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ON A BAHADUR–KIEFER REPRESENTATION OF VON MISES STATISTIC TYPE FOR INTERMEDIATE SAMPLE QUANTILES

BY

NADEZHDA GRIBKOVA* (ST. PETERSBURG) AND ROELOF HELMERS (AMSTERDAM)

Abstract. We investigate a Bahadur–Kiefer type representation for the p_n-th empirical quantile corresponding to a sample of n i.i.d. random variables when $p_n \in (0, 1)$ is a sequence which, in particular, may tend to zero or one, i.e., we consider the case of intermediate sample quantiles. We obtain an 'in probability' version of the Bahadur–Kiefer type representation for a k_n-th order statistic when $r_n = k_n \wedge (n - k_n) \to \infty$, $n \to \infty$, without any restrictions of the rate at which r_n tends to infinity. We give a bound for the remainder term in the representation with probability $1 - O(r_n^{-c})$ for arbitrary $c > 0$. We obtain also an 'almost sure' version under the additional assumption that $\log n / r_n \to 0$ as $n \to \infty$.

Finally, we establish a Bahadur–Kiefer type representation for the sum of order statistics lying between the population p_n-quantile and the corresponding intermediate sample quantile by a von Mises type statistic approximation, especially useful in establishing second order approximations for slightly trimmed sums.

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1. INTRODUCTION

Consider a sequence $X_1, X_2, \ldots$ of independent identically distributed (i.i.d.) real-valued random variables (r.v.) with common distribution function (*df*) F, and for each integer $n \geqslant 1$ let $X_{1:n} \leqslant \ldots \leqslant X_{n:n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Let $F^{-1}(u) = \inf\{x : F(x) \geqslant u\}$, $0 < u \leqslant 1$, $F^{-1}(0) = F^{-1}(0^+)$, denote the left-continuous inverse function of *df* F, and F_n, F_n^{-1} stand for the empirical *df* and its inverse, respectively. Denote by $f = F'$ a density of

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the underlying distribution when it exists. Let $\xi_p = F^{-1}(p)$ and $\xi_{pn:n} = F_n^{-1}(p)$ denote the p-th population and sample quantile, respectively.

For a fixed $p \in (0,1)$, assuming that F has at least two continuous derivatives in a neighborhood of ξ_p and $f(\xi_p) > 0$, Bahadur [1] was the first to establish the almost sure result:

$$\xi_{pn:n} = \xi_p - \frac{F_n(\xi_p) - p}{f(\xi_p)} + R_n(p), \tag{1.1}$$

where $R_n(p) = O_{\text{a.s.}}\left(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}\right)$ (a sequence of random variables R_n is said to be $O_{\text{a.s.}}(\tau_n)$ if R_n/τ_n is almost surely bounded). Kiefer [16]–[18] proved that if f' is bounded in a neighborhood of p and $f(\xi_p) > 0$, then

$$\limsup_{n\to\infty} \pm n^{3/4}(\log\log n)^{-3/4} R_n(p) = \frac{2^{5/4}3^{-3/4}\left(p(1-p)\right)^{1/4}}{f(\xi_p)} \quad \text{a.s.}$$

for either choice of sign.

Sample quantiles are closely related to empirical processes, and nowadays there is a well-developed theory of empirical processes; see, e.g., Shorack and Wellner [21], Einmahl and Mason [9], Deheuvels and Mason [8], Deheuvels [6].

In this paper we investigate the asymptotic behavior of the so-called intermediate sample quantile, i.e., of the k_n-th order statistic, $1 \leqslant k_n \leqslant n$, when $r_n := k_n \wedge (n - k_n) \to \infty$, $p_n := k_n/n \to 0$ (or $p_n \to 1$) as $n \to \infty$.

Part of our results can be compared with earlier results obtained by Chanda [4] and Watts [23], who established the 'almost sure' versions of a Bahadur–Kiefer representation for intermediate k_n-th order statistics under the following somewhat restrictive assumptions: $n^a/k_n \to 0$ for some $a > 0$ (cf. [4]) and $(\log n)^3/r_n \to 0$, $n \to \infty$ (cf. [23]), respectively. In addition, these authors assume that some strong regularity conditions on F are satisfied. An explicit 'almost sure' limit for the remainder term in the Bahadur–Kiefer representation for the uniform empirical processes under the condition $(\log n)/r_n \to 0$, $n \to \infty$, was obtained by Einmahl and Mason [9] (cf. Theorem 5 and Remark 3 therein).

In contrast, we obtain an 'in probability' version of the Bahadur–Kiefer type representations for intermediate sample quantiles without any assumptions on the rate at which r_n tends to infinity and under a mild regularity condition on F. For any fixed $c > 0$ we give a bound on the remainder term with probability $1 - O(r_n^{-c})$, under the additional assumption that $(\log n)/r_n \to 0$, $n \to \infty$; the bound holds true with probability $1 - O(n^{-c})$. We obtain an almost sure version as well.

In this paper we establish not only a Bahadur–Kiefer type representation for intermediate sample quantiles, but also derive a Bahadur–Kiefer representation for the sum of order statistics lying between the population p_n-th quantile and the corresponding intermediate sample quantile by a von Mises type statistic approximation.

Our interest in Bahadur–Kiefer type representations for intermediate empirical quantiles was first motivated by its uses in the second order asymptotic analysis of trimmed sums. It turns out (see Gribkova and Helmers [10]–[12]) that the Bahadur–Kiefer properties provide a very useful tool in the study of the asymptotic behavior of the distributions of trimmed sums of i.i.d. r.v.'s, slightly trimmed sums and their Studentized versions. In particular, the Bahadur–Kiefer representation allows us to construct a U-statistic type stochastic approximation for these statistics, which will enable us to establish Berry–Esseen type bounds and Edgeworth expansions for normalized and Studentized slightly trimmed sums.

We would like to emphasize that the Bahadur–Kiefer type representation which we obtain for a sum of order statistics lying between the p_n-th population quantile and the corresponding intermediate empirical quantile (cf. Theorem 2.2) is especially useful in the construction of a U-statistic type approximation for a (slightly) trimmed sum, as it provides a quadratic term of the desired U-statistic (cf. [10]–[12]). Note also that formally the representation (2.7) (cf. Theorem 2.2) can be obtained by integrating the corresponding Bahadur–Kiefer process on the interval $[\xi_{p_n n:n}, \xi_{p_n})$; however, we derive representation (2.7) rigorously for intermediate order statistics, i.e., when $p_n \to 0$ (or $p_n \to 1$). The remainder terms in our representations are shown to be of a suitable negligible order of magnitude.

We conclude this introduction by noting that some extensions of Bahadur's representation to dependent random variables have been obtained by Sen [20] and Wu [24]. The validity of Bahadur's representation for a bootstrapped p-quantile was proved (as an auxiliary result) in Gribkova and Helmers [11]. Deheuvels [7] established a multivariate Bahadur–Kiefer representation for the empirical copula process. The Bahadur–Kiefer theorems for uniform spacings processes were obtained by Beirlant et al. [2].

2. STATEMENT OF RESULTS

Assume that k_n is a sequence of integers such that $0 \leqslant k_n \leqslant n$, and $r_n = k_n \wedge (n-k_n) \to \infty$ as $n \to \infty$. Set $p_n = k_n/n$, and let $\xi_{p_n} = F^{-1}(p_n)$ and $\xi_{p_n\,n:n} = F_n^{-1}(p_n)$ denote the p_n-th population and empirical quantile, respectively.

Define two numbers

$$0 \leqslant a_1 = \liminf_{n\to\infty} p_n \leqslant a_2 = \limsup_{n\to\infty} p_n \leqslant 1. \tag{2.1}$$

We will assume throughout this paper that the following *smoothness condition* is satisfied.

$[A_1]$ *The function* F^{-1} *is differentiable in some open set* $U \subset (0,1)$, *i.e. the density* $f = F'$ *exists and is positive in* $F^{-1}(U)$; *moreover,*

(2.2)

$$U \supset \begin{cases} (0,\varepsilon) & \text{if } 0 = a_1 = a_2, \\ (0,a_2] & \text{if } 0 = a_1 < a_2, \\ [a_1,a_2] & \text{if } 0 < a_1 \leqslant a_2 < 1, \end{cases} \qquad U \supset \begin{cases} (1-\varepsilon,1) & \text{if } a_1 = a_2 = 1, \\ [a_1,1) & \text{if } 0 < a_1 < a_2 = 1, \\ (0,1) & \text{if } a_1 = 0, a_2 = 1, \end{cases}$$

for some $0 < \varepsilon \leqslant 1$ *in the cases given in the first line of* (2.2).

We will also need the following assumption to establish our 'almost sure' results:

$$[A_2] \qquad r_n^{-1} \log n \to 0 \quad \text{as } n \to \infty.$$

Let h be a real-valued function defined on the set $F^{-1}(U)$ (cf. (2.2)). Take an arbitrary $0 < C < \infty$ and for all sufficiently large n define

$$\Psi_{p_n,h}(C) = \sup_{|t| \leqslant C} \left| h \circ F^{-1}\left(p_n + t\sqrt{\frac{r_n \log r_n}{n^2}} \right) - h \circ F^{-1}(p_n) \right|, \tag{2.3}$$

where $h \circ F^{-1}(u) = h\left(F^{-1}(u)\right)$. Note that

$$p_n + t\sqrt{\frac{r_n \log r_n}{n^2}} = p_n + t\,\frac{r_n}{n}\sqrt{\frac{\log r_n}{r_n}} = p_n + t\,\frac{r_n}{n}\,o(1) \quad \text{as } n \to \infty.$$

In particular, this implies that the function introduced in (2.3) is well-defined for all sufficiently large n.

Next we define a function $\widehat{\Psi}_{p_n,h}(C)$ which is equal to $\Psi_{p_n,h}(C)$, where $\log r_n$ is replaced by $\log n$. As before we show that this function is well-defined for all sufficiently large n if the condition $[A_2]$ holds true.

We will obtain the Bahadur–Kiefer type representations for some smooth function of the empirical quantile, as it turned out (cf. [10]–[12]) that these extensions are very useful in the construction of the U-statistic type stochastic approximations for the Studentized (slightly) trimmed sums.

Let $G(x)$, $x \in R$, be a real-valued function, $g = G'$ its derivative when it exists, and let $(g/f)(x)$ and $(|g|/f)(x)$ denote the ratios $g(x)/f(x)$ and $|g(x)|/f(x)$, respectively.

THEOREM 2.1. (i) *Suppose that $r_n \to \infty$ as $n \to \infty$, the condition $[A_1]$ holds true, and G is differentiable on the set $F^{-1}(U)$. Then*

$$G(\xi_{p_n n:n}) - G(\xi_{p_n}) = -[F_n(\xi_{p_n}) - F(\xi_{p_n})]\frac{g}{f}(\xi_{p_n}) + R_n(p_n), \tag{2.4}$$

where for each $c > 0$

$$\mathbf{P}\left(|R_n(p_n)| > \Delta_n\right) = O(r_n^{-c}) \tag{2.5}$$

with

$$\Delta_n = A\left(p_n(1-p_n)\right)^{1/4}\left(\frac{\log r_n}{n}\right)^{3/4}\frac{|g|}{f}(\xi_{p_n}) + B\left(p_n(1-p_n)\right)^{1/2}\left(\frac{\log r_n}{n}\right)^{1/2}\Psi_{p_n,g/f}(C),$$

where A, B, and C are some positive constants which depend only on c.

(ii) *Moreover, if in addition the condition* $[A_2]$ *is also satisfied, then* (2.4) *holds true, with*

$$\mathbf{P}\big(|R_n(p_n)| > \widehat{\Delta}_n\big) = O(n^{-c}) \tag{2.6}$$

for each $c > 0$, *where*

$$\widehat{\Delta}_n = A\left(p_n(1-p_n)\right)^{1/4}\left(\frac{\log n}{n}\right)^{3/4}\frac{|g|}{f}(\xi_{p_n}) + B\left(p_n(1-p_n)\right)^{1/2}\left(\frac{\log n}{n}\right)^{1/2}\widehat{\Psi}_{p_n,g/f}(C)$$

for some positive constants A, B, *and* C *which depend only on* c.

Theorem 2.1 is a Bahadur–Kiefer type result. For the special case when $0 < p < 1$ is fixed it is stated in Lemma 3.1 in [10] (cf. also Lemma 4.1 in [11]).

REMARK 2.1. It is easy to see that if one compares the first term on the right-hand side of (2.4) and the orders of magnitude of the quantities Δ_n, $\widehat{\Delta}_n$ given in (2.5) and (2.6), the relation (2.4) provides a representation with a remainder term $R_n(p_n)$ of smaller order than the first term if and only if

$$\Psi_{p_n,g/f}(C) = o\bigg(\frac{|g|}{f}(\xi_{p_n})\bigg) \quad \text{and} \quad \widehat{\Psi}_{p_n,g/f}(C) = o\bigg(\frac{|g|}{f}(\xi_{p_n})\bigg)$$

for every fixed $C > 0$, as $n \to \infty$. The same remark also applies to the two assertions stated in Theorem 2.2 below.

We give the proofs of our results in Sections 3–5.

THEOREM 2.2. (i) *Suppose that* $r_n \to \infty$ *as* $n \to \infty$, *the condition* $[A_1]$ *holds true, and* G *is differentiable on the set* $F^{-1}(U)$. *Then*

$$\int_{\xi_{p_n n:n}}^{\xi_{p_n}} (G(x) - G(\xi_{p_n}))\, dF_n(x) = -\frac{1}{2}[F_n(\xi_{p_n}) - F(\xi_{p_n})]^2 \frac{g}{f}(\xi_{p_n}) + R_n(p_n), \tag{2.7}$$

where

$$\mathbf{P}\big(|R_n(p_n)| > \Delta_n\big) = O(r_n^{-c}) \tag{2.8}$$

for each $c > 0$ *with*

$$\Delta_n = A\big(p_n(1-p_n)\big)^{3/4}\left(\frac{\log r_n}{n}\right)^{5/4}\frac{|g|}{f}(\xi_{p_n}) + B\,p_n(1-p_n)\frac{\log r_n}{n}\Psi_{p_n,g/f}(C),$$

where A, B, *and* C *are some positive constants which depend only on* c.

(ii) *Moreover, if in addition the condition* $[A_2]$ *is also satisfied, then* (2.7) *holds true, with*

$$\mathbf{P}\big(|R_n(p_n)| > \widehat{\Delta}_n\big) = O(n^{-c}) \tag{2.9}$$

for each $c > 0$, *where*

$$\widehat{\Delta}_n = A\,\big(p_n(1-p_n)\big)^{3/4}\left(\frac{\log n}{n}\right)^{5/4}\frac{|g|}{f}(\xi_{p_n}) + B\,p_n(1-p_n)\frac{\log n}{n}\widehat{\Psi}_{p_n,g/f}(C)$$

for some positive constants A, B, *and* C *which depend only on* c.

Theorem 2.2 is our main result; the relation (2.7) provides us with a Bahadur–Kiefer type representation of von Mises type for a sum of order statistics (cf. the left-hand side of (2.7)). Theorem 2.2 also extends Lemma 4.3 from [11] (cf. also Lemma 3.2 in [10]), where (2.7) was established for a fixed p, to the more general case where p_n is a sequence which may tend to zero or one, as n gets large. Note also that if both the conditions $[A_1]$ and $[A_2]$ are satisfied, then Theorems 2.1 and 2.2 and an application of the Borel–Cantelli lemma imply an almost sure result, i.e. $R_n(p_n) = O_{\text{a.s.}}(\widehat{\Delta}_n)$ as $n \to \infty$.

Next we will state some consequences of Theorems 2.1 and 2.2 where the remainder terms are given in a simpler form. Our first two corollaries concern Bahadur–Kiefer type representations for central (not intermediate) order statistics.

COROLLARY 2.1. *Suppose that* $0 < a_1 \leqslant a_2 < 1$, *the condition* $[A_1]$ *holds true, and the functions* $f = F'$ *and* $g = G'$ *satisfy a Hölder condition of order* $\alpha \geqslant 1/2$ *on the set* $F^{-1}(U)$. *Then* (2.4) *is valid and*

$$\boldsymbol{P}\big(|R_n(p_n)| > A(\log n/n)^{3/4}\big) = O(n^{-c})$$

for each $c > 0$, *where* $A > 0$ *is some constant not depending on* n.

COROLLARY 2.2. *Suppose that the conditions of Corollary* 2.1 *are satisfied. Then* (2.7) *is valid and*

$$\boldsymbol{P}\big(|R_n(p_n)| > A(\log n/n)^{5/4}\big) = O(n^{-c})$$

for each $c > 0$, *where* $A > 0$ *is some constant not depending on* n.

To prove Corollaries 2.1 and 2.2 it suffices to note that the condition $0 < a_1 \leqslant a_2 < 1$ implies that $[A_2]$ is automatically satisfied. Moreover, due to the condition $[A_1]$ the density f is bounded away from zero on the set $F^{-1}([a_1-\delta,\, a_2+\delta])$ with some $\delta > 0$, and hence the ratio g/f satisfies a Hölder condition of order $\alpha \geqslant 1/2$ on this set. Then, an application of Hölder's condition to the function $\Psi_{p_n,g/f}(C)$ (cf. (2.3)) proves both corollaries.

Next we state several corollaries for the intermediate sample quantiles provided some regularity conditions are satisfied.

Note that the second terms of Δ_n and $\widehat{\Delta}_n$ in (2.5) and (2.6), and in (2.8) and (2.9), involving the functions $\Psi_{p_n,g/f}(C)$ and $\widehat{\Psi}_{p_n,g/f}(C)$, depend on the asymptotic properties of the ratio g/f, and we can describe some sets of conditions which enable us to absorb these second terms in the first ones. We will need the following conditions:

$$\text{(2.10)}\qquad \begin{aligned} &\text{(i)} \quad \Psi_{p_n,g/f}(C) = O\Big(\Big(\frac{\log r_n}{r_n}\Big)^{1/4}\frac{|g|}{f}(\xi_{p_n})\Big),\\ &\text{(ii)} \quad \widehat{\Psi}_{p_n,g/f}(C) = O\Big(\Big(\frac{\log n}{r_n}\Big)^{1/4}\frac{|g|}{f}(\xi_{p_n})\Big). \end{aligned}$$

Before describing some useful corollaries of Theorems 2.1 and 2.2 we state the following Theorems 2.3 and 2.4 which can be seen as simple consequences of those theorems, whenever the condition (2.10) is satisfied.

THEOREM 2.3. *Suppose that $r_n \to \infty$ as $n \to \infty$, the condition $[A_1]$ holds true, and G is differentiable on the set $F^{-1}(U)$. Assume, in addition, that the condition* (i) *in* (2.10) *holds true. Then the representation* (2.4) *and the relation* (2.5) *are valid together with*

$$\Delta_n = A\left(p_n(1-p_n)\right)^{1/4}\left(\frac{\log r_n}{n}\right)^{3/4}\frac{|g|}{f}(\xi_{p_n}),$$

where A is some positive constant not depending on n.

Moreover, if in addition the condition $[A_2]$ and the relation (ii) *in* (2.10) *are also satisfied, then* (2.4) *and* (2.6) *are valid with*

$$\widehat{\Delta}_n = A\left(p_n(1-p_n)\right)^{1/4}\left(\frac{\log n}{n}\right)^{3/4}\frac{|g|}{f}(\xi_{p_n}).$$

THEOREM 2.4. *Suppose that $r_n \to \infty$ as $n \to \infty$, the condition $[A_1]$ holds true, and G is differentiable on the set $F^{-1}(U)$. Assume, in addition, that the condition* (i) *in* (2.10) *holds true. Then the representation* (2.7) *and the relation* (2.8) *are valid together with*

$$\Delta_n = A\left(p_n(1-p_n)\right)^{3/4}\left(\frac{\log r_n}{n}\right)^{5/4}\frac{|g|}{f}(\xi_{p_n}),$$

where A is some positive constant not depending on n.

Moreover, if in addition the condition $[A_2]$ and the relation (ii) *in* (2.10) *are also satisfied, then* (2.7) *and* (2.9) *are valid with*

$$\widehat{\Delta}_n = A\left(p_n(1-p_n)\right)^{3/4}\left(\frac{\log n}{n}\right)^{5/4}\frac{|g|}{f}(\xi_{p_n}).$$

Now we present certain sets of conditions sufficient for the relations (2.10) to hold and, consequently, obtain some useful corollaries of Theorems 2.3 and 2.4.

Let $SRV_\rho^{+\infty}$ ($SRV_\rho^{-\infty}$) be a class of functions regularly varying in $+\infty$ ($-\infty$) defined as follows: $g \in SRV_\rho^{+\infty}$ ($g \in SRV_\rho^{-\infty}$) if and only if

(i) $g(x) = \pm|x|^\rho L(x)$ for $|x| > x_0$, with some $x_0 > 0$ ($x_0 < 0$), $\rho \in \mathbb{R}$, and $L(x)$ is a positive slowly varying function at $+\infty$ ($-\infty$);

(ii) the following second order regularity condition on the tails is satisfied:

$$|g(x + \triangle x) - g(x)| = O\left(|g(x)| \left|\frac{\triangle x}{x}\right|^{1/2}\right) \tag{2.11}$$

when $\triangle x = o(|x|)$ as $x \to +\infty$ ($x \to -\infty$).

Note that (2.11) holds true for g if

$$\left|\frac{L(x + \triangle x)}{L(x)} - 1\right| = O\left(\left|\frac{\triangle x}{x}\right|^{1/2}\right) \quad \text{as } x \to +\infty \ (x \to -\infty),$$

where L is the corresponding slowly varying function, and it is also satisfied (even with degree 1 instead of 1/2) if L is continuously differentiable for sufficiently large $|x|$ and $|L'(x)| = O\big(L(x)/|x|\big)$ as $x \to +\infty$ ($x \to -\infty$), which is valid, for instance, when L is some power of the logarithm.

COROLLARY 2.3. *Suppose that* $p_n \to 0$ ($p_n \to 1$), *the condition* $[A_1]$ *is satisfied,* $f \in SRV_\rho^{-\infty}$ ($f \in SRV_\rho^{+\infty}$), *where* $\rho = -(1+\gamma)$, $\gamma > 0$, *and* $g \in SRV_\rho^{-\infty}$ ($g \in SRV_\rho^{+\infty}$), *where* $\rho \in \mathbb{R}$. *Then the condition* (i) *in* (2.10) *is satisfied, and if in addition* $[A_2]$ *holds true, then the condition* (ii) *in* (2.10) *is also satisfied. Hence, both the assertions stated in Theorems 2.3 and 2.4 are valid.*

We postpone the proof of Corollary 2.3 to Section 5.

Our final corollary concerns the case when the *df* F and the function G are twice differentiable.

Let us define the function $\nu(u) := (g/f) \circ F^{-1}(u)$, $u \in (0, 1)$.

COROLLARY 2.4. *Suppose that* $p_n \to 0$ ($p_n \to 1$), *the condition* $[A_1]$ *is satisfied, and assume that the functions* f *and* g *are differentiable on the set* $F^{-1}(U)$. *In addition, suppose that*

$$\sup_{u \in U} \left|\frac{\nu'(u)\,[u \wedge (1-u)]}{\nu(u)}\right| < \infty, \tag{2.12}$$

and that

$$\limsup_{u \downarrow 0\ (u \uparrow 1)} \left|\frac{\nu\big(u + [u \wedge (1-u)]o(1)\big)}{\nu(u)}\right| < \infty, \tag{2.13}$$

where $o(1)$ *denotes any function tending to zero when* $u \downarrow 0$ ($u \uparrow 1$).

Then the condition (i) *in* (2.10) *is satisfied, and if, in addition,* $[A_2]$ *holds true, then the condition* (ii) *in* (2.10) *is also satisfied. Hence, both the assertions stated in Theorems* 2.3 *and* 2.4 *are valid.*

Proof. The proof of Corollary 2.4 is straightforward. Let us take an arbitrary $C > 0$, fix t such that $|t| < C$, and put $\alpha(n) = \sqrt{(\log r_n)/r_n}$ when we prove the relation (i) of (2.10), and $\alpha(n) = \sqrt{(\log n)/r_n}$ when we prove the relation (ii) of (2.10) (under the additional condition $[A_2]$). In both cases we have $\alpha(n) \to 0$ as $n \to \infty$. Consider now the quantity $\left|\nu\left(p_n + t[p_n \wedge (1-p_n)]\alpha(n)\right) - \nu(p_n)\right|$. Since, for all sufficiently large n, p_n and $p_n + t[p_n \wedge (1-p_n)]\alpha(n)$ take their values in the set U, the latter quantity is equal to

$$(2.14)\qquad |\nu(p_n)| \left| \frac{\nu'\left(p_n + \theta t[p_n \wedge (1-p_n)]\alpha(n)\right)}{\nu\left(p_n + \theta t[p_n \wedge (1-p_n)]\alpha(n)\right)}\, t\,[p_n \wedge (1-p_n)]\alpha(n) \right| \times \left| \frac{\nu\left(p_n + \theta t[p_n \wedge (1-p_n)]\alpha(n)\right)}{\nu(p_n)} \right| = O\left(|\nu(p_n)|\alpha(n)\right)$$

for some $0 < \theta < 1$, which, because of the conditions (2.12) and (2.13), directly yields (2.10). Thus the corollary is proved. ∎

REMARK 2.2. It should be noted that under the conditions of Corollary 2.4 we in fact have obtained somewhat stronger relations, namely:

$$\Psi_{p_n,g/f}(C) = O\left(\left(\frac{\log r_n}{r_n}\right)^{1/2} \cdot \frac{|g|}{f}(\xi_{p_n})\right)$$

and

$$\widehat{\Psi}_{p_n,g/f}(C) = O\left(\left(\frac{\log n}{r_n}\right)^{1/2} \cdot \frac{|g|}{f}(\xi_{p_n})\right)$$

(cf. (2.14)), which of course directly imply (2.10).

The following examples show that the conditions (2.12) and (2.13) hold true in a number of interesting cases.

EXAMPLE 2.1 (Gumbel). Consider the distribution $F(x) = \exp(-\exp(-x))$, $x \in \mathbb{R}$, and let $g(x) = x^k$, where $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. In this situation we have $f(x) = \exp(-x)\exp\left(-\exp(-x)\right)$, while the inverse function is equal to $F^{-1}(u) = -\log(-\log u)$, $u \in (0,1)$, whereas

$$f\left(F^{-1}(u)\right) = -u \log u, \quad \text{and} \quad \nu(u) = \frac{[-\log(-\log u)]^k}{-u \log u}.$$

After some simple computations we see that

$$(2.15)\qquad \frac{\nu'(u)\,[u \wedge (1-u)]}{\nu(u)} = -k \frac{u \wedge (1-u)}{-\log(-\log u)\, u \log u} + \frac{u \wedge (1-u)}{-u \log u}(1 + \log u).$$

If $u \to 0$, the first term on the right-hand side of (2.15) tends to zero and the second term tends to -1. When $u \to 1$, the first term is equivalent to

$$-k\frac{1-u}{-\log(-\log u)\,\log\big(1+(u-1)\big)} \sim k\frac{1}{\log(-\log u)} = o(1),$$

whereas, similarly, the second term is equivalent to

$$-\frac{1-u}{\log u} = \frac{u-1}{\log\big(1+(u-1)\big)} = 1 * o(1).$$

Thus, (2.12) is satisfied in both cases, i.e. $U = [0,\varepsilon]$ ($p_n \to 0$) and $U = [1-\varepsilon, 1]$ ($p_n \to 1$).

To check (2.13) we write

$$\frac{\nu\big(u+[u\wedge(1-u)]o(1)\big)}{\nu(u)} = \left[\frac{\log\big(-\log(u+[u\wedge(1-u)]o(1))\big)}{\log(-\log u)}\right]^k \times \frac{u}{u+[u\wedge(1-u)]o(1)}\,\frac{\log u}{\log\big(u+[u\wedge(1-u)]o(1)\big)},$$

and arguing as before we see that the latter quantity is $1+o(1)$, as $u \to 0$ and as $u \to 1$ as well.

EXAMPLE 2.2. Let $F(x) = \big(1-\exp(-x^\gamma)\big)\mathbb{1}(x \geqslant 0)$, $\gamma > 0$, and let $g(x) = x^\rho$, $\rho \in \mathbb{R}$. Now we get

$$F^{-1}(u) = [-\log(1-u)]^{1/\gamma}, \qquad u \in (0,1),$$

and

$$\nu(u) = \frac{[-\log(1-u)]^{\rho/\gamma}}{\gamma[-\log(1-u)]^{(\gamma-1)/\gamma}(1-u)} = \frac{1}{\gamma}[-\log(1-u)]^{(\rho+1)/\gamma-1}\frac{1}{1-u}.$$

Then we obtain

$$\frac{\nu'(u)\,[u\wedge(1-u)]}{\nu(u)} = \frac{\rho+1-\gamma}{\gamma}\,\frac{u\wedge(1-u)}{-(1-u)\log(1-u)} + \frac{u\wedge(1-u)}{1-u}. \tag{2.16}$$

The first term on the right-hand side in (2.16) tends to the constant $(\rho+1-\gamma)/\gamma$ as $u \to 0$ and it tends to zero as $u \to 1$, the second term tends to zero as $u \to 0$ and it tends to one as $u \to 1$. Thus, (2.12) is satisfied in both cases as in the previous example.

To check (2.13) we write

$$\frac{\nu\big(u+[u\wedge(1-u)]o(1)\big)}{\nu(u)} = \left[\frac{\log\big(1-u-[u\wedge(1-u)]o(1)\big)}{\log(1-u)}\right]^{(\rho+1)/\gamma-1} \times \frac{1-u}{1-u-[u\wedge(1-u)]o(1)}.$$

Some simple computations now show that both factors appearing in the latter quantity tend to one as $u \to 0$ or as $u \to 1$.

EXAMPLE 2.3 (Weibull). Let $F(x) = \exp(-x^{-\gamma})\mathbf{1}(x \geqslant 0)$, $\gamma > 0$, and let $g(x) = x^{\rho}$, $\rho \in \mathbb{R}$. Here we get

$$F^{-1}(u) = [-\log u]^{-1/\gamma}, \quad u \in (0,1), \qquad f\big(F^{-1}(u)\big) = \gamma(-\log u)^{(\gamma+1)/\gamma}\, u,$$

and

$$v(u) = \frac{[-\log u]^{-\rho/\gamma}}{\gamma[-\log u]^{(\gamma+1)/\gamma}\, u} = \frac{1}{\gamma}[-\log(1-u)]^{-(\rho+\gamma+1)/\gamma}\frac{1}{u}.$$

Then we obtain

$$\text{(2.17)} \qquad \frac{v'(u)\,[u \wedge (1-u)]}{v(u)} = -\frac{\rho+\gamma+1}{\gamma}\,\frac{u \wedge (1-u)}{u \log u} - \frac{u \wedge (1-u)}{u}.$$

If $u \to 0$, the first term on the right-hand side in (2.17) tends to zero and the second one tends to -1, and if $u \to 1$, the first term tends to the constant $(\rho+\gamma+1)/\gamma$ and the second one tends to zero. Thus, (2.12) is satisfied in both cases, i.e. $u \to 0$ and $u \to 1$. To check (2.13) we write

$$\frac{v\big(u + [u \wedge (1-u)]o(1)\big)}{v(u)} = \left[\frac{\log u}{\log\big(u + [u \wedge (1-u)]o(1)\big)}\right]^{(\rho+\gamma+1)/\gamma} \times \frac{u}{u + [u \wedge (1-u)]o(1)},$$

and simple evident arguments show that both factors here tend to one as $u \to 0$ and as $u \to 1$.

EXAMPLE 2.4. Let $f(x) = C_\gamma \exp(-|x|^\gamma)$, $\gamma > 0$, where C_γ is a constant depending only on γ, and let $g(x) = \pm|x|^\rho$, $\rho \in \mathbb{R}$. It is clear that the asymptotic behavior of the functions on the left-hand side of the conditions (2.12) and (2.13) is similar to that in Example 2.2 ($u \to 1$). Thus, these conditions are also satisfied. Note that if $\gamma = 2$, we are dealing with a normal law in this example.

EXAMPLE 2.5. Here we consider an example of a distribution with super heavy tails having no finite moments. In contrast to Examples 2.1–2.4, in this case Corollary 2.4 cannot be applied. In this situation, clearly outside the scope of Corollary 2.4, it is easily checked that the assumption (2.12) obviously fails to hold, whereas the condition (2.13) is also typically not satisfied either (the only exception would be the case where the term $o(1)/(1-u)$ appearing in (2.19) remains bounded as u tends to one). Nevertheless, the conditions of Theorems 2.3 and 2.4 hold true under the additional assumption (2.20) about the rate at which r_n tends to infinity, and the Bahadur–Kiefer representations (2.4)–(2.7) are still valid for the intermediate sample quantiles. We refer to Example 1 in Griffin and Pruitt [13] for a closely related discussion.

Let $F(x) = 1 - C/\log x$ for $x \geqslant x_0 > 0$, where $C > 0$ is some constant. Suppose for ease of presentation that $p_n \to 1$ as $n \to \infty$, while $r_n = n - k_n \to \infty$, and let us assume that $g(x) = x^\rho$, $\rho \in \mathbb{R}$. In this case

$$F^{-1}(u) = \exp\left(\frac{C}{1-u}\right),$$
$$f(F^{-1}(u)) = \frac{(1-u)^2}{C}\exp\left(-\frac{C}{1-u}\right),$$
$$\nu(u) = \exp\left((\rho+1)\frac{C}{1-u}\right)\frac{C}{(1-u)^2}.$$

Since $p_n \to 1$, we are interested only in the case $u \to 1$, so $u \wedge (1-u) = 1-u$, and after simple computations we obtain

$$\frac{\nu'(u)\,(1-u)}{\nu(u)} = \frac{C(\rho+1)}{1-u} + 2, \tag{2.18}$$

which is not bounded as $u \to 1$, and therefore (2.12) is clearly not satisfied. A simple analysis of the quantity on the left-hand side in (2.13) yields

$$\frac{\nu(u + [u\wedge(1-u)]o(1))}{\nu(u)} = \exp\left(C(\rho+1)\frac{o(1)}{1-u}\right)(1+o(1)). \tag{2.19}$$

We infer that (2.13) is satisfied only if $o(1)/(1-u)$ remains bounded as $u \to 1$. However, despite the fact that the conditions (2.12) and (2.13) are not satisfied in the present example, the relation (2.10) still holds true provided a stronger restriction on the rate at which r_n tends to infinity is imposed. Indeed, observe that in fact (cf. the proof of Corollary 2.4) we evaluate the functions appearing in (2.12) and (2.13) at the value $u = p_n$ and choose $o(1) = \alpha(n) = \sqrt{(\log r_n)/r_n}$, where in the present example $r_n = n - k_n$ for all sufficiently large n, since we assume that $p_n \to 1$. Then we see that (2.14), (2.18), and (2.19) together imply that (2.10) is satisfied whenever

$$\frac{\alpha(n)}{1-p_n} = O\left(\left(\frac{\log r_n}{r_n}\right)^{1/4}\right).$$

Observe that

$$\frac{\alpha(n)}{1-p_n} = \frac{n}{r_n}\left(\frac{\log r_n}{r_n}\right)^{1/2},$$

and the latter quantity is of order $O\left(((\log r_n)/r_n)^{1/4}\right)$, whenever the quantity

$$\frac{n}{r_n}\left(\frac{\log r_n}{r_n}\right)^{1/4} = \frac{n\,(\log r_n)^{1/4}}{r_n^{5/4}}$$

remains bounded as $n \to \infty$. Hence we can conclude that the conditions of Theorems 2.3 and 2.4 are satisfied provided

$$\limsup_{n\to\infty} \frac{n^{4/5}(\log r_n)^{1/5}}{r_n} < \infty. \tag{2.20}$$

Note that $[A_2]$ is satisfied automatically whenever (2.20) holds true.

Similarly, we see that a weaker relation (than (2.10)):

$$\Psi_{p_n,g/f}(C) = o\Big(\frac{|g|}{f}(\xi_{p_n})\Big)$$

(for every fixed $C > 0$), is satisfied provided $n^{2/3}(\log r_n)^{1/3}/r_n \to 0$ as $n \to \infty$, and this implies only the fact that (2.4) and (2.7) yield the representations (cf. Remark 2.1) with remainder terms of negligible order.

To conclude this section define a binomial random variable $N_p = \sharp\{i : X_i \leqslant \xi_p\}$, $0 < p < 1$. Our proof of Theorems 2.1 and 2.2 uses the following fact: conditionally on N_p the order statistics $X_{1:n}, \ldots, X_{N_p:n}$ are distributed as order statistics corresponding to a sample of N_p i.i.d. r.v.'s with distribution function $F(x)/p$, $x \leqslant \xi_p$. Though this fact is essentially known (cf., e.g., Theorem 12.4 in [15], and also [11]), we give a brief proof of it in Section 6.

3. PROOF OF THEOREM 2.1

We can assume without loss of generality that $a_2 \leqslant 1/2$. Then $k_n \leqslant n - k_n$ for all sufficiently large n, and so it is enough to prove (2.4) with

$$\Delta_n = A\, p_n^{1/4}\left(\frac{\log k_n}{n}\right)^{3/4}\frac{|g|}{f}(\xi_{p_n}) + B\, p_n^{1/2}\left(\frac{\log k_n}{n}\right)^{1/2}\Psi_{p_n,g/f}(C). \tag{3.1}$$

We begin with the proof of the first assertion of the theorem, where there is no restriction on the rate at which k_n approaches infinity.

Let $U_1, \ldots, U_n$ denote a sample of independent uniformly $(0,1)$ distributed r.v.'s, and $U_{1:n} \leqslant \ldots \leqslant U_{n:n}$ stand for the corresponding order statistics. Put

$$N_{p_n}^x = \sharp\{i : X_i \leqslant \xi_{p_n}\}, \qquad N_{p_n} = \sharp\{i : U_i \leqslant p_n\}, \tag{3.2}$$

and note that $\xi_{p_n n:n} = X_{k_n:n}$ (because $p_n = k_n/n$).

We have to prove that $\mathbf{P}\big(|R_n(p_n)| > \Delta_n\big) = O\left(k_n^{-c}\right)$ for each $c > 0$ (cf. (2.4)), and since the joint distribution of $X_{k_n:n}$, $N_{p_n}^x$ coincides with joint distribution of $F^{-1}(U_{k_n:n})$, N_{p_n}, it is sufficient to verify it for a remainder given by

$$R_n(p_n) = G\big(F^{-1}(U_{k_n:n})\big) - G\big(F^{-1}(p_n)\big) + \frac{N_{p_n} - p_n n}{n}\frac{g}{f}(\xi_{p_n}).$$

By Bernstein's inequality, $\mathbf{P}(U_{k_n:n} \notin U) = O\big(\exp(-\delta n)\big)$ for some $\delta > 0$ not depending on n (where U is the set defined in (2.2)), so we can rewrite $R_n(p_n)$ for all sufficiently large n as

$$\frac{g}{f}(\xi_{p_n})\, R_{n,1} + R_{n,2}, \tag{3.3}$$

where

$$R_{n,1} = U_{k_n:n} - p_n + \frac{N_{p_n} - p_n n}{n}, \tag{3.4}$$

and

$$R_{n,2} = \left[\frac{g}{f}\Big(F^{-1}(p_n + \theta(U_{k_n:n} - p_n))\Big) - \frac{g}{f}\left(F^{-1}(p_n)\right)\right](U_{k_n:n} - p_n), \tag{3.5}$$

$0 < \theta < 1$. Fix an arbitrary $c > 0$ and note that we can estimate $R_{n,j}$, $j = 1, 2$, on the set

$$E = \{\omega : |N_{p_n} - p_n n| < A_0 (p_n n \log k_n)^{1/2}\}, \tag{3.6}$$

where A_0 is a positive constant depending only on c, because by Bernstein's inequality $\mathbf{P}(\Omega \setminus E) = O(k_n^{-c})$ (in fact, we can take every A_0 such that $A_0^2 > 2c$). We will prove that

$$\mathbf{P}\Big(|R_{n,1}| > A_1 (p_n)^{1/4} \big((\log k_n)/n\big)^{3/4}\Big) = O(k_n^{-c}) \tag{3.7}$$

and that

$$\mathbf{P}\Big(|R_{n,2}| > A_2\, p_n\, \Psi_{p_n, g/f}(C) \big((\log k_n)/k_n\big)^{1/2}\Big) = O(k_n^{-c}). \tag{3.8}$$

Here and elsewhere A_i, $i = 1, 2, \ldots$, and C denote some positive constants depending only on c. The relations (3.3)–(3.8) imply (2.4) with Δ_n given in (3.1).

First we prove (3.7), using a similar argument conditioning on N_{p_n} to that in the proof of Lemmas 4.1 and 4.3 in [11].

Case 1. First let $k_n \leqslant N_{p_n}$. Then conditionally on N_{p_n} the order statistic $U_{k_n:n}$ is distributed as the k_n-th order statistic $U'_{k_n:N_{p_n}}$ of the sample $U'_1, \ldots, U'_{N_{p_n}}$ of independent $(0, p_n)$ uniformly distributed r.v.'s (cf. Lemma 6.1 in the Appendix). Its expectation is of the form

$$\mathbf{E}(U_{k_n:n} \mid N_{p_n},\ k_n \leqslant N_{p_n}) = p_n \frac{k_n}{N_{p_n} + 1},$$

and the conditional variance equals

$$V_{k_n}^2 = \frac{p_n^2}{N_{p_n} + 2} \frac{k_n}{N_{p_n} + 1} \left(1 - \frac{k_n}{N_{p_n} + 1}\right),$$

and on the set E we have an estimate $V_{k_n}^2 \leqslant A_0(p_n)^{1/2} n^{-3/2} \log^{1/2} n$. Then rewrite $R_{n,1}$ (whenever $k_n \leqslant N_{p_n}$) as

$$U_{k_n:n} - p_n \frac{k_n}{N_{p_n}+1} + R'_{n,1}, \tag{3.9}$$

where $R'_{n,1}$ equals

$$p_n \frac{k_n}{N_{p_n}+1} - p_n + \frac{N_{p_n} - p_n n}{n} = \frac{(N_{p_n}-k_n)^2}{n(N_{p_n}+1)} + \frac{N_{p_n}-k_n}{n(N_{p_n}+1)} - \frac{k_n}{n(N_{p_n}+1)},$$

and on the set E the latter quantity is of order $O\big((\log k_n)/n\big)$. Since $(\log k_n)/n = o\Big(p_n^{1/4}\big((\log k_n)/n\big)^{3/4}\Big)$, the remainder term $R'_{n,1}$ is of negligible order for our purposes. For the first two terms in (3.9) we have

(3.10)

$$\mathbf{P}\bigg(\bigg|U_{k_n:n} - p_n \frac{k_n}{N_{p_n}+1}\bigg| > A_1(p_n)^{1/4}\Big(\frac{\log k_n}{n}\Big)^{3/4} \,\Big|N_{p_n} : k_n \leqslant N_{p_n}\bigg)$$
$$= \mathbf{P}\bigg(\bigg|U'_{k_n:N_{p_n}} - p_n \frac{k_n}{N_{p_n}+1}\bigg| > A_1(p_n)^{1/4}\Big(\frac{\log k_n}{n}\Big)^{3/4}\bigg) = P_1 + P_2,$$

where N_{p_n} is fixed, $k_n \leqslant N_{p_n}$, A_1 is a constant which will be chosen later, and

$$\begin{aligned} P_1 &= \mathbf{P}\bigg(U'_{k_n:N_{p_n}} > p_n \frac{k_n}{N_{p_n}+1} + A_1(p_n)^{1/4}\Big(\frac{\log k_n}{n}\Big)^{3/4}\bigg), \\ P_2 &= \mathbf{P}\bigg(U'_{k_n:N_{p_n}} < p_n \frac{k_n}{N_{p_n}+1} - A_1(p_n)^{1/4}\Big(\frac{\log k_n}{n}\Big)^{3/4}\bigg). \end{aligned} \tag{3.11}$$

We evaluate P_1, the treatment for P_2 is similar. Consider a binomial r.v.

$$S'_n = \sum_{i=1}^{N_{p_n}} \mathbf{1}_{D_i}, \quad \text{where } D_i = \bigg\{U'_{i:N_{p_n}} \leqslant p_n \frac{k_n}{N_{p_n}+1} + A_1(p_n)^{1/4}\Big(\frac{\log k_n}{n}\Big)^{3/4}\bigg\},$$

with parameter (q_n, N_{p_n}), and

$$q_n = \min\Big(1, \frac{k_n}{N_{p_n}+1} + t_n\Big), \qquad t_n = A_1\Big(\frac{\log k_n}{k_n}\Big)^{3/4}.$$

If $q_n = 1$, then $P_1 = 0$ and the inequality we need is valid trivially. Let $q_n < 1$ and let $\overline{S'_n}$ denote the average S'_n/N_{p_n}. Then the probability P_1 is equal to

$$\mathbf{P}(S'_n < k_n) = \mathbf{P}\Big(\overline{S'_n} - q_n < \frac{k_n}{N_{p_n}} - \frac{k_n}{N_{p_n}+1} - t_n\Big). \tag{3.12}$$

Note that
$$\frac{k_n}{N_{p_n}} - \frac{k_n}{N_{p_n}+1} = \frac{k_n}{N_{p_n}(N_{p_n}+1)} < \frac{1}{N_{p_n}},$$
and since the latter quantity is of order $o(t_n k_n^{-1/4}) = o(t_n)$ on the set E, for our purposes this term can be omitted on the right-hand side of (3.12) in our analysis. To evaluate $\mathbf{P}(S_n' - q_n < -t_n)$ we note that $q_n - t_n = k_n/(N_{p_n}+1) \in (0,1)$, and that $q_n > 1/2$ for all sufficiently large n (and hence k_n and N_{p_n}) on the set E. So, we may apply the inequality (2.2) of Hoeffding [14] with $\mu = q_n$ and with $g(\mu) = 1/\big(2\mu(1-\mu)\big)$. Then we obtain

(3.13)
$$\mathbf{P}(S_n' < k_n) \leqslant \exp\big(-N_{p_n} t_n^2 g(q_n)\big) = \exp\left(-\frac{N_{p_n} A_1^2\big((\log k_n)/k_n\big)^{3/2}}{2q_n(1-q_n)}\right).$$

Finally, we note that
$$1 - q_n = 1 - \frac{k_n}{N_{p_n}+1} - A_1\left(\frac{\log k_n}{k_n}\right)^{3/4} \leqslant \frac{N_{p_n}+1-k_n}{N_{p_n}+1},$$
and on the set E the latter quantity is not greater than
$$\frac{A_0(k_n \log k_n)^{1/2}}{N_{p_n}}.$$

Then we can get a lower bound for the ratio on the right-hand side in (3.13):
$$\begin{aligned}\frac{N_{p_n} A_1^2\big((\log k_n)/k_n\big)^{3/2}}{2q_n(1-q_n)} &\geqslant \frac{A_1^2 N_{p_n}^2\big((\log k_n)/k_n\big)^{3/2}}{2A_0(k_n \log k_n)^{1/2}} \\ &= \frac{A_1^2}{2A_0}\log k_n\left(\frac{N_{p_n}}{k_n}\right)^2 = \frac{A_1^2}{2A_0}\log k_n\big(1+o(1)\big).\end{aligned}$$

This bound and (3.13) together yield that when $A_1^2/(2A_0) \geqslant c$, the desired relation $P_1 = O(k_n^{-c})$ holds true. The same argument is valid for P_2.

We note in passing that an application of a refinement of Hoeffding's inequality due to Talagrand [22] (cf. also Leon and Perron [19]) does not allow us to weaken the condition $A_1^2/(2A_0) \geqslant c$ which was needed to establish the desired estimates.

Case 2. In case $N_{p_n} < k_n$ we use the fact that conditionally on N_{p_n} the order statistic $U_{k_n:n}$ is distributed as the $(k_n - N_{p_n})$-th order statistic $U''_{k_n-N_{p_n}:n-N_{p_n}}$ of a sample $U_1'', \ldots, U''_{n-N_{p_n}}$ from $(1-p_n, 1)$ uniform distribution, its expectation is $p_n + (k_n - N_{p_n})/(n - N_{p_n}+1)$, and for the conditional variance we have the following upper bound:
$$V^2_{k_n-N_{p_n}} \leqslant A_0(p_n \log k_n)^{1/2} n^{-3/2}.$$

In this case we use a representation for $R_{n,1} = R''_{n,1} + R''_{n,2}$, where

$$R''_{n,1} = U_{k_n:n} - p_n - \frac{k_n - N_{p_n}}{n - N_{p_n} + 1}(1 - p_n),$$

and

$$R''_{n,2} = \frac{N_{p_n} - p_n n}{n} + \frac{k_n - N_{p_n}}{n - N_{p_n} + 1}(1 - p_n).$$

As in the first case we have that $R''_{n,2} = O\big((\log k_n)/n\big)$ with probability $1 - O(k_n^{-c})$, and this term is of negligible order for our purposes. Using Hoeffding's inequality we obtain for $R''_{n,1}$ a similar bound to that for $R'_{n,1}$. So (3.7) is proved.

It remains to prove (3.8). First note that by (3.7) on the set E with probability $1 - O(k_n^{-c})$ we have

$$\begin{aligned}|U_{k_n:n} - p_n| &\leqslant A_0 \frac{(k_n \log k_n)^{1/2}}{n} + A_1 p_n \left(\frac{\log k_n}{k_n}\right)^{3/4} \\ &= A_0 \left(p_n \frac{\log k_n}{n}\right)^{1/2} (1 + o(1)).\end{aligned}$$

Thus, there exists A_2, depending only on c, such that

$$|R_{n,2}| \leqslant A_2 \left(p_n \frac{\log k_n}{n}\right)^{1/2} \Psi_{p_n, g/f}(A_2)$$

with probability $1 - O(k_n^{-c})$. This implies (3.8). Thus, the first assertion of Theorem 2.1 is proved.

To prove the second assertion, it is sufficient to repeat our previous arguments replacing $\log k_n$ by $\log n$ throughout the proof, and applying the assumption $[A_2]$ instead of using the elementary fact that $(\log k_n)/k_n \to 0$ used before. Moreover, we should now use the function $\widehat{\Psi}_{p_n,h}(C)$ instead of $\Psi_{p_n,h}(C)$. These changes lead to bounds with probability $1 - O(n^{-c})$ for each $c > 0$. This completes the proof of the theorem. ■

4. PROOF OF THEOREM 2.2

We give a detailed proof of the first assertion of Theorem 2.2. To prove the second assertion it clearly suffices to make similar changes to those given in the proof of the corresponding part of Theorem 2.1, so we omit the proof of part (ii) of Theorem 2.2.

Let $N^x_{p_n}$ and N_{p_n} be given as in (3.2). Then we can rewrite the integral on the left-hand side of (2.7) as

$$\frac{\operatorname{sgn}(N^x_{p_n} - k_n)}{n} \sum_{i=(k_n \wedge N^x_{p_n})+1}^{k_n \vee N^x_{p_n}} \big(G(X_{i:n}) - G(\xi_{p_n})\big),$$

where $\mathrm{sgn}(x) = x/|x|$, $\mathrm{sgn}(0) = 0$. Let us adopt the following notation: for any integer k and m define a set

$$I_{(k,m)} := \{i : (k \wedge m) + 1 \leqslant i \leqslant k \vee m\}$$

and let

$$\sum_{i \in I_{(k,m)}} (\cdot)_i := \mathrm{sgn}(m-k) \sum_{i=(k\wedge m)+1}^{k \vee m} (\cdot)_i .$$

Then we have to estimate

$$R_n(p_n) = \frac{1}{n} \sum_{i \in I_{(k_n, N^x_{p_n})}} \big(G(X_{i:n}) - G(\xi_{p_n})\big) + \frac{(N^x_{p_n} - p_n n)^2}{2n^2} \frac{g}{f}(\xi_{p_n})$$

(cf. (2.7)), and as in the proof of Theorem 2.1 we can note that $R_n(p_n)$ is distributed as

$$\text{(4.1)}\quad \frac{1}{n} \sum_{i \in I_{(k_n, N_{p_n})}} \big(G \circ F^{-1}(U_{i:n}) - G \circ F^{-1}(p_n)\big) + \frac{(N_{p_n} - p_n n)^2}{2n^2} \frac{g}{f}(\xi_{p_n}) = \frac{g}{f}(\xi_{p_n}) R_{n,1} + R_{n,2},$$

where

$$R_{n,1} = \frac{1}{n} \sum_{i \in I_{(k_n, N_{p_n})}} (U_{i:n} - p_n) + \frac{\big(N_{p_n} - p_n n\big)^2}{2n^2},$$

$$R_{n,2} = \frac{1}{n} \sum_{i \in I_{(k_n, N_{p_n})}} \left[\frac{g}{f} \circ F^{-1}\big(p_n + \theta_i (U_{i:n} - p_n)\big) - \frac{g}{f} \circ F^{-1}(p_n)\right] (U_{i:n} - p_n),$$

$0 < \theta_i < 1$, $i \in I_{(k_n, N_{p_n})}$. As before (cf. the proof of Theorem 2.1) we can assume without loss of generality that $a_2 \leqslant 1/2$. Then we need to prove (2.7) with

$$\text{(4.2)}\qquad \Delta_n = A\, p_n^{3/4} \left(\frac{\log k_n}{n}\right)^{5/4} \frac{|g|}{f}(\xi_{p_n}) + B\, p_n \frac{\log k_n}{n} \Psi_{p_n, g/f}(C).$$

Fix an arbitrary $c > 0$ and prove that

$$\text{(4.3)}\qquad \mathbf{P}\left(|R_{n,1}| > A_1 (p_n)^{3/4} \left(\frac{\log k_n}{n}\right)^{5/4} \right) = O(k_n^{-c}),$$

$$\text{(4.4)}\qquad \mathbf{P}\left(|R_{n,2}| > A_2\, p_n \frac{\log k_n}{n} \Psi_{p_n, g/f}(A_2) \right) = O(k_n^{-c}),$$

where $A_i > 0$, $i = 1, 2, \ldots$, are some constants depending only on c. The relations (4.1), (4.3) and (4.4) imply (2.7) with Δ_n as in (4.2). As in the proof of Theorem 2.1 it is enough to estimate $R_{n,j}$, $j = 1, 2$, on the set

$$E = \{\omega : |N_{p_n} - p_n n| < A_0(p_n\, n \log k_n)^{1/2}\},$$

where $A_0 > 0$ is a constant, depending only on c, such that $\mathbf{P}(\Omega \setminus E) = O(k_n^{-c})$.

First we consider $R_{n,2}$. Note that

$$\max_{i \in I_{(k_n, N_{p_n})}} |U_{i:n} - p_n| = |U_{k_n:n} - p_n| \vee |U_{N_{p_n}:n} - p_n| \vee |U_{N_{p_n}+1:n} - p_n|,$$

$$\mathbf{P}\Big(|U_{k_n:n} - p_n| > A_0\big((p_n \log k_n)/n\big)^{1/2}\Big) = O(k_n^{-c})$$

(cf. the proof of Theorem 2.1), and for $j = N_{p_n:n}, N_{p_n:n} + 1$ simultaneously we have

$$\begin{aligned}\mathbf{P}\Big(|U_{j:n} - p_n| > A_1 \frac{\log k_n}{n}\Big) &\leqslant \mathbf{P}\Big(U_{N_{p_n}+1:n} - U_{N_{p_n}:n} > A_1 \frac{\log k_n}{n}\Big)\\ &= \mathbf{P}\Big(U_{1:n} > A_1 \frac{\log k_n}{n}\Big) = \Big(1 - A_1 \frac{\log k_n}{n}\Big)^n = O(k_n^{-c})\end{aligned}$$

for $A_1 > c$. Since $(\log k_n)/n = o\big((p_n \log k_n)/n\big)^{1/2}$, on the set E we obtain

$$\begin{aligned}|R_{n,2}| &\leqslant \frac{1}{n} \Psi_{p_n, g/f}(A_0) A_0^2 (p_n\, n \log k_n)^{1/2} \Big(\frac{p_n \log k_n}{n}\Big)^{1/2}\\ &= A_2 p_n \frac{\log k_n}{n} \Psi_{p_n, g/f}(A_0)\end{aligned}$$

with probability $1 - O(k_n^{-c})$, and (4.4) is proved.

Finally, consider $R_{n,1}$. Note that conditionally on N_{p_n}, $k_n < N_{p_n}$, the order statistics $U_{i:n}$, $k_n \leqslant i \leqslant N_{p_n}$, are distributed as the order statistics $U'_{i:N_{p_n}}$ from the uniform $(0, p_n)$ distribution (cf. the proof of Theorem 2.1), their conditional expectations are equal to $p_n[i/(N_{p_n} + 1)]$. Then in the case $k_n < N_{p_n}$ (the proof for the case $N_{p_n} \geqslant k_n$ with respect to the interval $(1 - p_n, 1)$ is similar to the proof of Theorem 2.1, and we omit the details) we rewrite $R_{n,1}$ as

$$R_{n,1} = \frac{1}{n} \sum_{i=k_n+1}^{N_{p_n}} \Big(U_{i:n} - p_n \frac{i}{N_{p_n} + 1}\Big) + R'_{n,1}, \tag{4.5}$$

where

$$R'_{n,1} = \frac{1}{n}\sum_{i=k_n+1}^{N_{p_n}} p_n\left(\frac{i}{N_{p_n}+1}-1\right) + \frac{(N_{p_n}-p_n n)^2}{2n^2}$$
$$= -\frac{k_n}{n^2}\frac{(N_{p_n}-k_n)(N_{p_n}-k_n-1)}{2(N_{p_n}+1)} + \frac{(N_{p_n}-k_n)^2}{2n^2}$$
$$= \frac{(N_{p_n}-k_n)^2(N_{p_n}+1-k_n)}{2(N_{p_n}+1)n^2} - \frac{k_n(N_{p_n}-k_n)}{2(N_{p_n}+1)n^2},$$

and on the set E the latter quantity is of the order

$$O\left(\frac{k_n^{1/2}(\log k_n)^{3/2}}{n^2}\right) = o\left((p_n)^{3/4}\left(\frac{\log k_n}{n}\right)^{5/4}\right),$$

i.e. $R'_{n,1}$ is of negligible order (cf. (4.3)) for our purposes.

It remains to evaluate the dominant first term on the right-hand side in (4.5). Fix an arbitrary $c_1 > c + 1/2$, and note that conditionally on N_{p_n} the variance of $U_{i:n}$ $(k_n + 1 \leqslant i \leqslant N_{p_n})$ is equal to

$$V_i^2 = (p_n)^2 \frac{1}{N_{p_n}+2}\frac{i}{N_{p_n}+1}\left(1-\frac{i}{N_{p_n}+1}\right),$$

and on the set E this quantity is less than

$$(p_n)^2\frac{A_0 k_n^{1/2}(\log k_n)^{1/2}}{N_{p_n}^2},$$

and

$$V_i \leqslant p_n A_0^{1/2} k_n^{1/4}(\log k_n)^{1/4}/N_{p_n} \leqslant A_0^{1/2} p_n k_n^{-3/4}(\log k_n)^{1/4}$$
$$\leqslant A_0^{1/2}(p_n)^{1/4} n^{-3/4}(\log k_n)^{1/4}.$$

Using Hoeffding's inequality (as in the proof of Theorem 2.1), we find that

$$\mathbf{P}\left(\left|U_{i:n} - p_n\frac{i}{N_{p_n}+1}\right| > A_1(p_n)^{1/4}\left(\frac{\log k_n}{n}\right)^{3/4}\Big|N_{p_n} : k_n \leqslant N_{p_n}\right) = O(k_n^{-c}),$$

where A_1 depends only on c_1 (in fact, this is true for every A_1 such that $A_1^2 > 2A_0c_1$). Thus

(4.6)
$$\mathbf{P}\left(\frac{1}{n}\left|\sum_{i=k_n}^{N_{p_n}}\left(U_{i:n} - p_n\frac{i}{N_{p_n}+1}\right)\right| > A_0A_1(p_n)^{3/4}\left(\frac{\log k_n}{n}\right)^{5/4}\Big|N_{p_n} : k_n \leqslant N_{p_n}\right)$$
$$\leqslant A_0(k_n \log k_n)^{1/2}\, O(k_n^{-c_1}) = O(k_n^{-c}).$$

Combining (4.5) and (4.6) and using similar estimates for the case $N_{p_n} < k_n$, we arrive at (4.3). Thus the theorem is proved. ■

5. PROOF OF COROLLARY 2.3

Suppose that $p_n \to 0$ as $n \to \infty$; the case where $p_n \to 1$ as $n \to \infty$ can be treated in a similar fashion and is therefore omitted. We will establish the relations:

$$\Psi_{p_n,g/f}(C) = O\left(\left(\frac{\log k_n}{k_n}\right)^{1/4}\frac{|g|}{f}(\xi_{p_n})\right),$$
$$\widehat{\Psi}_{p_n,g/f}(C) = O\left(\left(\frac{\log n}{k_n}\right)^{1/4}\frac{|g|}{f}(\xi_{p_n})\right). \tag{5.1}$$

Let $\log(\cdot)$ denote $\log k_n$ when we prove the first relation in (5.1), and $\log n$ when we prove the second one. Since we will need only the relation $\log(\cdot)/k_n \to 0$, which is evident in the first case and is valid by $[A_2]$ in the second case, this simple observation will allow us to prove both the assertions in (5.1) at the same time.

Define $x_n = F^{-1}(p_n)$ which approaches $-\infty$ as $n \to \infty$. Fix $C > 0$ and for a fixed t such that $|t| \leqslant C$, put

$$\triangle x_n = F^{-1}\left(p_n + t\sqrt{p_n\frac{\log(\cdot)}{n}}\right) - x_n = F^{-1}\left(p_n\left(1+t\sqrt{\frac{\log(\cdot)}{k_n}}\right)\right) - x_n.$$

First we prove that $(\triangle x_n)/x_n \to 0$ as $n \to \infty$. Due to the smoothness condition $[A_1]$, for all sufficiently large n we may write

$$\begin{aligned}\frac{\triangle x_n}{x_n} &= \frac{1}{x_n f\left(F^{-1}\left(p_n\left(1+\theta t\sqrt{k_n^{-1}\log(\cdot)}\right)\right)\right)}\,t\sqrt{p_n\frac{\log(\cdot)}{n}}\\ &= \frac{1}{x_n f(x_n)}\,\frac{f(F^{-1}(p_n))}{f\left(F^{-1}\left(p_n\left(1+\theta t\sqrt{k_n^{-1}\log(\cdot)}\right)\right)\right)}\,t\sqrt{p_n\frac{\log(\cdot)}{n}},\end{aligned}$$

where $0 < \theta < 1$, and since due to the condition of regular variation we have

$$f(x_n)x_n \sim -\gamma F(x_n) = -\gamma p_n \qquad \text{as } x_n \to -\infty$$

(cf., e.g., Bingham et al. [3]), the latter quantity is equivalent to

$$\begin{aligned}&-\frac{1}{\gamma p_n}\,\frac{f(F^{-1}(p_n))}{f\left(F^{-1}\left(p_n(1+o(1))\right)\right)}\,t\sqrt{p_n\frac{\log(\cdot)}{n}}\\ &\qquad\qquad = -\frac{1}{\gamma}\,\frac{f(F^{-1}(p_n))}{f\left(F^{-1}\left(p_n(1+o(1))\right)\right)}\,t\sqrt{\frac{\log(\cdot)}{k_n}}.\end{aligned}$$

It remains to show that

$$\frac{f(F^{-1}(p_n))}{f\left(F^{-1}\left(p_n(1+o(1))\right)\right)} = 1 + o(1).$$

Since $f \in SRV^{-\infty}_{-(1+\gamma)}$, for all $x < x_0 < 0$ we have $f(x) = |x|^{-(1+\gamma)}L(x)$, where $L(x)$ is a positive function slowly varying at $-\infty$. Moreover, the inverse function $F^{-1}(u)$ is regularly varying at zero, i.e. $F^{-1}(u) = u^{-1/\gamma}L_1(u)$, where $L_1(u)$ is a corresponding function slowly varying at zero. So, for sufficiently large n, we have

$$\begin{aligned}
&\frac{f(F^{-1}(p_n))}{f(F^{-1}(p_n(1+o(1))))} \\
&= \frac{[p_n^{-1/\gamma}L_1(p_n)]^{-(1+\gamma)}L(F^{-1}(p_n))}{\big[(p_n(1+o(1)))^{-1/\gamma}L_1(p_n(1+o(1)))\big]^{-(1+\gamma)}L(F^{-1}(p_n(1+o(1))))} \\
&\sim \frac{L(F^{-1}(p_n))}{L(F^{-1}(p_n(1+o(1))))} = \frac{L[p_n^{-1/\gamma}L_1(p_n)]}{L\big[(p_n(1+o(1)))^{-1/\gamma}L_1(p_n(1+o(1)))\big]} \sim 1.
\end{aligned}$$

Thus, $|(\Delta x_n)/x_n| = O\big(\sqrt{k_n^{-1}\log(\cdot)}\big)$.

Finally, we obtain a bound for $|(g/f)(x_n + \Delta x_n) - (g/f)(x_n)|$ for an arbitrary fixed $C > 0$ and $|t| \leqslant C$, as $n \to \infty$. Due to the relation (2.11) which holds true for the density f as well as for the function g we have

$$\begin{aligned}
&\left|\frac{g}{f}(x_n + \Delta x_n) - \frac{g}{f}(x_n)\right| \\
&= \left|\frac{f(x_n)[g(x_n+\Delta x_n) - g(x_n)] - g(x_n)[f(x_n+\Delta x_n) - f(x_n)]}{f(x_n+\Delta x_n)f(x_n)}\right| \\
&= O\left(\frac{|g|}{f}(x_n)\frac{f(x_n)}{f(x_n+\Delta x_n)}\left|\frac{\Delta x_n}{x_n}\right|^{1/2}\right) \\
&= O\left(\frac{|g|}{f}(x_n)\frac{f(x_n)}{f(x_n) + [f(x_n+\Delta x_n) - f(x_n)]}\left|\frac{\Delta x_n}{x_n}\right|^{1/2}\right) \\
&= O\left(\frac{|g|}{f}(x_n)\frac{1}{1+O(|(\Delta x_n)/x_n|^{1/2})}\left|\frac{\Delta x_n}{x_n}\right|^{1/2}\right) = O\left(\frac{|g|}{f}(x_n)\left|\frac{\Delta x_n}{x_n}\right|^{1/2}\right) \\
&= O\left(\frac{|g|}{f}(x_n)\left(\frac{\log(\cdot)}{k_n}\right)^{1/4}\right) \quad \text{as } n \to \infty.
\end{aligned}$$

The latter bound yields (5.1). Thus the corollary is proved. ■

6. APPENDIX

As before, let $N_p = \sharp\{i : X_i \leqslant \xi_p,\ i = 1, \dots, n\}$, where $0 < p < 1$ is fixed. We prove that conditionally on N_p the order statistics $X_{1:n}, \dots, X_{N_p:n}$ are distributed as the order statistics corresponding to a sample of N_p i.i.d. r.v.'s with distribution function $F(x)/p$, $x \leqslant \xi_p$. Let $U_1, \dots, U_n$ denote a sample of independent uniformly $(0,1)$ distributed r.v.'s, and $U_{1:n} \leqslant \ldots \leqslant U_{n:n}$ be the corresponding order statistics. Put $N_{p,u} = \sharp\{i \,:\, U_i \leqslant p,\ i = 1, \ldots, n\}$. Since the joint distribution of the pair $X_{i:n}$, N_p is the same as the joint distribution of $F^{-1}(U_{i:n})$, $N_{p,u}$, it suffices to prove the assertion for the uniform distribution.

LEMMA 6.1. *Conditionally given $N_{p,u}$, the order statistics $U_{1:n}, \ldots, U_{N_{p,u}:n}$ are distributed as the order statistics corresponding to a sample of $N_{p,u}$ independent $(0,p)$-uniformly distributed r.v.'s.*

Proof. (a) First consider the case where $N_{p,u} = n$. Take arbitrary $0 < u_1 \leqslant \ldots \leqslant u_n < p$ and note that

$$\mathbf{P}(U_{1:n} \leqslant u_1, \ldots, U_{N_{p,u}:n} \leqslant u_n \mid N_{p,u} = n) = \frac{\mathbf{P}(U_{1:n} \leqslant u_1, \ldots, U_{n:n} \leqslant u_n)}{p^n}$$
$$= \frac{n!}{p^n} \int_0^{u_1} \int_{u_1}^{u_2} \ldots \int_{u_{n-1}}^{u_n} dx_1\, dx_2 \ldots dx_n,$$

where the latter expression is nothing but the joint *df* of the order statistics corresponding to a sample of n independent $(0,p)$-uniformly distributed r.v.'s.

(b) Next consider the case where $N_{p,u} = k < n$. Let $F_{i,n}(u) = \mathbf{P}(U_{i:n} \leqslant u)$ be a *df* of i-th order statistic, and put

$$P_n(k) = \mathbf{P}(N_{p,u} = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Then we have

$$\text{(6.1)}\quad \mathbf{P}(U_{1:n} \leqslant u_1, \ldots, U_{N_{p,u}:n} \leqslant u_k \mid N_{p,u} = k)$$
$$= \frac{\mathbf{P}(U_{1:n} \leqslant u_1, \ldots, U_{k:n} \leqslant u_k, U_{k+1:n} > p)}{P_n(k)}.$$

The probability in the numerator on the right-hand side of (6.1) is equal to

$$\int_p^1 \mathbf{P}(U_{1:n} \leqslant u_1, \ldots, U_{k:n} \leqslant u_k | U_{k+1:n} = v)\, dF_{k+1,n}(v),$$

and by the Markov property of order statistics the latter quantity equals

$$\int_p^1 \left(\frac{k!}{v^k} \int_0^{u_1} \int_{u_1}^{u_2} \ldots \int_{u_{k-1}}^{u_k} dx_1\, dx_2 \ldots dx_k \right) dF_{k+1,n}(v)$$
$$= \frac{k!}{p^k} \left(\int_0^{u_1} \int_{u_1}^{u_2} \ldots \int_{u_{k-1}}^{u_k} dx_1\, dx_2 \ldots dx_k \right) p^k \int_p^1 \frac{1}{v^k}\, dF_{k+1,n}(v).$$

Since

$$p^k \int_p^1 \frac{1}{v^k}\, dF_{k+1,n}(v) = p^k \int_p^1 \frac{(1-v)^{n-k-1}}{B(k+1,n-k)}\, dv = \binom{n}{k} p^k (1-p)^{n-k} = P_n(k),$$

where $B(k+1, n-k) = k!(n-k-1)!/n!$, we see that the conditional probability in (6.1) is equal to

$$\frac{k!}{p^k} \int_0^{u_1} \int_{u_1}^{u_2} \dots \int_{u_{k-1}}^{u_k} dx_1\, dx_2 \dots dx_k,$$

which obviously corresponds to the $(0, p)$-uniform distribution. Thus the lemma is proved. ∎

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St. Petersburg State University
Mathematics and Mechanics Faculty
198504, St. Petersburg, Stary Peterhof
Universitetsky pr. 28
Russia
E-mail: nv.gribkova@gmail.com

Centre for Mathematics and Computer Science
P.O. Box 94079, 1090 GB
Amsterdam, The Netherlands
E-mail: R.Helmers@cwi.nl

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