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Introduction to Option Pricing in a Securities Market – II: Poisson Approximation

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In this part of the lecture notes on securities trading we aim at the limiting transition from a binary market of part I towards the Poisson market described in Section 4. The conditions for this are formulated in Section 3, and the results in Section 5. The Poisson model describes the situation when the stock price develops with sudden jumps of a constant amplitude at random instants.

1. Introduction

In this paper the material is used of the previous paper by Dzhaparidze and Van Zuijlen (1996), which is referred to below as part I. Most of this material is presented in Section 3 in the form aimed at the limiting transition in Section 5 towards the Poisson model (see Section 4 for details on this model). The presentation in this paper is kept at the same low technical level as in part I. A path by path approach pursued in these papers is based on certain unsophisticated algebraic considerations, in contrast with the usual treatment based on a probabilistic approach, namely on a martingale approach, cf. e.g. Aase (1988), Back (1991), Colwell, Elliott and Kopp (1991), Föllmer (1991), Harrison and Pliska (1981), Page and Sanders (1986). The results obtained in this manner in Section 5 are of an heuristic nature, for the full rigour would require higher technical level of the general theory of stochastic processes, see e.g. Duffie and Protter (1991) and the references therein, cf. also Willinger and Taqqu (1987, 1989, 1991).

As in part I, it is assumed that in a securities market two assets, called the bond and stock, are traded during the time interval $[0, T]$. New prices on both assets are announced at certain fixed trading times, say $t_0 < t_1 < \cdots < t_N$ where $t_0 = 0$ is the current date and $t_N = T$ the terminal date. Thus the whole time interval $[0, T]$ is divided in N trading periods by a grid $\{t_0, t_1, \ldots, t_N\}$. It is supposed throughout the present paper that the number N of the trading times is very large, and possibilities are sought for approximating the option pricing formulas of part I. To this end, we let $N \to \infty$. We can expect in the limit sensible results if only the grid $\{t_0, t_1, \ldots, t_N\}$ of trading times becomes finer and finer in the sense that the mesh size of the grid tends to zero as $N \to \infty$ (the mesh size is the maximal length of the trading periods) and if the asset prices are made dependent on the index N in a certain special manner. See Section 3.1 for the conditions under which the Poisson approximation of the present paper is obtained. Asymptotically, the cumulative return process on the bond is assumed to increase with a constant interest rate, see (3.1.10). The asymptotics of the returns on the stock is characterized by the displacements at certain random instants, upwards with a constant amplitude or downwards with an infinitesimal amplitude, cf. (3.1.17) or (3.1.18). To these displacements certain weights are assigned (called as in part I risk neutral probabilities, cf. (3.2.11) and (3.2.12)) so that under the conditions 3.1.1 and 3.1.2 the approximation (3.2.18) holds. (In the probabilistic interpretation, the upward displacements become rare events.) This leads to the Poisson approximation of Section 5.

In Section 4 the complete description of the Poisson model can be found (or Merton's model, as it is sometimes called, cf. MERTON (1990)). The price processes on the bond and the stock are given by (4.1.2) and (4.1.4), respectively (see (4.1.1) and (4.1.3) for the corresponding returns). As usual, the self-financing strategy is defined by the portfolio selection founded only on an initial endowment so that all changes in the portfolio values are due to capital gains during trading and no infusion or withdrawal of funds is allowed. It is shown that the value process of a self-financing strategy has the integral representation and, moreover, Clark's formula holds; cf. the propositions 3.2.2 and 3.2.6 in the binary case and the similar propositions 4.2.1 and 3.2.2 in the Poisson case (see OCONE and KARATZAS (1991) and NUALART (1995) for the genuine Clark formula). Next, it is shown in proposition 4.2.3 that this value process satisfies the differential equations (4.2.12) which play the same rôle in the Poisson case as equations (3.2.16) in the binary case. In particular, they entail the completeness of a Poisson market, see proposition 4.3.4. The hedging strategy against any desired wealth is explicitly defined by the portfolio components (4.3.9) and (4.3.10) in terms of the Poisson distribution (4.3.3) (in fact, the right hand side of (4.3.12) is a certain conditional expectation). This gives rise to the term *Poisson market*. Finally, the option pricing formulas are presented for a certain contingent claim (see (4.3.13) with a Poisson expectation on the right hand side) and for the European call option in particular, see proposition 4.3.5.

The integral representations (3.2.8) and (4.2.5) mentioned above involve the Riemann-Stieltjes integrals with respect to piecewise continuous functions. Certain elementary facts concerning this kind of functions and respective integrals are gathered in the next section.

2. Auxiliary results

2.1. Piecewise continuous functions

In the present paper the asset prices are supposed to evolve along piecewise continuous trajectories in a time period $[0, T]$. Therefore we will need some common facts concerning functions of this type. For the definitions below we make use of the indicator function $I_{\mathcal{T}}$ of a set $\mathcal{T} \subset [0, T]$ which is a function of time $t \in [0, T]$ such that

$$I_{\mathcal{T}}(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T} \\ 0 & \text{otherwise.} \end{cases}$$

Let F be a function of the same argument $t \in [0, T]$, discontinuous at certain instants $T_1, \ldots, T_n$ so that $0 < T_1 < \ldots < T_n \leq T$ and continuous in-between. Let be defined by means of certain continuous functions $\{f_k\}_{k=0,1,\ldots,n}$ so that

$$F(t) = \sum_{k=0}^{n} f_k(t) I_{[T_k, T_{k+1})}(t). \tag{2.1.1}$$

Here $T_0 = 0$ and $T_{n+1} \geq T$ for convenience. Note that F is a right-continuous function in the sense that by approaching an instant $t \in [0, T)$ from the right we get $\lim_{s \downarrow t} F(s) = F(t)$. With this function F another function F_- is associated by the following conventions: $F_-(0) = F(0)$ and $F_-(t) = F(t-) \equiv \lim_{s \uparrow t} F(s)$ for $t \in (0, T]$. By continuity of the components $\{f_k\}_{k=0,1,\ldots,n}$ we have

$$F_-(t) = f_0(t) I_{[T_0, T_1]}(t) + \sum_{k=1}^{n} f_k(t) I_{(T_k, T_{k+1}]}(t). \tag{2.1.2}$$

Obviously, F_- is a left-continuous function. We will write equally $F(t)$ or F_t, $F(t-)$ or F_{t-}, for the notation with the variable as a subscript is more widely used in stochastic calculus.

Further, the function ΔF of jumps of F is defined by $\Delta F = F - F_-$. In view of (2.1.1) and (2.1.2), ΔF takes on non-zero values only at the instants of discontinuity $T_1, \ldots, T_n$ when

$$\Delta F(T_k) = f_k(T_k) - f_{k-1}(T_k), \quad k = 1, \ldots, n. \tag{2.1.3}$$

2.2. Riemann-Stieltjes integrals

In the propositions 3.2.2 and 4.2.1 integral representations are asserted, in terms of Riemann-Stieltjes integrals with respect to piecewise continuous functions. The definition of such integrals is as follows (see e.g. Shiryaev (1984), Section II.6.10, for more details).

Let H be another piecewise continuous function defined by

$$H(t) = \sum_{k=0}^{n} h_k(t) I_{[T_k, T_{k+1})}(t).$$

We define the integral up to $t \in [0, T]$ of H_- with respect to F, using two alternative notations

$$\int_0^t H_{u-} dF_u = H_- \cdot F_t,$$

the latter being usual in stochastic calculus.

Let G be a function of the above type so that

$$G(t) = \sum_{k=0}^{n} g_k(t) I_{[T_k, T_{k+1})}(t)$$

where

$$g_k(t) = g_k(T_k) + \int_{T_k}^{t} h_k(u) df_k(u)$$

for $t \in [T_k, T_{k+1})$, with $g_0(0) = 0$ and

$$g_k(T_k) = \sum_{j=0}^{k-1} \int_{T_j}^{T_{j+1}} h_j(u) df_j(u) + \sum_{j=1}^{k} h_{j-1}(T_j) \Delta F(T_j)$$

for $k = 1, \ldots, n$. In order to give a proper meaning to the integrals just introduced, assume all $\{f_k\}_{k=0,1,\ldots,n}$ to be of bounded variation. Though this is truely superfluous, as in the present paper only continuously differentiable functions f_k will occur, with $df_k(u)$ to be understood as $f_k'(u)du$ where f_k' is the derivative of f_k.

The integral of H_- with respect to F is now defined by the identity

$$G = H_- \cdot F.$$

Note that for $k = 1, \ldots, n$ and $t \in [T_k, T_{k+1})$

$$g_k(t) - g_{k-1}(T_k) = h_{k-1}(T_k) \Delta F(T_k) + \int_{T_k}^{t} h_k(u) df_k(u).$$

Hence by (2.1.3)

$$\Delta(H_- \cdot F) = H_- \Delta F. \tag{2.2.1}$$

In Section 3 the trajectories of price development are certain piecewise constant functions. In this special case of F given by (2.1.1) with the constant components $f_k(t) \equiv f_k$, it follows from (2.2.1) that

$$H_- \cdot F_t = \sum_{u \in [0,t]} H_{u-} \Delta F_u. \tag{2.2.2}$$

2.3. Integration by parts

Integrals defined in the previous section allow for integrating by parts: for each $t \in [0, T]$

$$H_t F_t - H_0 F_0 = H_- \cdot F_t + F \cdot H_t$$

with the second integral on the right hand side to be understood as

$$F \cdot H_t = F_- \cdot H_t + \sum_{u \in [0,t]} \Delta H_u \Delta F_u$$

(see SHIRYAEV (1984), Section II.6.11 for more details). In the course of proving proposition 3.2.2 we will use the following consequence for this integration by parts formula.

STATEMENT 2.3.1. *Let H', F' and H'', F'' be piecewise continuous functions of the above type. The function*

$$F = F'H' + F''H''$$

has integral representation

$$F - F_0 = H' \cdot F' + H'' \cdot F''$$

if and only if

$$F'_- \cdot H' + F''_- \cdot H'' = 0.$$

2.4. Exponentials

The details on the material of present Section can be found in SHIRYAEV (1984), Section II.6.12. See also JACOD (1979), ELLIOTT (1982) or PROTTER (1990).

Obviously, in case of a continuous function F of bounded variation the solution of the integral equation

$$H_t = 1 + H_- \cdot F_t \tag{2.4.1}$$

is uniquely defined by

$$H_t = e^{F_t - F_0}.$$

In the another extreme case of a piecewise constant F

$$H_t = \prod_{u \in [0,t]} (1 + \Delta F_u).$$

In case of a piecewise continuous function F the above two cases are combined in the solution

$$H_t = e^{F_t - F_0} \prod_{u \in [0,t]} (1 + \Delta F_u) e^{-\Delta F_u}. \tag{2.4.2}$$

In stochastic calculus the solution to (2.4.1) is usually denoted by $H_t = \mathcal{E}(F)_t$ and called Doléans-Dade exponential (or stochastic exponential).

We apply (2.4.2) to the following special case. Let $\mathcal{N} = \{\mathcal{N}_t\}_{t\in[0,T]}$ be the so-called *counting process* with $\mathcal{N}(t)$ which counts the number of jumps observed up to time $t \in [0,T]$. By assumption $\mathcal{N}(0) = 0$. Further $\mathcal{N}(t) = 0$ if no jumps occur up to time t and $\mathcal{N}(t) = k$ if and only if $T_k \leq t$ for $k \in \{1, \dots, n\}$. Let

$$F(t) = a(\mathcal{N}(t) - \lambda t)$$

with certain positive numbers a and λ, cf. (4.1.6). Then

$$\Delta F_t = \begin{cases} a & \text{if } t \in \{T_1, \dots, T_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Hence (cf. (4.1.5))

$$\mathcal{E}(F)_t = (1+a)^{\mathcal{N}(t)} e^{-a\lambda t}.$$

3. Binary market

3.1. *Conditions on the bond and stock price processes*

Consider a binary securities market in which the bond and stock are traded during the time interval $[0,T]$ which is divided in N trading periods by a grid $\{t_0, t_1, \dots, t_N\}$. Unlike in part I, the prices on the bond and stock announced at the n^{th} trading time t_n with $n \in \{0, 1, \dots, N\}$ are now denoted by B_n^N and S_n^N, respectively, in order to express the dependence on N. Moreover, the corresponding price processes $B^N = \{B_t^N\}_{t\in[0,T]}$ and $S^N = \{S_t^N\}_{t\in[0,T]}$ are defined in the entire time interval $[0,T]$ by

(3.1.1)
$$B^N(t) = \sum_{n=0}^{N} B_n^N I_{[t_n, t_{n+1})}(t)$$

and

(3.1.2)
$$S^N(t) = \sum_{n=0}^{N} S_n^N I_{[t_n, t_{n+1})}(t).$$

As in (2.1.1), an additional instant $t_{N+1} \geq T$ is introduced for convenience. Put $B^N(0) = 1$ and $S^N(0) = s$ for simplicity, where s is a certain positive number. The discounted stock price process is denoted as in part I by $\dot{S}^N = \{\dot{S}_t^N\}_{t\in[0,T]}$ with

(3.1.3)
$$\dot{S}^N(t) = \frac{S^N(t)}{B^N(t)}.$$

The bond is a riskless asset and the price process B^N evolves along a prescribed piecewise constant trajectory, while the stock is a risky asset and the price process S^N is allowed to evolve along 2^N different piecewise constant trajectories. These trajectories are specified by the binary transition scheme of part I, Section 3.1. They all start from the same fixed state s, the current state of the stock price

$$s = s_{10}^N > 0.$$

Further, the whole price tree is uniquely determined by two offsprings at each trading time. If at t_{n-1} with $n = 1, \ldots, N$ the stock price was in state $s_{k,n-1}^N$, then at the consecutive trading time t_n it is announced either in state $s_{2k,n}^N$ or $s_{2k-1,n}^N$ with

$$s_{2k,n}^N > s_{2k-1,n}^N > 0.$$

Hence if $t \in [t_n, t_{n+1})$ with some $n = 0, 1, \ldots, N$ the stock price $S^N(t)$ may occupy one of the states $\{s_k^N(t)\}_{k=1,\ldots,2^n}$ with $s_k^N(t) \equiv s_{kn}^N$. Note that by definition (cf. (2.1.2))

$$S^N(t-) = sI_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} S_n^N I_{(t_n,t_{n+1}]}(t). \tag{3.1.4}$$

During the first period $[t_0, t_1)$, for instance, the stock price stays in the current state $s > 0$. At the terminal date $t_N = T$ the stock price $S^N(T)$ may occupy one of 2^N states $s_k^N(T)$ with some $k = 1, \ldots, 2^N$. In this case we also say that the stock price evolves along the k^{th} trajectory. In order to describe the stock price development along this particular trajectory, we specify the stock price state at each $t \in [t_n, t_{n+1})$ for $n = 1, \ldots, N$ by the identity $s_{k_n}^N(t) \equiv s_{k_n,n}^N$ where $k_n = k_n(k, N)$ is the smallest integer exceeding $\frac{k-1}{2^{N-n}}$. Cf. part I, definition (2.1.4), according to which

$$k_n(k, N) = 1 + \left[\frac{k-1}{2^{N-n}}\right]. \tag{3.1.5}$$

Here and elswhere below the largest integer not exceeding a number x is denoted by $[x]$.

We shall now formulate the conditions of the present paper which restrict the behaviour of asset prices in the market so as to allow for the limiting transition in Section 5 when the number of trading periods N increases unboundedly while the length of each trading period, say $\Delta t_n = t_n - t_{n-1}$ with $n \in \{1, \ldots, N\}$, tends to zero. For instance, think of the special case of markets where new prices are announced regularly so that the trading times are equidistant, given by (5.1.1) in Section 5, and the corresponding mesh is given by (5.1.2). In fact, all the entries $\{t_j\}_{j=0,1,\ldots,N}$ in the N^{th} grid depend on N and one should write $\{t_j^N\}_{j=0,1,\ldots,N}$ instead, but for simplicity the upper index is always suppressed.

Our conditions will be formulated in terms of returns on both assets. The cumulative return process on the bond $\mathcal{R}^N = \{\mathcal{R}_t^N\}_{t\in[0,T]}$ is defined as the sum of all previous returns. At the current date $t_0 = 0$ the return equals 0 by convention and at $t \in (0, T]$ it equals

$$\Delta\mathcal{R}^N(t) \equiv \frac{\Delta B_t^N}{B_{t-}^N} = \frac{B_t^N}{B_{t-}^N} - 1. \tag{3.1.6}$$

So

$$\mathcal{R}^N(t) = \sum_{s\in(0,t]} \frac{\Delta B_u^N}{B_{u-}^N} = \int_0^t \frac{dB_s^N}{B_{u-}^N} \tag{3.1.7}$$

and

$$B^N(t) = \prod_{u\in(0,t]} (1 + \Delta\mathcal{R}_u^N) = \mathcal{E}(\mathcal{R}^N)_t, \tag{3.1.8}$$

see Section 2.3.

CONDITION 3.1.1. As $N \to \infty$ the increase of the return process on the bond over each trading period becomes proportional to the length of this period: for each $n = 1, \ldots, N$

$$\frac{\mathcal{R}^N(t_n) - \mathcal{R}^N(t_{n-1})}{t_n - t_{n-1}} = r + \varrho_n^N,$$

where $r > 0$ is a positive constant, while ϱ_n^N is a negligible remainder term.

Obviously, condition 3.1.1 means that the return on the bond at the trading time t_n with $n = 1, \ldots, N$ (when it is non-zero according to definition (3.1.6)) is asymptotically proportional to the length of the preceding period:

$$\Delta\mathcal{R}^N(t_n) = (r + \varrho_n^N)\Delta t_n \sim r\Delta t_n. \tag{3.1.9}$$

Here and elsewhere below the sign $\sim$ indicates that the ratio of the two sides tends to unity. In view of (3.1.6) - (3.1.8), we have for each $t \in [0, T]$ that

$$\mathcal{R}^N(t) \sim rt \tag{3.1.10}$$

and

$$B^N(t) \sim e^{rt}. \tag{3.1.11}$$

Indeed, by (3.1.9)

$$\log B^N(t) \sim \sum_k \log(1 + r\Delta t_k) \sim r \sum_k \Delta t_k$$

(here $\log(1+x) \sim x$ is used) and

$$\mathcal{R}^N(t) \sim r \sum_k \Delta t_k \sim rt$$

where the summation extends over the lengths of all past periods up to time t.

The cumulative return process on the stock $R^N = \{R_t^N\}_{t\in[0,T]}$ is defined similarly as the sum of all previous returns. At the current date $t_0 = 0$ the return equals 0 and at $t \in (0,T]$ it equals

$$\Delta R^N(t) \equiv \frac{\Delta S_t^N}{S_{t-}^N} = \frac{S_t^N}{S_{t-}^N} - 1.$$

So

$$R^N(t) = \int_0^t \frac{dS_u^N}{S_{u-}^N} = \sum_{u\in(0,t]} \frac{\Delta S_u^N}{S_{u-}^N}$$

and

$$S^N(t) = s\mathcal{E}(R^N)_t = s \prod_{u\in(0,t]} (1+\Delta R_u^N), \tag{3.1.12}$$

analogously to (3.1.7) and (3.1.8). In part I, Section 2.3, we also introduced the return process $\grave{R}^N = \{\grave{R}_t^N\}_{t\in[0,T]}$ by

$$\grave{R}^N(t) = \int_0^t \frac{d\grave{S}_u^N}{\grave{S}_{u-}^N} = \sum_{u\in(0,t]} \frac{\Delta \grave{S}_u^N}{\grave{S}_{u-}^N},$$

with the discounted stock price process $\grave{S}^N$ defined by (3.1.3). Obviously,

$$\grave{S}^N(t) = s\mathcal{E}(\grave{R}^N)_t = s \prod_{u\in(0,t]} (1+\Delta \grave{R}_u^N).$$

As was shown in part I, Section 2.3, at each $t \in [0,T]$

$$\Delta \grave{R}^N(t) = \frac{\Delta \grave{S}_t^N}{\grave{S}_{t-}^N} = \frac{\Delta R^N(t) - \Delta \mathcal{R}^N(t)}{1+\Delta \mathcal{R}^N(t)}. \tag{3.1.13}$$

We now formulate conditions on the behaviour of the returns $\{\Delta R^N(t_n)\}_{n=1,\dots,N}$ in terms of their states

$$r_{kn}^N \equiv \frac{s_{kn}^N}{s_{k_{n-1},n-1}^N} - 1, \quad k = 1,\dots,2^n, \tag{3.1.14}$$

where $k_{n-1} = k_{n-1}(k,n) = \left[\frac{k+1}{2}\right]$, cf. (3.1.5).

CONDITION 3.1.2. At the trading time t_n with some $n = 1, \dots, N$ the return on the stock $\Delta R^N(t_n)$ is in one of the 2^n states

$$r^N_{2k,n} = a + \alpha^N_{2k,n}, \quad k = 1, \dots, 2^{n-1}, \tag{3.1.15}$$

or

$$r^N_{2k-1,n} = -(b + \beta^N_{2k-1,n})\Delta t_n, \quad k = 1, \dots, 2^{n-1}, \tag{3.1.16}$$

where a and b are some positive constants, while $\{\alpha^N_{2k,n}\}_{k=1,\dots,2^{n-1}}$ and $\{\beta^N_{2k-1,n}\}_{k=1,\dots,2^{n-1}}$ are negligible remainder terms as $N \to \infty$. In fact b can be negative but exceeding $-r$ with $r > 0$ of condition 3.1.1, to guarantee inequality (3.1.19) below.

Using the same sign $\sim$ as above we may express (3.1.15) and (3.1.16) in the following form

$$r^N_{kn} \sim \begin{cases} a & \text{if } k \text{ is even} \\ -b\Delta t_n & \text{if } k \text{ is odd.} \end{cases} \tag{3.1.17}$$

If condition 3.1.1 holds as well, then the states $\{\grave{r}^N_{kn}\}_{k=1,\dots,2^n}$ of the discounted return $\Delta \grave{R}^N(t_n)$ with $n \in \{1, \dots, N\}$ are approximated as follows. Due to (3.1.13) it follows from (3.1.10) and (3.1.17) that

$$\grave{r}^N_{kn} \sim \begin{cases} a & \text{if } k \text{ is even} \\ -a\lambda\Delta t_n & \text{if } k \text{ is odd} \end{cases} \tag{3.1.18}$$

where

$$\lambda = \frac{r + b}{a} > 0 \tag{3.1.19}$$

is a parameter which later on will play the rôle of the intensity of the Poisson distribution, cf. (4.3.3). Since the even state indices correspond to the upward displacements, and the odd indices to the downward displacements, the asymptotic relations (3.1.17) and (3.1.18) tell us that for N sufficiently large all the upward displacements are of the same order $a > 0$. We call this parameter a the *amplitude* of the upward displacements. On the other hand, all the downward displacements are infinitesimal, of magnitude Δt_n. Moreover, the range of the parameter b is restricted (see inequality (3.1.19)) so as to guarantee the negative downward displacements of the discounted returns in (3.1.18). This will allow us to assign the weights $\lambda\Delta t_n$ and $1 - \lambda\Delta t_n$ to the upward and downward displacements, respectively, which may be interpreted as risk neutral probabilities, see formula (3.2.18) and related comments. In the probabilistic interpretation the upward displacements become rare events with the probability of occurrence $\lambda\Delta t_n$ and this provides for the conditions under which the Poisson approximation of Section 5 is valid.

REMARK 3.1.3. In condition 3.1.1 the remainder term ϱ_n^N is called negligible, because it can be suppressed in the asymptotic relation (3.1.9), as well as in its consequences (3.1.10) and (3.1.11). This is achieved, in particular, if

$$\max_{n\in\{1,\ldots,N\}} |\varrho_n^N| \to 0$$

as $N \to \infty$. The negligibility of ϱ_n^N will be used several times below; see, for instance, lemma 3.2.7 and formula (3.2.20). The same applies to the remainder terms $\{\alpha_{2k,n}^N\}_{k=1,\ldots,2^{n-1}}$ and $\{\beta_{2k-1,n}^N\}_{k=1,\ldots,2^{n-1}}$ in condition 3.1.2, since they are negligible in the asymptotic relation (3.1.17) (cf. lemma 3.2.7 below, where yet another set of negligible remainder terms (3.2.20) occurs). For instance, let $\{\alpha_{2k,n}^N\}_{k=1,\ldots,2^{n-1}}$ to satisfy

$$\max_{n\in\{1,\ldots,N\}} \max_{k\in\{1,\ldots,2^{n-1}}\}|\alpha_{2k,n}^N| \to 0$$

as $N \to \infty$.

3.2. Self-financing strategies and value processes

Suppose that one invests an amount $v \geq 0$ in the two assets described in Section 3.1. Let Ψ_0 and Φ_0 denote the number of shares of the bond and stock, respectively, owned by the investor at the current date $t_0 = 0$. Since $B^N(0) = 1$ and $S^N(0) = s$, the investment equals

$$v = \Psi_0 + s\Phi_0. \tag{3.2.1}$$

Furthermore, let Ψ_n^N and Φ_n^N denote the number of shares of the bond and stock, respectively, owned by the investor at the consecutive trading times t_n, $n = 1,\ldots,N$. The couple (Ψ_n^N, Φ_n^N) is called the investor's *portfolio* at time t_n. Observe that the components Ψ_n^N and Φ_n^N of a portfolio may become negative, which has to be interpreted as short-selling the bond or stock.

Since the investor selects his portfolio at time t_n with $n = 1,\ldots,N$ on the basis of the history of the price development in the market, the number of shares Ψ_n^N and Φ_n^N of the bond and stock he owns at time t_n may depend on prices B_ν^N and S_ν^N with $\nu < n$, but not on prices not yet announced, e.g. B_n^N and S_n^N. In particular

$$(\Psi_0, \Phi_0) = (\Psi_1^N, \Phi_1^N),$$

which means that the currently selected portfolio is kept unchanged during the whole first period $[t_0, t_1]$. Afterwords, just after the stock price S_1^N is announced at time t_1 the portfolio turns into (Ψ_2^N, Φ_2^N) and stays unchanged during the whole period $(t_1, t_2]$. The investor proceeds further in the same manner, selecting last time his portfolio (Ψ_N^N, Φ_N^N) just after the announcement of the stock price S_{N-1}^N at time t_1 and keeping it until the terminal date $t_N = T$.

The process $\pi^N = (\Psi_t^N, \Phi_t^N)_{t\in[0,T]}$ with the bond and stock components

$$\Psi^N(t) = \Psi_1^N I_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} \Psi_{n+1}^N I_{(t_n,t_{n+1}]}(t) \tag{3.2.2}$$

and

$$\Phi^N(t) = \Phi_1^N I_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} \Phi_{n+1}^N I_{(t_n,t_{n+1}]}(t) \tag{3.2.3}$$

is called a *trading strategy*. Note that the possible trajectories of both components are piecewise constant functions of the same type as those of S_-^N (cf. (3.2.2), (3.2.3) and (3.1.4)). At each $t \in (0,T]$ the dependence of the portfolio only on the past prices means that if $t \in (t_n, t_{n+1}]$ and $S^N(t-)$ is in state s_{kn}^N for some $n = 0, 1, \ldots, N-1$ and $k = 1, \ldots, 2^n$, then (Ψ_t^N, Φ_t^N) is in state

$$(\Psi_{n+1}^N(s_{kn}^N), \Phi_{n+1}^N(s_{kn}^N)).$$

In stochastic calculus processes of this type are called *simple predictable*, cf. PROTTER (1990), p. 43.

With each trading strategy π^N we associate the process

$$V^N(\pi) = \{V_t^N(\pi)\}_{t\in[0,T]}$$

by

$$V^N(t;\pi) = \Psi^N(t)B^N(t) + \Phi^N(t)S^N(t)$$

so that $V^N(0;\pi) = v \geq 0$, cf. (3.2.1). This process is usually called the *value process* for a trading strategy π^N, since $V^N(t;\pi)$ represents the market value of the portfolio at time t held just before any changes are made in the portfolio. It will be shown below that the value process $V^N(\pi)$ is of special structure when π^N belongs to the following class of trading strategies:

DEFINITION 3.2.1. A trading strategy is said to be *self-financing* if the construction is founded only on the initial endowment so that all changes in the portfolio values due to capital gains during trading and no infusion or withdrawal of funds takes place. Then the corresponding portfolio satisfies the condition: for all $t \in [0,T]$

$$B_-^N \cdot \Psi_t^N + S_-^N \cdot \Phi_t^N = 0. \tag{3.2.4}$$

The notion just introduced is of universal use whenever the integrals in (3.2.4) are well-defined, which in the present special case of piecewise constant portfolio components (3.2.2) and (3.2.3) are particularly simple:

$$B_-^N \cdot \Psi_t^N = \sum_{u\in[0,t]} B_{u-}^N \Delta\Psi_u^N \tag{3.2.5}$$

and

$$S_-^N \cdot \Phi_t^N = \sum_{u\in[0,t]} S_{u-}^N \Delta\Phi_u^N, \tag{3.2.6}$$

cf. (2.2.2). Hence (3.2.4) is equivalent to

$$B_-^N \Delta\Psi^N + S_-^N \Delta\Phi^N = 0. \tag{3.2.7}$$

Using the integrating by parts formula of Section 2.3, we obtain exactly in the same manner as in part I, Section 3.2, the following characterization of self-financing strategies.

PROPOSITION 3.2.2. *A trading strategy π^N is self-financing if and only if its discounted value process $\grave{V}^N(\pi) = \{\grave{V}_t^N(\pi)\}_{t\in[0,T]}$ admits the following integral representation: at each $t \in [0,T]$*

$$\grave{V}^N(t;\pi) = v + \Phi^N \cdot \grave{S}_t^N. \tag{3.2.8}$$

PROOF. In view of (3.2.5) - (3.2.7), the integral representation (3.2.8) follows by the same arguments as in part I, proposition 3.2.1. But it is valid also in general whenever the integrals are well-defined and statement 2.3.1 holds. This is easily seen by taking into consideration that (3.2.4) is equivalent to

$$\Psi_t^N - \Psi_0 + \grave{S}_-^N \cdot \Phi_t^N = 0.$$

□

REMARK 3.2.3. It is important to notice that the value process for a self-financing strategy is a process of the same type as the stock price process, since it evolves along one of 2^N piecewise constant trajectories. In fact

$$\grave{V}^N(t;\pi) = \sum_{n=0}^{N} \grave{V}_n^N(\pi) I_{[t_n,t_{n+1})}(t)$$

where $\grave{V}_n^N(\pi)$ may occupy one of the states

$$\grave{v}_{kn}^N(\pi) = v + \sum_{\nu=1}^{n} \Phi_\nu^N(s_{k_{\nu-1},\nu-1}^N)(\grave{s}_{k_\nu,\nu}^N - \grave{s}_{k_{\nu-1},\nu-1}^N) \tag{3.2.9}$$

with $k = 1, \ldots, 2^n$, cf. part I, remark 3.2.3. Recall that $k_\nu = k_\nu(k,n)$ is given by (3.1.5).

REMARK 3.2.4. Suppose that currently an amount (3.2.1) is invested by selecting the portfolio (Ψ_0, Φ_0) which afterwards is kept unchanged. Clearly, this particular strategy of keeping the constant portfolio $(\Psi_t^N, \Phi_t^N) \equiv (\Psi_0, \Phi_0)$ all the time $t \in [0,T]$ is self-financing - this needs no infusion or withdrawal of funds (both terms on the left-hand side of (3.2.7) equal 0). The integral representation (3.2.8) gives

$$\grave{V}^N(t;\pi) - v = \Phi_0(\grave{S}^N(t) - s).$$

In the trivial case $(\Psi_0, \Phi_0) = (0,1)$ this reduces to the identity $V^N(\pi) \equiv S^N$.

In order to rewrite the integral representation (3.2.8) in the form of Clark's formula (cf. part I, Section 3.3), define the difference operator D in the state space that is applied to the stock price process S^N according to the following definition (cf. part I, definition 2.4.1).

DEFINITION 3.2.5. The process $DS^N = \{DS_t^N\}_{t\in[0,T]}$ is defined by

$$DS^N(t) = DS_1^N I_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} DS_{n+1}^N I_{(t_n,t_{n+1}]}(t)$$

which is of the same type as $S^N(t-)$, cf. (3.1.4). Its states are defined conditionally on those of $S^N(t-)$: if $S^N(t-)$ is in state s_{kn}^N with some $k = 1, \ldots, 2^n$, i.e. $t \in (t_n, t_{n+1}]$ according to (3.1.4), then $DS^N(t)$ is in the same state as DS_{n+1}^N which is $s_{2k,n+1}^N - s_{2k-1,n+1}^N > 0$. Taking into consideration remark 3.2.3, define similarly

$$DV^N(t;\pi) = DV_1^N(\pi) I_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} DV_{n+1}^N(\pi) I_{(t_n,t_{n+1}]}(t)$$

which is in state $v_{2k,n+1}^N(\pi) - v_{2k-1,n+1}^N(\pi)$, provided $S^N(t-)$ is in state s_{kn}^N. Introduce finally

$$\frac{DV^N(t;\pi)}{DS^N(t)} = \frac{DV_1^N(\pi)}{DS_1^N} I_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} \frac{DV_{n+1}^N(\pi)}{DS_{n+1}^N} I_{(t_n,t_{n+1}]}(t). \tag{3.2.10}$$

The results similar to proposition 3.3.1 and corollary 3.3.2 in part I are now formulated as follows:

PROPOSITION 3.2.6. *Under the self-financing condition* (3.2.4) *the stock component of the portfolio is given by* (3.2.10): *for each* $t \in [0, T]$

$$\Phi^N(t) = \frac{DV^N(t;\pi)}{DS^N(t)}$$

and therefore the integral representation (3.2.8) *takes the form*

$$\grave{V}^N(t;\pi) = v + \frac{DV^N(\pi)}{DS^N} \cdot \grave{S}_t^N.$$

In the sequel we focus our attention on markets excluding arbitrage opportunities (see part I, Section 6) in which

$$\grave{s}_{2k-1,n}^N < \grave{s}_{k,n-1}^N < \grave{s}_{2k,n}^N$$

for each $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$. This means that the numerical values of

$$p_{2k,n}^N = \frac{\grave{s}_{k,n-1}^N - \grave{s}_{2k-1,n}^N}{\grave{s}_{2k,n}^N - \grave{s}_{2k-1,n}^N} \tag{3.2.11}$$

and

$$(3.2.12) \qquad p_{2k-1,n}^N = \frac{\grave{s}_{2k,n}^N - \grave{s}_{k,n-1}^N}{\grave{s}_{2k,n}^N - \grave{s}_{2k-1,n}^N}$$

are positive and satisfy

$$(3.2.13) \qquad p_{2k,n}^N + p_{2k-1,n}^N = 1.$$

As is easily seen (cf. part I, Section 3.4), in this case every state $s_{k,n-1}^N$ at trading time t_{n-1} is expressed as a convex combination of two alternative states $\grave{s}_{2k-1,n}^N$ and $\grave{s}_{2k,n}^N$ at the next trading time t_n, i.e.

$$(3.2.14) \qquad \grave{s}_{k,n-1}^N = p_{2k,n}^N \grave{s}_{2k,n}^N + p_{2k-1,n}^N \grave{s}_{2k-1,n}^N.$$

By these considerations the numerical values of $p_{2k,n}^N$ and $p_{2k-1,n}^N$ are called in part I the *risk neutral probabilities*, see part I, remark 3.4.1. It is easily verified that in terms of the states of the discounted returns (3.1.13), relation (3.2.14) turns into

$$(3.2.15) \qquad 0 = p_{2k,n}^N \grave{r}_{2k,n}^N + p_{2k-1,n}^N \grave{r}_{2k-1,n}^N$$

which means that the weights (3.2.11) and (3.2.12) with property (3.2.13) are chosen so as to neutralize the upward displacements in the discounted returns by the downward displacements (corresponding to the even and odd space indices, respectively). The identity (3.2.15) is easily derived:

$$\begin{aligned} 0 &= p_{2k,n}^N \left(\grave{s}_{2k,n}^N - \grave{s}_{k,n-1}^N\right) + p_{2k-1,n}^N \left(\grave{s}_{2k-1,n}^N - \grave{s}_{k,n-1}^N\right) \\ &= p_{2k,n}^N \left(r_{2k,n}^N - \Delta\mathcal{R}^N(t_n)\right) + p_{2k-1,n}^N \left(r_{2k-1,n}^N - \Delta\mathcal{R}^N(t_n)\right) \\ &= p_{2k,n}^N \grave{r}_{2k,n}^N + p_{2k-1,n}^N \grave{r}_{2k-1,n}^N \end{aligned}$$

Furthermore, it is shown in part I, proposition 3.6.1, that the relations (3.2.14) extend to the states (3.2.9) of the value process $V^N(\pi)$ for any self-financing strategy π^N: for $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$ we also have

$$(3.2.16) \qquad \grave{v}_{k,n-1}^N(\pi) = p_{2k,n}^N \grave{v}_{2k,n}^N(\pi) + p_{2k-1,n}^N \grave{v}_{2k-1,n}^N(\pi).$$

Note that (3.2.16) reduces to (3.2.14) in the trivial case of remark 3.2.4 when only one share of the stock is kept all the time. Similarly, the relations of part I, corollary 3.6.2,

$$(3.2.17) \qquad p_{2k,n}^N = \frac{\grave{v}_{k,n-1}^N(\pi) - \grave{v}_{2k-1,n}^N(\pi)}{\grave{v}_{2k,n}^N(\pi) - \grave{v}_{2k-1,n}^N(\pi)}$$

and

$$p_{2k-1,n}^N = \frac{\grave{v}_{2k,n}^N(\pi) - \grave{v}_{k,n-1}^N(\pi)}{\grave{v}_{2k,n}^N(\pi) - \grave{v}_{2k-1,n}^N(\pi)}$$

reduce in this special case to (3.2.11) and (3.2.12).

It will be shown next that under the conditions 3.1.1 and 3.1.2 we have the following approximation to the risk neutral probabilities: for each $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$

$$p^N_{2k,n} \sim \lambda \Delta t_n. \tag{3.2.18}$$

By (3.1.19) the intensity λ is positive so that the expression on the right hand side of (3.2.18) may be interpreted as the risk neutral probability of the upward displacements. We have already mentioned this at the end of the previous section. By (3.1.18) and (3.2.18)

$$p^N_{2k,n}\hat{r}^N_{2k,n} + p^N_{2k-1,n}\hat{r}^N_{2k-1,n} \sim a\lambda\Delta t_n + (1 - \lambda\Delta t_n)(-a\lambda\Delta t_n) = a(\lambda\Delta t_n)^2,$$

which is of a lower magnitude than (3.2.18). Clearly, this meets (3.2.15), with 0 on the left hand side. Thus the approximation (3.2.18) ensures that the contribution of the approximate upward displacements in (3.1.18) is neutralized by the contribution of the downward displacements.

LEMMA 3.2.7. *Under the conditions* 3.1.1 *and* 3.1.2 *we have for each* $n = 1, \ldots, N$ *that*

$$p^N_{2k,n} = (\lambda + \lambda^N_{2k,n})\Delta t_n, \quad k = 1, \ldots, 2^{n-1}, \tag{3.2.19}$$

with negligible remainder terms $\{\lambda^N_{2k,n}\}_{k=1,\ldots,2^{n-1}}$.

PROOF. It will be shown that with the notations of the conditions 3.1.1 and 3.1.2

$$\lambda^N_{2k,n} = \frac{\varrho^N_n + \beta^N_{2k-1,n}}{r^N_{2k,n} - r^N_{2k-1,n}} - \lambda\frac{\alpha^N_{2k,n} - r^N_{2k-1,n}}{r^N_{2k,n} - r^N_{2k-1,n}}, \tag{3.2.20}$$

which is indeed negligible under these conditions (cf. remark 3.1.3). To this end, we first rewrite (3.2.19) in terms of the returns on both assets. By (3.1.3), (3.1.6) and (3.1.14)

$$p^N_{2k,n} = \frac{\Delta\mathcal{R}^N(t_n) - r^N_{2k-1,n}}{r^N_{2k,n} - r^N_{2k-1,n}}.$$

Due to (3.1.9), (3.1.15) and (3.1.16), it follows from the latter equality that

$$\frac{p^N_{2k,n} - \lambda\Delta t_n}{\Delta t_n} = \frac{\varrho^N_n + \beta^N_{2k-1,n}}{r^N_{2k,n} - r^N_{2k-1,n}} - \lambda\frac{\alpha^N_{2k,n} + (b + \beta^N_{2k-1,n})\Delta t_n}{r^N_{2k,n} - r^N_{2k-1,n}}.$$

Compare this with (3.2.19). It is easily seen that (3.2.20) holds. □

The approximate relations (3.2.22) asserted in the next proposition, may be viewed as a prelimiting version of equations (4.2.12) for the Poisson model to be discussed in Section 4.

PROPOSITION 3.2.8. *Let π^N be a self-financing strategy and $\grave{V}^N(\pi)$ its discounted value process.*

Then for each $n = 1, \ldots, N$ the states $\{\grave{v}^N_{kn}(\pi)\}_{k=1,\ldots,2^n}$ given by (3.2.9), *satisfy the following identities:*

$$\frac{\grave{v}^N_{2k-1,n}(\pi) - \grave{v}^N_{k,n-1}(\pi)}{t_n - t_{n-1}} = -(\lambda + \lambda^N_{2k,n})(\grave{v}^N_{2k,n}(\pi) - \grave{v}^N_{2k-1,n}(\pi)) \tag{3.2.21}$$

with λ and $\{\lambda^N_{2k,n}\}_{k=1,\ldots,2^{n-1}}$ given by (3.1.19) *and* (3.2.20), *respectively. If, moreover, the conditions* 3.1.1 *and* 3.1.2 *hold, then*

$$\frac{\grave{v}^N_{2k-1,n}(\pi) - \grave{v}^N_{k,n-1}(\pi)}{t_n - t_{n-1}} \sim -\lambda(\grave{v}^N_{2k,n}(\pi) - \grave{v}^N_{2k-1,n}(\pi)). \tag{3.2.22}$$

REMARK 3.2.9. It is easily verified that the equations (3.2.21) and (3.2.22) take the following undiscounted form:

$$\begin{aligned} \frac{v^N_{2k-1,n}(\pi) - v^N_{k,n-1}(\pi)}{t_n - t_{n-1}} &- (r + \varrho^N_n) v^N_{k,n-1}(\pi) \\ &= -(\lambda + \lambda^N_{2k,n})(v^N_{2k,n}(\pi) - v^N_{2k-1,n}(\pi)) \end{aligned} \tag{3.2.23}$$

and

$$\frac{v^N_{2k-1,n}(\pi) - v^N_{k,n-1}(\pi)}{t_n - t_{n-1}} - r v^N_{k,n-1}(\pi) \sim -\lambda(v^N_{2k,n}(\pi) - v^N_{2k-1,n}(\pi)). \tag{3.2.24}$$

PROOF. We prove at once the undiscounted equations (3.2.23) and (3.2.24), departing from the following undiscounted version of equation (3.2.17):

$$p^N_{2k,n} = \frac{(1 + \Delta\mathcal{R}^N(t_n)) v^N_{k,n-1}(\pi) - v^N_{2k-1,n}(\pi)}{v^N_{2k,n}(\pi) - v^N_{2k-1,n}(\pi)},$$

see (3.1.6) for the definition of $\Delta\mathcal{R}^N$. By (3.2.19)

$$\frac{(1 + \Delta\mathcal{R}^N(t_n)) v^N_{k,n-1}(\pi) - v^N_{2k-1,n}(\pi)}{v^N_{2k,n}(\pi) - v^N_{2k-1,n}(\pi)} = (\lambda + \lambda^N_{2k,n})\Delta t_n.$$

Due to (3.1.9), this is equivalent to

$$\begin{aligned} &v^N_{k,n-1}(\pi) - v^N_{2k-1,n}(\pi) + (r + \varrho^N_n) v^N_{k,n-1}(\pi)\Delta t_n \\ &= (v^N_{2k,n}(\pi) - v^N_{2k-1,n}(\pi))(\lambda + \lambda^N_{2k,n})\Delta t_n \end{aligned}$$

which yields (3.2.23). In conclusion, (3.2.23) implies (3.2.24), since ϱ^N_n and $\{\lambda^N_{2k,n}\}_{k=1,\ldots,2^{n-1}}$ are negligible remainder terms. □

3.3. Completeness, hedging strategy and option valuation
Given the states of the discounted stock price over the entire trading period $[0,T]$, the risk neutral probabilities $\{p_{kn}^N\}_{k=1,\ldots,2^n}$ are determined by (3.2.11) and (3.2.12). For each fixed $n \in \{1,\ldots,N\}$ and $k \in \{1,\ldots,2^N\}$, we define

$$P_{n|\nu}^N(k) = p_{k_{\nu+1},\nu+1}^N \cdots p_{k_n,n}^N, \quad \nu = 0,1,\ldots,n-1, \tag{3.3.1}$$

where $k_\nu = k_\nu(k,n)$ is given by (3.1.5). Put $P_{n|n}^N(k) \equiv 1$ for convenience. Denote $P_{kn}^N = P_{n|0}^N(k)$ so that

$$P_{kn}^N = p_{k_1,1}^N \cdots p_{k_n,n}^N. \tag{3.3.2}$$

Note that $p_{kn}^N = P_{n|n-1}^N(k)$. We usually write $P_{kN}^N = P_k^N(T)$. We use these notations to describe the solution of the system of recurrent equations (cf. (3.2.14) and (3.2.16))

$$\grave{x}_{k,n-1} = p_{2k,n}^N \grave{x}_{2k,n} + p_{2k-1,n}^N \grave{x}_{2k-1,n} \tag{3.3.3}$$

for $n = 1,\ldots,N$ and $k = 1,\ldots,2^{n-1}$, subject to the boundary conditions

$$\grave{x}_{kN} = \grave{w}_k^N(T), \quad k = 1,\ldots,2^N, \tag{3.3.4}$$

with given numbers $\{\grave{w}_k^N(T)\}_{k=1,\ldots,2^N}$. They are assumed to be given in the form

$$\grave{w}_k^N(T) = \frac{W(s_k^N(T))}{B^N(T)}.$$

For $n = 0,1,\ldots,N$ the solutions $\{\grave{x}_{k,N-n}\}_{k=1,\ldots,2^{N-n}}$ of these equations are obtained by

$$\grave{x}_{k,N-n} = \sum_{2^n(k-1)<j\le 2^n k} P_{N|N-n}^N(j)\grave{w}_j^N(T). \tag{3.3.5}$$

In particular

$$x_{10} = \sum_{j=1}^{2^N} P_j^N(T)\grave{w}_j^N(T). \tag{3.3.6}$$

In the trivial case of $W(x) = x$ the boundary conditions are

$$\grave{x}_{kN} = \grave{s}_k^N(T), \quad k = 1,\ldots,2^N$$

(cf. (3.3.4)) and the solutions (3.3.5) and (3.3.6) reduce to

(3.3.7) $$\dot{s}_{k,N-n} = \sum_{2^n(k-1)<j\leq 2^n k} P^N_{N|N-n}(j)\dot{s}^N_j(T)$$

and

(3.3.8) $$s = \sum_{j=1}^{2^N} P^N_j(T)\dot{s}^N_j(T).$$

A similar relationship is satisfied by the states of a value process for a self-financing strategy (cf. (3.2.16)), and this results in the *completeness* of a binary market in the sense to be described next.

Under the circumstances of Section 3.1, consider an investor who is willing to invest now (at $t = 0$) in the bond and the stock in order to attain at the terminal date T a certain wealth, say $W^N(T)$, by trading over N periods without infusion or withdrawal of funds. Knowing the conditions in the market, i.e. knowing the 2^N possible trajectories of the stock price development up to the terminal date T (which correspond as usual to the states $\{s^N_k(T)\}_{k=1,\dots,2^N}$ of the stock price $S^N(T)$), the investor determines the wealth he desires to attain at the terminal date T by evaluating each of these possibilities. In this way $W^N(T)$ is interpreted as a variable which is in one of the 2^N possible states: in state $w^N_k(T)$ say, if the stock price is in state $s^N_k(T)$. In other words, $W^N(T)$ is a certain function of $S^N(T)$, say $W^N(T) = W(S^N(T))$ and $w^N_k(T) = W(s^N_k(T))$ for $k = 1, \dots, 2^N$.

DEFINITION 3.3.1. A binary market is *complete* if any desired wealth $W^N(T)$ of the above type is attainable with a certain initial endowment: there is a self-financing strategy π^N whose value process at the terminal date attains the identity $V^N(T;\pi) = W^N(T)$. The necessary initial endowment is then $v = V^N(0;\pi)$.

As is shown in part I, proposition 4.3.3, the present market is indeed complete and, moreover, there exists a unique strategy, called the *hedging strategy* against $W^N(T)$, which attains this wealth. In part I, Section 4.3, one can find the detailed construction of such strategy. Here we only note that the procedure is based on the solution of the equations (3.3.3), subject to the boundary conditions (3.3.4) with the states of the discounted desired wealth on the right hand side. If $\dot{W}^N_n$ is a variable with the possible states $\{\dot{w}^N_{kn}\}_{k=1,\dots,2^n}$ which are identified with the solutions (3.3.5) (so that $\dot{x}_{kn} = \dot{w}^N_{kn}$), then a process $\dot{W}^N = \{\dot{W}^N_t\}_{t\in[0,T]}$ is formed by

$$\dot{W}^N(t) = \sum_{n=0}^{N} \dot{W}^N_n I_{[t_n,t_{n+1})}(t).$$

Obviously, at the terminal date T this process attains the desired wealth. According to (3.3.5) and (3.3.6), it starts from

$$w_{10}^N = \sum_{j=1}^{2^N} P_j^N(T)\dot{w}_j^N(T) \tag{3.3.9}$$

and then, for $n = 1, \ldots, N$,

$$\dot{w}_{kn}^N = \sum_{2^{N-n}(k-1)<j\leq 2^{N-n}k} P_{N|n}^N(j)\dot{w}_j^N(T), \quad k = 1, \ldots, 2^n. \tag{3.3.10}$$

The hedging strategy against $\dot{W}^N(T)$ is then a unique strategy π^N whose value process $V^N(\pi)$ coincides with the process W^N formed above.

In part I, Section 5, formula (3.3.9) is applied to the following problem of option pricing. Suppose that today, at time $t = 0$, we are going to sign a contract that gives us the right to buy one share of a stock at a specified price K, called the *exercise price*, and at a specified time T, called the *maturity* or *expiration time*. If the stock price $S^N(T)$ is below the exercise price, i.e. $S^N(T) \leq K$, then the contract is worthless to us. On the other hand, if $S^N(T) > K$, we can exercise our option: we can buy one share of the stock at the fixed price K and then sell it immediately in the market for the price $S^N(T)$. Thus this option, called the *European call option*, yields a profit at maturity T equal to

$$\max\{0, S^N(T) - K\} = (S^N(T) - K)^+. \tag{3.3.11}$$

The function (3.3.11) of the stock price $S^N(T)$ is called the *payoff function* for the European call option. A contract with some fixed payoff function $H^N(T) = H(S^N(T))$, where $H^N(T)$ is a nonnegative variable with possible states $H(s_k^N(T))$ (not necessarily of form (3.3.11)) is called a *contingent claim.* The European call option is thus a special contingent claim with payoff (3.3.11).

Now, how much would we be willing to pay at time $t = 0$ for a ticket which gives the right to buy at maturity $t = T$ one share of stock with exercise price K? To put this in another way, what is a fair price to pay at time $t = 0$ for the ticket? In order to determine the fire price of a contingent claim, consider the following procedure:

(i) construct the hedging strategy against the contingent claim in question, which duplicates the payoff;
(ii) determine the initial wealth needed for construction in (i);
(iii) equate this initial wealth to the fair price of the contingent claim.

In other words, construct the hedging strategy π^N against the contingent claim with a payoff function $H^N(T)$, whose value process $V^N(\pi)$ coincides with a process that is obtained exactly in the same manner as the process $\dot{W}^N$ by solving the equations (3.3.3), but now subject to the boundary conditions (3.3.4) with

$$\grave{h}_k^N(T) = \frac{H(s_k^N(T))}{B^N(T)}$$

instead of $\grave{w}_k^N(T)$. This strategy indeed duplicates the payoff. It requires the initial wealth that yields the fair price $C^N = C(H^N)$ of the contingent claim with the payoff function $H^N(T)$, which amounts to

$$C(H^N) = \sum_{j=1}^{2^N} P_j^N(T)\grave{h}_j^N(T).$$

The European call option (3.3.11), in particular, has a special payoff function depending only on the stock price at maturity $t_N = T$ and its fair price is

$$C^N = \sum_{j=1}^{2^N} P_j^N(T)(\grave{s}_j^N(T) - \grave{K})^+. \tag{3.3.12}$$

4. Poisson market

4.1. *Asset pricing*

In this Section we consider the limiting model for a securities market. According to (3.1.10) and (3.1.11), the model for the bond is defined by the linear return process $\mathcal{R}^\circ = \{\mathcal{R}_t^\circ\}_{t\in[0,T]}$ with

$$\mathcal{R}_t^\circ = rt \tag{4.1.1}$$

and the exponential price process $B^\circ = \{B_t^\circ\}_{t\in[0,T]}$ with

$$B_t^\circ = e^{rt}, \tag{4.1.2}$$

where $r > 0$ is a riskless interest rate on the bond. Note that $B^\circ = \mathcal{E}(\mathcal{R}^\circ)$ in the sense of Section 2.4.

The stock is again a risky asset and its return process $R^\circ = \{R_t^\circ\}_{t\in[0,T]}$ may jump unexpectedly at certain instants. Let m be a number of jumps, a nonnegative integer equal zero if no jumps occur. Otherwise, if $m > 0$ jumps occur, we denote by $T_1, \ldots, T_m$ the consecutive instants. We assume $T_k < T_{k+1}$ for all $k = 0, 1, \ldots, m$ by adding $T_0 = 0$ and $T_{m+1} \geq T$ for convenience. Thus $\Delta R^\circ(t) = 0$ for all $t \in [0, T]$ except for

$$\Delta R^\circ(T_k) = a > 0, \quad k = 1, \ldots, m.$$

In order to describe cumulative return process R° we define the so-called *counting process* $\mathcal{N} = \{\mathcal{N}_t\}_{t\in[0,T]}$ with $\mathcal{N}(t)$ which counts the number of jumps observed up to time $t \in [0, T]$. By assumption $\mathcal{N}(0) = 0$. Further $\mathcal{N}(t) = 0$ if no jumps occur up to time t and $\mathcal{N}(t) = k$ if $T_k \le t$ for $k \in \{1, \dots, m\}$. In particular $\mathcal{N}(T) = m$. We define next the limiting return process R° on the stock in accordance with the right-hand side of (3.1.17): the cumulative effect on the upward displacement yields $a\mathcal{N}(t)$ and that of the downward displacement yields $-bt$. This leads to the model

$$R^\circ(t) = a\mathcal{N}(t) - bt. \tag{4.1.3}$$

Consequently, the price process on the stock $S^\circ = \{S^\circ_t\}_{t\in[0,T]}$ is now defined by

$$\begin{aligned} S^\circ(t) &= s\mathcal{E}(R^\circ)_t \\ &= s(1+a)^{\mathcal{N}(t)}e^{-bt} \end{aligned} \tag{4.1.4}$$

(cf. Section 2.4) where $s > 0$ is a fixed current price on the stock $S^\circ(0) = s$.

By (4.1.2) and (4.1.4) the discounted stock price process is defined by

$$\begin{aligned} \grave{S}^\circ(t) &\equiv \frac{S^\circ(t)}{B^\circ(t)} \\ &= s(1+a)^{\mathcal{N}(t)}e^{-a\lambda t} \end{aligned} \tag{4.1.5}$$

and the corresponding return process $\grave{R}^\circ = \{\grave{R}^\circ_t\}_{t\in[0,T]}$ by

$$\grave{R}^\circ(t) = a(\mathcal{N}(t) - \lambda t), \tag{4.1.6}$$

cf. (3.1.18). The relation $\grave{S}^\circ = s\mathcal{E}(\grave{R}^\circ)$ is obtained in Section 2.4.

The price process on the stock may be presented alternatively by introducing the state $s^\circ_k(t)$ of this process in the interval $[T_k, T_{k+1}), k = 0, 1, \dots, m$. By (4.1.4)

$$s^\circ_k(t) = s(1+a)^k e^{-bt} \tag{4.1.7}$$

(note that in the present case no distinction is needed between states of the stock price and their numerical values, due to one to one correspondence) and at each $t \in [0, T]$

$$S^\circ(t) = \sum_{k=0}^{m} s^\circ_k(t) I_{[T_k, T_{k+1})}(t).$$

Similarly,

$$\dot{S}^\circ(t) = \sum_{k=0}^{m} \dot{s}_k^\circ(t) I_{[T_k, T_{k+1})}(t) \tag{4.1.8}$$

with the states

$$\dot{s}_k^\circ(t) = s(1+a)^k e^{-a\lambda t} \tag{4.1.9}$$

in the interval $[T_k, T_{k+1}), k = 0, 1, \ldots, m$.

Analogously to (3.1.4)

$$S^\circ(t-) = s_0^\circ(t) I_{[0,T_1]}(t) + \sum_{k=1}^{m} s_k^\circ(t) I_{(T_k, T_{k+1}]}(t). \tag{4.1.10}$$

Suppose that $S^\circ(t-)$ is in state $s_k^\circ(t)$. Then at time t the stock price either stays in state $s_k^\circ(t)$ or jump to state $s_{k+1}^\circ(t)$. This observation leads to the following definition of the difference operator $DS^\circ(t)$ in the state space of the present market: if $S^\circ(t-)$ is in state $s_k^\circ(t)$, then $DS^\circ(t)$ in the state

$$D_k(S_t^\circ) = s_k^\circ(t) - s_{k-1}^\circ(t). \tag{4.1.11}$$

Hence

DEFINITION 4.1.1. The process $DS^\circ = \{DS_t^\circ\}_{t \in [0,T]}$ is defined by

$$DS^\circ(t) = D_1(S_t^\circ) I_{[0,T_1]}(t) + \sum_{k=1}^{m} D_{k+1}(S_t^\circ) I_{(T_k, T_{k+1}]}(t) \tag{4.1.12}$$

with the states given by (4.1.11). The process $D\dot{S}^\circ = \{D\dot{S}_t^\circ\}_{t \in [0,T]}$ is defined similarly so that

$$D\dot{S}^\circ(t) = \frac{DS^\circ(t)}{B^\circ(t)}.$$

PROPOSITION 4.1.2. *The states* (4.1.7) *and* (4.1.9) *of the stock price process and its discounted version satisfy the following differential equations:*

$$\frac{ds_k^\circ(t)}{dt} - r s_k^\circ(t) = -\lambda D_{k+1}(S_t^\circ), \quad t \in (T_k, T_{k+1}], \tag{4.1.13}$$

and

(4.1.14) $$\frac{d\grave{s}_k^\circ(t)}{dt} = -\lambda D_{k+1}(\grave{S}_t^\circ), \quad t \in (T_k, T_{k+1}].$$

PROOF. By (4.1.7) it follows from (4.1.11) that

$$D_k(S_t^\circ) = as(1+a)^{k-1}e^{-bt} = as_{k-1}^\circ(t).$$

Therefore, by differentiating both sides of (4.1.7) we get

$$\frac{ds_k^\circ(t)}{dt} - rs_k^\circ(t) = -(r+b)s_k^\circ(t) = -\lambda a s_k^\circ(t) = -\lambda D_{k+1}(S_t^\circ),$$

which yields (4.1.13). By definition (4.1.12), equation (4.1.14) follows from (4.1.13). The proof is complete. □

4.2. Self-financing strategies

Consider an investor who invests an amount $v \geq 0$ in the present market and then follows a trading strategy $\pi = (\Psi, \Phi)$ with portfolio components $\Psi = \{\Psi_t\}_{t\in[0,T]}$ and $\Phi = \{\Phi_t\}_{t\in[0,T]}$ which yield the value process $V^\circ(\pi) = \Psi B^\circ + \Phi S^\circ$ defined at $t \in [0,T]$ by

(4.2.1) $$V^\circ(t;\pi) = \Psi(t)B^\circ(t) + \Phi(t)S^\circ(t).$$

Clearly, the initial condition is

$$v = V^\circ(0;\pi) = \Psi(0)B^\circ(0) + \Phi(0)S^\circ(0).$$

Since between two consecutive jumps the stock price process evolve smoothly, the investor selects both components as piecewise continuous functions of type

(4.2.2) $$\Psi(t) = \psi_1(t)I_{[0,T_1]}(t) + \sum_{k=1}^{m} \psi_{k+1}(t)I_{(T_k,T_{k+1}]}(t)$$

and

(4.2.3) $$\Phi(t) = \phi_1(t)I_{[0,T_1]}(t) + \sum_{k=1}^{m} \phi_{k+1}(t)I_{(T_k,T_{k+1}]}(t),$$

where ψ_k and ϕ_k are continuously differentiable functions with $\Psi(0) = \psi_1(0)$ and $\Phi(0) = \phi_1(0)$.

According to definition 3.2.1, a trading strategy $\pi = (\Psi, \Phi)$ is self-financing if for each $t \in (0,T]$

(4.2.4) $$B^\circ_- \cdot \Psi_t + S^\circ_- \cdot \Phi_t = 0$$

In the present case proposition 3.2.2 may be reformulated as follows:

PROPOSITION 4.2.1. *A trading strategy π is self-financing if and only if its discounted value process $\grave{V}^\circ(\pi) = \{\grave{V}^\circ_t(\pi)\}_{t\in[0,T]}$ admits the following integral representation: at each $t \in [0,T]$*

(4.2.5) $$\grave{V}^\circ(t;\pi) = v + \Phi \cdot \grave{S}^\circ_t.$$

It is important to notice that the process $\grave{V}^\circ(\pi)$ for a self-financing strategy π is of the same type as the stock price process, since at each $t \in [0,T]$ it may be represented similarly to (4.1.8) as follows:

$$\grave{V}^\circ(t;\pi) = \sum_{k=0}^{m} \grave{v}^\circ_k(t;\pi) I_{[T_k,T_{k+1})}(t).$$

In view of definitions in Section 2.2, the states $\{\grave{v}^\circ_k(t;\pi)\}_{k=0,1,\dots,m}$ at $t \in [T_k, T_{k+1})$ satisfy

(4.2.6) $$\grave{v}^\circ_k(t,\pi) - \grave{v}^\circ_k(T_k;\pi) = \int_{T_k}^{t} \phi_{k+1}(u) d\grave{s}^\circ_k(u)$$

with $\grave{v}^\circ_0(T_0;\pi) = v$ and

$$\grave{v}^\circ_k(T_k;\pi) = v + \sum_{j=1}^{k} \left\{ \int_{T_{j-1}}^{T_j} \phi_j(u) d\grave{s}^\circ_{j-1}(u) + \phi_j(T_j)\left(\grave{s}^\circ_j(T_j) - \grave{s}^\circ_{j-1}(T_j)\right) \right\}.$$

Note that $\grave{s}^\circ_j(T_j) - \grave{s}^\circ_{j-1}(T_j) = D_j(\grave{S}^\circ_{T_j})$, cf. (4.1.11).

Arguing as before, we define $DV^\circ(t;\pi)$ as follows. Suppose that $S^\circ(t-)$ is in state $s^\circ_k(t)$. Then $V^\circ(t-;\pi)$ is in state $v^\circ_k(t;\pi)$ and at time t the process $V^\circ(\pi)$ either stays in state $v^\circ_k(t;\pi)$ or jump to state $v^\circ_{k+1}(t;\pi)$. Therefore the corresponding state $D_k(V^\circ_t(\pi))$ of $DV^\circ(t;\pi)$ is defined by

(4.2.7) $$D_k(V^\circ_t(\pi)) = v^\circ_k(t;\pi) - v^\circ_{k-1}(t;\pi).$$

Hence

DEFINITION 4.2.2. The process $DV^\circ(\pi) = \{DV^\circ_t(\pi)\}_{t\in[0,T]}$ is defined by

$$DV^\circ(t;\pi) = D_1(V^\circ_t(\pi)) I_{[0,T_1]}(t) + \sum_{k=1}^{m} D_{k+1}(V^\circ_t(\pi)) I_{(T_k,T_{k+1}]}(t)$$

where $D_k(V_t^\circ(\pi))$ is given by (4.2.7). Obviously, $D\grave{V}^\circ(\pi) = DV^\circ(\pi)/B^\circ$. Define finally the process $DV^\circ(\pi)/DS^\circ$ by

$$\frac{DV^\circ(t;\pi)}{DS^\circ(t)} = \frac{D_1(V_t^\circ(\pi))}{D_1(S_t^\circ)} I_{[0,T_1]}(t) + \sum_{k=1}^{m} \frac{D_{k+1}(V_t^\circ(\pi))}{D_{k+1}(S_t^\circ)} I_{(T_k,T_{k+1}]}(t).$$

In the next proposition Clark's formula (4.2.9) is obtained.

PROPOSITION 4.2.2. *Under the self-financing condition* (4.2.4) *the stock component of the portfolio is given by*

$$\Phi(t) = \frac{DV^\circ(t;\pi)}{DS^\circ(t)}. \tag{4.2.8}$$

Therefore the integral representation (4.2.5) *takes the form*

$$\grave{V}^\circ(t;\pi) = v + \frac{DV^\circ(\pi)}{DS^\circ} \cdot \grave{S}_t^\circ. \tag{4.2.9}$$

PROOF. It suffices to prove (4.2.8). So, it is needed to verify that if $S^\circ(t-)$ is in state $s_k^\circ(t)$, i.e. $t \in (T_k, T_{k+1}]$, then

$$D_k(V_t^\circ(\pi)) = \phi_k(t) D_k(S_t^\circ). \tag{4.2.10}$$

But if $t \in (T_k, T_{k+1}]$, then in view of (4.2.1) we either have

$$v_k^\circ(t;\pi) = \psi_k(t)e^{rt} + \phi_k(t)s_k^\circ(t)$$

or

$$v_{k+1}^\circ(t;\pi) = \psi_k(t)e^{rt} + \phi_k(t)s_{k+1}^\circ(t).$$

These identities imply (4.2.10). The proof is complete. □

It will be proved in the next proposition that the value process for a self-financing strategy satisfies differential equations (4.2.11), similar to (4.1.13). Note that the discounted versions of these equations (4.1.14) and (4.2.12) play in the present market the same rôle as the equations (3.2.14) and (3.2.16) in the binary market.

PROPOSITION 4.2.3. *Under the self-financing condition* (4.2.4) *the states of the value process and its discounted version satisfy the following differential equations: at* $t \in (T_k, T_{k+1}]$

$$\frac{dv_k^\circ(t;\pi)}{dt} - rv_k^\circ(t) = -\lambda D_{k+1}(V_t^\circ(\pi)) \tag{4.2.11}$$

and

$$\frac{d\grave{v}_k^\circ(t)(\pi)}{dt} = -\lambda D_{k+1}(\grave{V}_t^\circ(\pi)). \tag{4.2.12}$$

By (4.2.6)

$$\frac{d\dot{v}_k^\circ(t;\pi)}{dt} = \phi_{k+1}(t)\frac{d\dot{s}_k^\circ(t)}{dt},$$

4.2.12) follows from (4.1.14) and (4.2.10). The equivalence of (4.2.11) 12) is straightforward. The proof is complete. □

ging strategies

0, T]. Consider the system of differential equations

$$\frac{d\dot{x}_k(t)}{dt} = -\lambda\left(\dot{x}_{k+1}(t) - \dot{x}_k(t)\right), \quad k = 0, 1, \ldots,$$

to the boundary conditions

$$\dot{x}_k(T) = \dot{h}_k(T), \quad k = 0, 1, \ldots,$$

en numbers $\{\dot{h}_k(T)\}_{k=0,1,\ldots}$. The parameter $\lambda > 0$ is the same as f. (4.1.14) or (4.2.12).

explicit solution of this system is expressed in terms of so-called *Poisson tion* with the *intensity* λ, defined by $\mathcal{P}_\lambda = \{p_j(\lambda)\}_{j=0,1,\ldots}$ with

$$p_j(\lambda) = \frac{\lambda^j}{j!}e^{-\lambda}.$$

positive numbers (4.3.3) sum up to 1, so that $\mathcal{P}_\lambda$ is a probability dis- n. Note that definition (4.3.3) extends to $\lambda = 0$ as follows:

$$p_0(0) = 1 \text{ and } p_j(0) = 0 \text{ for } j = 1, 2, \ldots$$

lowing property of the Poisson distribution is well-known.

. 4.3.1. *At each* $t \in (0, T]$ *the Poisson distribution* $\mathcal{P}_{t\lambda} = \{p_j(t\lambda)\}_{j=0,1,\ldots}$ *by* (4.3.3) *satisfies the following system of differential equations*

$$\frac{dp_j(t\lambda)}{dt} = -\lambda(p_j(t\lambda) - p_{j-1}(t\lambda)), \quad j = 0, 1, \ldots$$

$_{-1}(t\lambda) \equiv 0$ *and the initial conditions* (4.3.4).

. This is easily verified by the direct differentiation of (4.3.3). □

re details see, e.g. FELLER (1971), vol. 1, Section 17.2, or COX and R (1965), Section 4.1. Lemma 4.3.1 allows for the following explicit n of the system of equations (4.3.1).

PROPOSITION 4.3.2. *The system* (4.3.1) *of differential equations in the interval* $t \in [0,T]$, *subject to the boundary conditions* (4.3.2), *is satisfied by*

$$\grave{x}_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{h}_{k+j}(T), \quad k = 0,1,\ldots \tag{4.3.5}$$

provided that the numbers $\{\grave{h}_k(T)\}_{k=0,1,\ldots}$ *allow the differentiation under the summation sign. In particular*

$$\grave{x}_0(0) = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{h}_j(T). \tag{4.3.6}$$

PROOF. The boundary conditions (4.3.2) are satisfied due to property (4.3.4) of the Poisson distribution. Differentiating both sides of (4.3.5) we get by lemma 4.3.1 that

$$\begin{aligned}\frac{d\grave{x}_k(t)}{dt} &= -\lambda\left\{\sum_{j=1}^{\infty} p_{j-1}(\lambda(T-t))\grave{h}_{k+j}(T) - \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{h}_{k+j}(T)\right\} \\ &= -\lambda\left(\grave{x}_{k+1}(t) - \grave{x}_k(t)\right).\end{aligned}$$

This yields (4.3.1). The proof is complete. □

By comparing (4.1.14) and (4.3.1) we see that the states (4.1.9) of the discounted stock price process $\grave{S}^\circ$ satisfy the relations

$$\grave{s}_k^\circ(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{s}_{k+j}^\circ(T). \tag{4.3.7}$$

In particular

$$s = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{s}_j^\circ(T). \tag{4.3.8}$$

This can also be verified directly, since

$$\sum_{j=0}^{\infty} \frac{(\lambda(T-t))^j}{j!} e^{-\lambda(T-t)} s(1+a)^{k+j} e^{-a\lambda T} = s(1+a)^k e^{-a\lambda t}.$$

Cf. (4.3.7) and (4.3.8) with (3.3.7) and (3.3.8). The differential equations (4.3.1) and their solutions (4.3.5) and (4.3.6) play here the same rôle as equations (3.3.3) and their solutions (3.3.5) and (3.3.6) in a binary market. They yield, in particular, the completeness of a Poisson market to be shown next.

Let $W(T)$ be a wealth desired by an investor at the terminal date T. Sup-se that $W(T)$ may occupy one of the states $\{w_k(T)\}_{k=0,1,\dots}$. There is, say :ertain function W so that $W(T) = W(S^\circ(T))$ and $w_k(T) = W(s_k^\circ(T)), k = 1,\dots$. The definitions in Section 3.3 of the completeness and the hedging ·ategy extend straightforwardly to the present situation. However, we present ew the formulations, because of their importance.

ƎFINITION 4.3.3. A Poisson market is *complete* if any desired wealth $W(T)$ the above type is attainable with a certain initial endowment: there is a ıf-financing strategy π whose value process at the terminal date T attains the əntity $V(T;\pi) = W(T)$. The necessary initial endowment is then $v = V(0;\pi)$. ıis particular strategy is called the *hedging strategy* against $W(T)$.

milarly to proposition 4.3.3 in part I, we have

ROPOSITION 4.3.4. *A Poisson market is complete. The hedging strategy* π *ainst a desired wealth* $W(T)$ *of the above type is uniquely defined by the rtfolio components* (4.2.2) *and* (4.2.3) *with*

$$\psi_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t)) \frac{(1+a)\grave{w}_{j+k}(T) - \grave{w}_{j+k+1}(T)}{a} \tag{.3.9}$$

ıd

$$\phi_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t)) \frac{\grave{w}_{j+k+1}(T) - \grave{w}_{j+k}(T)}{a\grave{s}_k^\circ(t)}, \tag{:.3.10}$$

here $\{\grave{w}_k(T)\}_{k=0,1,\dots}$ *are the discounted states of the wealth* $W(T)$*, i.e.* $\grave{w}_k(T) = \frac{{}_k(T)}{{}^\circ(T)}$. *The initial endowment needed amounts to*

$$v = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{w}_j(T). \tag{1.3.11}$$

'ROOF. Let $t \in [T_k, T_{k+1})$. By definition (4.2.1) the discounted value process ɔr the present strategy π is in state

$$\grave{v}_k^\circ(t;\pi) = \psi_k(t) + \phi_k(t)\grave{s}_k^\circ(t)$$

'hich by (4.3.9) and (4.3.10) coincides with

$$\grave{w}_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{w}_{k+j}(T). \tag{4.3.12}$$

3y proposition 4.3.2, (4.3.12) solves the differential equation (4.3.1) subject to he boundary condition (4.3.2) with $\grave{w}_k(T)$ instead of $\grave{h}_k(T)$. Then by propo-ition 4.2.3 the present strategy is self-financing. Moreover, at the terminal

PROPOSITION 4.3.2. *The system* (4.3.1) *of differential equations in the interval* $t \in [0, T]$, *subject to the boundary conditions* (4.3.2), *is satisfied by*

$$\grave{x}_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{h}_{k+j}(T), \quad k = 0, 1, \ldots \tag{4.3.5}$$

provided that the numbers $\{\grave{h}_k(T)\}_{k=0,1,\ldots}$ *allow the differentiation under the summation sign. In particular*

$$\grave{x}_0(0) = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{h}_j(T). \tag{4.3.6}$$

PROOF. The boundary conditions (4.3.2) are satisfied due to property (4.3.4) of the Poisson distribution. Differentiating both sides of (4.3.5) we get by lemma 4.3.1 that

$$\begin{aligned} \frac{d\grave{x}_k(t)}{dt} &= -\lambda \left\{ \textstyle\sum_{j=1}^{\infty} p_{j-1}(\lambda(T-t))\grave{h}_{k+j}(T) - \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{h}_{k+j}(T) \right\} \\ &= -\lambda \left(\grave{x}_{k+1}(t) - \grave{x}_k(t) \right). \end{aligned}$$

This yields (4.3.1). The proof is complete. □

By comparing (4.1.14) and (4.3.1) we see that the states (4.1.9) of the discounted stock price process $\grave{S}^\circ$ satisfy the relations

$$\grave{s}_k^\circ(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{s}_{k+j}^\circ(T). \tag{4.3.7}$$

In particular

$$s = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{s}_j^\circ(T). \tag{4.3.8}$$

This can also be verified directly, since

$$\sum_{j=0}^{\infty} \frac{(\lambda(T-t))^j}{j!} e^{-\lambda(T-t)} s(1+a)^{k+j} e^{-a\lambda T} = s(1+a)^k e^{-a\lambda t}.$$

Cf. (4.3.7) and (4.3.8) with (3.3.7) and (3.3.8). The differential equations (4.3.1) and their solutions (4.3.5) and (4.3.6) play here the same rôle as equations (3.3.3) and their solutions (3.3.5) and (3.3.6) in a binary market. They yield, in particular, the completeness of a Poisson market to be shown next.

Let $W(T)$ be a wealth desired by an investor at the terminal date T. Suppose that $W(T)$ may occupy one of the states $\{w_k(T)\}_{k=0,1,\ldots}$. There is, say a certain function W so that $W(T) = W(S^\circ(T))$ and $w_k(T) = W(s_k^\circ(T)), k = 0, 1, \ldots$. The definitions in Section 3.3 of the completeness and the hedging strategy extend straightforwardly to the present situation. However, we present anew the formulations, because of their importance.

DEFINITION 4.3.3. A Poisson market is *complete* if any desired wealth $W(T)$ of the above type is attainable with a certain initial endowment: there is a self-financing strategy π whose value process at the terminal date T attains the identity $V(T;\pi) = W(T)$. The necessary initial endowment is then $v = V(0;\pi)$. This particular strategy is called the *hedging strategy* against $W(T)$.

Similarly to proposition 4.3.3 in part I, we have

PROPOSITION 4.3.4. *A Poisson market is complete. The hedging strategy π against a desired wealth $W(T)$ of the above type is uniquely defined by the portfolio components* (4.2.2) *and* (4.2.3) *with*

$$\psi_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t)) \frac{(1+a)\mathring{w}_{j+k}(T) - \mathring{w}_{j+k+1}(T)}{a} \tag{4.3.9}$$

and

$$\phi_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t)) \frac{\mathring{w}_{j+k+1}(T) - \mathring{w}_{j+k}(T)}{a\mathring{s}_k^\circ(t)}, \tag{4.3.10}$$

where $\{\mathring{w}_k(T)\}_{k=0,1,\ldots}$ are the discounted states of the wealth $W(T)$, i.e. $\mathring{w}_k(T) = \frac{w_k(T)}{B^\circ(T)}$. The initial endowment needed amounts to

$$v = \sum_{j=0}^{\infty} p_j(\lambda T)\mathring{w}_j(T). \tag{4.3.11}$$

PROOF. Let $t \in [T_k, T_{k+1})$. By definition (4.2.1) the discounted value process for the present strategy π is in state

$$\mathring{v}_k^\circ(t;\pi) = \psi_k(t) + \phi_k(t)\mathring{s}_k^\circ(t)$$

which by (4.3.9) and (4.3.10) coincides with

$$\mathring{w}_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\mathring{w}_{k+j}(T). \tag{4.3.12}$$

By proposition 4.3.2, (4.3.12) solves the differential equation (4.3.1) subject to the boundary condition (4.3.2) with $\mathring{w}_k(T)$ instead of $\mathring{h}_k(T)$. Then by proposition 4.2.3 the present strategy is self-financing. Moreover, at the terminal date T its value process attains the desired wealth $W(T)$. Thus π is the hedging strategy against $W(T)$. Since $\{\grave{w}_k(T)\}_{k=0,1,\ldots}$ in (4.3.12) are arbitrary, the Poisson market is complete. Finally, (4.3.11) is an easy consequence of (4.3.12). □

The application of proposition 4.3.4 to option pricing is streightforward. Let

$$H(T) = H(S^\circ(T))$$

be a payoff function of a contingent claim, with possible states $\{h_j(T)\}_{j=0,1,\ldots}$ where

$$h_j(T) = H(s_j^\circ(T)).$$

According to (4.3.11), the fair price of the contingent claim H is

$$C(H) = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{h}_j(T) \tag{4.3.13}$$

with $\grave{h}_j(T) = \frac{h_j(T)}{B^\circ(T)}$. In the special case of the European call option

$$h_j(T) = (s_j^\circ(T) - K)^+ \tag{4.3.14}$$

(with a certain exercise price K, cf. (3.3.11)), we have

PROPOSITION 4.3.5. *For a nonnegative integer j_0 denote*

$$F(j_0;\lambda) = \sum_{j>j_0} p_j(\lambda), \tag{4.3.15}$$

cf. (4.3.3). *Then the fair price C of the European call option with the payoff function* (4.3.14) *may be presented as follows:*

$$\begin{aligned} C &= sF\left(\left[\frac{\log\frac{K}{s}+bT}{\log(1+a)}\right];(1+a)\lambda T\right) \\ &\quad +e^{-rT}KF\left(\left[\frac{\log\frac{K}{s}+bT}{\log(1+a)}\right];\lambda T\right) \end{aligned} \tag{4.3.16}$$

PROOF. By (4.3.13) and (4.3.14)

$$C = \sum_{j=0}^{\infty} p_j(\lambda T)(\grave{s}_j^\circ(T) - \grave{K})^+ \tag{4.3.17}$$

where $\grave{K} = e^{-rT}K$, as usual. Therefore, by (4.3.3) and (4.1.9)

$$C = e^{-\lambda T}\sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}(s(1+a)^j e^{-a\lambda T} - \grave{K})^+. \tag{4.3.18}$$

Obviously, the terms with

$$j \leq j_0 = \left\lceil \frac{\log \frac{K}{s} + bT}{\log(1+a)} \right\rceil$$

equal zero so that (4.3.18) reduces to

$$C = e^{-\lambda T} \sum_{j > j_0} \frac{(\lambda T)^j}{j!} (s(1+a)^j e^{-a\lambda T} - \dot{K})$$

which yields (4.3.16) by definition (4.3.15). □

REMARK 4.3.6. Formula (4.3.17) is often called the Cox and Ross option pricing formula (see e.g. COX and ROSS (1976) or HARRISON and PLISKA (1981), Section 6.2; cf. also COX, ROSS and RUBINSTEIN (1979) and COX and RUBINSTEIN (1985)). Its probabilistic interpretation is as follows: the right hand side is the expectation with respect to the Poisson distribution $\mathcal{P}_{\lambda T}$ of a random variable taking on the value $(s_j^\circ(T) - K)^+$ with probability $p_j(\lambda T), j = 0, 1, \ldots$. In its specific form (4.3.16), this formula is comparable with the well-known Black-Sholes formula for the geometric Brownian motion model, see BLACK-SHOLES (1973), HARRISON and PLISKA (1981), formula (1.5), or KARATZAS and SHREVE (1988), Section 5.8.

5. ON THE POISSON APPROXIMATION

5.1. Approximation of the assets

In the present Section the link is sought between the binary model of Section 3 and the Poisson model of Section 4. By using certain heuristic arguments we show that under the conditions of Section 3.1 the Poisson model can serve as an approximation to the binary model. This is already visible by the simple comparison of equations (3.2.22) and (4.2.12) (or (3.2.24) and (4.2.11)), for the right hand side of (3.2.22) may be viewed as a prelimiting version of that of (4.2.12).

The limiting transition is carried out by letting the number of trading periods N to increase unboundedly and letting the lenght of each trading period $\Delta t_n = t_n - t_{n-1}$ to tend to zero for $n = 1, \ldots, N$. Furthermore, we restrict our attention to the special case of markets where new prices are announced regularly so that trading times are equidistant, given by

$$\left\{ t_n = \frac{nT}{N} \right\}_{n=0,1,\ldots,N}, \tag{5.1.1}$$

and the lenghts of the trading periods are all given by

$$\left\{ \Delta t_n = \frac{T}{N} \right\}_{n=1,\ldots,N}. \tag{5.1.2}$$

For, asymptotically, this makes no difference (in fact one can proceed without this specification, however at the expense of some details which we want to avoid here). At the same time, the formulations are somewhat simplified. Instead of (3.1.1) and (3.1.2), for instance, we may write

$$B^N(t) = B^N_{[tN/T]}$$

and

$$S^N(t) = S^N_{[tN/T]}.$$

As usual, $[x]$ is the largest integer not exceeding x.

Concerning the bond, the situation is simple, since condition 3.1.1 means that at each fixed $t \in [0, T]$

$$B^N(t) \sim B^\circ(t),$$

as we have already seen, cf. (3.1.11) and (4.1.2).

As for a risky asset, the stock, the desired statement that at each fixed $t \in [0, T]$

$$S^N(t) \sim S^\circ(t)$$

concerns the trajectories of the processes on the both sides. The idea behind it is quite simple, as will be explained below. Its exact formulation, however, would require probabilistic considerations that lay beyond the scope of the present paper (we intend to return to this subject in a latter part of these lecture notes).

At fixed $t \in [0, T]$

$$S^N(t) = s \prod_{n=0}^{[\frac{tN}{T}]} (1 + \Delta R^N_n) \tag{5.1.3}$$

(cf. (3.1.12)) may occupy one of the $2^{[\frac{tN}{T}]}$ states, i.e. up to time t the stock price may evolve along one of $2^{[\frac{tN}{T}]}$ trajectories. The states of the returns ΔR^N_n in (5.1.3), given by (3.1.14), are under condition 3.1.2 approximated by (3.1.17), with the right hand side independent of the index k. Therefore all trajectories of the stock price up to time $t \in [0, T]$ formed by the same number, say $j \in \{0, 1, \dots, [\frac{tN}{T}]\}$, of upward displacements (and $[\frac{tN}{T}] - j$ downward displacements), get the same approximation equal

$$s(1 + a)^j \left(1 - \frac{bT}{N}\right)^{[\frac{tN}{T}] - j}, \tag{5.1.4}$$

due to (5.1.3). The latter expression tends to $s^\circ_j(t)$ (cf. (4.1.7)), since for fixed $j \geq 0$ and $t \in [0, T]$

$$\lim_{N \to \infty} \left(1 - \frac{bT}{N}\right)^{[\frac{tN}{T}] - j} = \lim_{N \to \infty} \left(1 - \frac{bT}{N}\right)^{\frac{tN}{T}} = e^{-bt}.$$

As $N \to \infty$ the index j in (5.1.4) may take on any nonnegative integer value, so that the approximate states are indeed $\{s_j^\circ(t)\}_{j=0,1,\ldots}$.

5.2. Approximate option pricing

According to lemma 3.2.7, for each $n = 1, \ldots, N$ the risk neutral probabilities $\{p_{kn}^N\}_{k=1,\ldots,2^n}$ are approximated independently of the indices k and n by

(5.2.1) $$p_{2k,n}^N \sim \frac{\lambda T}{N}$$

and

(5.2.2) $$p_{2k-1,n}^N \sim 1 - \frac{\lambda T}{N}$$

(cf. (3.2.19) and (5.1.2)). This allows for the approximation of the probabilities $\{P_{kN}^N\}_{k=1,\ldots,2^N}$ defined by (3.3.2). Recall that each value of the index k corresponds to a certain trajectory of the stock price development. Let us pick out any index k which belongs to the set of indices corresponding to the set of all trajectories formed by j upward and $N-j$ downward displacements. Clearly, their number equals $\binom{N}{j}$. By (5.1.2) in all these $\binom{N}{j}$ cases we have the same approximation

(5.2.3) $$P_{kN}^N \sim \left(\frac{\lambda T}{N}\right)^j \left(1 - \frac{\lambda T}{N}\right)^{N-j}.$$

We shall apply (5.2.3) to the option pricing formula (3.3.12). Taking into consideration the approximation of Section 5.1 to the stock price, we obtain

(5.2.4) $$C^N \sim \sum_{j=0}^{N} \binom{N}{j} \left(\frac{\lambda T}{N}\right)^j \left(1 - \frac{\lambda T}{N}\right)^{N-j} (\dot{s}_j^\circ(T) - \dot{K})^+.$$

The expression on the right hand side may be simplified, since for each fixed integer $j \geq 0$

$$\binom{N}{j} \left(\frac{\lambda T}{N}\right)^j \left(1 - \frac{\lambda T}{N}\right)^{N-j} = \frac{(\lambda T)^j}{j!} \left(1 - \frac{\lambda T}{N}\right)^{N-j} \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{j-1}{N}\right) \to \frac{(\lambda T)^j}{j!} e^{-\lambda T}$$

as $N \to \infty$. The limit is $p_j(\lambda T)$, cf. (4.3.3). Thus (5.2.4) yields

$$C^N \sim \sum_{j=0}^{\infty} p_j(\lambda T)(\dot{s}_j^\circ(T) - \dot{K})^+,$$

cf. (4.3.17).

5.3. Approximate hedging strategy

In this Section the heuristic arguments of the previous Sections 5.1 and 5.2 are applied to the hedging strategy against the European call option. The construction of this strategy is based on the formula (3.3.10) with

$$\dot{w}_k^N(T) = \frac{(s_k^N(T) - K)^+}{B^N(T)}. \tag{5.3.1}$$

We already know from Section 5.1 how to approximate the states (5.3.1). The set of weights in formula (3.3.10)

$$\left\{P_{N|N-n}^N(2^n(k-1)+1), \ldots, P_{N|N-n}^N(2^n k)\right\} \tag{5.3.2}$$

has to be approximated as well, for all $k \in \{1, \ldots, 2^{N-n}\}$. By arguments similar to that of the previous Section, the approximation is free of the index k. Indeed, all entries in the subset of (5.3.2) corresponding to the subset of the trajectories formed by j upward and $n-j$ downward displacements, $j \in \{0, 1, \ldots, n\}$, are approximated by the same number

$$\left(\frac{\lambda T}{N}\right)^j \left(1 - \frac{\lambda T}{N}\right)^{n-j},$$

due to (3.3.1), (5.2.1) and (5.2.2). For each $j \in \{0, 1, \ldots, n\}$ this subset consists of $\binom{n}{j}$ entries. Consequently, the process W^N of Section 3.3 occupies at time $t \in [0, T]$ one of the states $\{\dot{w}_{k,[tN/T]}^N\}_{k=1,\ldots,2^{[tN/T]}}$ approximated as follows:

(5.3.3)
$$\dot{w}_{k,[tN/T]}^N \sim \sum_{j=0}^{N-[\frac{tN}{T}]} \binom{N - [\frac{tN}{T}]}{j} \left(\frac{\lambda T}{N}\right)^j \left(1 - \frac{\lambda T}{N}\right)^{N-[\frac{tN}{T}]-j} \dot{w}_{k+j}^N(T)$$

with $\dot{w}_j^N(T)$ given by (5.3.1). The expression on the right hand side may be simplified by the following considerations. Firstly, by the results of section 5.1 and by (5.3.1)

$$\dot{w}_j^N(T) \sim \dot{w}_j(T) \tag{5.3.4}$$

with

$$\dot{w}_j(T) = (\dot{s}_j^\circ(T) - \dot{K})^+.$$

Secondly, for each integer $j \geq 0$ and $t \in [0, T]$

$$\begin{aligned} \binom{N-[\frac{tN}{T}]}{j} \left(\frac{\lambda T}{N}\right)^j &= \frac{(\lambda T)^j}{j!}\left(1 - \frac{[\frac{tN}{T}]}{N}\right) \cdots \left(1 - \frac{[\frac{tN}{T}]-j}{N}\right) \\ &\to \frac{[\lambda(T-t)]^j}{j!} \end{aligned} \tag{5.3.5}$$

as $N \to \infty$. Finally,

$$\text{(5.3.6)} \quad \lim_{N\to\infty}\left(1-\frac{\lambda T}{N}\right)^{N-[\frac{tN}{T}]-j} = \lim_{N\to\infty}\left(1-\frac{\lambda T}{N}\right)^{N(1-\frac{t}{T})} = e^{-\lambda(T-t)}.$$

In view of (4.3.3), the equations (5.3.3) - (5.3.6) result in

$$\text{(5.3.7)} \qquad \grave{w}^N_{k,[tN/T]} \sim \grave{w}_k(t)$$

where

$$\grave{w}_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{w}_{k+j}(T)$$

(cf. (4.3.12)). Thus we have derived the relation (5.3.7) between the states of the processes $\grave{W}^N$ and $\grave{W}$ which yield the value processes for the hedging strategies against the European call option in the prelimiting binary market of Section 3 and the Poisson market of Section 4, respectively.

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