.9995, M = 2^m.$$

For a detailed description of this choice of parameters, we refer the reader to [Ort12b]. Proceeding this way, the number of coefficients used for the Haar basis is 2^m and for the B-spline basis is $2^{m+1} - 1$.

Although the WA method is fast and accurate and it can be used for different underlying processes, in general the COS method is capable of achieving a higher rate of convergence using fewer terms in the expansion for "regular" options. In this work, we wish to focus particularly on the *robustness* of both methods when dealing with extreme cases and compare them from that point of view. We wish to show here advantages of using a wavelet method instead of the COS method. When pricing call options with the COS method, the accuracy of the solution exhibits sensitivity regarding the size of the truncated domain (32). This holds specifically for call options under fat-tailed distributions, like under certain Lévy jump processes (for instance, the CGMY model), or for options with a very long time to maturity. A call payoff grows exponentially in log-asset price which may introduce cancellation errors for large domain sizes. The authors in [Fan08] use the put-call parity relation as a remedy for this, since the payoff value of a put option is bounded by the strike price. Here, we can avoid this roundoff error by means of a wavelet method without relying in the put-call parity relation. Due to the compact support of the basis functions, we can remove some of the largest payoff coefficients to achieve higher accuracy without affecting the accuracy of the remaining part of the density. Furthermore, we can use

the adaptively defined $\mathrm{WA}^{\mathbb{R}}$ method to compute the interval of the approximation, avoiding this way a strategy of fixing an interval based on cumulants.

In the following sections, we show by examples these two important issues. We use the $\mathrm{WA}^{[a,b]}$ method to price call options under GBM and the Heston model with long time to maturity and the $\mathrm{WA}^{\mathbb{R}}$ method to price a call option under the CGMY process and fat-tailed density function. It is important to underline that both wavelet methods can be applied to both situations.

5.1. GBM. A process following a GBM has a characteristic function given by

$$\varphi_{\mathrm{GBM}}(w) = \exp\left(-iwx - iw\left(r_{\mathrm{int}} - q - \frac{1}{2}\sigma^2\right)(T-t) - \frac{1}{2}\sigma^2 w^2 (T-t)\right).$$

Note that we provide a definition of the characteristic function consistent with the definition of the Fourier transform (6).

Table 1 shows the results when pricing a European call option with the $\mathrm{WA}^{[a,b]}$ method using Haar and B-splines of order 1. B-splines behave much better than Haar wavelets in terms of accuracy at higher scales of approximation. For shorter maturities (like the very short maturity $T = 0.0001$, which is about one hour) the density function becomes extremely peaked. In this case, Haar wavelets are better than B-splines. The possible explanation is that the support for Haar basis function is half the support for B-splines scaling functions. Moreover, the shorter the maturity is, the better the accuracy for Haar wavelets. This follows from Proposition 4, since when $T \to 0$, $b - a \to 0$.

From now on, we will consider the Haar basis instead of the B-splines basis for the WA method, as the $\mathrm{WA}^{[a,b]}$ method with the Haar basis also achieves engineering accuracy and is much easier to implement than the method with B-splines. It is worth mentioning that under GBM process, the COS method is extremely accurate and highly efficient, due to its exponential convergence.

We consider the $\mathrm{WA}^{\mathbb{R}}$ method to price a European call option with the Haar basis. We need an initial seed k_0 to start the algorithm. For this purpose we consider the following set of parameters:

$$k_0 = \lfloor 2^m \cdot a \rfloor, \epsilon_{\mathrm{tol}} = 1.e-15,$$

where a is defined in (32) and $L = 10$. For the sake of simplicity, we have taken this initial seed, although the $\mathrm{WA}^{\mathbb{R}}$ method does not rely on the a priori choice of the interval $[a, b]$, so we can take any (random) seed to start the algorithm. Accurate results are shown in Table 2. We can observe that at scales 7 and 8, the computed

Table 1

Absolute errors comparing the $\mathrm{WA}^{[a,b]}$ method using Haar and B-splines basis for a European call option under GBM. Parameters: $S_0 = 100, r_{\mathrm{int}} = 0.1, q = 0, \sigma = 0.25$. The reference value is computed with Black–Scholes closed formulae.

$K = 120, T = 0.1, L = 10$				$K = 80, T = 1, L = 7.5$			
Haar		B-splines		Haar		B-splines	
Scale	Error	Scale	Error	Scale	Error	Scale	Error
5	3.50e 104	4	2.60e 104	5	5.24e 104	4	1.35e 104
6	9.10e 105	5	1.48e 105	6	1.02e 104	5	6.72e 106
7	2.08e 105	6	3.21e 107	7	2.01e 105	6	3.61e 107
8	1.41e 106	7	4.93e 108	8	4.22e 106	7	5.12e 109

Table 2

Absolute errors using the $WA^{\mathbb{R}}$ *method for a European call option under GBM. Parameters:* $S_0 = 100, K = 120, r_{\text{int}} = 0.1, q = 0, \sigma = 0.25, T = 0.1$. *The reference value is computed with Black–Scholes closed formulae.*

Scale	k_1	$y = \frac{k_1}{2^m}$	$\lvert f_{m,j}(y)\rvert$	k_2	$y = \frac{k_2}{2^m}$	$\lvert f_{m,j}(y)\rvert$	Error
5	52	1.63	3.63e−16	21	0.66	2.49e−16	1.27e−04
6	103	1.61	2.23e−16	47	0.73	2.96e−16	7.87e−06
7	138	1.08	7.90e−16	64	0.50	7.59e−16	4.91e−07
8	271	1.06	7.78e−16	137	0.54	6.25e−16	3.07e−08

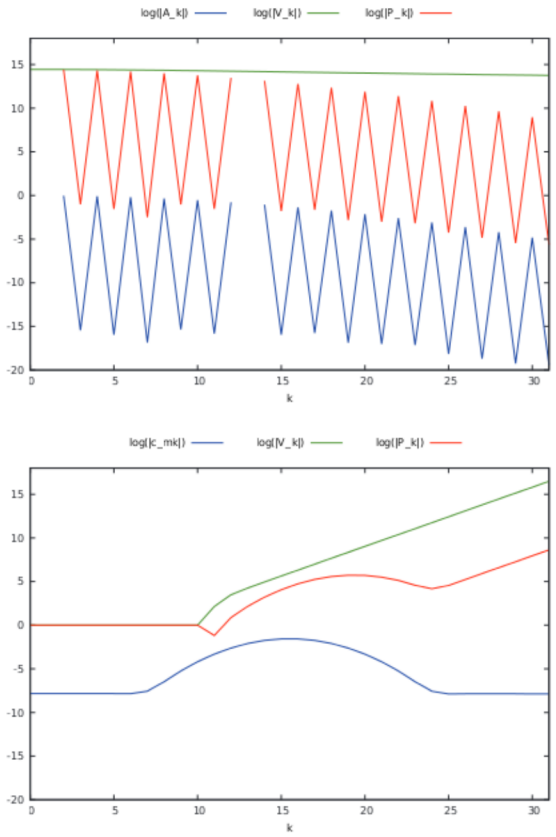

Fig. 2. *Payoff coefficients* (V_k) *and density coefficients* (A_k) *for pricing a call with the COS method (left) and coefficients* $c^0_{5,k}, V^{0,1}_{5,k}$ *for pricing a call with the* $WA^{[a,b]}$ *method (right). Parameters:* $S_0 = 100, K = 100, r_{\text{int}} = 0.1, q = 0, \sigma = 0.25, T = 100, L = 10$.

intervals for approximating the density function are almost equal. Notice that in this case the intervals are small and the density has thin tails, since it is bounded by 10^{-15} outside these intervals.

We consider a European call option with a very long maturity $T = 100$. (Long maturities may appear, for instance, in economics and real options problems.) First, we price the option by means of the COS method using $N = 32$ terms. The left plot in Figure 2 shows the payoff coefficients and the density coefficients. Due to the roundoff errors, the absolute error in the option price when comparing with the Black–Scholes closed formulae is almost 10^{+04}. We also employ a $\text{WA}^{[a,b]}$ method with the Haar

basis at scale 5 (using 32 coefficients). In principle, the absolute error is, like the COS method, about 10^{+04}. This is due to the very large payoff coefficients used in the final part of the interval. (See the right plot in Figure 2.) However, we can *remove* the last coefficients in the approximation, considering a locally truncated expansion, to achieve an accuracy of about 10^{-1} in the absolute error.

We can do this because each coefficient $c_{m,k}$ in the wavelet expansion only affects the points of the density lying in an interval $[\frac{k}{2^m}, \frac{k+1}{2^m}]$. Then, for each scale level m, we can consider a truncated series, i.e,

$$v(x,t) \simeq e^{-r(T-t)} \sum_{k=0}^{k(m)} c^j_{m,k}(x) \cdot V^{j,\alpha}_{m,k}, \tag{33}$$

instead of (21). We compute $k(m)$ so that $e^{-r(T-t)} \sum_{k=0}^{k(m)} c^j_{m,k}(x) \cdot V^{j,\alpha}_{m,k} < S_0$ (since S_0 is an upper bound for the value of a call) and $c^j_{m,k}(x) > 0$ whenever $V^{j,\alpha}_{m,k} > 0$ for all $k \le k(m)$. (When the density values are close to zero, we may find some negative coefficients $c^j_{m,k}(x)$ that may introduce inaccurate option values.)

Note that the same strategy can be followed for pricing call options with long maturities by means of the $\mathrm{WA}^{\mathbb{R}}$ method.

5.2. The Heston model. In this section, we focus on the Heston model [Hes93] for the underlying process. Under the Heston model, the volatility, denoted by $\sqrt{u_t}$, is modeled by a stochastic differential equation,

$$\begin{cases} dx_t &= (\mu - \frac{1}{2}u_t)dt + \sqrt{u_t}dW_{1t}, \\ du_t &= \lambda(\bar{u} - u_t)dt + \eta\sqrt{u_t}dW_{2t}, \end{cases}$$

where x_t denotes the log-asset price variable and u_t the variance of the asset price process. Parameters $\lambda \ge 0, \bar{u} \ge 0$, and $\eta \ge 0$ are called the speed of mean reversion, the mean level of variance, and the volatility of volatility, respectively. Furthermore, the Brownian motions W_{1t} and W_{2t} are assumed to be correlated with correlation coefficient ρ.

The characteristic function of the log-asset price reads

$$\begin{aligned} \varphi_{\mathrm{Heston}}(w) = \exp(-iwx) \\ \cdot \exp\left(-iw\mu(T-t) + \frac{u_0}{\eta^2}\left(\frac{1-e^{-D(T-t)}}{1-Ge^{-D(T-t)}}\right)(\lambda + i\rho\eta w - D)\right) \\ \cdot \exp\left(\frac{\lambda\bar{v}}{\eta^2}\left((\lambda + i\rho\eta w - D)(T-t) - 2\log\left(\frac{1-Ge^{-D(T-t)}}{1-G}\right)\right)\right). \end{aligned}$$

We present in Table 3 the results when pricing call options with long maturities under the Heston model by means of the $\mathrm{WA}^{[a,b]}$ method with the truncated series (33) and the COS method. We have selected maturities $T = 30$ and $T = 45$ which may correspond to pension or mortgage contracts. For $T = 30$ the COS method has very large errors. The $\mathrm{WA}^{[a,b]}$ method produces more accurate and robust results in the sense that when we range from scale 6 to scale 3, the absolute error increases up to 10^{+02}. For the COS method, the absolute error with the same number of terms is about 10^{+06}. For longer maturities like $T = 45$, the COS method with 32 terms has an absolute error about 10^{+04}, while the $\mathrm{WA}^{[a,b]}$ method is about half the unit. The recovered densities with the COS method are plotted in Figure 3.

Table 3

Absolute errors comparing the $WA^{[a,b]}$ *method and the COS method for a European call option under the Heston model. Parameters:* $S_0 = 100, K = 100, r_{\text{int}} = 0, q = 0, \lambda = 1.5768, \eta = 0.5751, \bar{u} = 0.0398, u_0 = 0.0175, \rho = \quad 0.5711, L = 12$. *The reference value is computed with the COS method using* 50000 *terms.*

$T = 30$				$T = 45$			
Haar		COS		Haar		COS	
Scale	Error	N	Error	Scale	Error	N	Error
3	$2.58e+02$	8	$1.72e+06$	3	$3.92e+02$	8	$3.19e+07$
4	$2.72e+00$	16	$2.75e+05$	4	$1.09e+01$	16	$5.45e+06$
5	$7.94e\ \ 01$	32	$2.19e+03$	5	$5.99e\ \ 01$	32	$3.10e+04$
6	$2.46e\ \ 03$	64	$3.37e\ \ 01$	6	$1.05e\ \ 02$	64	$9.68e\ \ 01$

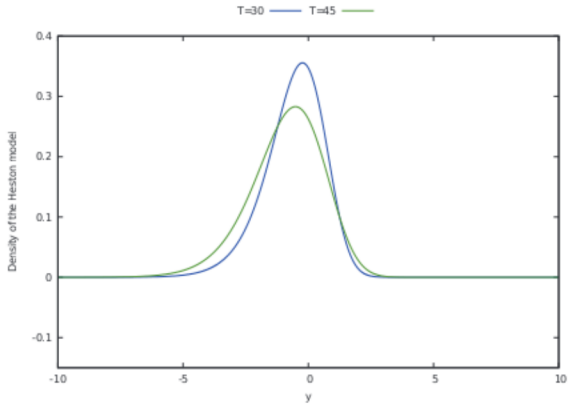

Fig. 3. *Recovered density funtion of the Heston model with the COS method.*

Remark 5. We can observe in Tables 1, 2, and 3 that the pricing errors decrease when we increase the scale of the approximation. This fact is in accordance with the theoretical error analysis from section 4, and also with the theory of MRA in section 2.2. However, the observed errors are smaller than the theoretical errors, which can be explained by the fact that the densities that we encounter are very regular, while one of our main assumptions in the error analysis is that the density function to be recovered is a C^2 function. We restrict ourselves to C^2 functions in the error analysis to make a somewhat general analysis, since we may not know the regularity of the density beforehand.

5.3. CGMY process. One problem with the GBM model is that it is not able to reproduce the volatility skew or smile present in most financial markets. Over the past few years it has been shown that several exponential Lévy models are, at least to some extent, able to reproduce the skew or the smile. One particular model is the CGMY model [Car02]. The characteristic function of the log-asset price reads

$$\varphi_{\text{CGMY}}(w) = \exp(-iwx) \cdot \exp\left(-iw(r-q+s)(T-t) - \frac{1}{2}w^2\sigma^2(T-t)\right) \cdot \exp\left(C\Gamma(-Y)\left((M+iw)^Y - M^Y + (G-iw)^Y - G^Y\right)(T-t)\right),$$

where $\Gamma(\cdot)$ represents the Gamma function, and

$$s = -C\Gamma(-Y)\left((M-1)^Y - M^Y + (G+1)^Y - G^Y\right).$$

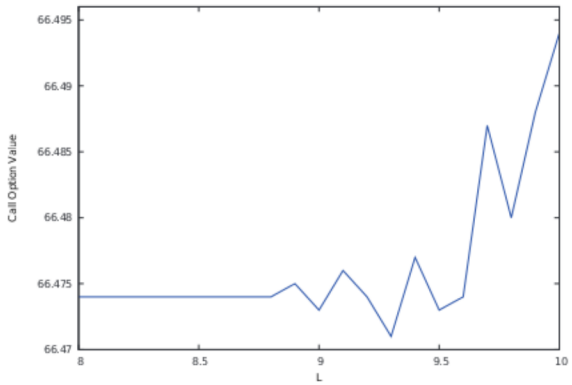

Fig. 4. *Call prices for the COS method under the CGMY process. Parameters:* $S_0 = 100, K = 110, r_{\mathsf{int}} = 0.1, q = 0.05, C = 1, G = 5, M = 5, Y = 1.5, T = 5$.

Table 4

Absolute errors using the $WA^{\mathbb{R}}$ *method for a European call option under the CGMY model. Parameters:* $S_0 = 100, K = 110, r_{\mathsf{int}} = 0.1, q = 0.05, C = 1, G = 5, M = 5, Y = 1.5, T = 5$. *The reference value is computed with the COS method using* 50000 *terms and* $L = 8$.

Scale	k_1	$y = \frac{k_1}{2^m}$	$\lvert f_{m,j}(y)\rvert$	k_2	$y = \frac{k_2}{2^m}$	$\lvert f_{m,j}(y)\rvert$	Error
2	109	27.25	6.29e−16	75	18.75	5.23e−16	2.83e−05

Here, we illustrate the sensitivity in the accuracy of the option value depending on the choice of the parameter L in (32), and hence, depending on the truncation range. For this purpose, we price a call option under the CGMY process with a fat-tailed density. Figure 4 shows the price of the call option for L ranging form 8 to 10 with the COS method with $N = 64$ terms. We can see the fluctuation in the prices, due to cancellation errors.[2]

If we consider the $\mathrm{WA}^{\mathbb{R}}$ method with $\epsilon_{\mathrm{tol}} = 1.e-15$ (for the approximation of the density), we get accurate results without relying on the choice of the interval of the approximation. The results are presented in Table 4.

5.4. Credit portfolio losses. Financial companies need to evaluate and to manage risks originating from their business activities. In particular, *credit risk* underlying a credit portfolio is often the largest risk in a bank and its measure is used to assign capital to absorb potential losses from the credit portfolio.

Consider a portfolio consisting of N obligors. The portfolio loss is defined as $L = \sum_{n=1}^{N} L_n$ with L_n the individual credit loss, defined as

$$L_n = \ell_n \cdot \mathbf{1}_{X_n < c_n}.$$

Here X_n represents an individual asset return value of an obligor. So, if a companies' asset return falls below default threshold c_n, a loss ℓ_n occurs.

Capital calculations for credit portfolios are typically based on the so-called Vasicek framework [Bas05]. The Vasicek model is called a *one-factor model*, because the

[2]By use of the put-call parity relation this may be avoided.

random part of the individual asset value is driven by one common economic factor, i.e.,

$$X_n = \sqrt{\rho_n} Y + \sqrt{1-\rho_n} Z_n,$$

where Y is a common factor and Z_n is the idiosyncratic factor for obligor n. The correlation between Y and X_n is ρ_n. All random variables follow standard normal distributions, and Y and Z_n are independent.

We define $c_n := \Phi^{-1}(P_n)$, where $\Phi^{-1}(x)$ denotes the inverse of the standard normal cumulative distribution function, and P_n is the default probability of obligor n. The probability of default of obligor n, conditional to a realization $Y = y$, is then given by

$$p_n(y) := \mathbb{P}(X_n < c_n \mid Y = y) = \Phi\left(\frac{c_n - \sqrt{\rho_n} y}{\sqrt{1-\rho_n}}\right).$$

In the case of full granularity and for $N \to \infty$, the portfolio loss is governed by common factor Y, because one then assumes that the idiosyncratic factors diversify away.

When a portfolio is *small* or has *name concentrations*, e.g., when there are a few big positions so that due to the obligor's default the portfolio loss will be significant, the portfolio loss is governed by the idiosyncratic factors of these obligors. In this situation, an analytic formula is not available, and the industrial standard is to employ Monte Carlo simulations. The overall distribution function is typically a staircase-like function.

Here, however, we show that a semianalytic formula based on the WA$^{[a,b]}$ method can be derived based on the characteristic function of the portfolio loss, which is defined as

$$\varphi_{\text{loss}}(w) = \mathbb{E}\left[e^{-iwL}\right]. \tag{34}$$

The starting point for the derivation of the characteristic function is the tower property to calculate the conditional expectation, conditional on variable Y,

$$\varphi_{\text{loss}}(w) := \mathbb{E}\left[\mathbb{E}\left[e^{-iwL}\middle| Y\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(-iw\sum_{n=1}^{N}\ell_n \cdot \mathbf{1}_{X_n < c_n}\right)|Y\right]\right].$$

We recall that in a one-factor model framework, if the systematic factor Y is fixed, default occurs independently since the only remaining uncertainty is the idiosyncratic risk. Then,

$$\begin{aligned}\varphi_{\text{loss}}(w) &= \mathbb{E}\left[\prod_{n=1}^{N}\mathbb{E}\left[e^{-iw\ell_n \cdot \mathbf{1}_{X_n < c_n}}|Y\right]\right] \\ &= \mathbb{E}\left[\prod_{n=1}^{N}\vartheta_n(w;y)\right] = \int_{\mathbb{R}} f_Y(y)\cdot\prod_{n=1}^{N}\vartheta_n(w;y)dy,\end{aligned} \tag{35}$$

where

$$\vartheta_n(w;y) := e^{-iw\ell_n} p_n(y) + 1 - p_n(y).$$

The conditional characteristic function for an individual obligor ϑ_n can be obtained analytically, and we only need to evaluate (35), the integral over y, numerically to find $\varphi_{\text{loss}}(w)$.

According to (34), the characteristic function φ_{loss} is the Fourier transform of the loss density f_L associated to the random variable L. Then,

$$\varphi_{\text{loss}}(w) = \int_{\mathbb{R}} e^{-iwl} f_L(l)dl = \int_{\mathbb{R}} e^{-iwl} F'_L(l)dl, \tag{36}$$

where F'_L is the derivative of distribution function F_L associated to the random variable L.

Without loss of generality, we can assume that $\sum_{n=1}^{N} \ell_n = 1$, and, therefore, we can consider

$$F_L(l) = \begin{cases} \overline{F}_L(l) & \text{if} \quad 0 \le l \le 1, \\ 1 & \text{if} \quad l > 1 \end{cases}$$

for certain $\overline{F}_L$ defined in $[0,1]$.

If we integrate by parts the expression (36), we have

$$\varphi_{\text{loss}}(w) = e^{-iw} + iw \int_0^1 e^{-iwl} \overline{F}_L(l)dl,$$

and then $(\varphi_{\text{loss}}(w) - e^{-iw})/(iw)$ is the Fourier transform of $\overline{F}_L$.

Since $\overline{F}_L \in L^2([0,1])$, according to the theory of MRA we can approximate $\overline{F}_L$ in $[0,1]$ by a sum of Haar scaling functions,

$$\overline{F}_L(l) \simeq \overline{F}_L^m(l), \quad \overline{F}_L^m(l) = \sum_{k=0}^{2^m - 1} c_{m,k}\phi_{m,k}(x) \tag{37}$$

with convergence in the L^2-norm. Finally, we can apply the $\text{WA}^{[a,b]}$ method in a bounded interval, as described in section 3.1, where $[a,b] = [0,1]$ in this case, to recover the coefficients of the approximation (37).

In summary, with the characteristic function of the portfolio loss determined, we can apply the $\text{WA}^{[a,b]}$ method to perform the inverse Fourier transformation to recover the density or the cumulative probability function of the portfolio loss efficiently. Value-at-risk values can then be easily extracted from the cumulative probability function.

We consider here a very small credit portfolio with parameters $N = 20, \ell_n = 1, P_n = 0.01$, and $\rho_n = 0.5$ for $n = 1, \ldots, N$. Figure 5 shows the tail probability of the loss random variable L, computed by the COS method with 1024 terms and the $\text{WA}^{[a,b]}$ method at scale 10. We use as a benchmark a Monte Carlo method with five million scenarios for the systematic factor Y. We can observe the high accuracy of the $\text{WA}^{[a,b]}$ method at low and high loss levels. On the contrary, the COS method exhibits oscillations without additional measures, and it is not capable to give reliable results at high loss levels, being often the quantiles of interest to calculate the economic capital to absorb future potential losses in the credit portfolio.

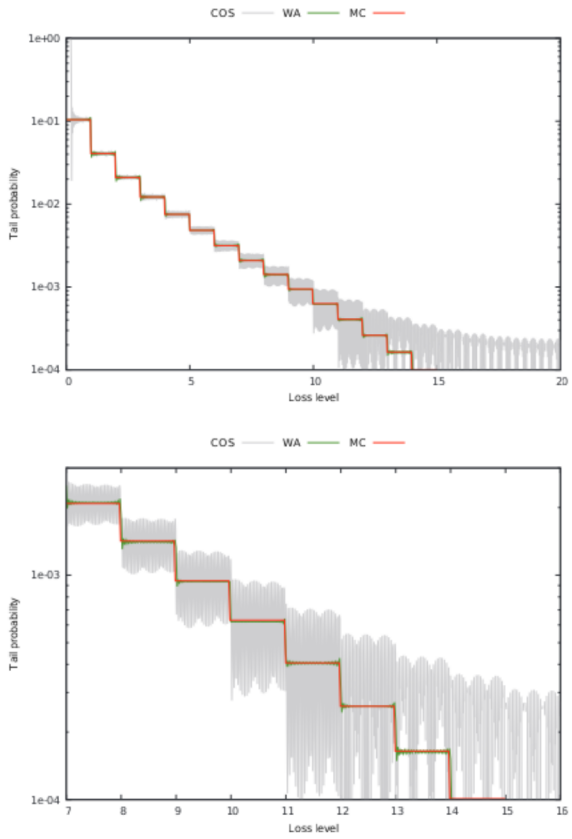

Fig. 5. *Tail probability of a small and diversified portfolio.*

6. Conclusions. We have investigated option pricing problems by means of the discounted expected payoff pricing formula and the wavelet theory.

We have presented two WA methods, the $\text{WA}^{[a,b]}$ and the $\text{WA}^{\mathbb{R}}$ method, for recovering the density function associated to the underlying asset price process, once the characteristic function is available. Technically, we approximate the density function by a finite combination of jth order B-splines basis functions and invert its Fourier transform to recover the coefficients. With the $\text{WA}^{[a,b]}$ method, we first truncate the integration range and then approximate the density function in a bounded interval. With the $\text{WA}^{\mathbb{R}}$ method, we adaptively compute the integration range in order to meet a predefined tolerance error for the density approximation. The second method may be more involved, but we avoid an a priori choice of the interval for the approximation. Regarding the selection of the basis, although in general B-splines may be somewhat more accurate than Haar wavelets, a Haar basis is more efficient and easier to implement than a B-splines basis.

We have compared the WA methods with the COS method which is based on a cosine expansion. Although the COS method is highly efficient and accurate, it may lack robustness in some cases. For long maturity contracts, the COS method may be inaccurate due to roundoff errors. With the WA method, we can truncate the series expansion conveniently to avoid roundoff errors. This is possible due to the compact support properties of the wavelets basis employed. The COS method also

shows certain sensitivity on the choice of the truncation interval when pricing call options. As a remedy for this, we have proposed a solution by means of the WA$^{\mathbb{R}}$ approach, avoiding the a priori selection of the interval. It is worth mentioning that for stepped functions, like the distribution functions arising in portfolio loss problems, Haar wavelets fit much better than Fourier-based methods.

An error analysis has been presented. We underline the local convergence property of the wavelets with compact support. Due to the local nature of theses wavelets, if the approximated function is non smooth in a small interval, this does not affect the convergence in the remaining part of the approximation domain, whereas the convergence of Fourier-based approximations is affected on the whole domain.

Appendix A. Proof of formula (12). We consider $P_{m,j}(z) = \sum_{l=0}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c_{m,l}^{j} z^{l}$. If we make the change of variables $z = re^{iu}$, then

$$P_{m,j}(re^{iu}) = \sum_{l=0}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c_{m,l}^{j} r^{l} e^{ilu}. \tag{38}$$

Taking the real part of (38) and multiplying by $\cos(ku)$ gives us

$$\begin{aligned}\Re\left(P_{m,j}(re^{iu})\right)\cos(ku) &= \sum_{l=0}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c_{m,l}^{j} r^{l} \cos(lu)\cos(ku) \\ &= \frac{1}{2} \sum_{l=0}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c_{m,l}^{j} r^{l} \left(\cos((l+k)u) + \cos((l-k)u)\right).\end{aligned}$$

We integrate between 0 and π and distinguish two cases:

- Case $k = 0$:

$$\begin{aligned}\int_0^{\pi} \Re\left(P_{m,j}(re^{iu})\right) &= \sum_{l=0}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c_{m,l}^{j} r^{l} \left(\int_0^{\pi} \cos(lu)du\right) \\ &= \pi c_{m,0}^{j} + \sum_{l=1}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c_{m,l}^{j} r^{l} \frac{1}{l} \sin(lu)\Big|_0^{\pi} = \pi c_{m,0}^{j}.\end{aligned}$$

- Case $k \neq 0$:

$$\begin{aligned}&\int_0^{\pi} \Re\left(P_{m,j}(re^{iu})\right)\cos(ku) \\ &= \frac{1}{2} \sum_{l=0}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c_{m,l}^{j} r^{l} \left(\int_0^{\pi} \cos((l+k)u)du + \int_0^{\pi} \cos((l-k)u)du\right) \\ &= \frac{\pi}{2} c_{m,k}^{j} r^{k} + \frac{1}{2} \sum_{\substack{l=0 \\ l\neq k}}^{(j+1)\cdot(2^{\mathrm{m}}-1)} c_{m,l}^{j} r^{l} \left(\frac{1}{l+k} \sin((l+k)u)\Big|_0^{\pi}\right. \\ &\qquad \left. + \frac{1}{l-k} \sin((l-k)u)\Big|_0^{\pi}\right) = \frac{\pi}{2} c_{m,k}^{j} r^{k}.\end{aligned}$$

Appendix B. Proof of Proposition 2. We consider three cases separately:

- Case $a < 0 < b$. We consider first the case that $\alpha = 1$. By definition,

$$V_{m,k}^{1,1} := \int_a^b \left[K\left(e^y - 1\right)\right]^+ \phi_{m,k}^1\left(2 \cdot \frac{y-a}{b-a}\right) dy$$
$$= \int_0^b K\left(e^y - 1\right) \phi_{m,k}^1\left(2 \cdot \frac{y-a}{b-a}\right) dy.$$

Taking into account that

$$\phi_{m,k}^1\left(2 \cdot \frac{y-a}{b-a}\right)$$
$$= \begin{cases} 2^{m/2}\left(1 - \left|2^{m+1} \cdot \frac{y-a}{b-a} - k - 1\right|\right), & \frac{k}{2^m} \le 2 \cdot \frac{y-a}{b-a} < \frac{k+2}{2^m}, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$V_{m,k}^{1,1} = \int_{[0,b]\cap[\bar{\beta}_k,\bar{\gamma}_k]} 2^{m/2} K\left(e^y - 1\right)\left(1 - \left|2^{m+1} \cdot \frac{y-a}{b-a} - k - 1\right|\right) dy$$
$$= \begin{cases} \int_{\bar{\delta}_k}^{\bar{\gamma}_k} 2^{m/2} K\left(e^y - 1\right)\left(1 - \left|2^{m+1} \cdot \frac{y-a}{b-a} - k - 1\right|\right) dy, & \bar{\gamma}_k > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we can split the above integral into two parts yielding

$$\int_{\bar{\delta}_k}^{\bar{\gamma}_k} 2^{m/2} K\left(e^y - 1\right)\left(1 - \left|2^{m+1} \cdot \frac{y-a}{b-a} - k - 1\right|\right) dy$$
$$= \int_{\bar{\delta}_k}^{\iota_k} 2^{m/2} K\left(e^y - 1\right)\left(2^{m+1} \cdot \frac{y-a}{b-a} - k\right) dy$$
$$+ \int_{\iota_k}^{\bar{\gamma}_k} 2^{m/2} K\left(e^y - 1\right)\left(1 - 2^{m+1} \cdot \frac{y-a}{b-a} + k + 1\right) dy.$$

After basic algebraic manipulation, we obtain the formulae for the coefficients $V_{m,k}^{1,1}$.

For $\alpha = -1$,

$$V_{m,k}^{1,-1} := \int_a^b \left[-K\left(e^y - 1\right)\right]^+ \phi_{m,k}^1\left(2 \cdot \frac{y-a}{b-a}\right) dy$$
$$= \int_a^0 -K\left(e^y - 1\right) \phi_{m,k}^1\left(2 \cdot \frac{y-a}{b-a}\right) dy.$$

Then,

$$V_{m,k}^{1,-1} = \int_{[a,0]\cap[\bar{\beta}_k,\bar{\gamma}_k]} -2^{m/2} K\left(e^y - 1\right)\left(1 - \left|2^{m+1} \cdot \frac{y-a}{b-a} - k - 1\right|\right) dy$$
$$= \begin{cases} \int_{\bar{\beta}_k}^{\bar{\zeta}_k} -2^{m/2} K\left(e^y - 1\right)\left(1 - \left|2^{m+1} \cdot \frac{y-a}{b-a} - k - 1\right|\right) dy, & \bar{\beta}_k < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we can split the above integral into two parts yielding

$$\begin{aligned}
&\int_{\bar{\beta}_k}^{\bar{\zeta}_k} -2^{m/2}K\left(e^y-1\right)\left(1-\left|2^{m+1}\cdot\frac{y-a}{b-a}-k-1\right|\right)dy \\
&\quad=\int_{\bar{\beta}_k}^{\lambda_k} -2^{m/2}K\left(e^y-1\right)\left(2^{m+1}\cdot\frac{y-a}{b-a}-k\right)dy \\
&\qquad+\int_{\lambda_k}^{\bar{\zeta}_k} -2^{m/2}K\left(e^y-1\right)\left(1-2^{m+1}\cdot\frac{y-a}{b-a}+k+1\right)dy.
\end{aligned}$$

After basic algebraic manipulation, we obtain the formulae for the coefficients $V_{m,k}^{1,-1}$.

- Case $a<b<0$. In this case, supp $\phi_{m,k}^1\subset(-\infty,0)$ and for call options $v(y,T)=0, y\le 0$. Then, $V_{m,k}^{1,1}=0, k=0,\dots,2^{m+1}-2$. For put options $v(y,T)>0, y\le 0$, and, therefore,

$$\begin{aligned}
V_{m,k}^{1,-1}&=\int_{[a,b]\cap[\bar{\beta}_k,\bar{\gamma}_k]} -2^{m/2}K\left(e^y-1\right)\left(1-\left|2^{m+1}\cdot\frac{y-a}{b-a}-k-1\right|\right)dy \\
&=\int_{\bar{\beta}_k}^{\bar{\gamma}_k} -2^{m/2}K\left(e^y-1\right)\left(1-\left|2^{m+1}\cdot\frac{y-a}{b-a}-k-1\right|\right)dy
\end{aligned}$$

for $k=0,\dots,2^{m+1}-2$.

- Case $0<a<b$. In this case, supp $\phi_{m,k}^1\subset(0,+\infty)$ and for call options $v(y,T)>0, y>0$. Then,

$$\begin{aligned}
V_{m,k}^{1,1}&=\int_{[a,b]\cap[\bar{\beta}_k,\bar{\gamma}_k]} 2^{m/2}K\left(e^y-1\right)\left(1-\left|2^{m+1}\cdot\frac{y-a}{b-a}-k-1\right|\right)dy \\
&=\int_{\bar{\beta}_k}^{\bar{\gamma}_k} 2^{m/2}K\left(e^y-1\right)\left(1-\left|2^{m+1}\cdot\frac{y-a}{b-a}-k-1\right|\right)dy
\end{aligned}$$

for $k=0,\dots,2^{m+1}-2$. For put options $v(y,T)=0, y\ge 0$, and, therefore, $V_{m,k}^{1,-1}=0, k=0,\dots,2^{m+1}-2$.

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