NICO M. TEMME

# Analytical methods for a Selection of elliptic Singular perturbation Problems

**Abstract**

We consider several model problems from a class of elliptic perturbation equations in two dimensions. The domains, the differential operators, the boundary conditions, and so on, are rather simple, and are chosen in a way that the solutions can be obtained in the form of integrals or Fourier series. By using several techniques from asymptotic analysis (saddle point methods, for instance) we try to construct asymptotic approximations with respect to the small parameter that multiplies the differential operator of highest order. In particular we consider approximations that hold uniformly in the so-called boundary layers. We also pay attention to how to obtain a few terms in the asymptotic expansion by using direct methods based on singular perturbation methods.

## 1. Introduction

A singular perturbation is a term or component in a differential equation existing of a derivative term (the highest order in the equation) with a small coefficient $\varepsilon$. There is an extensive literature on solving singularly perturbed differential equations (linear and nonlinear) by means of constructing perturbation expansions and by using matching principles. See for example Eckhaus (1973), (1979), Grasman (1971), O'Malley (1979) and Smith (1985). For a historical survey we refer to Shih and Kellogg (1987).

In this paper we consider several model problems from a class of elliptic perturbation equations in two dimensions. The domains, the differential operators, the boundary conditions, and so on, are rather simple, and are chosen in a way that the solutions can be obtained in the form of integrals or Fourier series. It is our aim to show how methods from uniform asymptotic analysis for integrals (saddle point methods, for instance) can be used to find the asymptotic behaviour of the solutions with respect to the small parameter that multiplies the differential operator of highest order. In particular we consider approximations that hold uniformly in the so-called boundary layers. We also pay attention how to obtain a few terms in the asymptotic expansion by using direct methods based on singular perturbation methods.

In general, the construction of the outer and inner expansions (for approximating the solutions outside and inside the boundary layers) by using singular perturbation methods is now a standard method, and does not cause any difficulties for the problems we are discussing. However, interesting problems arise when the equations are solved analytically, and for simple cases it is not always trivial how to obtain a uniform expansion, say from an integral, that is valid in and outside the boundary layer.

We consider the following type of singular perturbation problem:

$$\varepsilon\Delta\Phi(x,y) - \Phi_y(x,y) = f(x,y), \tag{1.1}$$

where $\varepsilon$ is a small positive parameter, in a domain $\Omega$ of the plane. The domain $\Omega$, the function $f$ and the boundary conditions will be chosen in a way that we can solve the equation in terms of an integral or a series expansion. We consider the following domains (not all in the same detail):

[1] The strip $\Omega = \left\{(x,y) \mid x \in \mathbb{R}, 0 \le y \le 1\right\}$,

[2] The quarter plane $\Omega = \left\{(x,y) \mid x \ge 0, y \ge 0\right\}$,

[3] The sector $\Omega = \left\{x = r\cos\theta, y = r\sin\theta \mid r \ge 0,\ 0 < \theta \le \alpha < 2\pi\right\}$,

[4] The exterior of the unit circle,

[5] The interior of the unit circle.

The problem is to find the asymptotic behaviour of $\Phi$ as $\varepsilon \to 0$. The solution to equation (1.1) has boundary layers at certain parts of the boundary $\partial\Omega$ of $\Omega$. In general, it is quite difficult to obtain uniform asymptotic approximations of $\Phi$ that are valid in the boundary layers and in the other parts of $\Omega$ as well.

We present for the analytic representation of $\Phi$ methods from asymptotic analysis (for instance saddle point methods) to indicate the kind of problems that arise when trying to obtain uniform approximations.

In Gold (1982), Roberts (1967), and Shercliff (1962) problems from mathematical physics are given that lead to the elliptic singular perturbation problem considered here. The equation (1.1) arises in magnetohydrodynamics, where $\varepsilon$ measures the importance of viscous force relative to the electromagnetic force, and in the theory of plate-membranes under tension in the $y$-direction, where $\varepsilon$ measures the bending stiffness.

## 2. An example from ODE's

To give a first impression of what is happening in the boundary layer we consider an example of a singular perturbation problem for an ordinary differential equation. The example is a simplification of (1.1), and gives insight in the boundary layer aspects of this equation.

The linear operator $\partial/\partial y$ plays a dominant role in (1.1). The interesting direction concerning the change in behaviour of the solution $\Phi$ is along the characteristic lines $x =$ constant of this operator. This means that the role of the term $\Phi_{xx}$ in (1.1) is not very great in the interior of $\Omega$, and therefore we omit in this example the second $x$-derivative in (1.1).

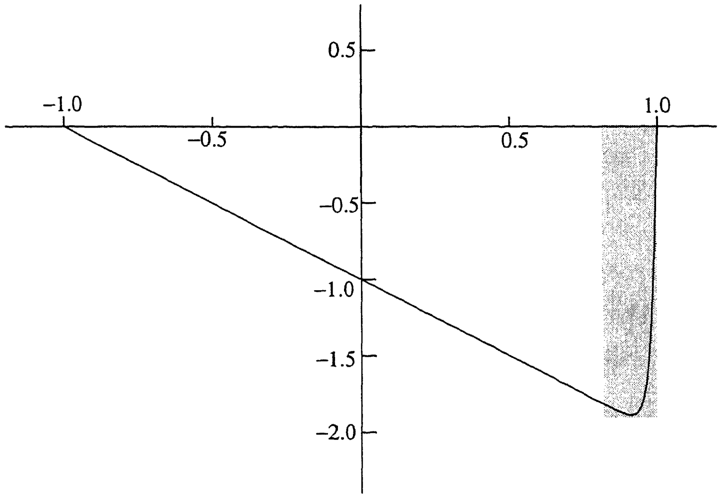


Figure 1: Boundary layer near $y = 1$ of the equation $\varepsilon w_{yy} - w_y = 1$ with boundary values $w(\pm 1) = 0$. The exact solution is given in (2.2).

The solution of the equation

$$\varepsilon \frac{d^2 w}{dy^2} - \frac{dw}{dy} = 1, \tag{2.1}$$

with the boundary values $w = 0$ if $y = \pm 1$, is given by

$$w(y) = -1 - y + \frac{e^{y/\varepsilon} - e^{-1/\varepsilon}}{\sinh 1/\varepsilon}. \tag{2.2}$$

We observe that on the interval $[-1, 1-\delta]$, where $\delta$ is a fixed small positive number, for small values of $\varepsilon$ the function $w$ is equal to $w_0(y) = -1 - y$ plus a function that is exponentially small. Near the boundary $y = 1$ the solution $w$ drops from the value $-2$ to its proper boundary value 0. In this example we see that the boundary layer occurs at $y = 1$.

A perturbation analysis of (2.1) starts with the expansion $w(y) \sim \sum_{n=0}^{\infty} \varepsilon^n w_n(y)$, which gives, when selecting equal powers of $\varepsilon$,

$$-w_0' = 1, \qquad w_{n+1}' = w_n'', \qquad n = 0, 1, 2, \ldots .$$

This gives,

$$w_0(y) = -y - 1, \qquad w_n(y) \equiv 0, \qquad n = 1, 2, 3, \ldots ,$$

when we take into account that all functions $w_n$ should vanish at $y = -1$.

When we want to analyze the situation near $y = 1$ we introduce the stretching variable $y = 1 - \varepsilon\eta$. This gives the differential equation $\ddot{w} + \dot{w} = \varepsilon$, where the dots indicate differentiation with respect to $\eta$. A first order approximation $v_0$ follows from the reduced equation $\ddot{v}_0 + \dot{v}_0 = 0$, with solution $v_0 = c_1 + c_2 e^{-\eta}$.

The constants $c_1$, $c_2$ follow from giving $w_0 + v_0$ the requested boundary value 0 at $y = 1$, giving $c_1 + c_2 = 2$. Another relation comes from a "matching" condition: if $v_0$ should vanish if $\eta \to \infty$ then we get $c_1 = 0$. This gives an approximation of the exact solution given in (2.2):

$$w(y) = -1 - y + 2e^{-(1-y)/\varepsilon} + \mathcal{O}(\varepsilon).$$

In the present case the error is exponentially small for all $y \in [-1, 1]$, because all $w_n$, $n > 0$, vanish. Selecting $c_1$, $c_2$ with the condition $w_0 + v_0 = 0$ at $y = -1$, we get the exact solution (2.2).

In general cases more functions $w_n$ and $v_n$ are needed to get an approximation of the form

$$w(y) = \sum_{n=0}^{m} \varepsilon^n w_n(y) + \sum_{n=0}^{m} \varepsilon^n v_n(y) + \mathcal{O}\left(\varepsilon^{m+1}\right), \quad \varepsilon \to 0,$$

where the $\mathcal{O}$-term holds uniformly with respect to $y \in [-1, 1]$.

### 2.1. The location of the boundary layer.

When we take $\varepsilon = 0$ in (2.1), we obtain the reduced equation $-w_y = 1$. The solutions of this equation cannot satisfy both boundary conditions $w = 0$ if $y = \pm 1$. We started the perturbation method by assuming (in fact knowing) that the boundary layer occurs at $y = 1$. The minus sign in front of the first derivative term in (2.1) is crucial here.

We might have started in a different way, by taking as a first approximation $w_0(y) = 1 - y$ and trying to match at $y = -1$. The stretching variable $\eta$ is now defined by $y = -1 + \varepsilon\eta$, which gives the differential equation $\ddot{w} - \dot{w} = \varepsilon$ with reduced equation $\ddot{v}_0 - \dot{v}_0 = 0$. Solutions are $v_0 = c_1 + c_2 \exp(\eta) = c_1 + c_2 \exp[(y+1)/\varepsilon]$, again with $c_1 + c_2 = 0$ to satisfy the boundary condition at $\eta = 0$, that is, at $y = -1$.

Observe that it is not possible now to use the matching principle to get a second relation for $c_1, c_2$. With the false start, assuming that the boundary layer occurs at $y = -1$, we have lost the matching principle. By the way, the exact solution can be written in the form

$$w(y) = 1 - y + \frac{e^{y/\varepsilon} - e^{1/\varepsilon}}{\sinh 1/\varepsilon}.$$

The term with the exponential functions is now relevant throughout the whole interval $[-1, 1]$.

As shown in Eckhaus and De Jager (1966) for elliptic problems, one can deduce from a maximum principle that $w(-1) = 0$ is the proper initial condition for the reduced equation of (2.1).

## 3. An equation in the strip $0 \le y \le 1$

We consider the equation

$$\varepsilon \Delta w(x,y) - \frac{\partial w}{\partial y}(x,y) = 0, \qquad 0 \le y \le 1, \tag{3.1}$$

with boundary conditions

$$w(x,0) = f(x), \qquad w(x,1) = g(x).$$

In this case the boundary layer occurs near $y = 1$.

A perturbation analysis starts with substituting an expansion

$$w(x,y) \sim \sum_{n=0}^{\infty} \varepsilon^n w_n(x,y)$$

in which the terms satisfy

$$w_0(x,y) = f(x), \quad \frac{\partial w_n(x,y)}{\partial y} = \Delta w_{n-1}(x,y), \quad n = 1, 2, \ldots . \tag{3.2}$$

This gives an approximation that holds outside the boundary layer. The first boundary layer correction $v_0$ to this approximation follows from introducing the local coordinate $\eta$ by writing $y = 1 - \varepsilon\eta$. As in the previous section it is given by

$$v_0(x,\eta) = c_1(x) + c_2(x)e^{-\eta}.$$

When we prescribe the matching condition $v_0(x,\eta) \to 0$ as $\eta \to \infty$, and the boundary condition $w_0 + v_0 = 0$ at $\eta = 0$, we obtain

$$w(x,y) = f(x) + [g(x) - f(x)]\, e^{-(1-y)/\varepsilon} + \mathcal{O}(\varepsilon), \quad \varepsilon \to 0.$$

When we prescribe that the approximation $w_0 + v_0$ assumes the matching and both boundary conditions, we obtain

$$w(x,y) = f(x) + [g(x) - f(x)]\frac{e^{-(1-y)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} + \mathcal{O}(\varepsilon).$$

We observe from the recursive scheme in (3.2) that the functions $w_n$ contain derivatives of $f$. When $f$ is discontinuous the scheme does not work, because $w_1$ is not defined for all $x$. The solution $w(x,y)$ of the elliptic differential equation (3.1) is smooth inside the strip $0 < y < 1$, also if $f$ and $g$ have integrable singularities. Therefore, it is of interest to consider an approach that is based on the exact solution.

To obtain an exact solution of (3.1) we assume that $f,\, g \in L^2\,(\mathbb{R})$. We take the Fourier transform of the differential equation with respect to $x$, that is, let

$$\widehat{w}(\xi,y) := \int_{-\infty}^{\infty} e^{-i\xi x}\, w(x,y)\, dx.$$

Then we obtain for any $\varepsilon > 0$ the solution of the transformed problem:

$$\widehat{w}(\xi, y) = e^{\omega y}\,\frac{\sinh(1-y)\rho}{\sinh\rho}\,\widehat{f}(\xi) + e^{\omega(y-1)}\,\frac{\sinh y\rho}{\sinh\rho}\,\widehat{g}(\xi),$$

where

$$\omega = \frac{1}{2\varepsilon}, \quad \rho = \sqrt{\omega^2 + \xi^2}.$$

The exact solution of the problem (3.1) is then the inverse

$$w(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\,\widehat{w}(\xi,y)\,d\xi. \tag{3.3}$$

Even for simple functions $f$, $g$ it is not an easy problem to obtain an asymptotic expansion of this integral. For example, if $g = 0$ and $f = \chi_{[-1,1]}$ is the characteristic function on $[-1,1]$, that is

$$f(x) = \begin{cases} 1, & \text{if } -1 \le x \le 1; \\ 0, & \text{otherwise,} \end{cases}$$

we obtain

$$w(x,y) = \frac{e^{\omega y}}{\pi}\int_{-\infty}^{\infty} e^{i\xi x}\,\frac{\sin\xi}{\xi}\,\frac{\sinh(1-y)\rho}{\sinh\rho}\,d\xi.$$

In this case it is of interest to investigate how the discontinuities of $f$ at $x = \pm 1$ are taken over by the smooth function $w(x,y)$ inside the strip. Also, $w(x,y)$ tends to zero exponentially fast if $y \to 1$. So, several examples of non-uniform behaviour can be observed.

From a first analysis we conclude that a uniform asymptotic approximation of $w(x,y)$ contains error functions (normal distribution functions) if $x$ crosses the values $\pm 1$, with $0 < y < 1$. We expect that it is even more difficult to investigate the behaviour of $w(x,y)$ in the boundary layer, in particular near the points $(\pm 1, 1)$.

In Cook and Ludford (1971) more details can be found on methods based on the exact solution, and with non-smooth boundary functions $f$ and $g$. Other related references are Mauss (1970) and Howes (1981).

## 4. An equation in the quarter plane $x \ge 0$, $y \ge 0$

We consider the elliptic partial differential equation

$$\varepsilon\Delta\Phi(x,y) - \frac{\partial\Phi}{\partial y}(x,y) = 0, \tag{4.1}$$

where $\varepsilon > 0$, in the quarter plane

$$\Omega = \{x > 0, \quad y > 0\}$$

with boundary conditions

$$\Phi(x, 0) = 0, \quad \Phi(0, y) = \phi(y).$$

In this case there is no upper boundary layer because the domain $\Omega$ is unbounded in the $y$-direction. However, along the $y$−axis a parabolic boundary layer occurs. In general, such type of boundary layers arise when the boundary contains segments that coincide with a characteristic of the first order operator $\partial/\partial y$ (that is, lines with $x =$ constant).

Solutions of the reduced equation $\frac{\partial \Phi_0}{\partial y}(x, y) = 0$ do not satisfy both given boundary conditions.

A local analysis in the parabolic boundary layer starts with the local coordinate

$$\xi = \frac{x}{\sqrt{\varepsilon}}.$$

Equation (4.1) transforms into

$$\frac{\partial^2 \Phi}{\partial \xi^2} - \frac{\partial \Phi}{\partial y} = -\varepsilon \frac{\partial \Phi}{\partial y}.$$

We define the function $w_0(\xi, y)$ as the solution of the reduced equation

$$\frac{\partial^2 w_0}{\partial \xi^2} - \frac{\partial w_0}{\partial y} = 0,$$

that satisfies the boundary conditions for $\Phi$. An explicit form of the solution reads:

$$w_0(\xi, y) = \sqrt{\frac{2}{\pi}} \int_{\xi/\sqrt{2y}}^{\infty} e^{-\frac{1}{2}t^2} \phi\left[y - \xi^2/(2t^2)\right] \, dt.$$

We continue with constructing an exact solution for the simple choice $\phi(y) \equiv 1$, and refer to Temme (1974) for the case of a general sector, and for a discussion on what happens if the sector becomes a quarter plane.

Put

$$\Phi(x, y) = e^{\omega y} F(x, y), \qquad \omega = \frac{1}{2\varepsilon},$$

then $F$ satisfies the Helmholtz equation

$$\Delta F(x, y) - \omega^2 F(x, y) = 0,$$

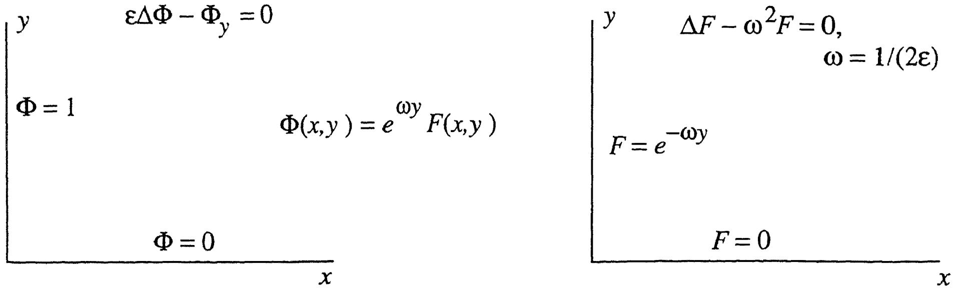


Figure 2: Boundary values of $\Phi$ and $F$ in the boundary value problems.

with boundary conditions

$$F(0,y)=e^{-\omega y}, \qquad F(x,0)=0.$$

Separating the variables we obtain the solution

$$F(x,y)=\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{ily-x\sqrt{l^2+\omega^2}}\,\frac{l\,dl}{l^2+\omega^2}.$$

Put $l=\omega\sinh t$ and introduce polar coordinates

$$x=r\cos\theta,\quad y=r\sin\theta,\quad 0\le\theta\le\frac{1}{2}\pi.$$

Then

$$F(x,y)=\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-\omega r\cosh(t-i\theta)}\,\frac{\sinh t}{\cosh t}\,dt.$$

Shift the contour up in the complex plane (assume that $\theta<\frac{1}{2}\pi$, and that, hence, we do not pass a pole of $\cosh t$):

$$F(x,y)=\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-\omega r\cosh t}\,\frac{\sinh(t+i\theta)}{\cosh(t+i\theta)}\,dt.$$

Write $\alpha=\frac{1}{2}\pi-\theta$. Then

$$F(x,y)=\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-\omega r\cosh t}\,\frac{\cosh(t-i\alpha)}{\sinh(t-i\alpha)}\,dt.$$

We want to obtain an asymptotic expansion of this integral for large values of $\omega$, in particular near the $y-$axis where the boundary layer occurs.

The asymptotic features of the integral are:

[1] there is a saddle point at $t=0$;

[2] when $\theta \to \frac{1}{2}\pi$ (or $\alpha \to 0$) the saddle point and the pole coalesce.
A standard procedure to deal with this kind of asymptotic phenomena is based on using the error function (cf. Wong (1989)). We use the error function representation (the proof is left as an exercise)

$$e^{-r\cos\alpha}\mathrm{erfc}\left(\sqrt{2r}\,\sin\frac{1}{2}\alpha\right) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-r\cosh t}\,\frac{dt}{\sinh\frac{1}{2}(t-i\alpha)},$$

where $0 < \alpha < 2\pi$, and split off the pole at $t = i\alpha$:

$$\frac{\cosh(t-i\alpha)}{\sinh(t-i\alpha)} = \left[\frac{\cosh(t-i\alpha)}{\sinh(t-i\alpha)} - \frac{1}{2\sinh\frac{1}{2}(t-i\alpha)}\right] + \frac{1}{2\sinh\frac{1}{2}(t-i\alpha)}.$$

This gives

$$F(x,y) = e^{-\omega y}\mathrm{erfc}\sqrt{\omega(r-y)} + \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\omega r\cosh t} f(t)\,dt,$$

where

$$f(t) = \frac{1}{i}\,\frac{\cosh(t-i\alpha)}{\sinh(t-i\alpha)} - \frac{1}{2i\sinh\frac{1}{2}(t-i\alpha)}.$$

We expand

$$f(t) = \cosh\frac{1}{2}t\sum_{n=0}^{\infty} c_n\left(\sinh\frac{1}{2}t\right)^n,$$

and obtain the asymptotic expansion

$$\Phi(x,y) \sim \mathrm{erfc}\sqrt{\omega(r-y)} + \frac{2}{\pi}e^{-\omega(r-y)}\sum_{n=0}^{\infty} c_{2n}\,\frac{\Gamma(n+\frac{1}{2})}{(2\omega r)^{n+\frac{1}{2}}},$$

as $\omega \to \infty$, uniformly in the quarter plane; $r$ should be bounded away from 0. In fact, we need $\omega r \to \infty$. The first coefficient reads

$$c_0 = \frac{\cos\alpha}{\sin\alpha} - \frac{1}{2\sin\frac{1}{2}\alpha}, \qquad \alpha = \frac{1}{2}\pi - \theta.$$

For small values of $\alpha$ we have

$$c_0 = -\frac{3}{8}\alpha - \frac{3}{128}\alpha^3 + \mathcal{O}\left(\alpha^5\right).$$

Because $f(t)$ is odd if $\alpha = 0$, all coefficients $c_{2n}$ vanish if $\alpha = 0$.

The problem of this section is treated in Temme (1971). For more details on the parabolic boundary layer we refer to Eckhaus and De Jager (1966). Other references

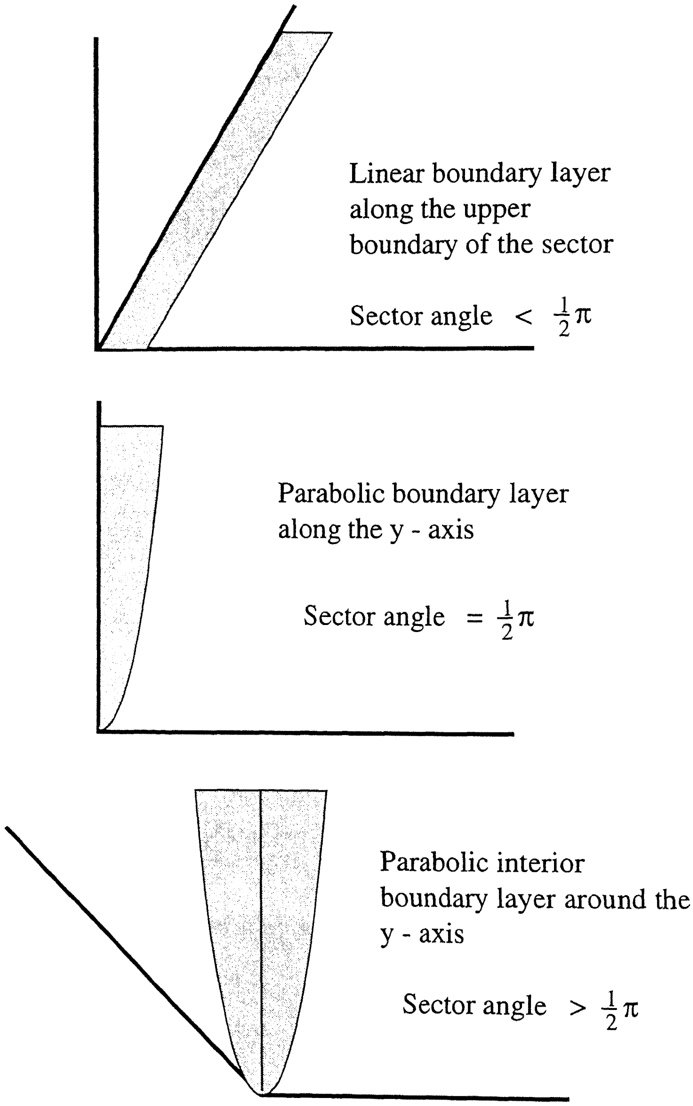


Figure 3: Boundary layers in sector problems.

for the elliptic problems having a characteristic boundary include Knowles and Messick (1964), Grasman (1968), Cook and Ludford (1973), Hedstrom and Osterheld (1980), and Shih and Kellogg (1987). In particular, Grasman obtained an asymptotic expansion from an integral representation of the exact solution of a quarter plane, based on which Shih and Kellogg devised a method of matched asymptotic expansions for an elliptic problem defined in a rectangular region. Hedstrom and Osterheld used a combination of a Bleistein transformation and the treatment of poles of Van der Waerden (1951) (see also Wong (1989, Chapter 7)) to construct an asymptotic expansion for an integral representation of the exact solution of an elliptic problem defined in the quarter plane, which is of the same type as one given in the present section.

## 5. The exterior of the circle

The model problem

$$\varepsilon\Delta\Phi(x,y) - \Phi_y(x,y) = 0, \tag{5.1}$$

on the region $\Omega = \{(x,y) | x^2+y^2 \geq 1\} \subset \mathbb{R}^2$ satisfying the boundary condition

$$\Phi(x,y) = 1 \qquad \text{for} \qquad x^2+y^2 = 1$$

and $\Phi \to 0$ at infinity, is discussed in great detail in Waechter (1968). See also Hemker (1996) for numerical aspects and a summary of Waechter's approach.

The exact solution can be written in terms of a Fourier series, in which quotients of Bessel functions arise as coefficients. The series is transformed into an integral in the complex plane, and the integration is with respect to the complex order of the Bessel functions (a Watson transform).

The domain $\Omega$ is divided into several subdomains in order to describe the rather complicated asymptotic behaviour. In a certain subdomain residue series are constructed by using zeros (with respect to the order) of modified Bessel functions. In another subdomain saddle point methods are used.

The exact solution for this problem reads

$$\Phi(x,y) = \sum_{n=-\infty}^{\infty} \frac{K_n(\omega)}{I_n(\omega)} K_n(\omega r) \cos n(\theta + \pi/2), \quad \omega = \frac{1}{2\varepsilon},$$

where $r, \theta$ are the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $r \geq 1$. For further details we refer to the cited literature, and to the next section for a similar problem.

## 6. The interior of the circle

Consider the model problem

$$\varepsilon\Delta\Phi(x,y) - \frac{\partial\Phi}{\partial y}(x,y) = 1, \quad x^2+y^2 < 1. \tag{6.1}$$

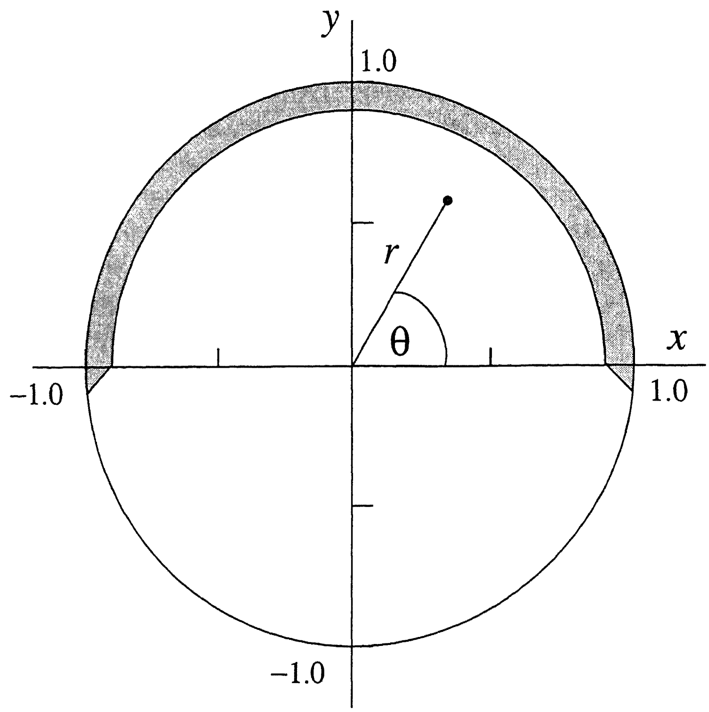


Figure 4: Boundary layer inside the circle along the upper boundary $r = 1$, $y > 0$ and near the points $(\pm 1, 0)$.

The boundary condition reads $\Phi(\cos\theta, \sin\theta) = 0$ on the boundary of the circle $r = 1$, where $r$, $\theta$ are the polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 \le r \le 1, \quad -\pi < \theta \le \pi.$$

Again, the problem is to find the asymptotic behaviour of $\Phi$ as $\varepsilon \to 0$. The solution to equation (6.1) has a boundary layer at the boundary where $y$ is positive. In particular it is of interest to find the behaviour of $\Phi$ in small neighborhoods of the points $(x, y) = (\pm 1, 0)$, the places of birth of the boundary layer.

### 6.1. Singular perturbation methods.

We give a few steps on the construction of the asymptotic solution of the singular perturbation problem by substituting an asymptotic expansion. We recall:

$$\begin{aligned} \varepsilon\Delta\Phi(x,y) - \frac{\partial\Phi}{\partial y}(x,y) &= 1, \quad x^2 + y^2 < 1, \\ \Phi(\cos\theta, \sin\theta) &= 0. \end{aligned} \tag{6.2}$$

The boundary layer occurs at the upper semi-circle, which is in agreement with the case of a similar ordinary differential equation. Inside the circle the solution of the circle problem can be approximated by

$$w_0(x,y) = -y - \sqrt{1-x^2},$$

which satisfies the condition on the lower semi-circle, but not on the upper semi-circle.

When we substitute the formal series

$$\Phi(x,y) \sim \sum_{n=0}^{\infty} \varepsilon^n w_m(x,y)$$

into (6.2) and equate equal powers of $\varepsilon$, we find

$$\frac{\partial w_0(x,y)}{\partial y} = -1, \quad \frac{\partial w_n(x,y)}{\partial y} = \Delta w_{n-1}(x,y), \quad n = 1,2,\dots,$$

and all $w_n$ should vanish at the lower part of the unit circle.

This gives $w_0$ given above and

$$w_n(x,y) = \int_{-\sqrt{1-x^2}}^{y} \Delta w_{n-1}(x,\eta)\, d\eta, \quad n = 1,2,\dots .$$

It is easily verified that

$$w_1(x,y) = \frac{y+\sqrt{1-x^2}}{(1-x^2)^{3/2}}.$$

We observe that $w_1$ becomes singular at the points $(\pm 1, 0)$ and that the functions $w_n$ do not satisfy the boundary condition $w_n = 0$ on the upper part of the unit circle.

To satisfy the boundary conditions along the upper part of the unit circle so-called boundary layer terms are introduced. These functions have the property of being of order $\mathcal{O}(\varepsilon^n)$ for all $n$ everywhere inside the unit circle except for a small neighborhood of the upper part of the circle.

Following the construction of the boundary layer term given in Eckhaus and De Jager (1966), we can write

$$\Phi(x,y) = -y - \sqrt{1-x^2} + 2\sin\theta\, e^{-\frac{1}{\varepsilon}(1-r)\sin\theta}\, \psi(x,y) + \varepsilon\, z_0(x,y),$$

where $z_0(x,y) = \mathcal{O}(1)$, uniformly inside the unit disk, with the exception of small neighborhoods of the points $(\pm 1, 0)$. The function $\psi$ is a smoothing factor, which equals unity on a neighborhood of the upper part of the circle, and $\psi$ vanishes in the lower part of the disc.

In Roberts (1967) the circle problem is considered with the same simple differential equation and boundary condition as in our case. A detailed analysis is given for the boundary layer near the points $(\pm 1, 0)$ by using boundary layers coordinates. Integrals of ratios of Airy functions are used to obtain the approximations. See also Grasman (1971).

### 6.2. The solution of the boundary value problem

We recall:

$$\varepsilon\Delta\Phi(x,y) - \frac{\partial\Phi}{\partial y}(x,y) = 1, \quad x^2+y^2<1, \tag{6.3}$$
$$\Phi(\cos\theta,\sin\theta) = 0.$$

We construct the exact solution of this equation. A first substitution

$$\Phi(x,y) = -y - e^{\omega y}F(x,y),$$

gives the problem

$$\Delta F(x,y) - \omega^2 F(x,y) = 0, \quad \omega = \frac{1}{2\varepsilon},$$

with boundary condition

$$F(\cos\theta,\sin\theta) = -\sin\theta\, e^{-\omega\sin\theta}.$$

The Helmholtz equation can be solved in terms of modified Bessel functions by using the polar coordinates and by separating the variables.

We have a solution of the form

$$F(x,y) = \sum_{n=-\infty}^{\infty} a_n I_n(\omega r)e^{in\theta},$$

where the coefficients $a_n$ follow from the well-known Bessel function series

$$e^{z\cos t} = \sum_{n=-\infty}^{\infty} I_n(z)\cos nt.$$

This gives

$$F(x,y) = \sum_{n=-\infty}^{\infty} \frac{I_n'(\omega)}{I_n(\omega)} I_n(\omega r)\ \cos n(\theta+\pi/2),$$
$$\Phi(x,y) = -y - e^{\omega r\sin\theta} \sum_{n=-\infty}^{\infty} \frac{I_n'(\omega)}{I_n(\omega)} I_n(\omega r)\ \cos n(\theta+\pi/2).$$

## 6.3. Transforming the Fourier series

We apply the Poisson summation formula to the Fourier series. We have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \widehat{f}(2\pi m),$$

where $\widehat{f}$ is the Fourier transform of $f$:

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)\, e^{ixy}\, dx.$$

This result holds if $f$ is of bounded variation and absolutely integrable on $\mathbb{R}$ (cf. Zygmund (1959, p.68)). For cosine transforms we have (by assuming that $f$ is even)

$$\sum_{n=0}^{\infty}{}' f(n) = \sum_{m=0}^{\infty}{}' \widehat{f}(2\pi m), \quad \widehat{f}(y) = 2\int_0^{\infty} f(x)\, \cos(xy)\, dx.$$

Applying this to series for $F(x,y)$ we obtain

$$F(x,y) = 2\sum_{m=0}^{\infty}{}' F_m(x,y),$$
$$F_m(x,y) = 2\int_0^{\infty} \frac{I'_\nu(\omega)}{I_\nu(\omega)}\, I_\nu(\omega r)\, \cos \nu(\theta+\pi/2)\, \cos(2\pi m\nu)\, d\nu.$$

The function $I_\nu(z)$ is an analytic function of $\nu$, with the following asymptotic behaviour:

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\left[1+\mathcal{O}(\nu^{-1})\right], \quad \nu\to\infty,$$

with $z$ fixed.

$I_\nu(z)$ is positive if $z$ and $\nu$ are positive.

It follows that all functions $F_m(x,y)$ in are well-defined, and that the Poisson summation formula can be applied. The integrals that define $F_m(x,y)$ converge fast for fixed values of $\omega$, as follows from the estimate for $I_\nu(z)$.

## 6.4. The asymptotic behaviour of $F_0(x,y)$

To investigate the asymptotic behaviour of $F_0(x,y)$ we use the Debye uniform approximation of $I_\nu(\omega)$.

We have

$$\frac{I'_\nu(\omega)}{I_\nu(\omega)}\, I_\nu(\omega r) = \frac{\sqrt{\nu^2+\omega^2}}{\sqrt{2\pi\omega}}\, \frac{e^{\nu\eta}}{(\nu^2+\omega^2 r^2)^{1/4}}\left[1+\mathcal{O}(1/\omega)\right],$$

as $\omega \to \infty$, uniformly with respect to $\nu \in [0, \infty)$. The quantity $\eta$ is given by

$$\eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}, \quad z = \frac{\omega r}{\nu}.$$

Using this in $F_0$, putting $\nu = \omega r \sinh t$, replacing the cosine by an exponential function, we obtain

$$F_0(x,y) \sim \sqrt{\frac{\omega r}{2\pi}} \int_{-\infty}^{\infty} \sqrt{1 + r^2 \sinh^2 t}\, \sqrt{\cosh t}\, e^{-\omega r f(t)}\, dt,$$

where

$$f(t) = (t - i\theta - i\pi/2) \sinh t - \cosh t.$$

The saddle points follow from the equation

$$f'(t) = (t - i\theta - i\pi/2) \cosh t = 0,$$

giving

$$t_0 = \left(\theta + \frac{1}{2}\pi\right) i, \quad t_1 = \frac{1}{2}\pi i.$$

Several interesting aspects can be observed.

• when $\theta \to 0$, the two saddle points and a singularity of the integrand coalesce.

• when $\theta \to 0$ and $r \to 1$, another singularity coalesces with the two coalescing saddle points.

There is no standard method in uniform asymptotic methods for integrals available to handle the second case. In the first case Airy functions can be used.

This first orientation in the asymptotic phenomena demonstrates the complicated situation that arises in the points $(\pm 1, 0)$.

Another approach is based on replacing the Fourier series with an integral in the complex plane, where we integrate with respect to complex orders of the Bessel functions.

For example, we can write:

$$F(x,y) = -i \int_C \frac{I'_\nu(\omega)}{I_\nu(\omega)}\, I_\nu(\omega r)\, \frac{\cos \nu(\theta - \pi/2)}{\sin \nu\pi}\, d\nu, \tag{6.4}$$

where $C$ is a contour around the poles of $1/\sin \nu\pi$; see Figure 5.

For this approach it is needed to investigate the location of the zeros of the modified Bessel function $I_\nu(\omega)$, and the possibility of using these zeros for obtaining an expansion in the form of a residue series. Again, we have to replace the Bessel functions in (6.4) with their asymptotic approximations.

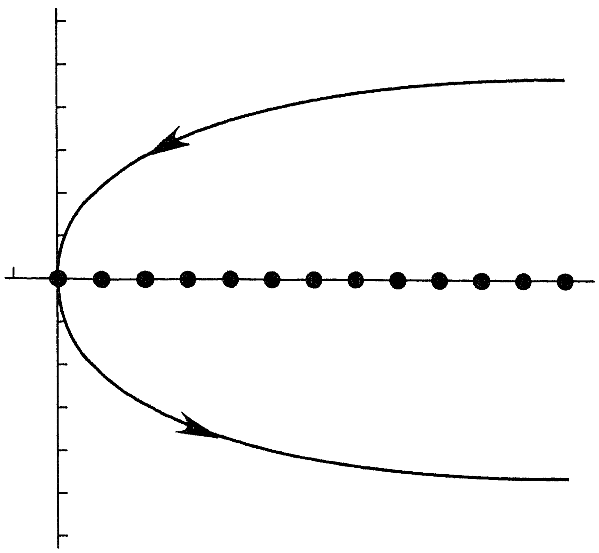

Figure 5: Contour of integration in (6.4) around the poles of $1/\sin \nu\pi$.

More details on the circle problem are given Temme (1997), where we show that use of the Watson transform (6.4) yields the uniform asymptotic approximation that is valid in the boundary layer. In Mauss (1969) a similar problem has been investigated.

**Acknowledgment.** I thank the referee for her/his comments on the first version of the paper, in particular for mentioning the references [1], [7], [9], [11], [12], [13], [14], [19], and [21]; some of these were known to me, but the referee underlined the importance of these references within the framework of this paper.

## References

[1] L.P. Cook and G.S.S. Ludford, *The behavior as* $\varepsilon \to 0$ *of solutions to* $\varepsilon\nabla^2 w = \partial w/\partial y$ *on the rectangle* $0 \le x \le l$, $|y| \le 1$, SIAM J. Math. Anal. **2** (1971), 567–594.

[2] L.P. Cook and G.S.S. Ludford, *The behavior as* $\varepsilon \to 0$ *of solutions to* $\varepsilon\nabla^2 w = \partial w/\partial y$ *in* $|y| \le 1$ *for discontinuous boundary data*, SIAM J. Math. Anal. **4** (1973), 161–184.

[3] W. Eckhaus, *Matched Asymptotic Expansions and Singular Perturbations*, North-Holland, Amsterdam, 1973.

[4] W. Eckhaus, *Asymptotic Analysis of Singular Perturbations*, North-Holland, Amsterdam, 1979.

[5] W. Eckhaus and E.M. de Jager, *Asymptotic solutions of singular perturbation problems for linear differential operators of elliptic type*, Arch. Rat. Mech. Anal. **23** (1966), 26–86.

[6] R.R. Gold, *Magnetohydrodynamic pipe flow. Part I*, J. Fluid Mech. **13** (1962), 505–512.

[7] J. Grasman, *On singular perturbations and parabolic boundary layers*, J. Engrg. Math. **2** (1968), 163–172.

[8] J. Grasman, *On the birth of boundary layers*, MC Tract No. 36, Mathematisch Centrum, Amsterdam, 1971.

[9] G.W. Hedstrom and A. Osterheld, *The effect of cell Reynolds numbers on the computation of a boundary layer*, J. Comput. Phys. **37** (1980), 399–421.

[10] P.W. Hemker, *A singularly perturbed model problem for numerical computation*, J. Comput. Appl. Math. **76** (1996), 277–285.

[11] F.A. Howes, *Perturbed boundary value problems whose reduced solutions are nonsmooth*, Indiana Univ. Math. J. **30** (1981), 267–280.

[12] J.K. Knowles and R.E. Messick, *On a class of singular perturbation problems*, J. Math. Anal. Appl. **9** (1964), 42–58.

[13] J. Mauss, *Approximation asymptotique uniforme de la solution d'un problème de perturbation singulière de type elliptique*, J. Méchanique **8** (1969), 373–391.

[14] J. Mauss, *Comportement asymptotique des solutions de problèmes de perturbation singulière de type elliptique*, J. Méchanique **9** (1970), 523–556.

[15] F.W.J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974; Reprinted in 1997 by A.K. Peters.

[16] R.E. O'Malley, Jr., *Introduction to singular perturbation*, Academic Press, New York, 1974.

[17] P.H. Roberts, *Singularities of Hartman layers*, Proc. R. Soc. Lond., Ser. A **300** (1967), 94–107.

[18] J.A. Shercliff, *Magnetohydrodynamic pipe flow. Part II. High Hartmann number*, J. Fluid Mech. **13** (1962), 513–518.

[19] S.-D. Shih and R.B. Kellogg, *Asymptotic analysis of a singular perturbation problem*, SIAM J. Math. Anal. **18** (1987), 1467–1511.

[20] D.R. Smith, *Singular perturbation Theory*, Cambridge University Press, 1985.

[21] N.M. Temme, *Analytical methods for a singular perturbation problem. The quarter plane*, Report TW 125/71, Mathematisch Centrum, Amsterdam, 1971.

[22] N.M. Temme, *Analytical methods for a singular perturbation problem in a sector*, SIAM J. Math. Anal. **5** (1974), 876–877.

[23] N.M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, Wiley, New York, 1996.

[24] N.M. Temme, *Analytical methods for a singular perturbation problem in a circle*, Manuscript, 1997.

[25] R.T. Waechter, *Steady longitudinal motion of an insulating cylinder in a conducting fluid*, Proc. Camb. Philos. Soc. **64** (1968), 1165–1201.

[26] B.L. van der Waerden, *On the method of saddle points*, Appl. Sci. Res. B **2** (1951), 33–45.

[27] R. Wong, *Asymptotic approximations of integrals*, Academic Press, New York, 1989.

Centrum voor Wiskunde en Informatica, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

E-mail: nicot@cwi.nl