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ABSTRACT

In this paper we extend several results concerning the termination of the disjoint union of term rewriting systems to conditional term rewriting systems. The first termination property we study is strong normalization (there are no infinite reduction sequences) and we show that sufficient conditions for the strong normalization of the disjoint union of two strongly normalizing term rewriting systems given by Rusinowitch and Middeldorp extend naturally to conditional term rewriting systems. Weak normalization (every term reduces to a normal form) is the second property we are interested in. We show that for conditional term rewriting systems weak normalization is not preserved under disjoint unions. This is rather surprising because the disjoint union of weakly normalizing unconditional term rewriting systems is weakly normalizing. Besides giving sufficient conditions for the weak normalization of disjoint unions of weakly normalizing conditional term rewriting systems, we will also give a much simpler proof of the recent approach to weak normalization due to Kurihara and Kaji[†].

[†] Added in print: recently, Kurihara and Kaji gave a revised proof of their theorem which is essentially the same as ours.

Introduction

Conditional term rewriting systems arise naturally in the algebraic specification of abstract data types. They have been studied by Bergstra and Klop [1], Kaplan [14] and Zhang and Rémy [24] from this point of view. Conditional term rewriting systems are also important for integrating the functional and logic programming paradigms. Several authors recognized that conditional term rewriting provides a natural computational mechanism for this integration; see Dershowitz and Plaisted [8, 9], Fribourg [11] and Goguen and Meseguer [12]. In both uses of conditional term rewriting systems, establishing confluence and termination is of great importance.

For unconditional term rewriting systems several methods have been developed for proving these properties. One of the methods that have been investigated consists of partitioning a term rewriting system into smaller term rewriting systems such that the validity of a certain property for the given system can be inferred from the validity of that property for the smaller systems. For 'disjoint' decompositions of term rewriting systems several positive results have been obtained. For instance, Toyama [21] proved that a term rewriting system is confluent if it can be partitioned into confluent systems with disjoint alphabets. In [19] we extended this result to conditional term rewriting systems. The present paper continues this line of research by extending results about the termination behaviour of disjoint unions of term rewriting systems to conditional term rewriting systems.

The paper is organized as follows. Conditional term rewriting is introduced in the next section. In Section 2 we give an overview of previous work on disjoint unions of (conditional) term rewriting systems. Section 2 also contains the necessary technical definitions and notations for dealing with disjoint unions. Section 3 is devoted to strong normalization. We extend sufficient conditions for the strong normalization of the disjoint union of strongly normalizing term rewriting systems (Rusinowitch [20], Middeldorp [18]) to conditional term rewriting systems. Weak normalization with respect to disjoint unions is studied in Section 4. We show that disjoint unions of weakly normalizing conditional term rewriting systems are generally not weakly normalizing. For two important subclasses of conditional term rewriting systems we obtain the preservation of weak normalization under disjoint unions. In Section 4 we also give an account of the interesting approach to disjoint unions by Kurihara and Kaji [16]. Section 5 contains some final remarks.

1. Conditional Term Rewriting Systems: Preliminaries

This introduction to conditional term rewriting is supposed to be self-contained, though familiarity with term rewriting systems (Klop [15], Dershowitz and Jouannaud [4]) might prove helpful.

Let $\mathcal{V}$ be a countably infinite set of *variables*. A *conditional term rewriting system* (CTRS for short) is a pair $(\mathcal{F}, \mathcal{R})$. The set $\mathcal{F}$ consists of *function symbols*; associated to every $f \in \mathcal{F}$ is its arity $n \geq 0$. The set of terms built from $\mathcal{F}$ and $\mathcal{V}$, notation $\mathcal{T}(\mathcal{F}, \mathcal{V})$, is the smallest set such that:

- $\mathcal{V} \subset \mathcal{T}(\mathcal{F}, \mathcal{V})$,
- if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

The set of variables occurring in a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is denoted by $V(t)$. Identity (syntactic equality) of terms is denoted by $\equiv$. The set $\mathcal{R}$ consists of *conditional rewrite rules*. Each conditional rewrite rule has the form

$$l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$$

for some $n \geq 0$ and terms $l, r, s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ subject to two constraints:

(1) the left-hand side l is not a variable ($l \notin \mathcal{V}$),

(2) the variables which occur in the right-hand side r also occur in l ($V(r) \subseteq V(l)$).

Unconditional rewrite rules do not have conditions (i.e. $n = 0$). They will be written as $l \rightarrow r$ instead of $l \rightarrow r \Leftarrow$. A *term rewriting system* (TRS) is a CTRS containing only unconditional rewrite rules. We

usually present a CTRS as a set of rewrite rules, without making explicit the set of function symbols. A rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ is *left-linear* if l does not contain multiple occurrences of the same variable. The rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ is *collapsing* if r is a single variable and it is *duplicating* if r contains more occurrences of some variable than l does.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that the set $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. This set is called the *domain* of σ and will be denoted by $\mathcal{D}(\sigma)$. Substitutions are extended to morphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, i.e. $\sigma(f(t_1, \ldots, t_n)) \equiv f(\sigma(t_1), \ldots, \sigma(t_n))$ for every n-ary function symbol f and terms $t_1, \ldots, t_n$. We call $\sigma(t)$ an *instance* of t. An instance of a left-hand side of a rewrite rule is a *redex* (reducible expression). Occasionally we present a substitution σ as $\sigma = \{x \to \sigma(x) \mid x \in \mathcal{D}(\sigma)\}$. The *empty* substitution will be denoted by ε (here $\mathcal{D}(\varepsilon) = \varnothing$).

A *context* $C[\,,\ldots,\,]$ is a 'term' which contains at least one occurrence of a special symbol $\square$. If $C[\,,\ldots,\,]$ is a context with n occurrences of $\square$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the result of replacing from left to right the occurrences of $\square$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\square$ is denoted by $C[\,]$. A term s is a *subterm* of a term t if there exists a context $C[\,]$ such that $t \equiv C[s]$.

The *rewrite relation* $\to_{\mathcal{R}}$ and *joinability relation* $\downarrow_{\mathcal{R}}$ are simultaneously defined as follows: $s \to_{\mathcal{R}} t$ if there exists a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow_{\mathcal{R}} \sigma(t_i)$ for $i \in \{1, \ldots, n\}$; $s \downarrow_{\mathcal{R}} t$ if there exists a term u such that $s \twoheadrightarrow_{\mathcal{R}} u$ and $t \twoheadrightarrow_{\mathcal{R}} u$ with $\twoheadrightarrow_{\mathcal{R}}$ denoting the transitive-reflexive closure of $\to_{\mathcal{R}}$. Such a term u is called a *common reduct* of s and t. In Definition 1.2 below an inductive definition of $\to_{\mathcal{R}}$ is given. We write $s \leftarrow_{\mathcal{R}} t$ if $t \to_{\mathcal{R}} s$; likewise for $s \twoheadleftarrow_{\mathcal{R}} t$. The transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to_{\mathcal{R}}^+$ and the symmetric closure of $\to_{\mathcal{R}}$ is denoted by $\leftrightarrow_{\mathcal{R}}$ (so $\leftrightarrow_{\mathcal{R}} = \to_{\mathcal{R}} \cup \leftarrow_{\mathcal{R}}$). The transitive-reflexive closure of $\leftrightarrow_{\mathcal{R}}$ is called *conversion* and denoted by $=_{\mathcal{R}}$. If $s =_{\mathcal{R}} t$ then s and t are *convertible*. Two terms s, t are *joinable* if $s \downarrow_{\mathcal{R}} t$ and we say that s *reduces* to t if $s \twoheadrightarrow_{\mathcal{R}} t$. We often omit the subscript $\mathcal{R}$.

A term s is a *normal form* if there are no terms t with $s \to t$. The set of normal forms of a CTRS $\mathcal{R}$ is denoted by NF$(\mathcal{R})$. A CTRS is *strongly normalizing (terminating)* if there are no infinite reduction sequences $t_1 \to t_2 \to t_3 \to \ldots$. In other words, every reduction sequence eventually ends in a normal form. A CTRS is *weakly normalizing* if every term reduces to a normal form. Clearly every strongly normalizing CTRS is also weakly normalizing. A CTRS is *confluent* or has the *Church-Rosser* property if for all terms s, t_1, t_2 with $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$ we have $t_1 \downarrow t_2$. A well-known equivalent formulation of confluence is that every pair of convertible terms is joinable ($t_1 = t_2 \Rightarrow t_1 \downarrow t_2$).

Sufficient conditions for strong normalization of CTRS's were given by Kaplan [14], Jouannaud and Waldmann [13] and Dershowitz, Okada and Sivakumar [7]. Sufficient conditions for confluence can be found in Bergstra and Klop [1] and Dershowitz, Okada and Sivakumar [6]. Dershowitz [3] contains an extensive survey of methods for proving strong normalization of TRS's.

EXAMPLE 1.1. Consider the CTRS $\mathcal{R}$ of Table 1. We have $even(S(0)) \to odd(0)$ by the second rule.

$even(0)$	$\to$	$true$		
$even(S(x))$	$\to$	$odd(x)$		
$odd(x)$	$\to$	$true$	$\Leftarrow$	$even(x) \downarrow false$
$odd(x)$	$\to$	$false$	$\Leftarrow$	$even(x) \downarrow true$

TABLE 1.

The term $odd(0)$ can be further reduced to *false* by application of the last rule, using the first rule to satisfy the condition $even(0) \downarrow true$. One easily shows that $\mathcal{R}$ is both strongly normalizing and

confluent.

The following definition of $\to_{\mathcal{R}}$ is fundamental for analyzing the behaviour of CTRS's (cf. [1], [6], [7], [14], [19]).

DEFINITION 1.2. Let $\mathcal{R}$ be a CTRS. We inductively define TRS's $\mathcal{R}_i$ for $i \geq 0$ as follows:

$$\mathcal{R}_0 = \{s \to t \mid s \equiv C[\sigma(l)] \text{ and } t \equiv C[\sigma(r)] \text{ for some context } C[\ ], \text{ substitution } \sigma$$
$$\text{and unconditional rewrite rule } l \to r \in \mathcal{R}\},$$

$$\mathcal{R}_{i+1} = \{s \to t \mid s \equiv C[\sigma(l)] \text{ and } t \equiv C[\sigma(r)] \text{ for some context } C[\ ], \text{ substitution } \sigma$$
$$\text{and rewrite rule } l \to r \Leftarrow s_1 \downarrow t_1, \dots, s_n \downarrow t_n \in \mathcal{R} \text{ such that}$$
$$\sigma(s_j) \downarrow_{\mathcal{R}_i} \sigma(t_j) \text{ for } j = 1, \dots, n\}.$$

We have $s \to_{\mathcal{R}} t$ if and only if $s \to_{\mathcal{R}_i} t$ for some $i \geq 0$. The *depth* of $s \to_{\mathcal{R}} t$ is defined as the minimum i such that $s \to_{\mathcal{R}_i} t$. Depths of reduction sequences $s \twoheadrightarrow_{\mathcal{R}} t$ and 'valleys' $s \downarrow_{\mathcal{R}} t$ are similarly defined.

EXAMPLE 1.3. Consider again the CTRS $\mathcal{R}$ of Table 1. The depth of $even(0) \to true$ is 0, the depth of $even(S(0)) \to false$ is 1 and, more generally, the depth of the reduction sequence from $even(S^n(0))$ to normal form equals n for all $n \geq 0$.

2. Modular Properties

In this section we recall some of the results that have been obtained with respect to disjoint unions of CTRS's. We also introduce notations for dealing with disjoint unions of CTRS's. These are consistent with [21], [23], [19].

DEFINITION 2.1. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be CTRS's with disjoint alphabets (i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \varnothing$). The *disjoint union* $\mathcal{R}_1 \oplus \mathcal{R}_2$ of $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ is the CTRS $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2)$.

DEFINITION 2.2. A property $\mathcal{P}$ of CTRS's is called *modular* if for all CTRS's $\mathcal{R}_1$, $\mathcal{R}_2$ the following equivalence holds:

$$\mathcal{R}_1 \oplus \mathcal{R}_2 \text{ has the property } \mathcal{P} \quad \Leftrightarrow \quad \text{both } \mathcal{R}_1 \text{ and } \mathcal{R}_2 \text{ have the property } \mathcal{P}.$$

Previous research on modularity can be characterized by the phrase "simple statements, complicated proofs". Apart from [19], this research is only concerned with unconditional TRS's. Confluence was the first property for which the modularity has been established.

THEOREM 2.3 (Toyama [21]). *Confluence is a modular property of TRS's.* $\square$

Toyama also gave the following simple example showing that strong normalization is not modular.

EXAMPLE 2.4 (Toyama [22]). Let $\mathcal{R}_1 = \{F(0, 1, x) \to F(x, x, x)\}$ and

$$\mathcal{R}_2 = \begin{cases} or(x, y) \to x, \\ or(x, y) \to y. \end{cases}$$

Both systems are terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits the following cyclic reduction:

$$F(or(0, 1), or(0, 1), or(0, 1)) \to F(0, or(0, 1), or(0, 1))$$
$$\to F(0, 1, or(0, 1))$$
$$\to F(or(0, 1), or(0, 1), or(0, 1)).$$

Notice that $\mathcal{R}_1$ contains a duplicating rule, $\mathcal{R}_2$ contains collapsing rules and $\mathcal{R}_2$ is not confluent.

The next theorem states sufficient conditions for the strong normalization of $\mathcal{R}_1 \oplus \mathcal{R}_2$ in terms of the distribution of collapsing and duplicating rules among $\mathcal{R}_1$ and $\mathcal{R}_2$. The first two conditions were independently obtained by Rusinowitch [20] and Drosten [10]. The sufficiency of the third condition is a positive answer by the present author [18] to a question raised in Rusinowitch [20]. In the next section we extend these results to CTRS's.

THEOREM 2.5. *Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly normalizing TRS's.*
(1)   *If neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains collapsing rules then $\mathcal{R}_1 \oplus \mathcal{R}_2$ is strongly normalizing.*
(2)   *If neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains duplicating rules then $\mathcal{R}_1 \oplus \mathcal{R}_2$ is strongly normalizing.*
(3)   *If one of the systems $\mathcal{R}_1, \mathcal{R}_2$ contains neither collapsing nor duplicating rules then $\mathcal{R}_1 \oplus \mathcal{R}_2$ is strongly normalizing.*
$\square$

In view of Example 2.4, Toyama conjectured the modularity of the combination of confluence and strong normalization, but Barendregt and Klop constructed a counterexample involving a non-left-linear TRS (see [22]). A simpler counterexample can be found in Drosten [10]. Toyama, Klop and Barendregt [23] gave an extremely complicated proof showing the modularity of the combination of confluence and strong normalization for the restricted class of left-linear TRS's. Modular aspects of properties related to unicity of normal forms have been studied by the present author [17].

For a discussion of the next theorem we refer to Section 4.

THEOREM 2.6. *Weak normalization is a modular property of TRS's.* $\square$

In [19] we extended Toyama's confluence result for disjoint unions of TRS's (Theorem 2.3) to CTRS's.

THEOREM 2.7 (Middeldorp [19]). *Confluence is a modular property of CTRS's.* $\square$

Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be CTRS's with disjoint alphabets. Every term $t \in \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ can be viewed as an alternation of $\mathcal{F}_1$-parts and $\mathcal{F}_2$-parts. This layered structure is formalized in Definition 2.8, see Figure 1.

NOTATION. We abbreviate $\mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ to $\mathcal{T}$ and we will use $\mathcal{T}_i$ as a shorthand for $\mathcal{T}(\mathcal{F}_i, \mathcal{V})$ $(i = 1, 2)$. When writing $\to$ ($\downarrow$, $\twoheadrightarrow$) without a subscript, we will always mean $\to_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ ($\downarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$, $\twoheadrightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$).

DEFINITION 2.8. Let $t \in \mathcal{T}$.
(1)   The *root symbol* of t, notation $root(t)$, is defined by

$$root(t) = \begin{cases} F & \text{if } t \equiv F(t_1, \ldots, t_n), \\ t & \text{otherwise.} \end{cases}$$

(2) Let $t \equiv C[t_1, ..., t_n]$ with $C[, ...,] \not\equiv \square$. We write $t \equiv C[\![t_1, ..., t_n]\!]$ if $C[, ...,]$ is a $\mathcal{F}_a$-context and $root(t_i) \in \mathcal{F}_b$ with $a \neq b$ for $i = 1, ..., n$ ($a, b \in \{1, 2\}$). The t_i's are the *principal* subterms of t.

(3) The *rank* of t is defined by

$$rank(t) = \begin{cases} 1 & \text{if } t \in \mathcal{F}_1 \cup \mathcal{F}_2, \\ 1 + \max\{rank(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, ..., t_n]\!]. \end{cases}$$

(4) The multiset $S(t)$ of *special* subterms of t is defined as follows:

$$S_1(t) = [t]^\dagger,$$

$$S_{n+1}(t) = \begin{cases} [] & \text{if } rank(t) = 1, \\ S_n(t_1) \cup ... \cup S_n(t_m) & \text{if } t \equiv C[\![t_1, ..., t_m]\!], \end{cases}$$
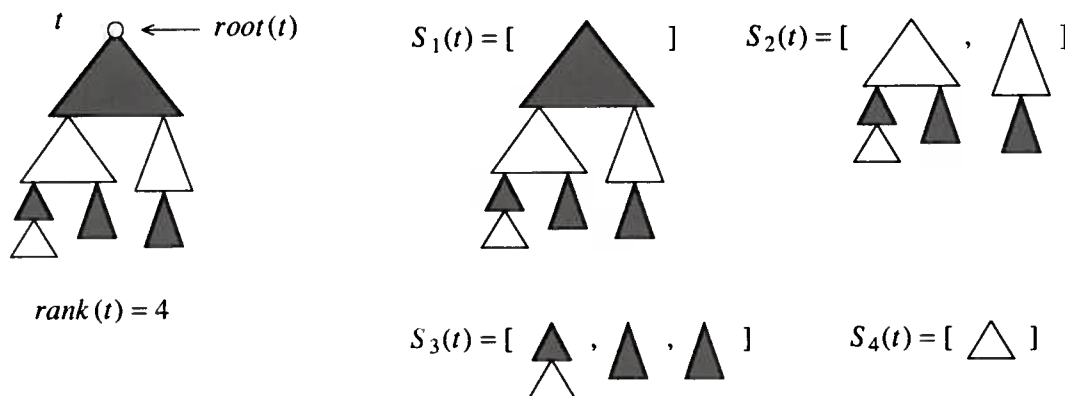
$$S(t) = \bigcup_{i \geq 1} S_i(t).$$

FIGURE 1.

NOTATION. We will use $S_{>1}(t)$ as a shorthand for $\bigcup_{i > 1} S_i(t)$. The set $\{t \in \mathcal{J} \mid rank(t) = n\}$ is abbreviated to $\mathcal{J}^n$ and $\mathcal{J}^{<n}$ denotes the set of all terms with rank less than n.

To achieve better readability we will call the function symbols of $\mathcal{F}_1$ *black* and those of $\mathcal{F}_2$ *white*. Variables have no colour. A black (white) term does not contain white (black) function symbols, but may contain variables. In examples, black symbols will be printed as capitals and white symbols in lower case. This convention was already used in Example 2.4.

The next proposition states some frequently used properties of special subterms. The trivial proofs have been omitted.

PROPOSITION 2.9. *Let* $t \in \mathcal{J}$.

(1) $S_n(t) = [] \Leftrightarrow n > rank(t)$.

(2) $S(t) = S_1(t) \cup S_{>1}(t)$.

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† To distinguish between sets and multisets, we use brackets instead of braces for the latter.

(3) *If $s \in S_n(t)$ then $rank(s) \le rank(t) - n + 1$.*

(4) $s \in S_2(t) \iff s$ *is a principal subterm of t.*

$\square$

PROPOSITION 2.10. *If $s \twoheadrightarrow t$ then $rank(s) \ge rank(t)$.*

PROOF. Straightforward. $\square$

The following definition is illustrated in Figure 2.

DEFINITION 2.11. Let $s \to t$ by application of a rewrite rule A. We write $s \to^i t$ if $s \equiv C[\![s_1, \ldots, s_n]\!]$ and A is being applied in one of the s_j's and we write $s \to^o t$ otherwise. The relation $\to^i$ is called *inner* reduction and $\to^o$ is called *outer* reduction.
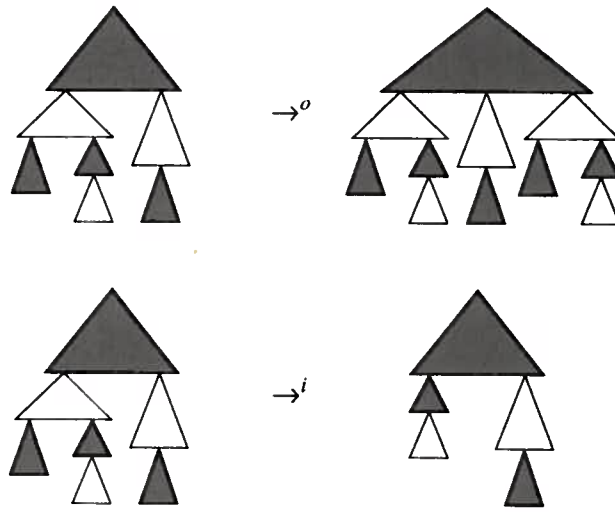

FIGURE 2.

Notice that the outer reduction step in Figure 2 uses a duplicating rule from $\mathcal{R}_1$ and the inner reduction step uses a collapsing rule from $\mathcal{R}_2$. The remaining definitions and propositions of this section were introduced in [19] to handle CTRS's.

DEFINITION 2.12. Suppose σ and τ are substitutions. We write $\sigma \propto \tau$ if $\sigma(x) \equiv \sigma(y)$ implies $\tau(x) \equiv \tau(y)$ for all $x, y \in \mathcal{V}$. Notice that $\sigma \propto \varepsilon$ if and only if σ is injective. We write $\sigma \twoheadrightarrow \tau$ if $\sigma(x) \twoheadrightarrow \tau(x)$ for all $x \in \mathcal{V}$. Clearly $\sigma(t) \twoheadrightarrow \tau(t)$ whenever $\sigma \twoheadrightarrow \tau$.

DEFINITION 2.13. A substitution σ is called *black* (*white*) if $\sigma(x)$ is a black (white) term for every $x \in \mathcal{D}(\sigma)$. We call σ *top black* (*top white*) if the root symbol of $\sigma(x)$ is black (white) for every $x \in \mathcal{D}(\sigma)$.

PROPOSITION 2.14. *Every substitution σ can be decomposed into $\sigma_2 \circ \sigma_1$ such that σ_1 is black (white), σ_2 is top white (top black) and $\sigma_2 \propto \varepsilon$.* $\square$

In the remainder of this paper we only state propositions for a single colour situation (usually: ... black term ... top white substitution ...) without mentioning the reverse situation between parentheses.

3. Strong Normalization

In this section we extend Theorem 2.5 to CTRS's. The proofs given in Rusinowitch [20] and Middeldorp [18] carry over to CTRS's easily. The only complication is the increased complexity of Proposition 3.11 below. We start with a short review of multiset orderings.

NOTATION. The set of all finite multisets over a set S is denoted by $\mathcal{M}(S)$.

DEFINITION 3.1. Let $>$ be a binary relation on a set S. The *multiset extension* $\gg$ of $>$ is a binary relation on $\mathcal{M}(S)$ defined as follows: $M_1 \gg M_2$ if there exist multisets $X, Y \in \mathcal{M}(S)$ satisfying

(1) $[\,] \neq X \subseteq M_1$,

(2) $M_2 = (M_1 - X) \cup Y$,

(3) $\forall y \in Y \; \exists x \in X$ such that $x > y$.

Occasionally we write $>^m$ instead of $\gg$.

THEOREM 3.2 (Dershowitz and Manna [5]). *A relation $>$ is terminating on a set S if and only if the multiset extension $\gg$ of $>$ is terminating on $\mathcal{M}(S)$.* $\square$

From now on we assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly normalizing CTRS's with disjoint alphabets.

DEFINITION 3.3. Let $t \in \mathcal{T}$. The *topmost homogeneous part* of t, notation $top(t)$, is the result of replacing all principal subterms of t by $\square$, i.e.

$$top(t) = \begin{cases} t & \text{if } rank(t) = 1, \\ C[\,,\dots,\,] & \text{if } t \equiv C[\![\, t_1,\dots,t_n \,]\!]. \end{cases}$$

NOTATION. We abbreviate $\mathcal{T}(\mathcal{F}_1 \cup \{\square\}, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2 \cup \{\square\}, \mathcal{V})$ to $\mathcal{T}_{top}$. The 'restriction' of the rewrite relation $\to_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ to $\mathcal{T}_{top}$ is denoted by $\Rightarrow$.

PROPOSITION 3.4. *The relation $\Rightarrow$ is terminating.*

PROOF. If $\Rightarrow$ is not terminating then there exists an infinite sequence $t_1 \Rightarrow t_2 \Rightarrow t_3 \dots$. It is easy to see that either $t_1 \to_{\mathcal{R}_1} t_2 \to_{\mathcal{R}_1} t_3 \to_{\mathcal{R}_1} \dots$ or $t_1 \to_{\mathcal{R}_2} t_2 \to_{\mathcal{R}_2} t_3 \to_{\mathcal{R}_2} \dots$, contradicting the strong normalization of either $\mathcal{R}_1$ or $\mathcal{R}_2$. $\square$

DEFINITION 3.5. Let σ be a substitution. The substitution $\sigma^{\square}$ is defined as $\{x \to \square \mid x \in \mathcal{D}(\sigma)\}$.

The next proposition is only used in the proof of Proposition 3.7. Its technical proof is omitted.

PROPOSITION 3.6. *Suppose $\sigma(s) \to s_1 \downarrow \sigma(t)$ with s, t black terms and σ a top white substitution. There exists a black term s_2 and a top white substitution τ such that $\sigma(s) \equiv \tau(s)$, $s_1 \equiv \tau(s_2)$ and $\sigma(t) \equiv \tau(t)$.* $\square$

The proof of Proposition 3.7 is very similar to the proof of Proposition 3.5 from [19].

PROPOSITION 3.7. *Let s and t be black terms with $s \notin \mathcal{V}$. If σ is a top white substitution with $\sigma(s) \to^o \sigma(t)$ then $\sigma^{\square}(s) \Rrightarrow \sigma^{\square}(t)$.* $\square$

PROOF. We use induction on the depth of $\sigma(s) \to^o \sigma(t)$. The case of zero depth is straightforward. If the depth of $\sigma(s) \to^o \sigma(t)$ equals $n+1$ ($n \geq 0$) then there exists a context $C[\ ]$, a substitution ρ and a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_m \downarrow t_m$ in $\mathcal{R}_1$ such that $\sigma(s) \equiv C[\rho(l)]$, $\sigma(t) \equiv C[\rho(r)]$ and $\rho(s_i) \downarrow_1^o \rho(t_i)$ for $i = 1, \ldots, m$ with depth less than or equal to n. Proposition 2.14 yields a decomposition $\rho_2 \circ \rho_1$ of ρ such that ρ_1 is black and ρ_2 is top white. The situation is illustrated in Figure 3. Using
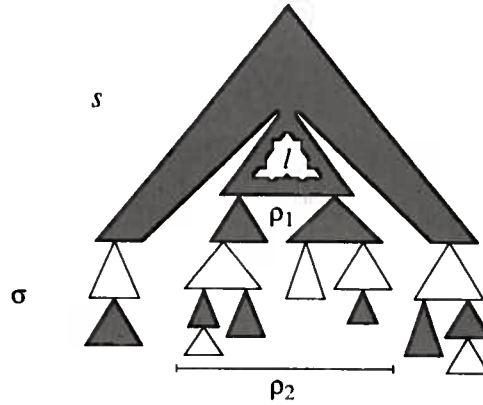

FIGURE 3.

the induction hypothesis and Proposition 3.6 we obtain the joinability of $\rho_2^{\square}(\rho_1(s_i))$ and $\rho_2^{\square}(\rho_1(t_i))$ with respect to $\Rrightarrow$ by a straightforward induction on the length of the valley $\rho_2(\rho_1(s_i)) \downarrow \rho_2(\rho_1(t_i))$ for $i = 1, \ldots, m$. Hence $\rho_2^{\square}(\rho_1(l)) \Rrightarrow \rho_2^{\square}(\rho_1(r))$. Let $C^*[\ ]$ be the context obtained from $C[\ ]$ by replacing all principal subterms by $\square$. Because $\sigma^{\square}(s) \equiv C^*[\rho_2^{\square}(\rho_1(l))]$ and $\sigma^{\square}(t) \equiv C^*[\rho_2^{\square}(\rho_1(r))]$ we conclude that $\sigma^{\square}(s) \Rrightarrow \sigma^{\square}(t)$. $\square$

DEFINITION 3.8. We say that a rewrite step $s \to t$ is *destructive at level* 1 if the root symbols of s and t have different colours. The rewrite step $s \to t$ is *destructive at level* $n+1$ if $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to^i C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s_j \to t_j$ destructive at level n.

Notice that if $s \to t$ is destructive at level 1 then s is top white, $s \to^o t$ and either $t \in S_2(s)$ or $t \in V(top(s))$. Notice furthermore that a rewrite step can only be destructive if the used rewrite rule is collapsing. The inner reduction step in Figure 2 is destructive at level 2. The following two propositions are very intuitive, see Figure 4 and the second part of Figure 2. Formal proofs are omitted.

PROPOSITION 3.9. *If $s \to^o t$ is a non-destructive rewrite step then the set inclusion $\{u \mid u \in S_2(t)\} \subseteq \{u \mid u \in S_2(s)\}$ holds. If the applied rewrite rule is not duplicating, we even have the multiset inclusion $S_2(t) \subseteq S_2(s)$.* $\square$

PROPOSITION 3.10. *If $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to^i C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ is destructive at level 2 then $S_2(t) = S_2(s) - [s_j] \cup S_2(t_j)$.* $\square$

PROPOSITION 3.11.

(1) *If $s \to^o t$ is not destructive at level 1 then $top(s) \Rrightarrow top(t)$.*

(2) *If $s \to^i t$ is not destructive at level 2 then $top(s) \equiv top(t)$.*

PROOF.

(1) If $s \to^o t$ is not destructive then we may write $s \equiv C[\![s_1, \ldots, s_n]\!]^\dagger$ and $t \equiv C^*[\![s_{i_1}, \ldots, s_{i_m}]\!]^\dagger$. Choose distinct fresh variables $x_1, \ldots, x_n$ and define terms $s' \equiv C[x_1, \ldots, x_n]$, $t' \equiv C^*[x_{i_1}, \ldots, x_{i_m}]$ and substitution $\sigma \equiv \{x_i \to s_i \mid 1 \leq i \leq n\}$. Clearly $s \equiv \sigma(s') \to^o \sigma(t') \equiv t$. Applying Proposition 3.7 yields $\sigma^\square(s') \Rrightarrow \sigma^\square(t')$ and because $\sigma^\square(s') \equiv top(s)$ and $\sigma^\square \equiv top(t)$ we are done.

(2) We have $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C[\![s_1, \ldots, t_j, \ldots, s_n]\!] \equiv t$ with $s_j \to t_j$. Clearly $top(s) \equiv C[\ , \ldots, \ ] \equiv top(t)$.

$\square$

$l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ is a duplicating rule

$l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ is not a duplicating rule
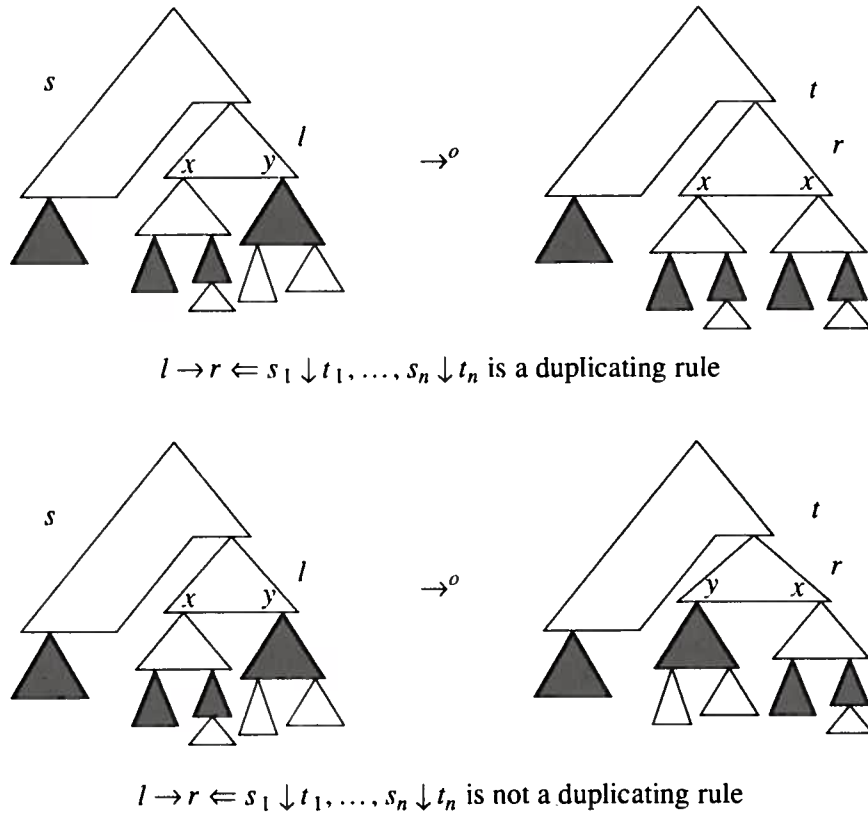
FIGURE 4.

We are now ready to prove the sufficiency of the conditions expressed in Theorem 2.5 for the strong normalization of $\mathcal{R}_1 \oplus \mathcal{R}_2$. We first consider the condition that neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains collapsing rules.

DEFINITION 3.12. We define a relation $>_1$ on $\mathcal{I}$ as follows: $s >_1 t$ if

(1) $rank(s) \geq rank(t)$,

(2) $top(s) \Rrightarrow top(t)$ or
    $top(s) \equiv top(t)$ and $S_2(s) \gg_1 S_2(t)$.

PROPOSITION 3.13. *The relation $>_1$ is terminating.*

PROOF. If $>_1$ is not terminating on $\mathcal{I}$, then there exists an infinite sequence $t_1 >_1 t_2 >_1 t_3 >_1 \ldots$. We

---

† In order to avoid an explosion of cases to be considered, we allow for $n = 0$ and $m = 0$.

will show by induction on $rank(t_1)$ that this is impossible. If $rank(t_1)=1$ then we have $t_1 \Rightarrow t_2 \Rightarrow t_3 \Rightarrow ...$, contradicting Proposition 3.4. Let $rank(t_1)=n$ with $n>1$. The induction hypothesis states that $>_1$ is terminating on $\mathcal{I}^i$ for all $i<n$. Because $s >_1 t$ implies $rank(s) \geq rank(t)$, the relation $>_1$ is also terminating on $\mathcal{I}^{<n}$. Theorem 3.2 yields the termination of $\gg_1$ on $\mathcal{M}(\mathcal{I}^{<n})$. From the definition of $>_1$ and Proposition 3.4 we know that there exists an index i such that

$$S_2(t_i) \gg_1 S_2(t_{i+1}) \gg_1 S_2(t_{i+2}) \gg_1 ....$$

We obtain a contradiction since $S_2(t_j) \in \mathcal{M}(\mathcal{I}^{<n})$ for all $j \geq i$. $\square$

PROPOSITION 3.14. *If $s \rightarrow t$ then $s >_1 t$.*

PROOF. Proposition 3.4 yields $rank(s) \geq rank(t)$, so we only have to show that $top(s) \Rightarrow top(t)$ or $top(s) \equiv top(t)$ and $S_2(s) \gg_1 S_2(t)$. This will be established by induction on $rank(s)$. If $rank(s)=1$ then $top(s) \equiv s \Rightarrow t \equiv top(t)$. Let $rank(s)=n$ with $n>1$. If $s \rightarrow^o t$ then $top(s) \Rightarrow top(t)$ by Proposition 3.11(1). If $s \rightarrow^i t$ then $top(s) \equiv top(t)$ by Proposition 3.11(2) and we may write $s \equiv C[\![s_1, ..., s_j, ..., s_m]\!] \rightarrow C[\![s_1, ..., t_j, ..., s_m]\!] \equiv t$ with $s_j \rightarrow t_j$. The induction hypothesis yields $s_j >_1 t_j$. Hence $S_2(s) = [s_1, ..., s_j, ..., s_m] \gg_1 [s_1, ..., t_j, ..., s_m] = S_2(t)$. $\square$

THEOREM 3.15. *Strong normalization is a modular property of CTRS's without collapsing rules.*

PROOF. Immediate consequence of Proposition 3.13 and 3.14. $\square$

Next we assume that neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ contains duplicating rules.

DEFINITION 3.16. Let $t \in \mathcal{I}$. We define $\#t$ as the cardinality of the multiset $S(t)$, provided t is not a variable. If $t \in \mathcal{V}$ then $\#t=0$.

Notice that $\#t$ denotes the number of black and white parts in t. The special treatment of variables enables a more elegant formulation of certain properties.

NOTATION. The multiset $[top(u) \mid u \in S(t)]$ is denoted by $\Delta(t)$.

DEFINITION 3.17. We define a relation $>_2$ on $\mathcal{I}$ as follows: $s >_2 t$ if $\#s > \#t$ or $\#s = \#t$ and $\Delta(s) \Rightarrow^m \Delta(t)$.

PROPOSITION 3.18. *The relation $>_2$ is terminating.*

PROOF. Suppose $>_2$ is not terminating. It is easy to show that there exists an infinite sequence

$$t_1 >_2 t_2 >_2 t_3 >_2 ...$$

in which all terms have the same number of black and white parts. Hence we have the infinite sequence

$$\Delta(t_1) \Rightarrow^m \Delta(t_2) \Rightarrow^m \Delta(t_3) \Rightarrow^m ....$$

But this is impossible, since combining Proposition 3.4 and Theorem 3.2 yields the termination of $\Rightarrow^m$. $\square$

PROPOSITION 3.19. *If $s \rightarrow t$ then $s >_2 t$.*

PROOF. We will show by induction on $rank(s)$ that either $\#s > \#t$ or $\#s = \#t$ and $\Delta(s) \Rightarrow^m \Delta(t)$. First assume that $rank(s)=1$. If $s \rightarrow t$ is destructive then $\#s = 1 > 0 = \#t$. Otherwise $\#s = \#t = 1$ and

$top(s) \equiv s \Rightarrow t \equiv top(t)$. Now let $rank(s) = n$ with $n > 1$. We distinguish two cases:

(1) If $s \to^o t$ is destructive then either $t \in V(top(s))$ or $t \in S_2(s)$. In both cases we clearly have $\#s > \#t$. If $s \to^o t$ is not destructive then $S_2(t) \subseteq S_2(s)$ by Proposition 3.9 and therefore $S_i(t) \subseteq S_i(s)$ for all $i \geq 2$. Proposition 3.11(1) yields $top(s) \Rightarrow top(t)$. Hence

$$\Delta(s) = [top(s)] \cup [top(u) \mid u \in S_{>1}(s)] \Rightarrow^m [top(t)] \cup [top(u) \mid u \in S_{>1}(t)] = \Delta(t).$$

(2) If $s \to^i t$ is destructive at level 2 then we easily obtain $\#s > \#t$. Otherwise we may write $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_m]\!] \to^i C[\![s_1, \ldots, t_j, \ldots, s_m]\!] \equiv t$ with $s_j \to t_j$. The induction hypothesis yields $s_j >_2 t_j$. If $\#s_j > \#t_j$ then $\#s > \#t$. If $\#s_j = \#t_j$ and $\Delta(s_j) \Rightarrow^m \Delta(t_j)$ then also $\#s = \#t$ and $\Delta(s) \Rightarrow^m \Delta(t)$.

$\square$

THEOREM 3.20. *Strong normalization is a modular property of CTRS's without duplicating rules.*

PROOF. Immediate consequence of Proposition 3.18 and 3.19. $\square$

Finally we consider the condition that one of $\mathcal{R}_1, \mathcal{R}_2$ contains neither collapsing nor duplicating rules. Without loss of generality we assume that $\mathcal{R}_1$ contains neither collapsing nor duplicating rules. The next definition is motivated in [18].

DEFINITION 3.21. To each term $t \in \mathcal{T}$ we assign a weight $\| t \|$ as follows:

$$\| t \| = \begin{cases} 0 & \text{if } t \in \mathcal{V}, \\ \sum_{s \in S_2(t)} \| s \| & \text{if } t \text{ is top black,} \\ 1 + \max_{s \in S_2(t)} \| s \| & \text{if } t \text{ is top white.} \end{cases}$$

EXAMPLE 3.22. Let

$$\mathcal{R}_1 = \begin{cases} F(x, y, z) \to G(z) & \Leftarrow x \downarrow y \\ G(A) \to F(A, B, A) \end{cases}$$

and

$$\mathcal{R}_2 = \begin{cases} e(x) \to f(x, x) \\ f(x, y) \to x. \end{cases}$$

In the reduction sequence

$$e(F(f(G(A), B), G(A), e(B)))$$
$$\to f(F(f(G(A), B), G(A), e(B)), F(f(G(A), B), G(A), e(B)))$$
$$\to f(F(G(A), G(A), e(B)), F(f(G(A), B), G(A), e(B)))$$
$$\to F(G(A), G(A), e(B))$$
$$\to G(e(B))$$
$$\to G(f(B, B))$$
$$\to G(B)$$

we have the weights 3, 3, 3, 1, 1, 1 and 0, respectively.

PROPOSITION 3.23. *If $s \to t$ is destructive at level 1 then $\|s\| > \|t\|$.*

PROOF. We either have $s \equiv C[\![s_1, \ldots, s_n]\!] \to s_k \equiv t$ or $s \to x \equiv t$ for some variable $x \in V(top(s))$. In the former case we obtain

$$\|s\| = 1 + \max \{ \|s_i\| \mid 1 \le i \le n \} > \|s_k\| = \|t\|$$

because s is top white and in the latter case we clearly have $\|s\| > 0 = \|t\|$. $\square$

PROPOSITION 3.24. *If $s \to t$ is destructive at level 2 then $\|s\| > \|t\|$.*

PROOF. We have $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s_j \to t_j$ destructive at level 1. From Proposition 3.23 we obtain $\|s_j\| > \|t_j\|$. Notice that s and t are top black. Hence $\|s\| = \sum_{i=1}^{n} \|s_i\|$ and $\|t\| = \|s\| - \|s_j\| + \sum_{u \in S_2(t_j)} \|u\|$ by Proposition 3.10. We only have to show that $\|s_j\| > \sum_{u \in S_2(t_j)} \|u\|$. Because $s_j \to t_j$ is destructive at level 1 we either have $t_j \in V(top(s_j))$ or $t_j \in S_2(s_j)$. In the first case we clearly have $\|s_j\| > 0 = \sum_{u \in [\,]} \|u\|$ and in the second case we obtain

$$\|s_j\| > \|t_j\| = \sum_{u \in S_2(t_j)} \|u\|$$

since t_j is top black. $\square$

The second step in the reduction sequence of Example 3.22 shows that the previous propositions do not generalize to destructive rewrite steps at a level greater than 2.

PROPOSITION 3.25. *If $s \to t$ then $\|s\| \ge \|t\|$.*

PROOF. Using Proposition 3.23 and 3.24 we may assume that $s \to t$ is not destructive at level 1 or 2. We will use induction on $rank(s)$. If $rank(s) = 1$ then $rank(t) = 1$ by Proposition 2.10. Because t is not a variable (otherwise $s \to t$ would be destructive at level 1) we have $\|s\| = 1 = \|t\|$ by definition. Assume the statement is true for all terms with rank less than n ($n > 1$) and let $rank(s) = n$. We distinguish two cases:

(1) If $s \to^o t$ then $\{u \mid u \in S_2(t)\} \subseteq \{u \mid u \in S_2(s)\}$ by Proposition 3.9. If the applied rewrite rule is duplicating then s and t are top white and

$$\|s\| = 1 + \max_{u \in S_2(s)} \|u\| \ge 1 + \max_{u \in S_2(t)} \|u\| = \|t\|.$$

If the applied rewrite rule is not duplicating, we obtain the multiset inclusion $S_2(t) \subseteq S_2(s)$ from Proposition 3.9. Therefore

$$\sum_{u \in S_2(s)} \|u\| \ge \sum_{u \in S_2(t)} \|u\|$$

and

$$1 + \max_{u \in S_2(s)} \|u\| \ge 1 + \max_{u \in S_2(t)} \|u\|,$$

so we always have $\|s\| \ge \|t\|$.

(2) If $s \to^i t$ then we may write $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_m]\!] \to C[\![s_1, \ldots, t_j, \ldots, s_m]\!] \equiv t$ with $s_j \to t_j$. The induction hypothesis yields $\|s_j\| \ge \|t_j\|$. Clearly $S_2(t) = S_2(s) - [s_j] \cup [t_j]$. So again we have both

$$\sum_{u \in S_2(s)} \|u\| \ge \sum_{u \in S_2(t)} \|u\|$$

and

$$1 + \max_{u \in S_2(s)} \|u\| \geq 1 + \max_{u \in S_2(t)} \|u\|.$$

Hence $\|s\| \geq \|t\|$.

$\square$

DEFINITION 3.26. We define a relation $>_3$ on $\mathcal{I}$ as follows: $s >_3 t$ if

(1) $rank(s) \geq rank(t)$,

(2) $\|s\| > \|t\|$ or

$\|s\| = \|t\|$ and $top(s) \Rrightarrow top(t)$ or

$\|s\| = \|t\|$, $top(s) \equiv top(t)$ and $S_2(s) \gg_3 S_2(t)$.

PROPOSITION 3.27. *The relation $>_3$ is terminating.*

PROOF. Similar to the proof of Proposition 3.13. $\square$

PROPOSITION 3.28. *If $s \to t$ then $s >_3 t$.*

PROOF. Since $rank(s) \geq rank(t)$ by Proposition 3.4, we only have to show that $\|s\| > \|t\|$ or $\|s\| = \|t\|$ and $top(s) \Rrightarrow top(t)$ or $\|s\| = \|t\|$, $top(s) \equiv top(t)$ and $S_2(s) \gg_3 S_2(t)$. This will be done using induction on $rank(s)$. First we consider the case $rank(s) = 1$. If $s \to t$ is destructive at level 1 then $\|s\| > \|t\|$ by Proposition 3.23. Otherwise $\|s\| = \|t\| = 1$ and $top(s) \Rrightarrow top(t)$ by Proposition 3.11(1). We now assume that $rank(s) = n$ with $n > 1$, Proposition 3.25 yields $\|s\| \geq \|t\|$. We distinguish two cases:

(1) If $s \to^o t$ is destructive at level 1 then $\|s\| > \|t\|$ by Proposition 3.23 and if $s \to^o t$ is not destructive then $top(s) \Rrightarrow top(t)$ by Proposition 3.11(1).

(2) If $s \to^i t$ is destructive at level 2 then the result follows from Proposition 3.24. If $s \to^i t$ is not destructive at level 2 then $top(s) \equiv top(t)$ by Proposition 3.11(2) and we may write $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_m]\!] \to C[\![s_1, \ldots, t_j, \ldots, s_m]\!] \equiv t$ with $s_j \to t_j$. From the induction hypothesis we obtain $s_j >_3 t_j$. Therefore $S_2(s) = [s_1, \ldots, s_j, \ldots, s_m] \gg_3 [s_1, \ldots, t_j, \ldots, s_m] = S_2(t)$.

$\square$

THEOREM 3.29. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly normalizing CTRS's with disjoint alphabets such that one of $\mathcal{R}_1, \mathcal{R}_2$ contains neither duplicating nor collapsing rules, then $\mathcal{R}_1 \oplus \mathcal{R}_2$ strongly normalizing.*

PROOF. Immediate consequence of Proposition 3.27 and 3.28. $\square$

4. Weak Normalization

Contrary to strong normalization, weak normalization is a modular property of TRS's. This has been independently observed by several authors (Bergstra, Klop and Middeldorp [2], Drosten [10], Kurihara and Kaji [16], Toyama, Klop and Barendregt [23]). Two approaches can be identified in establishing the weak normalization of the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ of two weakly normalizing TRS's $\mathcal{R}_1, \mathcal{R}_2$:

(1) Every term $t \in \mathcal{I}$ can be normalized using 'innermost' rewriting, i.e. first the bottom layer of t is reduced to normal form, then the layer above the bottom layer is normalized and working steadily upwards we eventually normalize t. This is the method of [2], [10] and [23].

(2) A term $t \in \mathcal{I}$ can also be normalized by the following recipe: First we normalize t with respect to $\mathcal{R}_1$ with result, say, t_1. The term t_1 is then normalized with respect to $\mathcal{R}_2$ giving t_2. Now we use

again $\mathscr{R}_1$ to normalize t_2 and continuing in this manner we eventually arrive at a $\mathscr{R}_1 \oplus \mathscr{R}_2$-normal form of t. The termination of this process is guaranteed by an interesting result of Kurihara and Kaji [16].

Before studying weak normalization with respect to disjoint unions of CTRS's, we give an account of the work of Kurihara and Kaji. Instead of allowing an arbitrary interleaving of $\mathscr{R}_1$-reduction steps and $\mathscr{R}_2$-reduction steps, they adopt the obligation to use as long as possible the rewrite rules of the same TRS. So, if a rule of $\mathscr{R}_1$ ($\mathscr{R}_2$) is applied to a term t, we must first normalize t with respect to $\mathscr{R}_1$ ($\mathscr{R}_2$), before applying rules of $\mathscr{R}_2$ ($\mathscr{R}_1$). In particular, the reduction sequence in Example 2.4 is no longer allowed. The reader is referred to [16] for a comprehensive motivation of this approach.

DEFINITION 4.1. We write $s \twoheadrightarrow_{\mathscr{R}} t$ if $s \rightarrow^+_{\mathscr{R}} t$ and t is a normal form with respect to $\mathscr{R}$.

The main result of Kurihara and Kaji can be stated as follows.

THEOREM 4.2 (Kurihara and Kaji [16]). *If $\mathscr{R}_1, \ldots, \mathscr{R}_n$ are disjoint TRS's then $\twoheadrightarrow_{\mathscr{R}_1} \cup \ldots \cup \twoheadrightarrow_{\mathscr{R}_n}$ is a terminating relation.* $\square$

Notice that $\mathscr{R}_1, \ldots, \mathscr{R}_n$ are not required to be strongly (weakly) normalizing. The modularity of weak normalization for TRS's is an easy corollary of this theorem. The reader is invited to verify that the above result is not a consequence of the same result for $n = 2$. The proof of Kurihara and Kaji is rather complicated and not very intuitive[†]. We now give an easier proof which does not rely on multiset orderings. We assume that $\mathscr{R}_1, \ldots, \mathscr{R}_n$ are TRS's with pairwise disjoint alphabets.

NOTATION. Let $X \subseteq \{1, \ldots, n\}$. The relation $\bigcup_{x \in X} \twoheadrightarrow_{\mathscr{R}_x}$ is denoted by $\twoheadrightarrow_X$ and we use $\rightarrow_X$ as a shorthand for $\bigcup_{x \in X} \rightarrow_{\mathscr{R}_x}$. The relation $\twoheadrightarrow_{\{1, \ldots, n\}}$ is further abbreviated to $\twoheadrightarrow$ and we write $\twoheadrightarrow_x$ instead of $\twoheadrightarrow_X$ whenever X is a singleton set $\{x\}$.

The concepts introduced in Definition 2.8 carry over easily to the situation in which we are dealing with n (C)TRS's instead of two (C)TRS's. The only difference is that a layer in a term may have more than one colour.

DEFINITION 4.3. Let $X \subseteq \{1, \ldots, n\}$. To each term $t \in \mathscr{T}$ we assign a number $\|t\|_X$ denoting the deepest layer in t that contains a $\rightarrow_X$-redex, and a set $\phi_X(t)$ consisting of all $i \in X$ with the property that layer $\|t\|_X$ of t contains a $\rightarrow_{\mathscr{R}_x}$-redex:

$$\|t\|_X = \begin{cases} 0 & \text{if } t \text{ is not } \rightarrow_X\text{-reducible,} \\[2ex] \max\{i \mid S_i(t) \text{ contains a } \rightarrow_X\text{-reducible term}\} & \text{otherwise,} \end{cases}$$

$$\phi_X(t) = \begin{cases} \varnothing & \text{if } \|t\|_X = 0, \\[2ex] \{x \mid S_{\|t\|_X}(t) \text{ contains a } \rightarrow_{\mathscr{R}_x}\text{-reducible term}\} & \text{otherwise.} \end{cases}$$

---

† See however the footnote on the title-page.

The following properties are used in the sequel. Their trivial proofs have been omitted.

PROPOSITION 4.4. *Let* $X \subseteq \{1, \ldots, n\}$ *and* $t \in \mathcal{T}$.

(1) $\|t\|_X \le rank(t)$.

(2) *If* $s \in S_m(t)$ *and* $m > \|t\|_X$ *then* s *is not* $\to_X$-*reducible*.

(3) $\phi_X(t) \subseteq X$.

(4) $\phi_X(t) = \varnothing \Leftrightarrow \|t\|_X = 0 \Leftrightarrow t$ *is not* $\to_X$-*reducible*.

$\square$

PROPOSITION 4.5. *Let* $x \in X \subseteq \{1, \ldots, n\}$, $s \to_{\mathcal{R}_x} t$ *and* $m = \|s\|_X$. *For every* $t' \in S_m(t)$ *there exists a term* $s' \in S_m(s)$ *such that* $t' \in S(s')$ *or* $s' \to_{\mathcal{R}_x}^0 t'$.

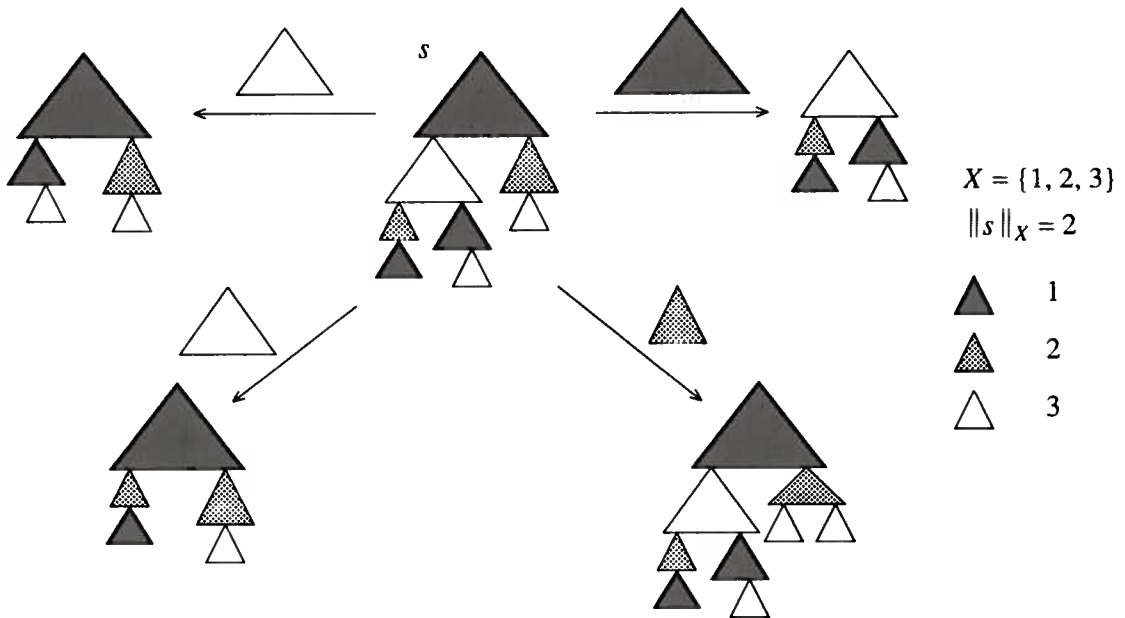PROOF. Figure 5 shows some typical cases. A formal proof is omitted. $\square$

$X = \{1, 2, 3\}$

$\|s\|_X = 2$

FIGURE 5.

PROPOSITION 4.6. *Let* $x \in X \subseteq \{1, \ldots, n\}$, $s \to_{\mathcal{R}_x} t$ *and* $m > \|s\|_X$. *For every* $t' \in S_m(t)$ *there exists a term* $s' \in S_m(s)$ *such that* $t' \in S(s')$.

PROOF. Straightforward application of the previous proposition. $\square$

PROPOSITION 4.7. *If* $x \in X \subseteq \{1, \ldots, n\}$ *and* $s \to_{\mathcal{R}_x} t$ *then* $\|s\|_X \ge \|t\|_X$.

PROOF. If $\|t\|_X = m > \|s\|_X$ then there exists a term $t' \in S_m(t)$ which is $\to_X$-reducible. Proposition 4.6 yields a term $s' \in S_m(s)$ such that $t' \in S(s')$. Because s' is a normal form with respect to $\to_X$, t' can not be $\to_X$-reducible. We conclude that $\|t\|_X \le \|s\|_X$. $\square$

The next example shows that the condition $x \in X$ is necessary.

EXAMPLE 4.8. Let $\mathcal{R}_1 = \{F(x, x) \to x\}$, $\mathcal{R}_2 = \{a \to b\}$ and $s \equiv F(a, b)$. We have $\|s\|_{\{1\}} = 0$ because s is a normal form with respect to $\mathcal{R}_1$, $s \to_2 t \equiv F(b, b)$ and $\|t\|_{\{1\}} = 1$.

PROPOSITION 4.9. *Let* $x \in X \subseteq \{1, \ldots, n\}$, $s \twoheadrightarrow_{\mathcal{R}_x} t$ *and* $m \geq \|s\|_X$. *For every* $t' \in S_m(t)$ *there exists a term* $s' \in S_m(s)$ *such that* $t' \in S(s')$ *or* $s' \twoheadrightarrow^o_{\mathcal{R}_x} t'$.

PROOF. Induction on the length of $s \twoheadrightarrow_{\mathcal{R}_x} t$. The case of zero length is trivial. Assume $s \to_{\mathcal{R}_x} s_1 \twoheadrightarrow_{\mathcal{R}_x} t$ and let $t' \in S_m(t)$. Because Proposition 4.7 gives us $m \geq \|s_1\|_X$, we can apply the induction hypothesis, yielding a term $s_1' \in S_m(s_1)$ such that $t' \in S(s_1')$ or $s_1' \twoheadrightarrow^o_{\mathcal{R}_x} t'$. From Proposition 4.5 or 4.6 we obtain a term $s' \in S_m(s)$ with $s_1' \in S(s')$ or $s' \twoheadrightarrow^o_{\mathcal{R}_x} s_1'$. By distinguishing four cases we will show that $t' \in S(s')$ or $s' \twoheadrightarrow^o_{\mathcal{R}_x} t'$.

(1) If $t' \in S(s_1')$ and $s_1' \in S(s')$ then $t' \in S(s')$.

(2) If $t' \in S(s_1')$ and $s' \twoheadrightarrow^o_{\mathcal{R}_x} s_1'$ then $\|s'\|_X = 1$. Applying Proposition 4.5 or 4.6 yields a term $s'' \in S(s')$ such that $t' \in S(s'')$ or $s'' \twoheadrightarrow^o_{\mathcal{R}_x} t'$. In the former case we obtain $t' \in S(s')$ and the latter case is only possible if $s'' \equiv s'$ from which we derive $s' \twoheadrightarrow^o_{\mathcal{R}_x} t'$.

(3) Suppose $s_1' \twoheadrightarrow^o_{\mathcal{R}_x} t'$ and $s_1' \in S(s')$. If $s_1' \in S_{>1}(s')$ then s_1' is a normal form with respect to $\to_X$ and hence $t' \equiv s_1' \in S(s')$. If $s_1' \equiv s'$ then clearly $s' \twoheadrightarrow^o_{\mathcal{R}_x} t'$.

(4) If $s_1' \twoheadrightarrow^o_{\mathcal{R}_x} t'$ and $s' \twoheadrightarrow^o_{\mathcal{R}_x} s_1'$ then $s' \twoheadrightarrow^o_{\mathcal{R}_x} t'$.

□

PROPOSITION 4.10. *Let* $x \in X \subseteq \{1, \ldots, n\}$. *If* $s \twoheadrightarrow_x t$ *then either* $\|s\|_X > \|t\|_X$ *or* $\|s\|_X = \|t\|_X$ *and* $\phi_X(t) \subseteq \phi_X(s) - \{x\}$.

PROOF. We have $s \to^+_{\mathcal{R}_x} t$ with t a normal form with respect to $\mathcal{R}_x$. Repeated application of Proposition 4.7 yields $\|s\|_X \geq \|t\|_X$. Assume $\|s\|_X = \|t\|_X = m$ and let $y \in \phi_X(t)$. Clearly $y \neq x$. By definition there exists a term $t' \in S_m(t)$ such that t' is reducible with respect to $\to^o_{\mathcal{R}_y}$. According to Proposition 4.9 we can find a term $s' \in S_m(s)$ such that $t' \in S(s')$ or $s' \twoheadrightarrow^o_{\mathcal{R}_x} t'$. If $t' \equiv s'$ then $y \in \phi_X(s)$ by definition. The remaining cases are easily shown to be contradictory to the assumption that t' is $\to^o_{\mathcal{R}_y}$-reducible. □

EXAMPLE 4.11. Consider the TRS's of Table 2 and let $X = \{1, 2, 3\}$ and $t \equiv F(\underline{e}(B, B), g(A, B))$.

$F(A, x)$	$\to$	x	$\underline{e}(x, y)$	$\to$	x	$g(x, y)$	$\to$	$g(y, x)$
B	$\to$	A	$\underline{e}(x, y)$	$\to$	y	$g(x, x)$	$\to$	x
$\mathcal{R}_1$			$\mathcal{R}_2$			$\mathcal{R}_3$		

TABLE 2.

Table 3 summarizes the changes in $\|\cdot\|_X$ and $\phi(\cdot)_X$ for the following two $\twoheadrightarrow$-reduction sequences:

$$t \twoheadrightarrow_1 F(\underline{e}(A, A), g(A, A)) \equiv t_1$$
$$\twoheadrightarrow_3 F(\underline{e}(A, A), A) \equiv t_2$$
$$\twoheadrightarrow_2 F(A, A) \equiv t_3$$
$$\twoheadrightarrow_1 A \equiv t_4,$$
$$t \twoheadrightarrow_2 F(B, g(B, A)) \equiv t_5$$
$$\twoheadrightarrow_1 g(A, A) \equiv t_6$$
$$\twoheadrightarrow_3 A \equiv t_7.$$

	t	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\|\cdot\|_X$	3	2	2	1	0	3	1	0
$\phi(\cdot)_X$	$\{1\}$	$\{2,3\}$	$\{2\}$	$\{1\}$	$\varnothing$	$\{1\}$	$\{3\}$	$\varnothing$

TABLE 3.

THEOREM 4.12. *Let* $\mathcal{R}_1, \ldots, \mathcal{R}_n$ *be disjoint* TRS's. *The relation* $\twoheadrightarrow_X$ *is terminating for all* $X \subseteq \{1, \ldots, n\}$.

PROOF. Induction (1) on the size of X. The case of empty X is trivial and if X contains only one element then the result follows by definition. Let X contain at least two elements and suppose there exists an infinite $\twoheadrightarrow_X$-sequence. By Proposition 4.10 there exists an infinite $\twoheadrightarrow_X$-sequence

$$t_1 \twoheadrightarrow_X t_2 \twoheadrightarrow_X t_3 \twoheadrightarrow_X \cdots$$

in which all terms have the same $\|\cdot\|_X$ norm. We will show by induction (2) on the size of $\phi_X(t_1)$ that this is impossible. If $\phi_X(t_1) = \varnothing$ then t_1 is a normal form with respect to $\rightarrow_X$ and this excludes the possibility of an infinite $\twoheadrightarrow_X$-sequence. Let us now assume that $\phi_X(t_1) \neq \varnothing$ and define $Y = X - \phi_X(t_1)$. Because Y is smaller than X, we know from induction hypothesis (1) that $\twoheadrightarrow_Y$ is terminating. Hence there exists an index $i \geq 1$ such that

$$t_1 \twoheadrightarrow_Y \cdots \twoheadrightarrow_Y t_i \twoheadrightarrow_X t_{i+1}$$

for some $x \in \phi_X(t_1)$. Proposition 4.10 yields $\phi_X(t_{i+1}) \subseteq \phi_X(t_1) - \{x\}$ and from induction hypothesis (2) we obtain the impossibility of an infinite $\twoheadrightarrow_X$-sequence starting at t_{i+1}. Therefore any $\twoheadrightarrow_X$-sequence starting at t_1 is finite. We conclude that $\twoheadrightarrow_X$ is indeed terminating. $\square$

COROLLARY 4.13. *Weak normalization is a modular property of* TRS's.

PROOF. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be disjoint TRS's. If $\mathcal{R}_1 \oplus \mathcal{R}_2$ is weakly normalizing then clearly both $\mathcal{R}_1$ and $\mathcal{R}_2$ are weakly normalizing. Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are weakly normalizing. One easily shows that the set of normal forms with respect to $\twoheadrightarrow$ $(= \twoheadrightarrow_{\{1,2\}})$ coincides with $\mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2)$. Theorem 4.12 yields the weak normalization of $\mathcal{R}_1 \oplus \mathcal{R}_2$. $\square$

The next example shows that $\twoheadrightarrow$ does not need to be a confluent relation, even if the participating TRS's are confluent. This answers a question of Kurihara and Kaji [16].

EXAMPLE 4.14. Let

$$\mathcal{R}_1 = \begin{cases} F(x, x) & \rightarrow & F(x, x) \\ A & \rightarrow & B \end{cases}$$

and $\mathcal{R}_2 = \{e(x) \rightarrow x\}$. Both systems are clearly confluent. We have $F(e(A), A) \twoheadrightarrow_1 F(e(B), B) \twoheadrightarrow_2 F(B, B)$ and $F(e(A), A) \twoheadrightarrow_2 F(A, A)$. The terms $F(A, A)$ and $F(B, B)$ are different normal forms with respect to $\twoheadrightarrow$. Therefore $\twoheadrightarrow$ is not confluent.

COROLLARY 4.15. (Kurihara and Kaji [16]) *If* $\mathcal{R}_1, \ldots, \mathcal{R}_n$ *are disjoint confluent and weakly normalizing* TRS's *then* $\twoheadrightarrow$ *is a confluent relation.*

PROOF. Easy consequence of Theorem 2.3 and Theorem 4.12, using the observation that every $\twoheadrightarrow$-normal form is a normal form with respect to $\mathcal{R}_1 \oplus \ldots \oplus \mathcal{R}_n$. $\square$

A careful inspection of the proofs of Theorem 4.12 and its preceding propositions reveals that Theorem 4.12 immediately extends to CTRS's.

THEOREM 4.16. *If $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are disjoint CTRS's then $\twoheadrightarrow$ is a terminating relation.* $\square$

However, this result cannot be used to obtain the modularity of weak normalization for CTRS's. The reason is that a $\twoheadrightarrow$-normal form does not need to be a normal form with respect to $\mathcal{R}_1 \oplus \ldots \oplus \mathcal{R}_n$, notwithstanding the fact that every $\twoheadrightarrow$-normal form is a normal form with respect to $\mathcal{R}_i$ ($i = 1, \ldots, n$) for weakly normalizing CTRS's $\mathcal{R}_1, \ldots, \mathcal{R}_n$.

EXAMPLE 4.17. Let $\mathcal{R}_1 = \{F(x, y) \to C \Leftarrow x \downarrow z, z \downarrow y\}$ and

$$\mathcal{R}_2 = \begin{cases} a & \to & b, \\ a & \to & c. \end{cases}$$

We have $F(b, c) \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} C$ because $b \downarrow_{\mathcal{R}_2} a$ and $a \downarrow_{\mathcal{R}_2} c$, but $F(b, c)$ is neither $\mathcal{R}_1$-reducible nor $\mathcal{R}_2$-reducible. Notice that the rewrite rule of $\mathcal{R}_1$ contains an extra variable (z) in the conditions and $\mathcal{R}_2$ is not confluent.

A slight modification of this example shows that weak normalization is not a modular property of CTRS's.

EXAMPLE 4.18. Let

$$\mathcal{R}_1 = \begin{cases} F(x, x) & \to & C \\ F(x, y) & \to & F(x, y) \Leftarrow x \downarrow z, z \downarrow y \end{cases}$$

and

$$\mathcal{R}_2 = \begin{cases} a & \to & b \\ a & \to & c. \end{cases}$$

One easily shows that $\mathcal{R}_1$ is confluent. From this we obtain the weak normalization of $\mathcal{R}_1$ by a routine argument. Clearly $\mathcal{R}_2$ is weakly normalizing. However, $\mathcal{R}_1 \oplus \mathcal{R}_2$ is not weakly normalizing: the term $F(b, c)$ reduces only to itself.

Example 4.17 suggests two sufficient conditions for the equality of $NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$ and $NF(\mathcal{R}_1) \cap NF(\mathcal{R}_2)$, and hence for the modularity of weak normalization for CTRS's.

LEMMA 4.19. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint CTRS's without extra variables in the conditions then $NF(\mathcal{R}_1 \oplus \mathcal{R}_2) = NF(\mathcal{R}_1) \cap NF(\mathcal{R}_2)$.*

PROOF.

$\subseteq$ Trivial.

$\supseteq$ If $NF(\mathcal{R}_1) \cap NF(\mathcal{R}_2)$ is not a subset of $NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$ then there exists a smallest term t such that $t \in NF(\mathcal{R}_1) \cap NF(\mathcal{R}_2)$ and $t \notin NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$. Clearly t must be a redex, so there is a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1 \oplus \mathcal{R}_2$ and a substitution σ such that $t \equiv \sigma(l)$ and $\sigma(s_i) \downarrow \sigma(t_i)$ for $i = 1, \ldots, n$. Assume without loss of generality that the rewrite rule stems from $\mathcal{R}_1$. Because $V(u) \subseteq V(l)$ for all $u \in \{s_1, \ldots, s_n, t_1, \ldots, t_n\}$ we may assume that $\mathcal{D}(\sigma) \subseteq V(l)$. Due to the minimality of t, $\sigma(x) \in NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$ for every $x \in \mathcal{D}(\sigma)$. Using this fact, we can easily show that $\sigma(s_i) \downarrow_{\mathcal{R}_1} \sigma(t_i)$ for $i = 1, \ldots, n$. But then $\sigma(l) \to_{\mathcal{R}_1} \sigma(r)$, contradicting the assumption $t \in NF(\mathcal{R}_1)$.

$\square$

COROLLARY 4.20. *Weak normalization is a modular property of* CTRS'*s without extra variables in the conditions of the rewrite rules.*

PROOF. Immediate consequence of Theorem 4.16 and Lemma 4.19. □

The sufficiency of confluence for the equality of $NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$ and $NF(\mathcal{R}_1) \cap NF(\mathcal{R}_2)$ is based on the following definitions and propositions from [19].

DEFINITION 4.21. The rewrite relation $\rightarrow_1$ is defined as follows: $s \rightarrow_1 t$ if there exists a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a context $C[\ ]$ and a substitution σ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ for $i = 1, \ldots, n$, where the superscript o in $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ means that $\sigma(s_i)$ and $\sigma(t_i)$ are joinable using only *outer* $\rightarrow_1$-reduction steps. Notice that the restrictions of $\rightarrow_1$ and $\rightarrow_{\mathcal{R}_1}$ to $\mathcal{J}_1 \times \mathcal{J}_1$ coincide. The relation $\rightarrow_2$ is defined similarly.

NOTATION. The union of $\rightarrow_1$ and $\rightarrow_2$ is denoted by $\rightarrow_{1,2}$.

EXAMPLE 4.22. Let

$$\mathcal{R}_1 = \begin{cases} F(x, y) & \rightarrow & G(x) & \Leftarrow & x \downarrow y \\ A & \rightarrow & B \end{cases}$$

and suppose $\mathcal{R}_2$ contains an unary function symbol g. We have $F(g(A), g(B)) \rightarrow_{\mathcal{R}_1} G(g(A))$ because $g(A) \rightarrow_{\mathcal{R}_1} g(B)$, but we do not have $F(g(A), g(B)) \rightarrow_1 G(g(A))$ since $g(A)$ and $g(B)$ are different normal forms with respect to $\rightarrow_1^o$. The only possible $\rightarrow_1$-reduction sequence starting at $F(g(A), g(B))$ is as follows:

$$F(g(A), g(B)) \rightarrow_1 F(g(B), g(B)) \rightarrow_1 G(g(B)).$$

PROPOSITION 4.23. *The rewrite relation* $\rightarrow_{1,2}$ *is confluent.* □

PROPOSITION 4.24. *Let* $s_1, \ldots, s_n, t_1, \ldots, t_n$ *be black terms. For every substitution* σ *with* $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ $(i = 1, \ldots, n)$ *there exists a substitution* τ *such that* $\sigma \twoheadrightarrow_{1,2} \tau$ *and* $\tau(s_i) \downarrow_1^o \tau(t_i)$ $(i = 1, \ldots, n)$. □

PROPOSITION 4.25. *If* $s \rightarrow t$ *then* $s \downarrow_{1,2} t$. □

PROPOSITION 4.26. *For all terms* s, t *we have* $s \downarrow t$ *if and only if* $s \downarrow_{1,2} t$.
PROOF. Easy consequence of Proposition 4.23 and 4.25. □

LEMMA 4.27. *If* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are disjoint confluent* CTRS'*s then* $NF(\mathcal{R}_1 \oplus \mathcal{R}_2) = NF(\mathcal{R}_1) \cap NF(\mathcal{R}_2)$.
PROOF.
$\subseteq$ Trivial.
$\supseteq$ If $NF(\mathcal{R}_1) \cap NF(\mathcal{R}_2)$ is not a subset of $NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$ then there exists a smallest term t such that $t \in NF(\mathcal{R}_1) \cap NF(\mathcal{R}_2)$ and $t \notin NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$. Clearly t must be a redex, so there is a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1 \oplus \mathcal{R}_2$ and a substitution σ such that $t \equiv \sigma(l)$ and $\sigma(s_i) \downarrow \sigma(t_i)$ for $i = 1, \ldots, n$. Notice that $\sigma(x) \in NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$ for every $x \in \mathcal{D}(\sigma) \cap V(l)$, due to the minimality of t. Without loss of generality we assume that the rewrite rule stems from $\mathcal{R}_1$. We obtain $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ $(i = 1, \ldots, n)$ from Proposition 4.26 and Proposition 4.24 yields a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $\tau(s_i) \downarrow_1^o \tau(t_i)$ $(i = 1, \ldots, n)$. Because $\tau(x) \equiv \sigma(x)$ for all $x \in V(l)$, we have

$t \equiv \sigma(l) \equiv \tau(l) \to_1 \tau(r)$, which contradicts the assumption $t \in \text{NF}(\mathcal{R}_1)$.

$\square$

COROLLARY 4.28. *Weak normalization is a modular property of confluent CTRS's.*

PROOF. Immediate consequence of Theorem 4.16 and Lemma 4.27. $\square$

In Example 4.18 we have seen that the Corollary 4.13 is not valid for CTRS's. Due to Lemma 4.27, Corollary 4.15 does extend to CTRS's.

COROLLARY 4.29. *If $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are disjoint confluent and weakly normalizing CTRS's then $\twoheadrightarrow$ is a confluent relation.*

PROOF. Easy consequence of Theorem 2.7, Theorem 4.16 and Lemma 4.27. $\square$

Finally we show that method (1) can also be used to obtain the sufficiency of confluence and "no extra variables in the conditions" for the modularity of weak normalization. The proof of the next proposition is similar to the proof of Proposition 3.7 and therefore not included.

PROPOSITION 4.30. *If s and t are black terms and σ a top white substitution with $\sigma \propto \varepsilon$ such that $\sigma(s) \to^o_{\mathcal{R}_1} \sigma(t)$, then $s \to^o_{\mathcal{R}_1} t$.* $\square$

DEFINITION 4.31. A substitution σ is *normalized* if $\sigma(x) \in \text{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2)$ for every $x \in \mathcal{D}(\sigma)$.

PROPOSITION 4.32. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be disjoint CTRS's with $\text{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2) = \text{NF}(\mathcal{R}_1) \cap \text{NF}(\mathcal{R}_2)$. If t is a black normal form and σ a top white normalized substitution with $\sigma \propto \varepsilon$, then $\sigma(t)$ is a normal form.*

PROOF. Suppose $\sigma(t)$ is not a normal form. Then we have $\sigma(t) \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} t'$ for some term t'. Because $\text{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2) = \text{NF}(\mathcal{R}_1) \cap \text{NF}(\mathcal{R}_2)$, t is black and σ is top white normalized, we obtain $\sigma(t) \to^o_{\mathcal{R}_1} t'$. Clearly $t' \equiv \sigma(t'')$ for some black term t''. Proposition 4.30 yields $t \to^o_{\mathcal{R}_1} t''$, contradicting the assumption that t is a normal form. $\square$

LEMMA 4.33. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint weakly normalizing CTRS's such that $\text{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2) = \text{NF}(\mathcal{R}_1) \cap \text{NF}(\mathcal{R}_2)$, then $\mathcal{R}_1 \oplus \mathcal{R}_2$ is weakly normalizing.*

PROOF. We will show by induction on $rank(t)$ that every term $t \in \mathcal{T}$ has a normal form with respect to $\mathcal{R}_1 \oplus \mathcal{R}_2$. If $rank(t) = 1$ then the result follows from the assumption that $\mathcal{R}_1$ and $\mathcal{R}_2$ are weakly normalizing. Let $t \equiv C[\![t_1, \ldots, t_n]\!]$. Without loss of generality we assume that t is top black. Applying the induction hypothesis to $t_1, \ldots, t_n$ yields normal forms $s_1, \ldots, s_n$ such that $t_i \twoheadrightarrow s_i$ for $i = 1, \ldots, n$. Choose mutually different fresh variables $x_1, \ldots, x_n$ and define $s \equiv C[x_1, \ldots, x_n]$ and $\sigma = \{x_i \to s_i \mid 1 \leq i \leq n\}$. According to Proposition 2.14 we can decompose σ into $\sigma_2 \circ \sigma_1$ such that σ_1 is black, σ_2 is top white and $\sigma_2 \propto \varepsilon$. Notice that σ_2 is normalized. Because $rank(\sigma_1(s)) = 1$ we can apply the induction hypothesis to obtain a normal form s' with $\sigma_1(s) \twoheadrightarrow s'$. By Proposition 4.32, $\sigma_2(s')$ also is a normal form. Hence t has a normal form:

$$t \equiv C[\![t_1, \ldots, t_n]\!] \twoheadrightarrow C[s_1, \ldots, s_n] \equiv \sigma_2(\sigma_1(s)) \twoheadrightarrow \sigma_2(s').$$

$\square$

Corollaries 4.20 and 4.28 are immediate consequences of this lemma.

5. Concluding Remarks

In this paper we extended several results concerning the termination behaviour of TRS's to CTRS's. We restricted ourselves to the so-called *join* CTRS's, meaning that the conditions of the rewrite rules are expressed in terms of joinability. Two other ways of interpreting the conditions of rewrite rules should be mentioned. In a *semi-equational* CTRS the conditions of the rewrite rules are expressed in terms of conversion. So a rewrite rule

$$l \rightarrow r \Leftarrow s_1 = t_1, \dots, s_n = t_n$$

of a semi-equational CTRS $\mathcal{R}$ is applicable with substitution σ if $\sigma(s_i)$ and $\sigma(t_i)$ are convertible with respect to the rewrite relation being defined ($i = 1, \dots, n$). The rewrite rules of a *normal* CTRS have the form

$$l \rightarrow r \Leftarrow s_1 \twoheadrightarrow t_1, \dots, s_n \twoheadrightarrow t_n$$

with $t_1, \dots, t_n$ ground (i.e. $V(t_i) = \varnothing$) normal forms with respect to the induced rewrite relation. Several other ways of expressing the conditions of rewrite rules are known, see [7] for an overview. In [19] we showed that confluence not only is a modular property of join CTRS's, but also of semi-equational and normal CTRS's. We believe that the main results of this paper (Theorems 3.15, 3.20, 3.29, 4.16 and Lemma 4.33) carry over easily to semi-equational and normal CTRS's. This does not mean that all caution should be cast to the winds when extending results obtained for join CTRS's to other kinds of CTRS's. For instance, Corollary 4.20 is not valid for semi-equational CTRS's as can be seen from the systems

$$\mathcal{R}_1 = \begin{cases} F(x, x) & \rightarrow & C \\ F(x, y) & \rightarrow & F(x, y) & \Leftarrow & x = y \end{cases}$$

and

$$\mathcal{R}_2 = \begin{cases} a & \rightarrow & b \\ a & \rightarrow & c. \end{cases}$$

Both CTRS's are easily shown to be weakly normalizing, but in $\mathcal{R}_1 \oplus \mathcal{R}_2$ the term $F(b, c)$ reduces only to itself.

From a programming point of view the syntactic restrictions imposed on the distribution of variables in the rewrite rules is too restrictive. It is desirable to extend our results to CTRS's like ([6])

$$\mathcal{R} = \begin{cases} Fib(0) & \rightarrow & <0, 1> \\ Fib(x+1) & \rightarrow & <z, y+z> & \Leftarrow & Fib(x) \downarrow <y, z>, \end{cases}$$

containing variables in the right-hand side of a rewrite rules which do not occur in the corresponding left-hand side. However, as already observed in [19], it is not clear at all how this should be carried out.

References

1.   J.A. Bergstra and J.W. Klop, *Conditional Rewrite Rules: Confluence and Termination*, Journal of Computer and System Sciences **32**(3), pp. 323-362, 1986.

2.   J.A. Bergstra, J.W. Klop and A. Middeldorp, *Termherschrijfsystemen*, Kluwer Bedrijfswetenschappen, Deventer, 1989 (in Dutch).

3. N. Dershowitz, *Termination of Rewriting*, Journal of Symbolic Computation 3(1), pp. 69-116, 1987.

4. N. Dershowitz and J.-P. Jouannaud, *Rewrite Systems*, Rapport de Recherche **478**, LRI, Orsay, 1989. (To appear in: Handbook of Theoretical Computer Science, Vol. B (ed. J. van Leeuwen), North-Holland, 1989.)

5. N. Dershowitz and Z. Manna, *Proving Termination with Multiset Orderings*, Communications of the ACM **22**(8), pp. 465-476, 1979.

6. N. Dershowitz, M. Okada and G. Sivakumar, *Confluence of Conditional Rewrite Systems*, Proceedings of the 1st International Workshop on Conditional Term Rewriting Systems, Orsay, Lecture Notes in Computer Science **308**, pp. 31-44, 1987.

7. N. Dershowitz, M. Okada and G. Sivakumar, *Canonical Conditional Rewrite Systems*, Proceedings of the 9th Conference on Automated Deduction, Argonne, Lecture Notes in Computer Science **310**, pp. 538-549, 1988.

8. N. Dershowitz and D.A. Plaisted, *Logic Programming cum Applicative Programming*, Proceedings of the IEEE Symposium on Logic Programming, Boston, pp. 54-66, 1985.

9. N. Dershowitz and D.A. Plaisted, *Equational Programming*, in: Machine Intelligence **11** (eds. J.E. Hayes, D. Michie and J. Richards), Oxford University Press, pp. 21-56, 1987.

10. K. Drosten, *Termersetzungssysteme*, Informatik-Fachberichte **210**, Springer, 1989 (in German).

11. L. Fribourg, *SLOG: A Logic Programming Language Interpreter Based on Clausal Superposition and Rewriting*, Proceedings of the 2nd IEEE Symposium on Logic Programming, Boston, pp. 172-184, 1985.

12. J.A. Goguen and J. Meseguer, *EQLOG: Equality, Types and Generic Modules for Logic Programming*, in: Logic Programming: Functions, Relations and Equations (eds. D. DeGroot and G. Lindstrom), Prentice-Hall, pp. 295-363, 1986.

13. J.-P. Jouannaud and B. Waldmann, *Reductive Conditional Term Rewriting Systems*, Proceedings of the 3rd IFIP Working Conference on Formal Description of Programming Concepts, Ebberup, pp. 223-244, 1986.

14. S. Kaplan, *Fair Conditional Term Rewriting Systems: Unification, Termination and Confluence*, Rapport de Recherche **194**, LRI, Orsay, 1984.

15. J.W. Klop, *Term Rewriting Systems*, to appear in: Handbook of Logic in Computer Science, Vol. I (eds. S. Abramsky, D. Gabbay and T. Maibaum), Oxford University Press, 1989.

16. M. Kurihara and I. Kaji, *Modular Term Rewriting Systems: Termination, Confluence and Strategies*, Report, Hokkaido University, Sapporo, 1988.

17. A. Middeldorp, *Modular Aspects of Properties of Term Rewriting Systems Related to Normal Forms*, Proceedings of the 3rd International Conference on Rewriting Techniques and Applications, Chapel Hill, Lecture Notes in Computer Science **355**, pp. 263-277, 1989.

18. A. Middeldorp, *A Sufficient Condition for the Termination of the Direct Sum of Term Rewriting Systems*, Proceedings of the 4th IEEE Symposium on Logic in Computer Science, Pacific Grove, pp. 396-401, 1989.

19. A. Middeldorp, *Confluence of the Disjoint Union of Conditional Term Rewriting Systems*, Report CS-R89..., Centre for Mathematics and Computer Science, Amsterdam, 1989.

20. M. Rusinowitch, *On Termination of the Direct Sum of Term Rewriting Systems*, Information Processing Letters **26**, pp. 65-70, 1987.

21. Y. Toyama, *On the Church-Rosser Property for the Direct Sum of Term Rewriting Systems*, Journal of the ACM **34**(1), pp. 128-143, 1987.

22. Y. Toyama, *Counterexamples to Termination for the Direct Sum of Term Rewriting Systems*, Information Processing Letters **25**, pp. 141-143, 1987.

23. Y. Toyama, J.W. Klop and H.P. Barendregt, *Termination for the Direct Sum of Left-Linear Term Rewriting Systems* (preliminary draft), Proceedings of the 3rd International Conference on Rewriting Techniques and Applications, Chapel Hill, Lecture Notes in Computer Science **355**, pp. 477-491, 1989.

24. H. Zhang and J.L. Rémy, *Contextual Rewriting*, Proceedings of the 1st International Conference on Rewriting Techniques and Applications, Dijon, Lecture Notes in Computer Science **202**, pp. 46-62, 1985.