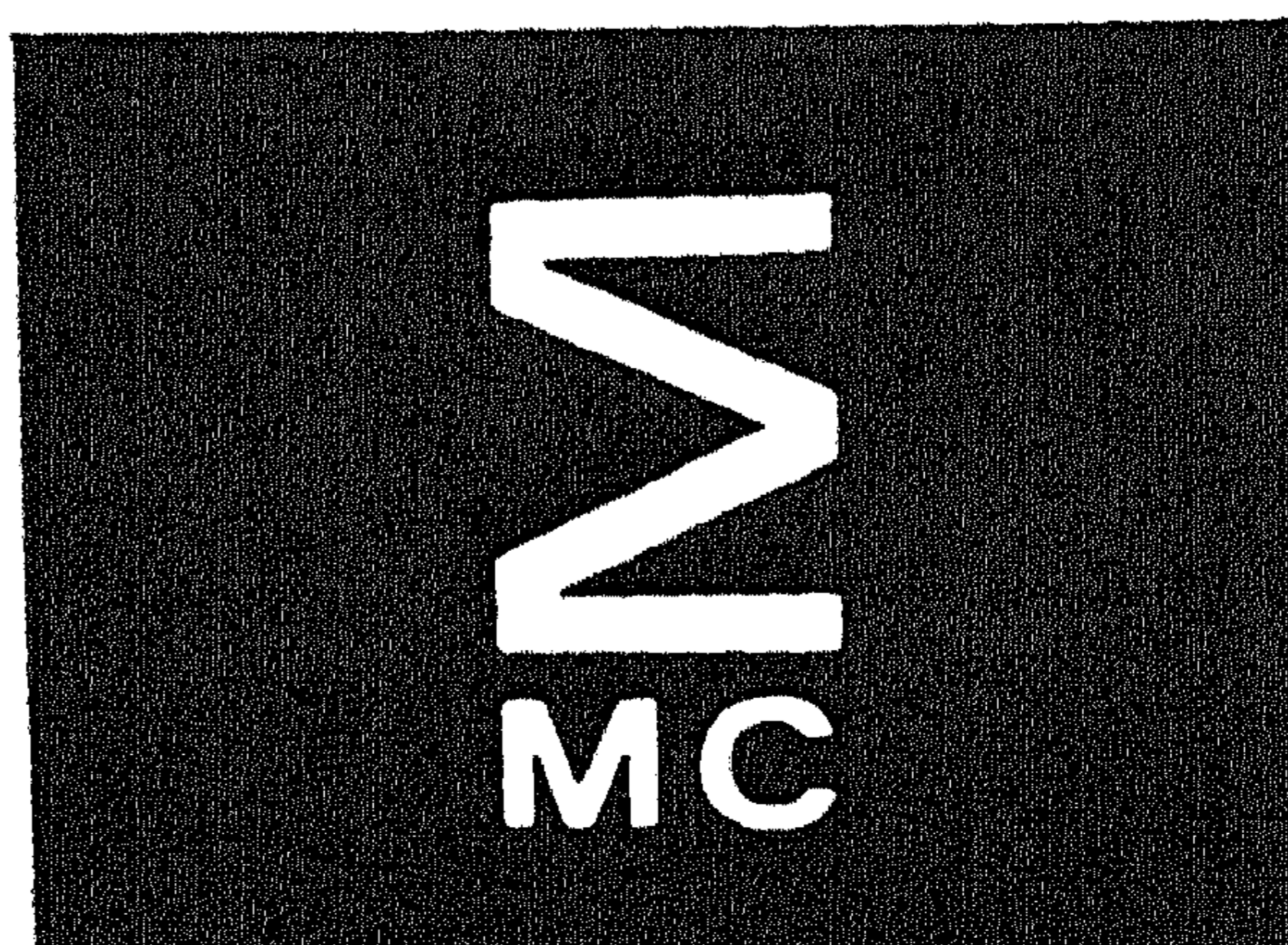
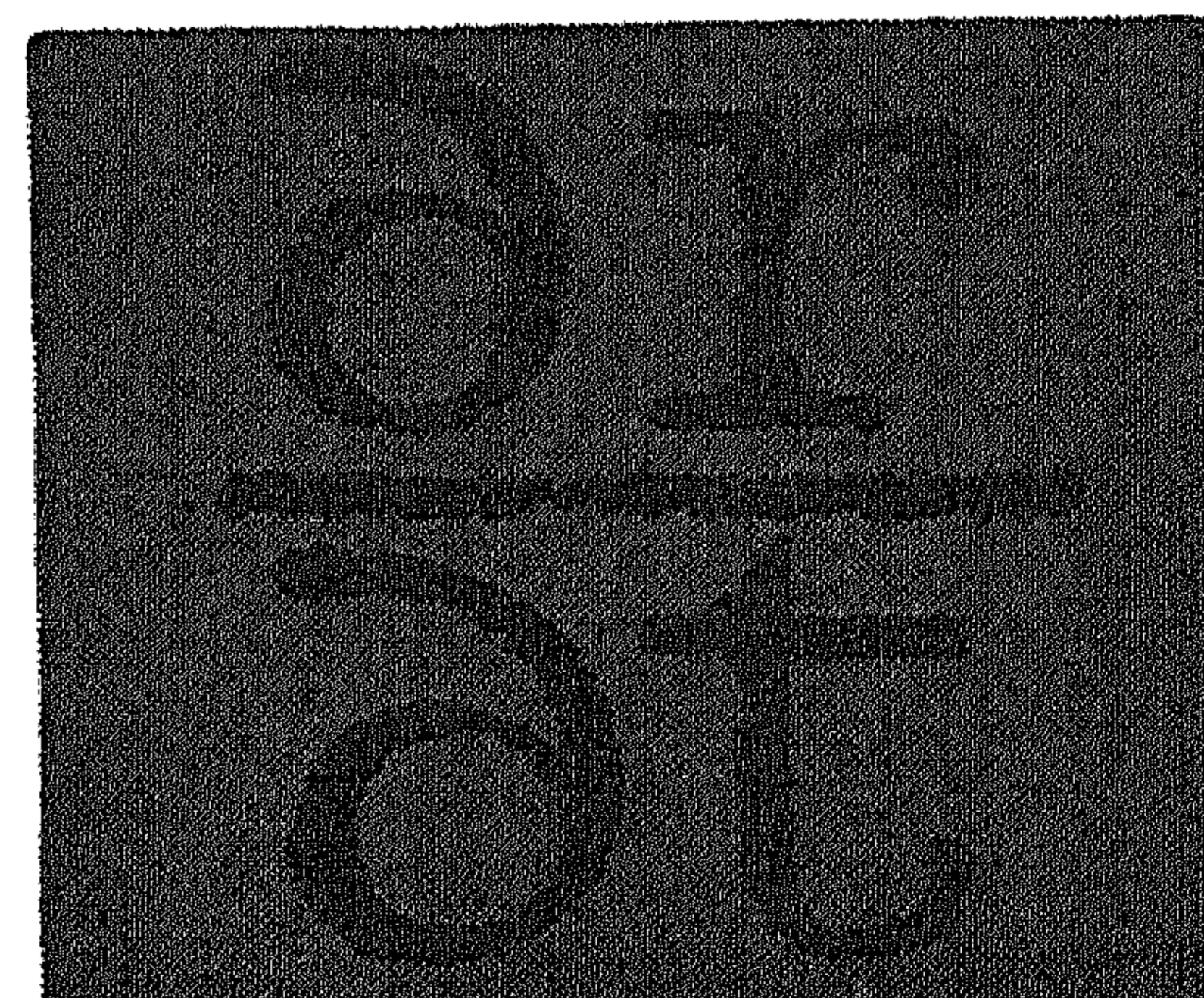
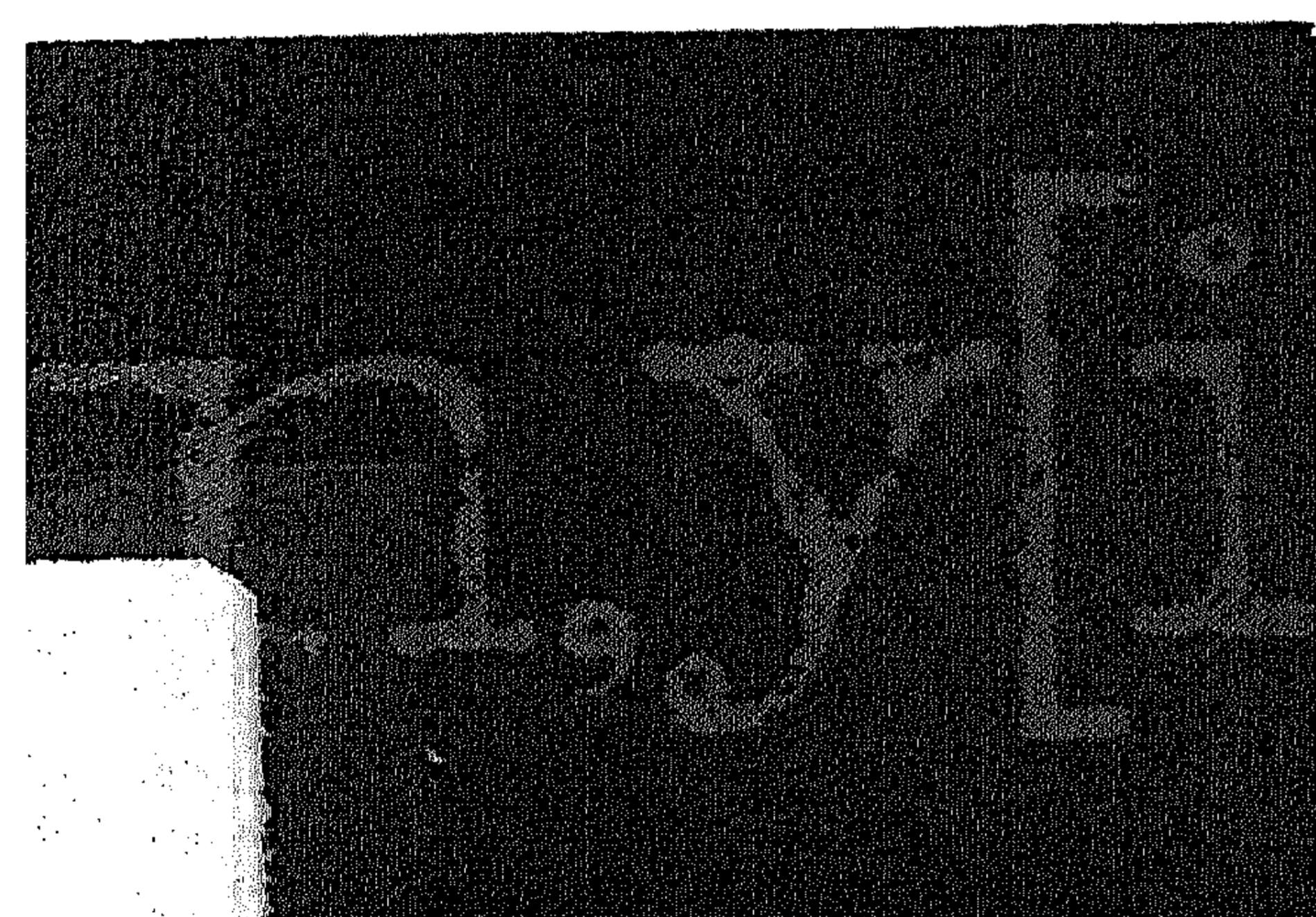
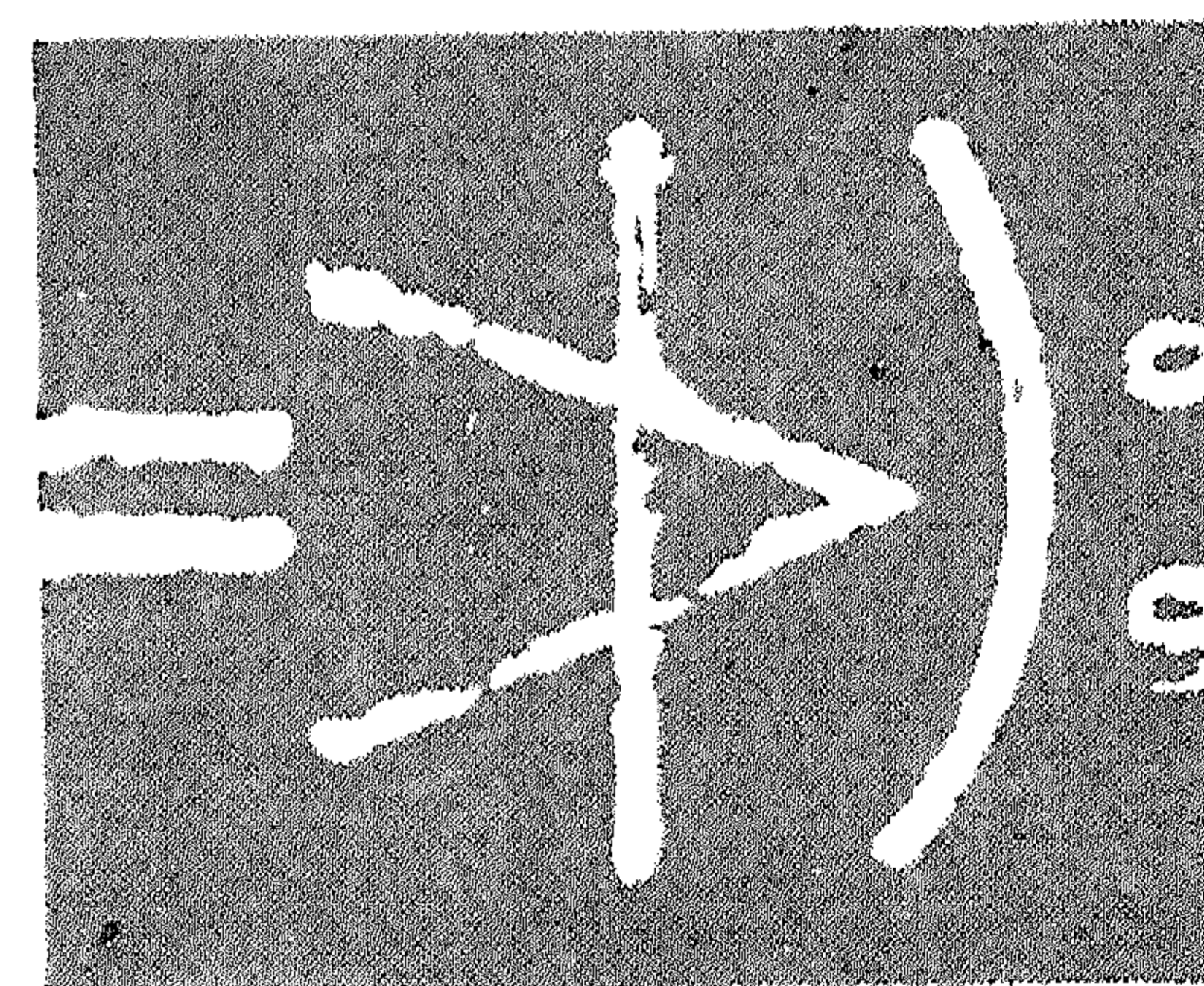
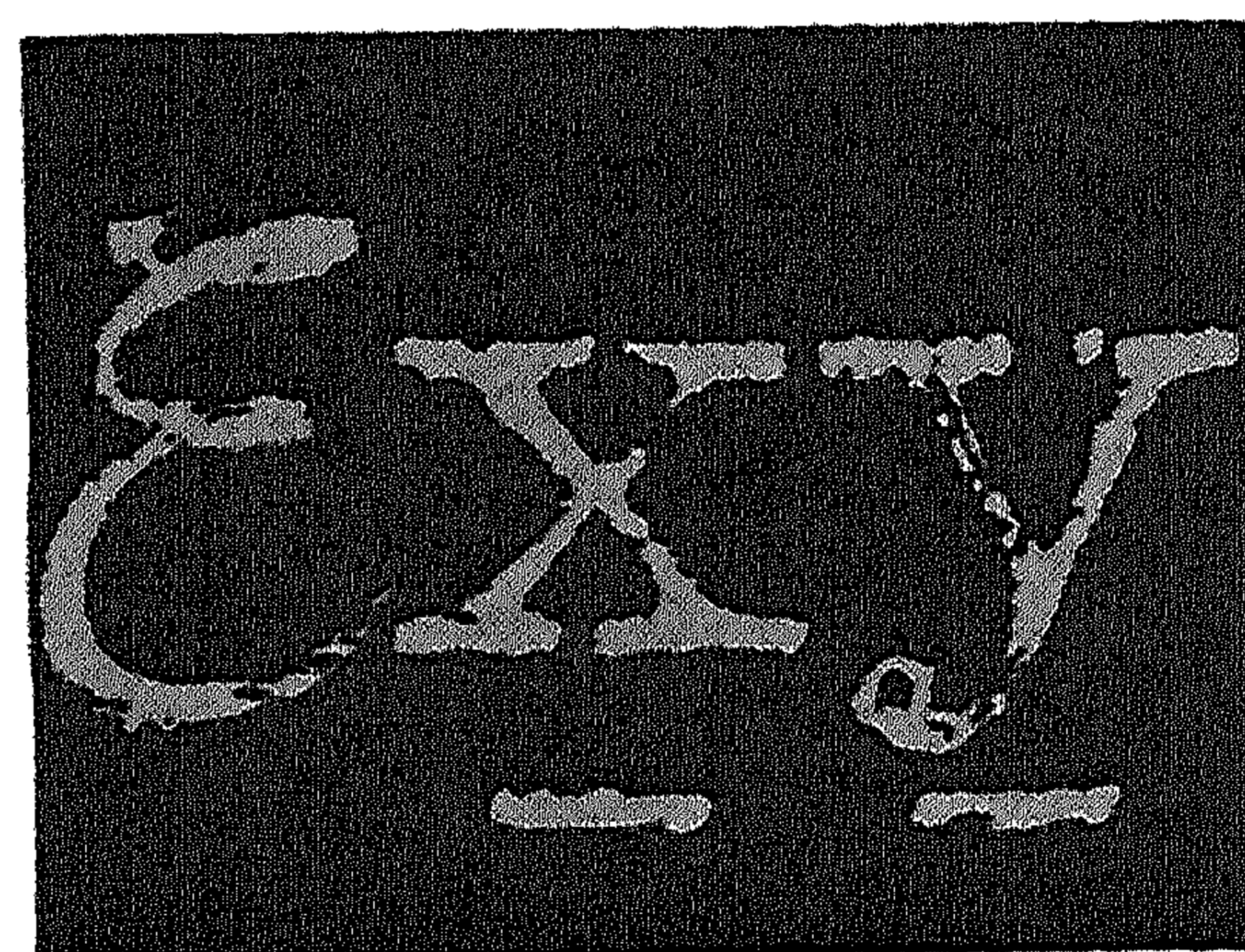
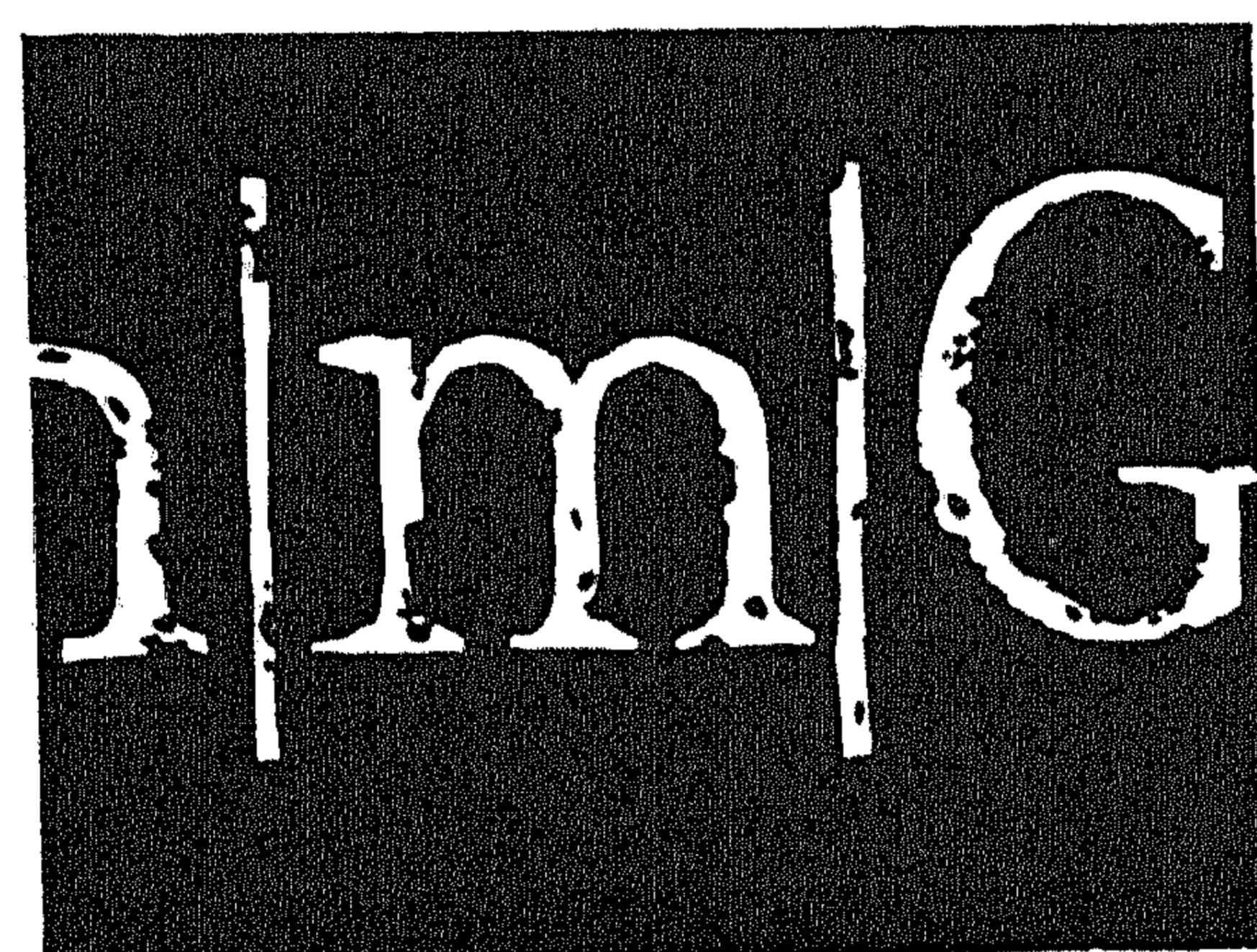
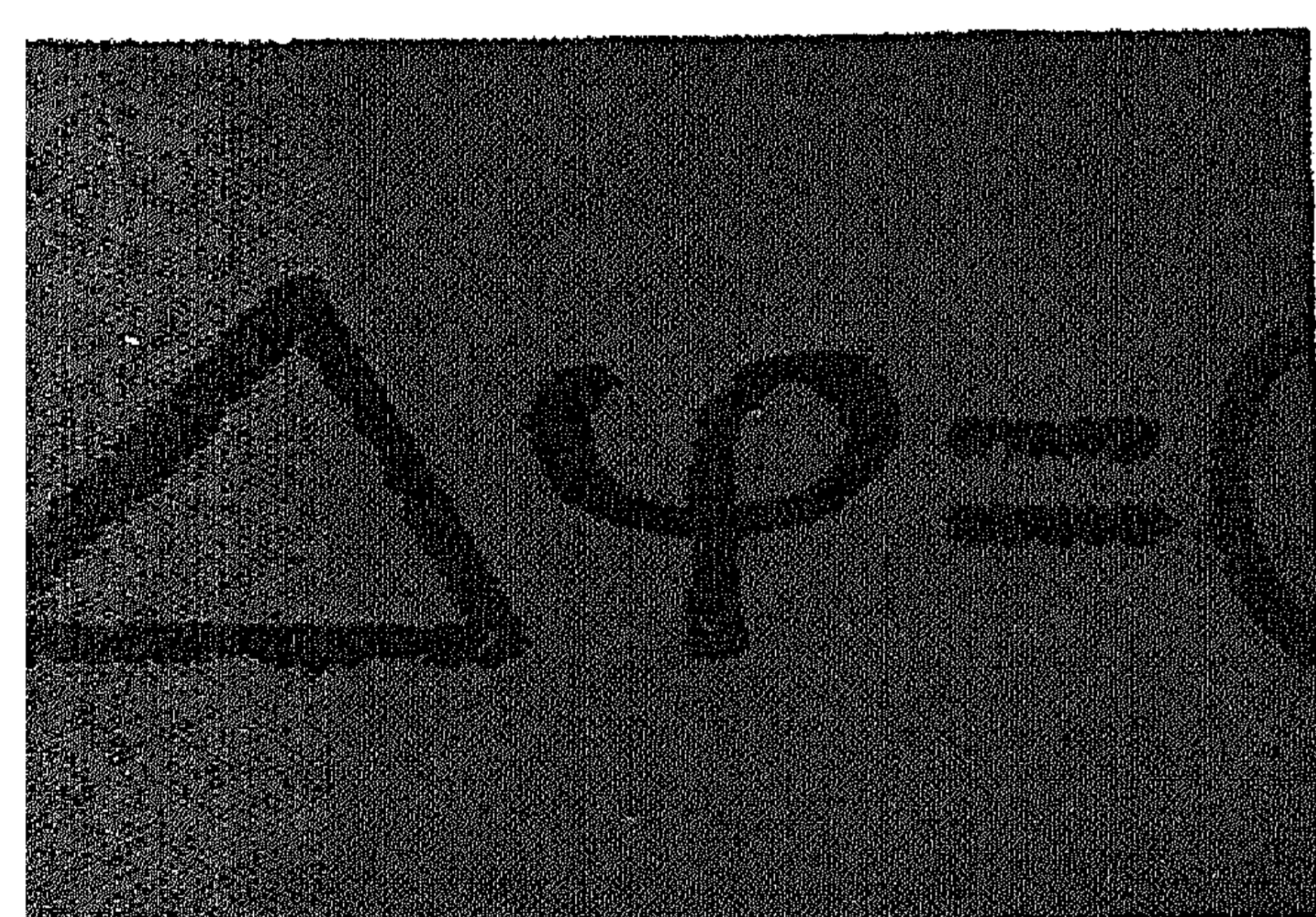
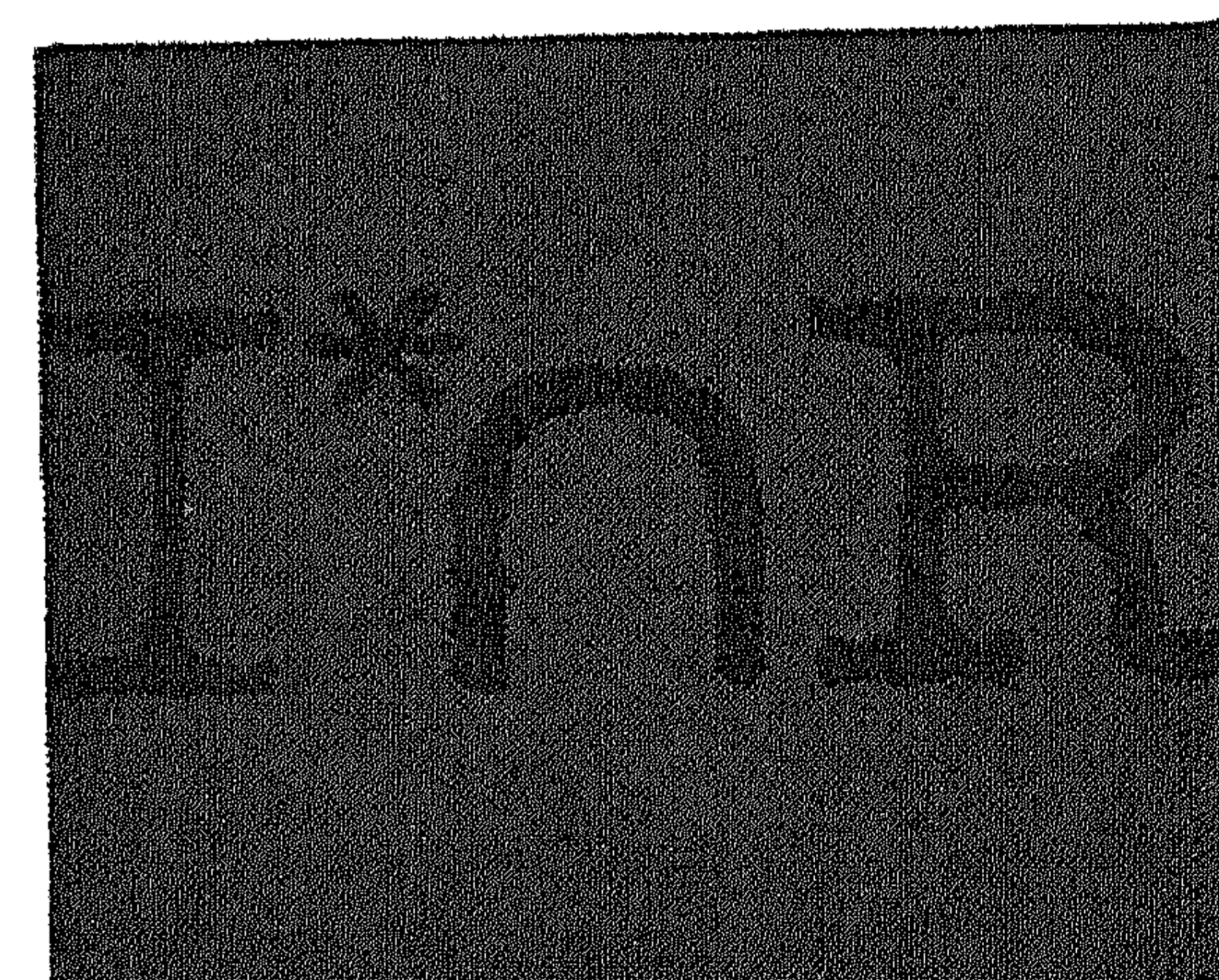
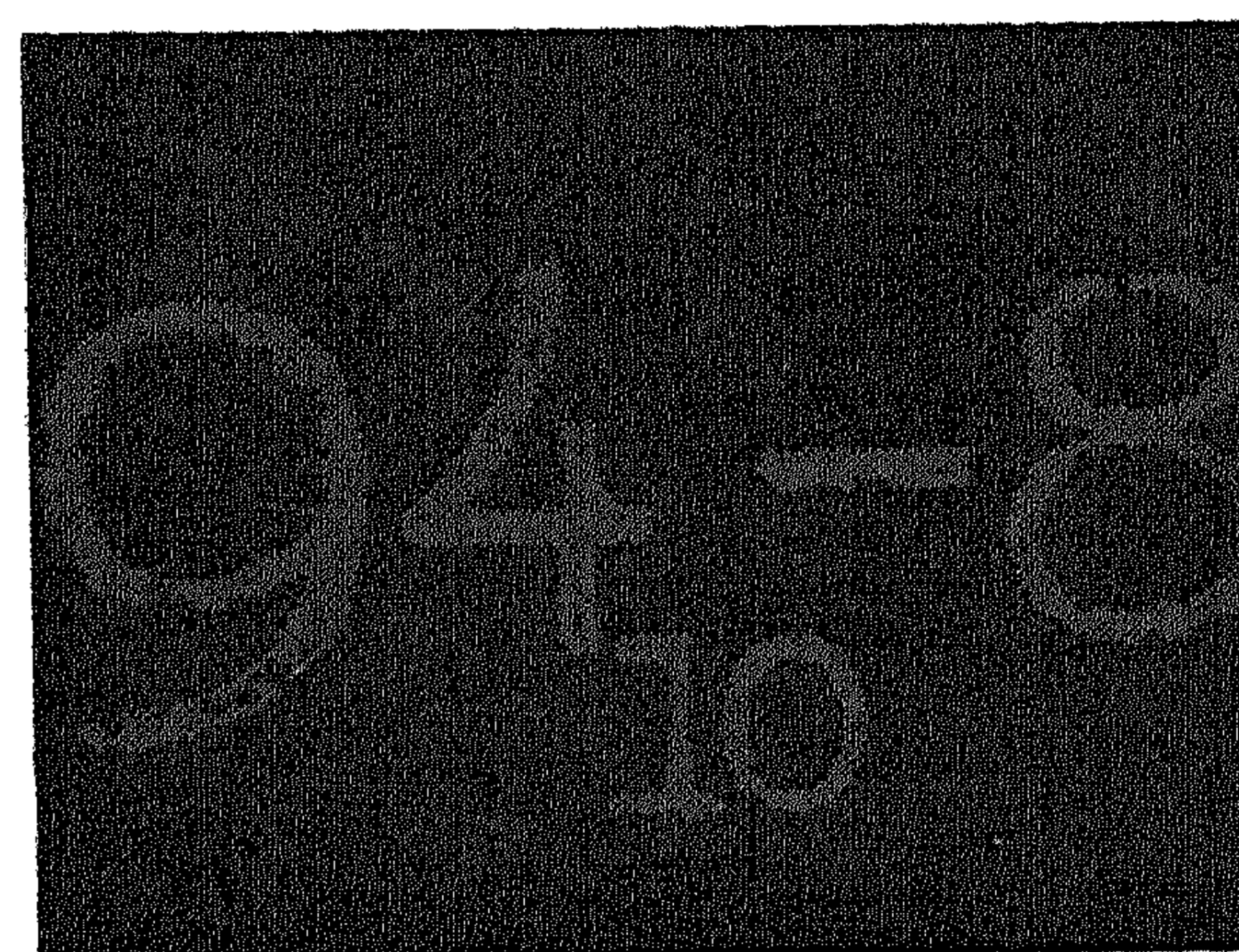
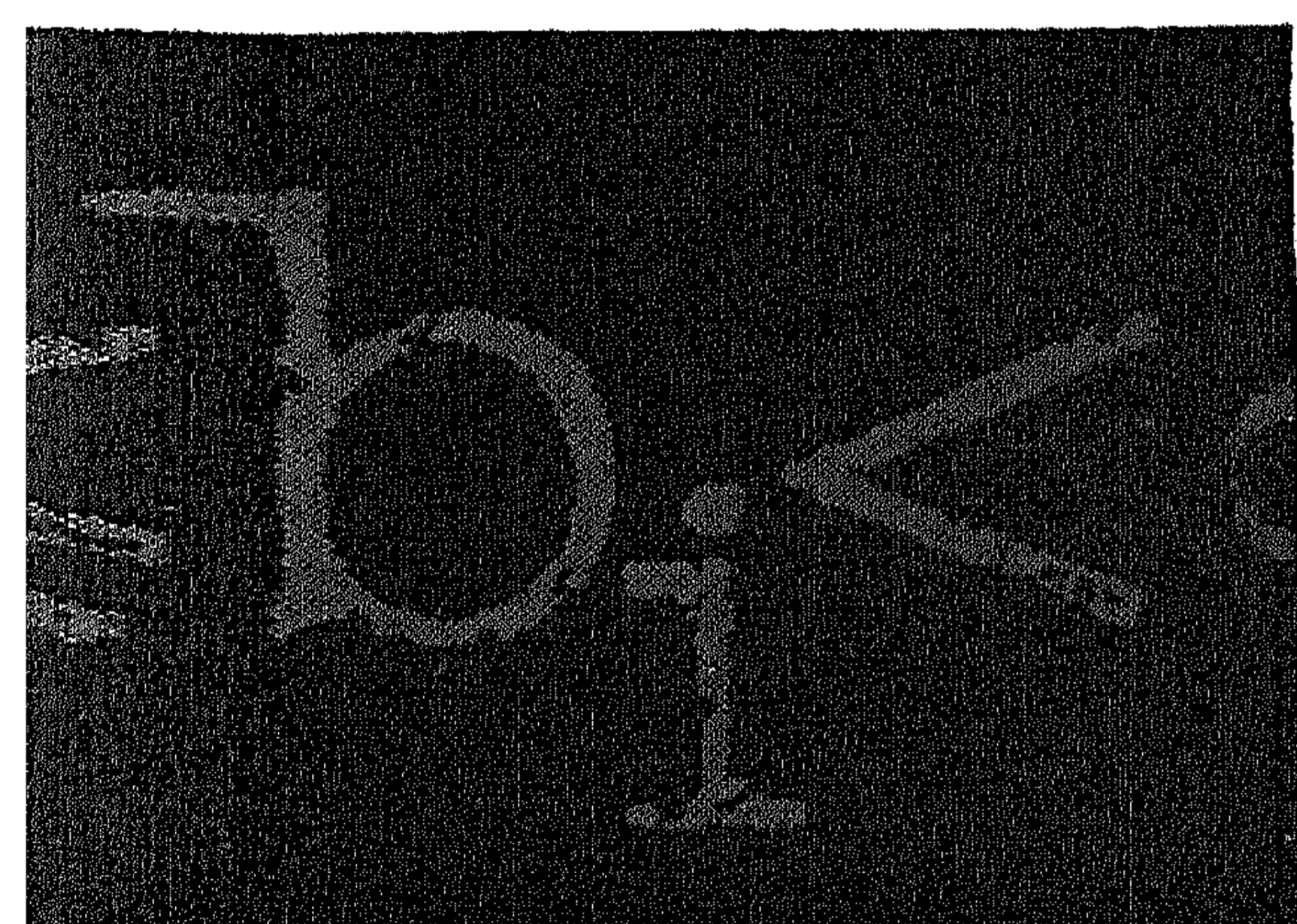


COMBINATORICS

Part 2: Graph theory; foundations, partitions and combinatorial geometry

M. HALL, Jr. (ed.)

J.H. VAN LINT (ed.)



MATHEMATICAL CENTRE TRACTS 56

M. HALL, Jr. (ed.)

J.H. VAN LINT (ed.)

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**Part 2: Graph theory; foundations, partitions and
combinatorial geometry**

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PREFACE

Combinatorics has come of age. It had its beginnings in a number of puzzles which have still not lost their charm. Among these are EULER's problem of the 36 officers and the KÖNIGSBERG bridge problem, BACHET's problem of the weights, and the Reverend T.P. KIRKMAN's problem of the schoolgirls. Many of the topics treated in ROUSE BALL's *Recreational Mathematics* belong to combinatorial theory.

All of this has now changed. The solution of the puzzles has led to a large and sophisticated theory with many complex ramifications. And it seems probable that the four color problem will only be solved in terms of as yet undiscovered deep results in graph theory. Combinatorics and the theory of numbers have much in common. In both theories there are many problems which are easy to state in terms understandable by the layman, but whose solution depends on complicated and abstruse methods. And there are now interconnections between these theories in terms of which each enriches the other.

Combinatorics includes a diversity of topics which do however have interrelations in superficially unexpected ways. The instructional lectures included in these proceedings have been divided into six major areas: 1. *Theory of designs*; 2. *Graph theory*; 3. *Combinatorial group theory*; 4. *Finite geometry*; 5. *Foundations, partitions and combinatorial geometry*; 6. *Coding theory*. They are designed to give an overview of the classical foundations of the subjects treated and also some indication of the present frontiers of research.

Without the generous support of the North Atlantic Treaty Organization, this *Advanced Study Institute on Combinatorics* would not have been possible, and we thank them sincerely. Thanks are also due to the National Science Foundation for the support of some advanced students, in addition to the support of those with their own NSF grants. The IBM Corporation has kindly given us financial support to supplement the NATO grant. The Xerox Corporation has helped with donations of material and equipment.

Finally we must acknowledge the extensive activities of the Mathematical Centre of Amsterdam in making all the arrangements necessary for holding this conference and preparing these proceedings.

M. HALL, Jr.

J.H. VAN LINT

GRAPH THEORY

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ISOMORPHISM PROBLEMS FOR HYPERGRAPHS

C. BERGE

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1. INTRODUCTION

A *hypergraph* $H = (X, E) = (E_1, E_2, \dots, E_m) = (E_i : i \in M)$ is a family E of subsets E_i of a set $X = \{x_j : j \in N\}$ of *vertices*. The sets E_i are called *edges*.

The *rank* $r(H)$ of a hypergraph H is the maximum cardinality of the edges. If all edges have the same cardinality, the hypergraph is said to be *uniform*. The *subhypergraph* induced by a subset A of X is the hypergraph $H_A = (E_i \cap A : i \in M, E_i \cap A \neq \emptyset)$.

If $I \subseteq M$, the *partial hypergraph* generated by I is the hypergraph $(E_i : i \in I)$. The *section hypergraph* is the partial hypergraph $H \times A = (E_i : i \in M, E_i \subseteq A \subseteq X)$.

The *dual* H^* of H is a hypergraph with vertex set $E = \{e_1, \dots, e_m\}$, and having edges which are certain subsets of E , namely edges X_j where $X_j = \{e_i : i \in M, x_j \in E_i\}$.

Consider two hypergraphs $H = (E_1, \dots, E_m)$ and $H' = (F_1, \dots, F_m)$. H is *equivalent* to H' ($H \equiv H'$) if the mapping $\phi: X \rightarrow Y$, $\phi(x_i) = y_i$, satisfies $\phi(E_i) = F_{\pi i}$ ($i \in M$) for some permutation π of M .

H is *equal* to H' (or $H = H'$) if the permutation π in the above definition can be the identity.

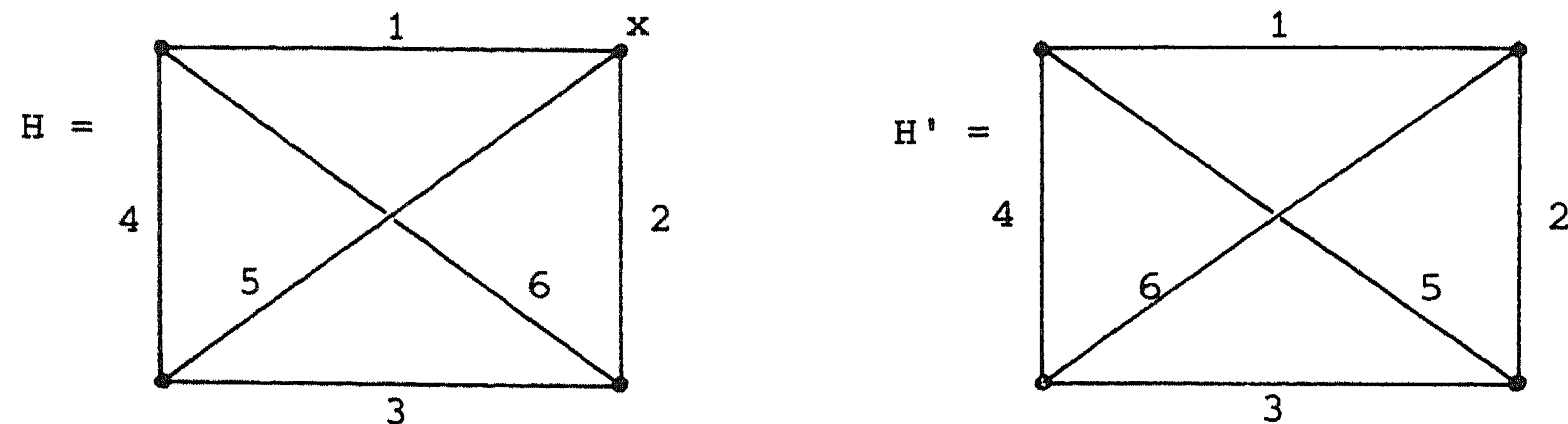
H is *isomorphic* to H' (or $H \cong H'$) if there is a bijection $\phi: X \rightarrow Y$ and if there is a permutation π of M such that $\phi(E_i) = F_{\pi i}$ ($i \in M$). The bijection ϕ is called an *isomorphism*.

H is *strongly isomorphic* to H' (or $H \cong H'$) if there is a bijection $\phi: X \rightarrow Y$ for which $\phi(E_i) = F_i$ for all $i \in M$.

Observe that equality implies the other three relations and any of the relations imply isomorphism.

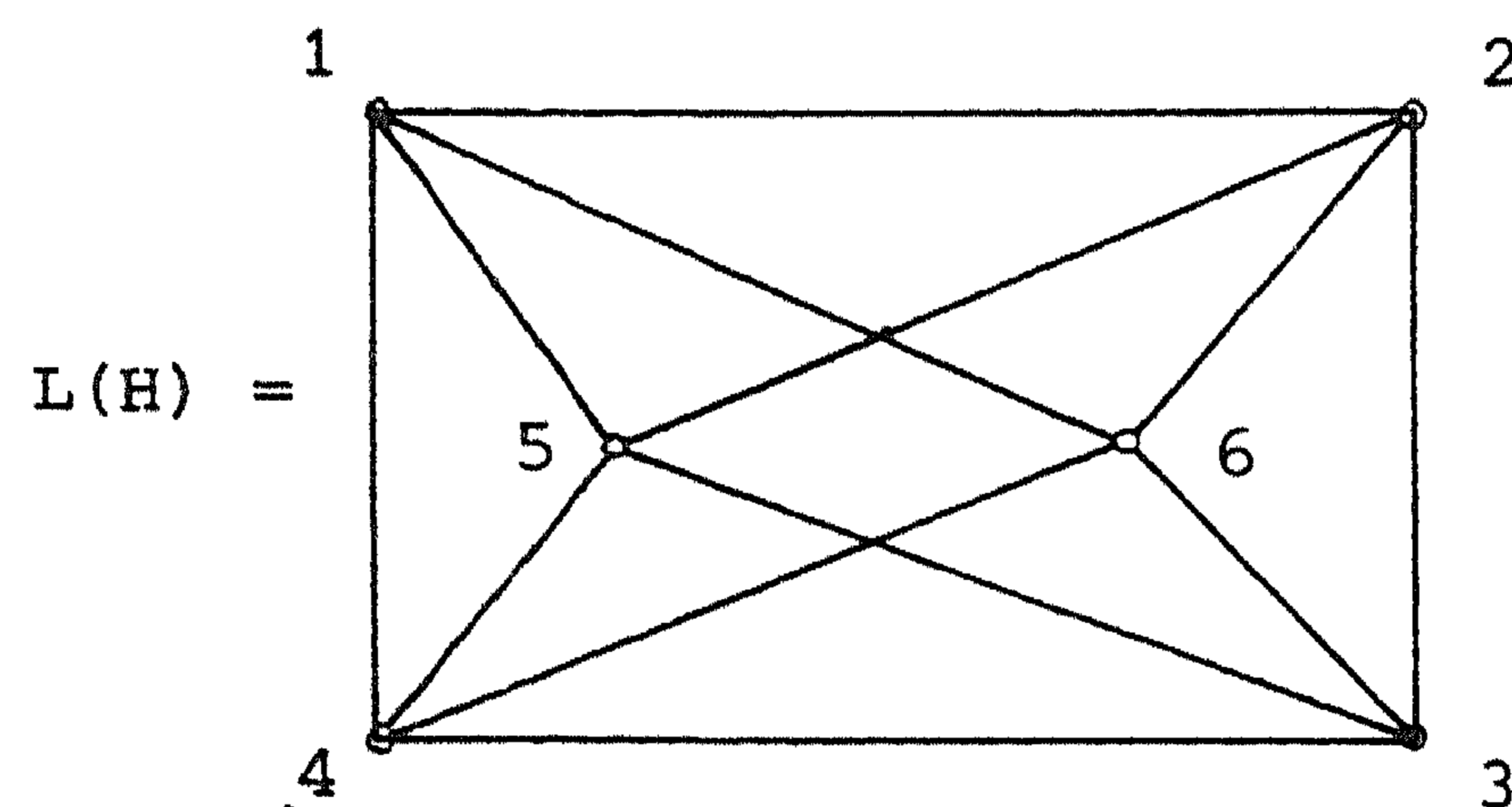
We give some examples:

EXAMPLE. Consider the following:



Observe that $H \simeq H'$, but $H \not\cong H'$, since if $H \cong H'$, the vertex x would map to the non-existent vertex meeting edges 1, 2 and 5 in H' .

EXAMPLE. Consider the line graph $L(H)$ of the graph H above:



Observe that $L(H) = L(H')$, but since the edges are unlabeled here, equality is meaningless.

Our purpose in this paper is to present some general results concerning isomorphisms and other relations among hypergraphs.

A *multigraph* is a hypergraph with $|E_i| \leq 2$ for all $i \in M$.

PROPOSITION 1. If $H = (X, (E_i)_{i \in M})$ and $H' = (Y, (F_i)_{i \in M})$ are multigraphs, and if $\phi: X \rightarrow Y$ is a bijection, then the following are equivalent:

- (i) ϕ is an isomorphism;
- (ii) $m_H(x, y) = m_{H'}(\phi(x), \phi(y))$ for all $x, y \in X$, where $m_H(x, y)$ is the number of edges joining x and y in H .

PROPOSITION 2. If H is a hypergraph, then $(H^*)^* = H$.

PROPOSITION 3. If H, H' are hypergraphs, then $H \simeq H'$ if and only if $H^* \simeq H'^*$.

PROPOSITION 4. If H, H' are hypergraphs, then $H \equiv H'$ if and only if $H^* \equiv H'^*$.

PROPOSITION 5. The dual of the partial hypergraph of H generated by $(E_i : i \in I)$ equals the subhypergraph of H^* induced by $\{e_i : i \in I\}$.

PROPOSITION 5*. The dual of the subhypergraph $H_A = (E_i \cap A : i \in M, E_i \cap A \neq \emptyset)$ with $A = \{x_j : j \in J\}$ equals the partial hypergraph of H^* generated by $(x_j : j \in J)$.

PROPOSITION 6. The dual of the section hypergraph $H \times A, A = \{x_j : j \in J\}$, equals the subhypergraph of H^* induced by $\bigcup_{j \in N} x_j - \bigcup_{j \in N-J} x_j$.

2. TRANSITIVE HYPERGRAPHS

Let $H = (X, E)$ be a hypergraph. Two vertices x and y of H are *symmetric* if there exists an automorphism ϕ of H such that $\phi(x) = y$. Two edges E_i and E_j are *symmetric* if there exists an automorphism ϕ of H such that $\phi(E_i) = E_j$.

H is said to be *vertex-transitive* (resp. *edge-transitive*) if any two vertices (resp. edges) are symmetric. A hypergraph that is both vertex-transitive and edge-symmetric is said to be *transitive*. Because of the duality principle for hypergraphs, the study of vertex-transitive hypergraphs reduces to the study of edge-transitive hypergraphs.

The following result is a generalization of a theorem for graphs due to E. DAUBER [3].

THEOREM 1. For an edge-transitive hypergraph $H = (X, E)$, there exists a partition (X_1, X_2, \dots, X_k) of X such that

- (i) $\sum_{\lambda} r(X_{\lambda}) = r(X)$, where $r(A)$ denotes the rank of H_A ,
- (ii) $H_{X_{\lambda}}$ is transitive for all λ .

Since H is edge-transitive, $|E_i| = h$ for all i . Let $E_1 = \{x_1, x_2, \dots, x_h\}$. For $i \in M$, let ϕ_i be an automorphism such that $\phi_i(E_1) = E_i$.

Let $Y_p = \{\phi_i(x_p) : i \in M\}$, ($p=1, 2, \dots, h$). Then, $\bar{H} = (Y_1, \dots, Y_h)$ is a hypergraph on X , because

$$\bigcup_p Y_p = \bigcup_{i \in M} \phi_i(E_1) = \bigcup_{i \in M} E_i = X.$$

Let X_1, X_2, \dots, X_k be the connected components of \bar{H} .

PROOF OF (i). Let $E_1^\lambda = \{x_p : p \leq h, Y_p \subseteq X_\lambda\}$ for $\lambda = 1, 2, \dots, k$.
For $x_p \in E_1^\lambda$,

$$\phi_i(x_p) \in E_i \cap Y_p \subseteq E_i \cap X_\lambda .$$

Hence

$$\phi_i(E_1^\lambda) \subseteq E_i \cap X_\lambda .$$

Thus,

$$h = \sum_{\lambda} |E_1^\lambda| = \sum_{\lambda} |\phi_i(E_1^\lambda)| \leq \sum_{\lambda} |E_i \cap X_\lambda| = h .$$

Hence the equality holds, and

$$E_i \cap X_\lambda = \phi_i(E_1^\lambda) .$$

Hence

$$|E_i \cap X_\lambda| = |E_1^\lambda| , \quad (i \in M) .$$

This shows that H_{X_λ} is uniform with rank $|E_1^\lambda|$, and furthermore,

$$\sum_{\lambda} r(X_\lambda) = \sum_{\lambda} |E_1^\lambda| = \sum_{\lambda} |E_i \cap X_\lambda| = h .$$

PROOF OF (ii). In H_{X_λ} the edges $E_i \cap X_\lambda$ and $E_j \cap X_\lambda$ are symmetric, since

$$\phi_j \phi_i^{-1}(E_i \cap X_\lambda) = \phi_j(E_1^\lambda) = E_j \cap X_\lambda .$$

Hence H_{X_λ} is edge-transitive.

Furthermore, two vertices $x, y \in Y_p$ are symmetric, since

$$\left. \begin{array}{l} x = \phi_i(x_p) \\ y = \phi_j(x_p) \end{array} \right\} \text{ implies } y = \phi_j \phi_i^{-1}(x) .$$

Now consider two vertices x, x' in X_λ with $x \in Y_p, x' \in Y_q$. There exists a sequence $(Y_p, Y_{p_1}, Y_{p_2}, \dots, Y_q)$ such that any two consecutive sets of the sequence intersect. Let $x_k \in Y_{p_{k-1}} \cap Y_{p_k}$. In the sequence $(x, x_1, x_2, \dots, x_q = x')$, any two consecutive vertices are symmetric. Therefore

x and x' are symmetric.

Thus, H_{X_λ} is both edge-transitive and vertex-transitive. \square

COROLLARY 1. *If H is an edge-transitive hypergraph that is not vertex-transitive, then H is bicolourable.*

PROOF. This follows, since the partition of X has at least two classes, and they are both transversal sets of H . \square

COROLLARY 2. (DAUBER). *If H is an edge-transitive graph that is not vertex-transitive, then H is bipartite.*

PROOF. This follows from corollary 1. \square

3. EXTENSIONS OF THE WHITNEY THEOREM

Let $G = (E_i : i \in M)$ and $G' = (F_i : i \in M)$ be two connected simple graphs with $|M| = m > 2$. H. WHITNEY [6] has shown that $|E_i \cap E_j| = |F_i \cap F_j|$ for all i, j implies that $G \cong G'$, unless $G = K_3$ and $G' = K_{1,3}$, or vice versa.

An easy corollary of WHITNEY's theorem states that if G and G' are two simple graphs different from K_3 and $K_{1,3}$, then $G - E_i \cong G' - F_i$ for all i implies that $G \cong G'$. (The weak reconstruction conjecture states only that $n(G) > 4$, $G - E_i \cong G' - F_i$ for all i , implies that $G \cong G'$).

In [2], BERGE & RADO have proved several extensions of these theorems for hypergraphs.

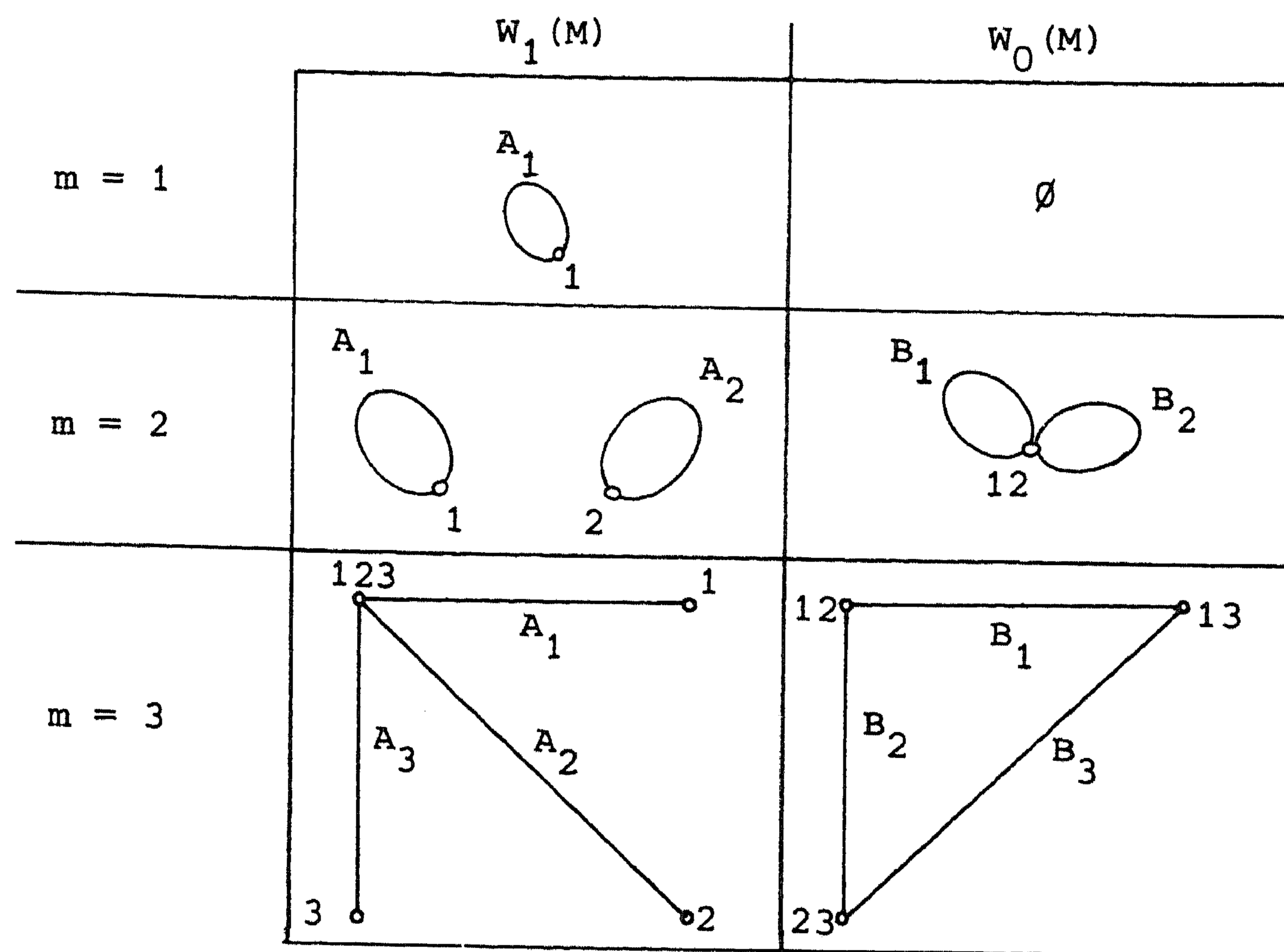
Denote by $\mathcal{P}(M)$ the set of all subsets of $M = \{1, 2, \dots, m\}$, by $\mathcal{P}_1(M)$ the set of all subsets $I \subseteq M$ such that $|I| \equiv 1$ modulo 2, and by $\mathcal{P}_0(M)$ the set of all subsets $J \subseteq M$ such that $|J| \equiv 0$ modulo 2 and $J \neq \emptyset$. Clearly, $|\mathcal{P}_1(M)| = 2^{m-1}$ and $|\mathcal{P}_0(M)| = 2^{m-1} - 1$ (because the regular bipartite graph whose vertex-sets are $\mathcal{P}_1(M)$ and $\mathcal{P}_0(M) \cup \{\emptyset\}$ and where (S, T) is an edge iff $-1 \leq |S| - |T| \leq 1$, has a perfect matching).

The two *Whitney hypergraphs* $W_1(M)$ and $W_0(M)$ are defined as follows: The vertex set of $W_1(M)$ is $\mathcal{P}_1(M)$, and its edges are

$$A_i = \{I : I \in \mathcal{P}_1(M), I \ni i\}, \quad (i \in M).$$

The vertex set of $W_0(M)$ is $\mathcal{P}_0(M)$, and its edges are

$$B_i = \{J : J \in \mathcal{P}_0(M), J \ni i\}, \quad (i \in M).$$



PROPOSITION 7. For $m \geq 2$, the Whitney hypergraphs $W_1(M) = (A_i : i \in M)$ and $W_0(M) = (B_i : i \in M)$ are two uniform hypergraphs of rank 2^{m-2} ; their boolean atoms have cardinality one, and they are not isomorphic. However, they satisfy

$$W_1(M) - A_i \cong W_0(M) - B_i, \quad (i \in M).$$

PROOF. For $K \subseteq M$, $K \neq \emptyset$, put

$$A_K = \bigcup_{i \in K} A_i,$$

$$A_{[K]} = \bigcap_{i \in K} A_i,$$

$$A_\emptyset = \emptyset.$$

Clearly, $W_1(M)$ and $W_0(M)$ are not isomorphic, since $|A_M| = 2^{m-1}$ and $|B_M| = 2^{m-1} - 1$.

For $K \subseteq M$, we have

$$\begin{aligned} A_{[K]} - A_{M-K} &= \{I : I \in \mathcal{P}_1(M), I \supseteq K\} - \{I : I \in \mathcal{P}_1(M), I \cap (M-K) \neq \emptyset\} = \\ &= \{K\} \text{ or } \emptyset. \end{aligned}$$

If this set of vertices is not empty, it has cardinality one, and it is a boolean atom of $W_1(M)$. Therefore, all the boolean atoms have cardinality one.

Now, let $N = \{2, 3, \dots, m\}$, and let us show that $(A_i : i \in N)$ and $(B_i : i \in N)$ are strongly isomorphic.

If $K \subseteq N$, $|K| \equiv 1$ modulo 2, we have

$$A_{[K]} - A_{N-K} = \{K\} ,$$

$$B_{[K]} - B_{N-K} = \{K \cup \{1\}\}$$

(and vice versa if $|K| \equiv 0$ modulo 2). Hence, for all $K \subseteq N$, $K \neq \emptyset$, we have

$$|A_{[K]} - A_{N-K}| = |B_{[K]} - B_{N-K}| = 1 .$$

This shows that $(A_i : i \in N) \cong (B_i : i \in N)$. \square

A converse of this proposition is:

THEOREM 2. Let $H = (E_i : i \in M)$ and $H' = (F_i : i \in M)$ be two families of subsets $E_i \subseteq X$ and $F_i \subseteq Y$ (with possibly empty edges or infinite edges), with at least one finite edge, such that

$$(3.1) \quad \begin{cases} \text{for all } k, \text{ there exists a bijection } \phi_k : X \rightarrow Y \text{ such that} \\ \phi_k(E_i) = F_i \quad (i \in M, i \neq k). \end{cases}$$

Then $H \cong H'$, unless there exist two sets $A \subseteq X$ and $B \subseteq Y$ with $|A| = |B|$ such that $(A \cap E_i : i \in M) \cong W_\varepsilon(M)$ and $(B \cap F_i : i \in M) \cong W_{1-\varepsilon}(M)$.

PROOF. The proof, by induction on m , is the same as in [2, theorem 2]. For the finite case, a direct proof, shorter than our original one, was found recently by LOVÁSZ [5]. \square

Note that the statement of theorem 2 would not be true if there is no finite edge: take four infinite sets D_0, D_1, D_2, D_3 of the same cardinality, and put $X = Y = D_0 \cup D_1 \cup D_2 \cup D_3$, $E_1 = D_1 \cup D_2 \cup D_3$, $F_1 = D_2 \cup D_3$, $E_2 = F_2 = D_2$, $E_3 = F_3 = D_3$. $H = (E_1, E_2, E_3)$ and $H' = (F_1, F_2, F_3)$ satisfy (1), and there is no $A \subseteq X$ such that $(A \cap E_i) \cong W_1(1, 2, 3)$ and no B such that $(B \cap F_i) \cong W_1(1, 2, 3)$. However, $H \not\cong H'$.

COROLLARY 1. Let $|M| \geq p \geq 2$, and let $H = (E_i : i \in M)$ and $H' = (F_i : i \in M)$ be two hypergraphs such that

$$(3.2) \quad (E_i : i \in I) \cong (F_i : i \in I) , \quad (I \subseteq M, |I| = p-1).$$

Then $H \cong H'$, unless there exist sets $A \subseteq E_M$, $B \subseteq F_M$ and $P \subseteq M$ such that $|P| = p$ and

$$(3.3) \quad \begin{cases} (A \cap E_i : i \in P) \cong W_\varepsilon(P) , \\ (B \cap F_i : i \in P) \cong W_{1-\varepsilon}(P) . \end{cases}$$

PROOF. For $m = p$, consider two hypergraphs H and H' with m edges which satisfy (3.2) and not (3.3).

Let us show first that $|E_M| = |F_M|$; if we have for instance $|E_M| < |F_M|$, consider a set X obtained from E_M by adding $|F_M| - |E_M|$ additional points, and put $Y = F_M$. By theorem 2, there exists a bijection $\phi: X \rightarrow Y$ such that $\phi(E_i) = F_i$, and therefore

$$\phi(E_M) = \cup F_i = F_M .$$

This shows that $|E_M| = |F_M|$ which is a contradiction. Thus, $|E_M| = |F_M|$, and theorem 2, applied with $X = E_M$, $Y = F_M$ shows that $H \cong H'$.

Now, let $m = p+t$, $t \geq 1$, and assume that the statement of this corollary is true for hypergraphs with $p+t-1$ edges. Consider two hypergraphs $H = (E_i : i \in M)$, $H' = (F_i : i \in M)$, with $M = \{1, 2, \dots, m\}$, satisfying (3.2) but not (3.3). By the induction hypothesis, we have, for $k \in M$,

$$(E_i : i \in M - \{k\}) \cong (F_i : i \in M - \{k\}) .$$

On the other hand, there exist no sets $A_0 \subseteq E_M$, $B_0 \subseteq F_M$, such that

$$(A_0 \cap E_i : i \in M) \cong W_\varepsilon(M) , (B_0 \cap F_i : i \in M) \cong W_{1-\varepsilon}(M) ,$$

because this would imply the existence of two sets A and B satisfying (3.3). Since the theorem is true for $p = m$, we have also $H \cong H'$. \square

COROLLARY 2. Let $H = (E_i : i \in M)$ and $H' = (F_i : i \in M)$ be two hypergraphs of rank $h < 2^{p-2}$. If, for every $J \subseteq M$ with $|J| = p-1$, we have $(E_i : i \in J) \cong (F_i : i \in J)$, then $H \cong H'$.

PROOF. The proof follows immediately from corollary 1. \square

COROLLARY 3. Let $H = (E_i : i \in M)$ and $H' = (F_i : i \in M)$ be two multigraphs such that

$$(3.4) \quad |E_i \cap E_j| = |F_i \cap F_j| \quad \text{for all } i, j \in M,$$

$$(3.5) \quad \begin{cases} H, H' \text{ do not contain as partial graphs} \\ W_\varepsilon(i, j, k) \text{ and } W_{1-\varepsilon}(i, j, k), \text{ respectively.} \end{cases}$$

Then $H \cong H'$.

PROOF. This follows from corollary 1 with $p = 3$. \square

The Whitney theorem follows easily from corollary 3 because, if H and H' are connected, of order >4 , and if, say, $(E_i, E_j, E_k) \cong W_1(i, j, k)$ and $(F_i, F_j, F_k) \cong W_0(i, j, k)$, then there exists an edge E_q which has exactly one endpoint in $W_1(i, j, k)$; hence

$$\begin{aligned} & |F_i \cap F_q| + |F_j \cap F_q| + |F_k \cap F_q| = \\ & = |E_i \cap E_q| + |E_j \cap E_q| + |E_k \cap E_q| = 0 \text{ or } 3, \end{aligned}$$

which is impossible. If H is of order 4 with more than three edges, it is easy to check that $H \cong H'$.

The following result is in fact due to LOVÁSZ [4], who stated it only for graphs.

THEOREM 3. Let $H = \{E_i : i \in M\}$ and $H' = \{F_i : i \in M\}$ be r -uniform simple hypergraphs of order n with $|M| = m > \frac{1}{2} \binom{n}{r}$, such that

$$H - E_i \cong H' - F_i, \quad (i \in M).$$

Then $H \cong H'$.

PROOF. Denote by $\bar{H} = P_r(X) - H$ the complement hypergraph of H , whose number of edges is

$$m(\bar{H}) = \binom{n}{r} - m < m.$$

We may assume that $X = Y$. If $S \subseteq P_r(X)$, denote by $\alpha(S, H')$ the number of isomorphisms $\pi: X \rightarrow Y$ such that $\{\pi S : S \in S\} \subseteq \{F_i : i \in M\}$. By the sieve formula,

$$\alpha(H, H') = \sum_{k=0}^m (-1)^k \sum_{\substack{I \subseteq M \\ |I|=k}} \alpha(\{E_i : i \in I\}, \bar{H}').$$

Since the terms with $|I| > m(\bar{H})$ are null,

$$(3.6) \quad \alpha(H, H') = \sum_{k=0}^{m-1} (-1)^k \sum_{\substack{I \subseteq M \\ |I|=k}} \alpha(\{E_i : i \in I\}, \bar{H}')$$

and

$$(3.7) \quad \alpha(H', H') = \sum_{k=0}^{m-1} (-1)^k \sum_{\substack{J \subseteq M \\ |J|=k}} \alpha(\{F_j : j \in J\}, \bar{H}').$$

Since, by hypothesis, H and H' have the same proper partial hypergraphs, the terms in (3.6) and in (3.7) are equal, hence:

$$\alpha(H, H') = \alpha(H', H') \geq 1. \quad \square$$

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EXTREMAL PROBLEMS FOR HYPERGRAPHS

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By a *hypergraph* we mean a pair (V, A) , where V is a finite set, and $A = \{A_1, \dots, A_m\}$ is a family of its different subsets. $|V|$ means the number of elements of V ; this is usually denoted simply by n . Similarly, $|A| = m$. The elements of V are called *vertices*, the elements of A are the *edges*.

We use the term hypergraph, because it becomes more and more familiar, but the questions concerned here did not develop directly from the theory of graphs (with some exceptions); the particular cases of these theorems give usually trivialities for graphs.

A hypergraph is a *k-graph* if $|A| = k$ holds for all $A \in A$. (V, A) is a *complete k-graph* if A consists of all the k -tuples of V .

In this paper we try to give a survey of some extremal problems of hypergraphs, namely, the problems developed from SPERNER's [74] theorem. We shall mention briefly some other areas, too. On the other hand we give some remarks on the possible generalizations for more general structures.

We have the feeling, that the classification of the problems in this paper is not good. However, the various questions are connected in many ways, thus the only proper way of classification would be a graph whose vertices are the problems and the "connected" problems are connected. (The most interesting question concerning this graph would be "how to get nice new vertices?")

For the interested readers it is suggested to read the survey paper of ERDÖS & KLEITMAN [21] on this subject, since our paper contains it only partly.

1. $|V|$ IS FIXED, MAXIMIZE $|A|$

The typical problem of this type: A set of conditions is given on A , and we are interested in determining the maximum (minimum) of $m = |A|$ if

$n = |V|$ is fixed and (V, A) runs over all the possible A 's satisfying the given conditions.

The origin of these theorems is the well-known theorem of SPERNER [74].

THEOREM 1. *If (V, A) satisfies $A_i \not\subset A_j$ ($i \neq j$), then*

$$(1) \quad m \leq \binom{n}{\lfloor n/2 \rfloor},$$

where equality holds for the complete $\lfloor n/2 \rfloor$ -graph.

The following beautiful proof is due to LUBELL.

PROOF. $C = \{C_0, \dots, C_n\}$ is called a *complete chain*, if $C_0 \subset C_1 \subset \dots \subset C_n$. (\subset denotes inclusion without $=$); ($|C_i| = i$ follows). Let us count in two different ways the number of pairs (C, A_i) , where $A_i \in A$ and $A_i = C_j \in C$ for some j . For a given A_i , C_j must be equal to $C_{|A_i|}$, we have $|A_i|!$ possibilities in choosing $C_0, C_1, \dots, C_{|A_i|-1}$, and $(n - |A_i|)!$ possibilities for $C_{|A_i|+1}, \dots, C_n$. The number of possible C 's is $|A_i|! (n - |A_i|)!$, and the total number of pairs (C, A_i) is $\sum_{i=1}^m |A_i|! (n - |A_i|)!$. On the other hand, fixing C , there is at most one A_i since $A_i = C_j \subset C_k = A_1$ would contradict the condition given on A . Thus, the number of pairs (C, A_i) is at most $n!$, the total number of C 's. We obtain the inequality

$$\sum_{i=1}^m |A_i|! (n - |A_i|)! \leq n!$$

or

$$(2) \quad \sum_{i=1}^m \frac{1}{\binom{n}{|A_i|}} \leq 1.$$

(1) follows from (2) easily, using

$$\binom{n}{|A_i|} \leq \binom{n}{\lfloor n/2 \rfloor}.$$

The proof is completed. \square

Equation (2) (which was discovered by LUBELL [67], MESHALKIN [68] and YAMAMOTO [77]) is perhaps more important than (1) itself. If $\sum_{i=1}^m f(|A_i|)$, where f is an arbitrary function, is maximized, then the maximum is attained

by the complete k -graph, where k is defined by

$$f(k) \binom{n}{k} = \max_{0 \leq i \leq n} f(i) \binom{n}{i}.$$

The proof of this statement (cf. [58,45]) easily follows from (2)

$$1 \geq \sum_{i=1}^m \frac{1}{\binom{n}{|A_i|}} = \sum_{i=1}^m \frac{f(|A_i|)}{f(|A_i|) \binom{n}{|A_i|}} \geq \sum_{i=1}^m \frac{f(|A_i|)}{f(k) \binom{n}{k}},$$

that is,

$$\sum_{i=1}^m f(|A_i|) \leq \binom{n}{k} f(k).$$

In some other cases LUBELL's method works again. In order to show, what properties of \mathcal{C} are used in general, (may be) it is worthwhile to formulate the method as a separate lemma. (W', \mathcal{B}') is called a *sub-hypergraph* of (W, \mathcal{B}) if $W' \subseteq W$ and $\mathcal{B}' \subseteq \mathcal{B}$. (W', \mathcal{B}) is a *spanned sub-hypergraph* if $\mathcal{B} = \{B: B \subseteq W', B \in \mathcal{B}\}$. We say that U is an *independent set* in (W, \mathcal{B}) if $U \subseteq W$, and there is no $B \in \mathcal{B}$ such that $B \subseteq U$.

LEMMA 1. Let $(W_1, \mathcal{B}_1), \dots, (W_z, \mathcal{B}_z)$ be spanned sub-hypergraphs of (W, \mathcal{B}) , the maximal number of independent elements being f_1, \dots, f_z and f , respectively. Then

$$(3) \quad f \leq \frac{\sum_{i=1}^z f_i}{\min_{w \in W} |\{i: w \in W_i\}|}.$$

If, additionally, $|W_1| = \dots = |W_z|$, $(W_1, \mathcal{B}_1), \dots, (W_z, \mathcal{B}_z)$ are isomorphic, and $|\{i: w \in W_i\}|$ does not depend on w , then

$$(4) \quad \frac{f}{|W|} \leq \frac{f_i}{|W_i|}.$$

PROOF. Let $F \subseteq W$ ($|F| = f$) be an independent set in (W, \mathcal{B}) . Let us count in two different ways the number of pairs $((W_i, \mathcal{B}_i), w)$ where $w \in F$ and $w \in W_i$. For a given $w \in F$ there are $|\{i: w \in W_i\}|$ sub-hypergraphs, thus the total number is $\sum_{w \in F} |\{i: w \in W_i\}|$. On the other hand, fixing a sub-hypergraph (W_i, \mathcal{B}_i) ,

the maximal number of w 's satisfying $w \in W_i$ can be f_i . Thus the number of pairs is at most $\sum_{i=1}^z f_i$. The resulting inequality

$$(5) \quad \sum_{w \in F} |\{i: w \in W_i\}| \leq \sum_{i=1}^z f_i.$$

However, since

$$f \min_{w \in W} |\{i: w \in W_i\}| \leq \sum_{w \in F} |\{i: w \in W_i\}|,$$

the inequality (3) follows from (5).

Using the additional suppositions

$$\sum_{w \in W} |\{i: w \in W_i\}| = \sum_{i=1}^z |W_i| = z|W_i|,$$

and

$$|\{i: w \in W_i\}| = \frac{z|W_i|}{|W|}.$$

On the other hand $\sum_{i=1}^z f_i = zf_i$. Substituting this result into (3) the inequality (4) is obtained, which completes the proof. \square

How to apply this lemma to our problems? W equals 2^V (the power set of V) and B consists of the subsets of 2^V which are excluded by the given condition. If the conditions exclude only elements and pairs of elements of 2^V , then (W, B) is a simple graph. For instance, in the case of SPERNER's theorem: two vertices $A_i, A_j \in W$ are connected iff $A_i \subset A_j$ or $A_j \subset A_i$. For W_1, \dots, W_z we choose all possible chains C given in LUBELL's proof. In this case (5) leads to (2), and (3) leads to (1).

The next natural condition (see [19]) for A is

$$(6) \quad A_i \cap A_j \neq \emptyset.$$

This question is, however, trivial: A can contain at most one of the sets $A, V-A$, thus $|A| \leq$ half the number of all subsets of V :

$$(7) \quad |A| \leq \frac{2^n}{2} = 2^{n-1}.$$

(The application of lemma 1 gives the same if we take $W_i = \{A_i, V-A_i\}$ for all $A_i \in 2^V$; then (4) gives (7).) This is the best possible bound:

$\hat{A} = \{A: v \in A, v \in V, v \text{ fixed}\}$ gives equality in (7).

The other classical theorem (ERDÖS, KO & RADO [19]) solves the problem for a combined condition, with a small modification.

THEOREM 2. *If (V, A) is a hypergraph satisfying the condition*

$$(8) \quad A_i \not\subset A_j, A_i \cap A_j \neq \emptyset, |A_i| \leq k \text{ if } A_i, A_j \in A \text{ (} i \neq j \text{),}$$

where $k \leq \frac{n}{2}$, then

$$(9) \quad m \leq \binom{n-1}{k-1},$$

and this is the best possible bound.

PROOF. First the constructions concerning (9):

$$A = \{A: |A|=k, v \in A, v \in V, v \text{ fixed}\}.$$

In the proof lemma 1 is used again. W consists of all elements of 2^V having at least k elements. (W, \mathcal{B}) is a simple graph. Two different vertices A, A' are connected iff $A \subset A'$, $A \supset A'$ or $A \cap A' = \emptyset$. W_i 's are defined in the following way. Let us consider all possible cyclic orderings of V . W_i consists of all subsets of V with size $\leq k$, and with consecutive elements according to the i -th ordering. The (W_i, \mathcal{B}_i) 's are isomorphic, f_i does not depend on i .

We shall show that $f_i \leq k$ if $k \leq n/2$. Fix the i -th cyclic ordering v_1, \dots, v_n (the indices are mod n), and suppose w_1, \dots, w_{f_i} are independent vertices in (W_i, \mathcal{B}_i) . By the symmetry we can suppose $w_1 = \{v_1, \dots, v_n\}$. If the first and last elements of a w_j are outside w_1 then either $w_j \supset w_1$, or $w_j \cap w_1 = \emptyset$ holds. Then the first or last element of each w_j is in w_1 . Fix an l , ($1 \leq l < r$), and consider all sets $A \in W_i$, the last element of which is v_l or the first element of which is v_{l+1} . These vertices are all connected in (W, \mathcal{B}) (or in (W_i, \mathcal{B}_i)), thus there is at most one w_j among them. Altogether, we have at most $(r-1)$ w_j 's with last element from v_1, \dots, v_{r-1} or with first element from v_2, \dots, v_r . v_1 can be the first element of w_1 , only. (Other $A \in W_i$ with this property either contain or are contained in w_1 .)

The same holds for the w_j 's having v_1 as a last element. We obtained $f_i \leq r \leq k$.

We need $|\{i: A \in W_i\}| = |A|! (n - |A|)!$. This is simply the number of cyclic orderings in which A has consecutive members. (5) gives the following inequality:

$$(10) \quad \sum_{i=1}^m \frac{1}{\binom{n}{|A_i|}} \leq \frac{k}{n} \quad \text{if } k \leq \frac{n}{2}$$

and hence, using that in the case $|A_i| \leq k \leq n/2$,

$$\binom{n}{|A_i|} \leq \binom{n}{k}$$

holds, we obtain (9), and the proof is completed. \square

This proof is a stronger version of the proof given in [42]. By (10) it is also easy to determine $\max \sum_{i=1}^m f(A_i)$ under (8).

An obvious question: what happens if the condition $|A_i| \leq k$ is omitted (or more generally, $n/2 < k \leq n$). If n is odd, then theorem 1 gives the estimation $\binom{n}{(n+1)/2}$, and the complete $\frac{n+1}{2}$ -graph satisfies the conditions. The case of even n is solved by BRACE & DAYKIN [2].

Another type of conditions is $A_i \cup A_j \neq V$. This does not seem to be a new condition, since it is equivalent to $(V - A_i) \cap (V - A_j) \neq \emptyset$. However, in some combinations of conditions we can not use the complement sets. For instance if

$$A_i \cap A_j \neq \emptyset \quad \text{and} \quad A_i \cup A_j \neq V,$$

this is the case. Under this condition $m \leq 2^{n-2}$, as DAYKIN & LOVÁSZ [12] proved; equality holds with $A = \{A: v \in A, w \notin A, \text{ where } v \neq w \text{ are fixed elements of } V\}$.

The next type of conditions is the constraint on the sizes of $A_i \cap A_j$ or $A_i \cup A_j$ ($i \neq j$) (perhaps of $A_i \cap A_j \cap A_l$, and so on). An example: in [19] the following condition is considered

$$(11) \quad |A_i| = k, \quad |A_i \cap A_j| \geq 1, \quad (k \geq 1).$$

The result [19]: if n is large enough (relatively to k and 1), then

$$(12) \quad m \leq \binom{n-1}{k-1},$$

where equality holds for $A = \{A: L \subset A \text{ where } |L| = 1, L \text{ a fixed subset of } V\}$. The result does not hold for small n , as the following example shows (given by MIN): $n=8, k=4, l=2, A = \{A: |A|=4, |A \cap \{1,2,3,4\}|=3\}, m = 16 > \binom{6}{2}$. This result gives a good example for the case, that sometimes the exact formulas are valid only for large values.

There is a large class of problems, where the solution (the extremal hypergraph) can be constructed by finite geometries or block designs. We shall not consider these problems, because their methods are completely different from the problems treated here. Thus, we do not investigate (with some exceptions) the conditions of such type, where $|A_i \cap A_j|$ has to be small, or $|A_i - A_j|$ has to be large. However, the questions (11)-(12) give an opportunity for a glimpse at the connections between the two areas. Consider the case $k=3, l=2$ (in this simple case (12) holds if $n \geq 6$ [39]). A Steiner triple system is a 3-graph (V, C) with the property, that each pair $v, w \in V$ ($v \neq w$) is contained by exactly one $C \in C$. It is well known [71], that such a system exists iff $n \equiv 1$ or $3 \pmod{6}$. Use lemma 1; W consists of all the triples of V ; w_1 and w_2 are connected in B iff $|w_1 \cap w_2| < 2$. W_i consists of the triples arising from a fixed Steiner triple system by the i -th permutation of V . It is easy to see, that (W_i, B_i) is a complete graph, so $f_i = 1$. Trivially, $|W_i| = \binom{n}{2}/3, |W| = \binom{n}{3}$, thus (4) gives $f \leq n-2$, and this is (12) for $k=3, l=2$.

By the combinations of the above conditions we obtain a lot of problems. We try to list some of them.

If

$$(13) \quad A_i \neq A_j, \quad A_i \cap A_j \neq \emptyset, \quad A_i \cup A_j \neq V,$$

then [2] (see also [45,59]) gives

$$(14) \quad m \leq \binom{n-1}{\lceil (n-2)/2 \rceil}.$$

If

$$(15) \quad |A_i \cap A_j| \geq 1,$$

then [39] gives

$$(16) \quad m \leq \sum_{i=\frac{1+n}{2}}^n \binom{n}{i} \quad \text{if } n+1 \text{ is even}$$

and

$$(17) \quad m \leq \binom{n-1}{\frac{n+1-1}{2}} + \sum_{i=\frac{n+1+1}{2}}^n \binom{n}{i} \quad \text{if } n+1 \text{ is odd.}$$

If

$$(18) \quad A_i \not\subset A_j, \quad |A_i \cap A_j| \geq 1,$$

then [69] gives

$$(19) \quad m \leq \binom{n}{\lfloor (n+1)/2 \rfloor}.$$

Let $1 \leq k \leq n$ and $1 \leq h \leq \min(k, n-k)$, and suppose

$$(20) \quad A_i \cap A_j \neq \emptyset, \quad h \leq |A_i| \leq k,$$

then [36] gives

$$(21) \quad m \leq \sum_{i=h}^k \binom{m-1}{i-1}.$$

If $1 \leq k \leq n$, and there is no pair $i \neq j$ such that

$$(22) \quad A_i \supset A_j \quad \text{and} \quad |A_i - A_j| \geq k,$$

then [17] gives

$$(23) \quad m \leq (\text{the sum of } k \text{ largest binomial coefficients of order } n).$$

Conversely, if there is no pair satisfying

$$(24) \quad A_i \supset A_j \quad \text{and} \quad |A_i - A_j| < k,$$

then [43] gives

$$(25) \quad m \leq \sum_{i \equiv [n/2] \pmod{k}} \binom{n}{i}.$$

Concerning the combinations which are missing, three cases can happen.

- 1) It is an easy consequence of another one.
- 2) The author of this paper does not know the result.
- 3) It is a nice open problem.

An example for case 3):

If $|A_i \cap A_j| \geq 1$ but there is no pair with $A_i \cup A_j = V$, then probably the inequalities (16) and (17) hold with $n-1$ rather than n . (We can not give examples for case 2).)

2. CONDITIONS VARYING ON A WIDER SCALE

In this section we consider the same kind of problems as in section 1, but the *conditions vary on a wider scale*.

The most general form of theorem 2 (and (11)-(12)) is the following theorem of HAJNAL & ROTHSCHILD [29].

If

$$(26) \quad \begin{cases} |A_i| = k, \text{ and for any } i_1, \dots, i_{r+1} \\ \text{there are } i_j \text{ and } i_h \text{ with } |A_{i_j} \cap A_{i_h}| \geq 1, \end{cases}$$

then

$$(27) \quad m \leq \sum_{i=1}^r (-1)^{i+1} \binom{r}{i} \binom{n-1}{k-1},$$

provided n is large enough ($n \geq n(k, r, 1)$).

What are the best values for $n(k, r, 1)$? By theorem 2, $n(k, 1, 1) = 2k$. For the cases of $n(k, 1, 1)$ we can not expect a nice smallest value. The estimations of [19] are improved in [37]. The hopeful case is $n(k, r, 1)$. For instance, $n(k, 2, 1) = 3k+1$ might be true.

The same question without $|A_i| = k$, and only for $l=1$ is solved by KLEITMAN [55]. So, if for any i_1, \dots, i_{r+1} there is a pair i_j, i_h such that

$$(28) \quad A_{i_j} \cap A_{i_h} \neq \emptyset$$

and $n = (r+1)q$, then

$$(29) \quad m \leq \sum_{i=q+1}^{(r+1)q} \binom{(r+1)q}{i} + \binom{(r+1)q}{q} \frac{r}{r+1}.$$

If $n = (r+1)q-1$, another exact estimation is given. For other n 's there is a small gap between the estimations and the constructions [55].

An obvious open question is the case $l > 1$ ($|A_{i_j} \cap A_{i_h}| \geq 1$). This is solved only for $r=1$ (see (15)-(17)).

A third variant of these questions was posed by D. PETZ and solved by P. FRANKL [27] (students in Budapest):

If

$$(30) \quad A_i \not\supseteq A_j \quad \text{and} \quad |A_{i_1} \cup \dots \cup A_{i_r}| \leq qr+s$$

where $0 \leq s < r$, then

$$(31) \quad m \leq \sum_{i=0}^{\min(q, s/2)} \binom{n-s}{q-i} \binom{s}{[s/2]+i},$$

provided n is large enough depending on r and $qr+s$. The construction: let $C \subset V$, $|C|=s$, then $A = \{A: |A|=q+[s/2], |A \cap (V-C)| \leq q\}$. The cases $s=0$ and $s=1$ are solved independently by E. BOROS. Observe that (30)-(31) is a generalization of (18)-(19) using the complement set. In (18)-(19) $r=2$, $s=0$ or 1 .

It seems that in (9) equality can hold ($k < n/2$) only for the given extremal hypergraph (all the A 's containing a given $v \in V$). In [19] it is asked, what happens, if we exclude this extremal hypergraph, or suppose $\bigcap_{i=1}^m A_i = \emptyset$. HILTON & MILNER [33] have given the answer:

$$(32) \quad m \leq 1 + \binom{n-1}{k-1} + \binom{m-k-1}{k-1}.$$

They have more general theorems, too: If $1 \leq \min(3, s+1) \leq k \leq n/2$, and $|A_i| \leq k$, $A_i \not\supseteq A_j$ ($i \neq j$), $A_i \cap A_j \neq \emptyset$,

$$(33) \quad A_{i_1} \cap \dots \cap A_{i_{m-s+1}} = \emptyset$$

for any different indices i_1, \dots, i_{m-s+1} , then

$$(34) \quad m \leq \begin{cases} \binom{n-1}{k-1} - \binom{n-k}{k-1} + n-k & \text{if } 2 < k \leq s+2, \\ s + \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-s}{k-s-1} & \text{if } k \leq 2 \text{ or } k \geq s+2. \end{cases}$$

A combination of (32) and (20)-(21) is given in [36]:

Let $1 \leq k \leq n-1$, $1 \leq h \leq \min(k, n-k)$. If

$$(35) \quad A_i \cap A_j \neq \emptyset, h \leq |A_i| \leq k \quad \text{and} \quad \bigcap_{i=1}^m A_i = \emptyset,$$

then

$$(36) \quad m \leq 1 + \sum_{i=h}^k \left[\binom{n-1}{i-1} + \binom{n-k-1}{i-1} \right].$$

The following three results [36] are modifications of the above ones, when besides A_1, \dots, A_m there is an additional edge B of our hypergraph with slightly different conditions. Let h and k satisfy $1 \leq k \leq n/2$, $1 \leq h \leq n-1$. If $A_i \cap A_j \neq \emptyset$, $A_i \cap B \neq \emptyset$, $|A_i| \leq k$, $|B| = h$, $A_i \not\subseteq A_j$, $A_i \not\subseteq B$ ($B \subset A_i$ is not excluded), then

$$m \leq \begin{cases} \binom{n-1}{k-1} - \binom{n-h-1}{k-1} & \text{if } h \geq n/2, \\ \binom{n-1}{k-1} - \binom{h-1}{k-1} & \text{if } h < n/2. \end{cases}$$

If $A_i \cap A_j \neq \emptyset$, $A_i \cap B \neq \emptyset$, $|A_i| \leq k$, $|B| = h$, $A_i \not\subseteq A_j$, $B \cap A_1 \cap \dots \cap A_m = \emptyset$, then

$$m \leq \begin{cases} \binom{n-1}{k-1} - \binom{n-h-1}{k-1} & \text{if } k \leq h, \\ 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1} & \text{if } h < k. \end{cases}$$

Finally, if $A_i \cap A_j \neq \emptyset$, $A_i \cap B \neq \emptyset$, $g \leq |A_i| \leq k$, $|B| = h$, $B \cap A_1 \cap \dots \cap A_n = \emptyset$, then

$$m \leq \begin{cases} \sum_{i=g}^k \left[\binom{n-1}{i-1} - \binom{n-h-1}{i-1} \right] & \text{if } k \leq h, \\ 1 + \sum_{i=g}^k \left[\binom{m-1}{i-1} - \binom{m-k-1}{i-1} \right] & \text{if } h < k. \end{cases}$$

In a paper of HILTON [35] the concept of the *simultaneously disjoint pairs of edges* is defined by

$$A_{i_1} \cap A_{j_1} = \dots = A_{i_s} \cap A_{j_s} = \emptyset.$$

Let $2 \leq 2k \leq m$ and $s \leq \binom{n-k-1}{k-1} - 1$. If

$$(37) \quad \begin{cases} |A_i| \leq k, A_i \not\subset A_j \text{ and there are no } s+1 \\ \text{simultaneously disjoint pairs of edges,} \end{cases}$$

then (cf. [35])

$$(38) \quad m \leq \binom{n-1}{k-1} + s.$$

If $A_i \subset A_j$ is allowed, then we obtain (cf. [35])

$$m \leq \sum_{i=1}^k \binom{n-1}{i-1} + s.$$

Or more generally [32], if $h \leq |A_i| \leq k$, then

$$m \leq \sum_{i=h}^k \binom{n-1}{i-1} + s.$$

(For a recent result of this type see [11].)

Similar results are obtained in [32] in case we exclude the existence of $s+1$ simultaneously disjoint r -tuples of edges. For $s = 0$ it was solved earlier by ERDÖS & GALLAI [18] for 2-graphs and later by ERDÖS [20] for k -graphs. ERDÖS' case is also included by (26)-(27), but the common

generalization of HAJNAL & ROTHSCHILD [29] and HILTON [32] is still open.

3. WEAKENING THE CONDITIONS

Could we weaken the conditions of our theorems with the same conclusions?

In this section we give examples for that.

First KLEITMAN [49] and KATONA [40] independently observed, that if we fix a partition $V_0 \cup V_1 = V$, ($V_0 \cap V_1 = \emptyset$) of V and we exclude the edges satisfying

$$(39) \quad A_i \cap V_\delta = A_j \cap V_\delta \quad \text{and} \quad A_i \cap V_{1-\delta} \subset A_j \cap V_{1-\delta}$$

(instead of $A_i \subset A_j$), then under this weaker condition the conclusion

$$(40) \quad m \leq \binom{n}{\lfloor n/2 \rfloor}$$

remains the same.

A natural question: what happens for the partition $V_0 \cup V_1 \cup V_2 = V$ (V_0, V_1, V_2 are pairwise disjoint), if we exclude edges equal in two V_i 's and containing each other in the third? The answer is disappointing: m can be larger than $\binom{n}{\lfloor n/2 \rfloor}$. In [47] an additional condition is given, under which (40) remains true. This additional condition is rather complicated. It excludes some 4-tuples of edges of the hypergraph. Recently, GREENE & KLEITMAN [28] determined weak conditions from the symmetric chain method (see [3]).

A combination of (39) and (22) is given in [44], and a combination of (39) and (24) in [43]. Recent generalizations of this type can be found in [60].

A question: how could we weaken the conditions of theorem 2 with the same conclusion?

4. ONE CONDITION CONTAINING MORE OPERATIONS OR RELATIONS

In this section we treat the problems where *one condition contains more operations or relations*.

Probably the oldest result of this type is due to KLEITMAN [56]. If

there is no triple satisfying

$$A_i \cap A_j = \emptyset \quad \text{and} \quad A_i \cup A_j = A_h$$

simultaneously, then

$$m \leq \sum_{i=r+1}^{2r+1} \binom{n}{i},$$

provided $n = 3r+1$, and this is the best estimation. For $n = 3r$ and $n = 3r+2$ the results are near best possible.

Another problem: there are no 4 different edges in the hypergraph satisfying both

$$A_i \cup A_j = A_k \quad \text{and} \quad A_i \cap A_j = A_l.$$

ERDÖS & KLEITMAN [24] have constructed $c_1 \frac{2^n}{n^4}$ edges with this condition and they proved that

$$m \leq c_2 \frac{2^n}{n^4}$$

but $c_1 < c_2$.

Many obvious general questions can be asked.

In the next problem $|V|$ is not fixed, but we list it here, because its character is similar to the other problems treated here. Now $|A| = m$ is fixed and $f(m)$ is the largest number such that there are always $f(m)$ edges in the hypergraph no three different ones of them having the property

$$A_i \cup B_i = C_i.$$

The first result is given by KLEITMAN [50]:

$$f(m) \leq cm\sqrt{\log m}.$$

J. RIDDEL proved $\sqrt{m} < f(m)$, and finally ERDÖS & KOMLÓS [22] determined

$$f(m) < 2\sqrt{2n+4}.$$

BOLLOBÁS proved for 3-graphs that if

$$(41) \quad \text{there are no three different edges } A_h \supseteq (A_i - A_j) \cup (A_j - A_i),$$

then

$$m \leq \left(\frac{n}{3}\right)^3$$

if $3|n$. The hypergraph with equality: V is divided into 3 equal parts, and we choose the edges having exactly one vertex from each part.

It is conjectured also by BOLLOBÁS, that a similar for k -graphs. For 2-graphs it is a particular case of TURÁN's graph theorem [76]. A conjecture of ERDŐS & KATONA: Under the condition (41) (without size restrictions) the best hypergraph can be constructed in the following way. Divide V into $\lfloor \frac{n}{3} \rfloor$ classes of 3 and 2 elements, and choose those edges which contain exactly one vertex from each class.

5. MISCELLANY

We will treat three further problems which do not really fit into any of these sections. The first question was proposed by RÉNYI [70]. The edges of the hypergraph are called *qualitatively independent* if

$$(42) \quad A_i \cap A_j, \quad A_i \cap \bar{A}_j, \quad \bar{A}_i \cap A_j, \quad \bar{A}_i \cap \bar{A}_j$$

are all non-empty. What is the maximum of m under this condition? The answer is

$$m \leq \binom{n-1}{\lfloor (n-2)/2 \rfloor}.$$

This is an easy consequence of theorems 1 and 2, as it is pointed out by KLEITMAN & SPENCER [59] and independently in [45]. (Observe, that (13) and (42) are equivalent, thus [2] also gives the solution.) In [59] a harder problem is also considered. We say, the edges are *k-qualitatively independent* if

$$A_{i_1}^{\delta_1} \cap \dots \cap A_{i_k}^{\delta_k} \neq \emptyset$$

for any different i_1, \dots, i_k , where A^δ is either A or $\bar{A} = V - A$. Under this condition

$$m \leq 2^{\frac{cn}{2^k}}$$

and a hypergraph is constructed with

$$2^{\frac{dn}{k2^k}}$$

edges, where c and d are constants, k fixed and $n \rightarrow \infty$.

An unsolved question: maximize m under the condition that any of (42) has a size $\geq r$.

The density of a hypergraph was defined by ERDÖS. It is the largest s such that there is a $U \subseteq V$ such that $|U| = s$ and $|A \cap U| = 2^s$. SAUER [72] proved, that supposing

$$s \leq k,$$

we obtain

$$m \leq \sum_{i=0}^k \binom{n}{i}.$$

A similar problem of ERDÖS & KATONA: what is the maximum of m under the condition that $|A_i \cap A_j|$ are all different ($1 \leq i < j \leq m$)?

A new area of problems is considered in [2]. The valency $v = v(V, A)$ of a hypergraph is the minimal valency of its vertices. In [2] the maximum of v is asked for under several conditions.

If $A_i \cap A_j \neq \emptyset$, then

$$2^{n-2} + \frac{1}{2} \binom{n-1}{(n-1)/2} \quad \text{if } n \text{ is odd,}$$

$$v \leq$$

$$2^{n-2} + \left[\frac{1}{2} \binom{n}{n/2} \right] \quad \text{if } n \text{ is even.}$$

If $A_i \cap A_j = \emptyset$, then

$$v \leq \left[\frac{n-1}{(n-1)/2} \right]$$

and if $A_i \& A_j$, $A_i \cap A_j = \emptyset$, then the same holds.

6. THE PROBLEMS WE SHALL NOT CONSIDER HERE

These problems -although they have many points in common with our subject- require different methods, and are approached from various points of view. These problems are also extremal problems for hypergraphs, but this concept is too wide.

- 1) If $A_i \cap A_j$ is small, $A_i - A_j$ or $(A_i - A_j) \cup (A_j - A_i)$ are large, the problems are usually *coding problems*. Their methods are closer to block designs and finite geometries.
- 2) *Covering problems*. Usually a smallest family of edges is sought under some conditions, covering all the edges of a given hypergraph. In 1) and 2) the solutions give hypergraphs where the edges are "far" from each other, in our cases they are "close".
- 3) *Ramsey type theorems*. See the paper of GRAHAM & ROTHSCHILD in this tract (pp. 61-76).
- 4) *Turán type theorems*. Certain generalizations are very near (see [46]).
- 5) *Combinatorial search problems*. They are closely related to the coding problems (see [46]).
- 6) We did not touch the question of the number of optimal hypergraphs. In many cases there is only one. In some other cases it is an open problem how many of them exist. A closely related problem: how many hypergraphs do we have under several conditions? For these questions see [21].
- 7) *Δ -systems and B-property*. A hypergraph is a *strong Δ -system* if $A_i \cap A_j$ ($i \neq j$) does not depend on i and j . In the case of a *weak Δ -system* $|A_i \cap A_j|$ ($i \neq j$) is independent of i and j . $f_s(k,1)$ denotes the minimum of $|A|$ with the property that in the case $|A_i| = k$, ($1 \leq i \leq m$), there are always 1 A_i 's forming a strong Δ -system. $f_w(k,1)$ denotes the same for weak Δ -systems. There are lower and upper estimations for $f_s(k,1)$ and $f_w(k,1)$. We say that (V, A) has *property B*, if there is a set $B \subset V$ such that $|B \cap A_i| \geq 1$ but $B \supset A_i$ ($1 \leq i \leq m$). The questions concerning Δ -systems and the B-property are closely related to our problems; however, ERDÖS [25] has recently published a survey paper on this subject.

7. $|A|$ IS FIXED

Perhaps, the main feature of the problems in this section is not $|A|$ being fixed, because in many cases we obtain an inequality and in an inequality usually it is not important, which variable is fixed and which one is not. However, the problems treated here -as we shall see- have a definitely different character.

SPERNER's theorem says, if we have $\binom{n}{[n/2]} + 1$ edges in a hypergraph (with $|V| = n$), then there is a pair of different edges $A_i \subset A_j$. Observe, however, that adding one edge to the complete $[n/2]$ -graph there are always more pairs with $A_i \subset A_j$. What is the minimum? More generally, if m and n are fixed, what is the minimal number of pairs $A_i \subset A_j$? The solution is given by KLEITMAN [51]. The optimal hypergraph is constructed easily. Order all subsets of V , first take all $[n/2]$ -tuples, then all $[n/2]+1$ -tuples, all $[n/2]-1$ -tuples, all $[n/2]+2$ -tuples, and so on. The edges of the optimal hypergraph are the first m subsets according to this order.

The corresponding question is not solved yet, not even for the case of (15)-(16). This latter one can not be too hard for $l=1$. The optimal hypergraph could be constructed by taking the subsets of V according to their sizes, starting from n . (For the case of theorem 2 see later in this section.)

Let (V, A) now be a k -graph, and let $C(A)$ denote the family of subsets $C: C \subset A$ for an $A \in A$ and $|C| = k-1$. SPERNER [74] used in his proof the easy fact

$$|C(A)| \geq \frac{|A| \cdot k}{n-k+1} .$$

The question arises, what is $\min |C(A)|$ if n, k, m are fixed ($m \leq \binom{n}{k}$). The construction of the optimal k -graph is as follows. Fix an order v_1, \dots, v_n of the vertices in V . Form a sequence of 0's and 1's in the usual way from each k -set of V . The first m sequences in the lexicographic order give the optimal k -graph. A formula can also be given for $\min |C(A)|$. There is a unique expression of the form

$$(43) \quad m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_t}{t} ,$$

where $t \geq 1$, $a_k > a_{k-1} > \dots > a_t$ and $a_i \geq i$. Then

$$(44) \quad \min |C(A)| = \binom{a_k}{k-1} + \dots + \binom{a_t}{t-1} = f_k(m).$$

The result is more clear if m has the form $\binom{a_k}{k}$. Then we have for the optimum a complete k -graph in $V' \subset V$ where $|V'| = a_k$. An interesting thing: (44) does not depend on n . (44) was first proved by KRUSKAL [63]. Some years later it was rediscovered in [41]. Then CLEMENTS & LINDSTRÖM [4] proved a more general theorem by a different method. They also proved the theorem independently, but they found [41] and [63] before publishing it. HANSEL [30] also has a paper, and recently DAYKIN [13] found a relatively short proof.

A similar result was found earlier by KLEITMAN [52]: If (V, A) is a hypergraph with $A_i \cap A_j = \emptyset$ and $|A| = \binom{n}{k}$ then the number of different sets C for which there exists an $A_i \in A$ with $C \subseteq A_i$ is at least

$$(45) \quad \sum_{i=0}^k \binom{n}{i}.$$

This question was solved for any m by CLEMENTS [9], using (44). In this solution only an algorithm is given determining the optimal A , no formula of type (45) is given for the minimum in general. This remains open. [9] also contains useful inequalities concerning (44).

There are a lot of other consequences of (44). E.g., recently DAYKIN [14] observed that theorem 2 (ERDÖS, KO & RADO) follows from (44). Now we give some examples, where (44) is used in the proof.

Let (V, A) be a k -graph. A $(k-1)$ -representation of (V, A) is a set $\{B_1, \dots, B_m\}$ of $(k-1)$ -tuples such that $B_i \subset A_i$ ($1 \leq i \leq m$). ERDÖS asked what is the maximal m for which any (V, A) with $|A| = m$ has a $(k-1)$ -representation. The answer [41] is

$$m = \binom{2k-1}{k} + \binom{2(k-1)-1}{k-1} + \dots + \binom{1}{1}.$$

From inequality (2) it is trivial that if we modify the conditions of theorem 1 in such a way that $|A_i| = n/2$ (let n be even) is excluded, then $m \leq \binom{n}{(n/2)-1}$ and this is the best. However, if we describe the number of edges A_i with $|A_i| = n/2$ (and this number is > 0 , but $< \binom{n}{n/2}$), then usually we do not obtain an exact estimation for m . This question was solved in

[10] more generally: $p_j = |\{A: A \in \mathcal{A}, |A|=j\}|$ are called the *parameters* of a hypergraph. Let $0 \leq i_0 < n$ and the parameters $p_{i_0} \neq 0, p_{i_0+1}, \dots, p_n$ be fixed. $\max |A|$ is determined under this condition, provided $A_i \not\supset A_j$.

Another formulation is given independently by DAYKIN, GODFREY & HILTON [15]: If $p_0 = 0, p_1, \dots, p_g > 0$ are given integers, then the least integer n such that there exists a hypergraph (V, \mathcal{A}) with $|V| = n, A_i \not\supset A_j$ and with the parameters p_0, p_1, \dots, p_g is

$$n = p_1 + f_2(p_2 + f_3(p_3 + \dots + f_g(p_g) \dots)),$$

where f_i is defined in (44).

[15] solves a conjecture of KLEITMAN & MILNER, too: If (V, \mathcal{A}) satisfies $A_i \not\supset A_j$ and has the parameters p_0, p_1, \dots, p_n , then there is an other hypergraph (V, \mathcal{A}') satisfying $A_i' \not\supset A_j'$ and with parameters $0, \dots, 0, p_{n/2}, p_{n/2+1} + p_{n/2-1}, \dots, p_n + p_0$ (if n is odd, then the middle is: $\dots, 0, 0, p_{\frac{1}{2}(n+1)} + p_{\frac{1}{2}(n-1)}, p_{\frac{1}{2}(n+3)} + p_{\frac{1}{2}(n-3)}, \dots$).

Let the parameters p_1, \dots, p_{1+r} be fixed. What is the minimal number of $(1-1)$ -tuples contained in any edge $A \in \mathcal{A}$? This is answered in [8].

CLEMENTS [11] dealt with the problem what happens in theorem 2 if we take more edges than $\binom{n-1}{k-1}$. However, he did not minimize the number of disjoint pairs, but maximized the number of edges meeting all other edges of (V, \mathcal{A}) .

As is clear from the examples, (44) is almost necessary if containment is involved and the optimal arrangement does not consist of complete i -graphs. We had to write "almost", because KLEITMAN's result in [51] is an exception.

Another type of problems where $|A|$ is fixed: what is the maximal number of pairs $A_i \supset A_j, |A_i - A_j| = 1$? An extremal hypergraph can be constructed by choosing the first $m = |A|$ edges according to the lexicographic order. This is proved in [1, 31, 65]. However, as CLEMENTS [6] pointed out it is an easy consequence of (44).

$\min |C(\mathcal{A})|$ can be asked for under several conditions. For instance in [39] it is tried to do this supposing $|A_i \cap A_j| \geq 1$ ($|A_i| = k$ remains true, $1 < k$). However, only $\frac{|C(\mathcal{A})|}{|A|}$ is minimized. The optimal hypergraph is a complete k -graph on a $(2k-1)$ -element subset of V . For fixed $|A|$, the hypergraph minimizing $|C(\mathcal{A})|$ seems rather complicated, but it is regular enough to have some hope for the solution.

P. FRANKL asked the following question of similar type. If $|A|$ is fixed,

$|A| = k$ for $A \in \mathcal{A}$, what is the minimum of $(2k-1)$ -tuples contained in a union $A_i \cup A_j$ ($A_i, A_j \in \mathcal{A}$)?

8. MORE HYPERGRAPHS

In these problems we have more hypergraphs with the same vertex set. Usually it is supposed that the hypergraphs do not have common edges. The conditions and the questions are usually similar to those in the above sections.

The first result was achieved by ERDÖS [17]. If the hypergraphs $(V, \mathcal{A}_1), \dots, (V, \mathcal{A}_d)$ satisfy the condition

$$A_i \not\supset A_j, \quad A_i, A_j \in \mathcal{A}_h \quad (1 \leq h \leq d)$$

(and $A_i \cap A_j = \emptyset$ ($i \neq j$)), then

$$(46) \quad \sum_{i=1}^d |A_i| \leq (\text{the sum of the } d \text{ largest binomial coefficients of order } n).$$

By the same proof as in the case of theorem 1 we obtain the inequality

$$(47) \quad \sum_{i=1}^d \sum_{A \in \mathcal{A}_i} 1/\binom{n}{|A|} \leq d,$$

where simply the hypergraph $(V, \bigcup_{i=1}^d \mathcal{A}_i)$ was considered; thus one chain \mathcal{C} can contain at most d A 's. (47) is equivalent to $\sum_{k=0}^n x_k / \binom{n}{k} \leq d$, where x_k denotes the number of A 's with $|A| = k$. It is clear, that under this inequality $\sum_{k=0}^n x_k$ is maximal if we take the maximal values of the x_k 's with minimal coefficients, thus $x_k = \binom{n}{k}$ for the d middle k 's and 0 otherwise.

The next question, what is $\max \sum_{i=1}^d |A_i|$, if the \mathcal{A}_i 's are disjoint and $A_j \cap A_h \neq \emptyset$, $A_j, A_h \in \mathcal{A}_i$ ($1 \leq i \leq d$). The answer was found by KLEITMAN [53]:

$$\sum_{i=1}^d |A_i| \leq 2^n - 2^{n-d}.$$

The corresponding question for theorem 2 is unsolved. A problem of KNESER [62] is the following. If $(V, \mathcal{A}_1), \dots, (V, \mathcal{A}_d)$ are k -graphs ($k < n/2$) $A_i \cap A_j \neq \emptyset$ for $A_i, A_j \in \mathcal{A}_h$, $A_i \cap A_j = \emptyset$ and $(V, \bigcup_{h=1}^d \mathcal{A}_h)$ is the complete k -graph,

what is the minimum of d under these conditions?

Another line was started by HILTON & MILNER [33]. Let (V, A) and (V, B) be two hypergraphs such that

$$|A_i| \leq k, |B_i| \leq k, A_i \cap B_j \neq \emptyset, A_i \not\subset A_j, B_i \not\subset B_j;$$

then supposing $p \leq |A|, |B|$ and $1 \leq \min(2, p) \leq k \leq n/2$,

$$|A| + |B| \leq \begin{cases} \binom{n}{k} - \binom{n-k+1}{k} + n - k + 1 & \text{if } 1 < k \leq p + 1, \\ p + \binom{n}{k} - \binom{n-k+1}{k} + \binom{n-k-p+1}{k-p} & \text{otherwise,} \end{cases}$$

holds. HILTON [34] generalized, for the case $|B_i| \leq 1 \neq k$, KLEITMAN's result [57] on the same subject: (V, A) and (V, B) are hypergraphs satisfying

$$|A_i| = k, |B_i| = 1, k+1 \leq n, A_i \cap B_j \neq \emptyset$$

then either

$$|A| \leq \binom{n-1}{k-1}$$

or

$$|B| \leq \binom{n-1}{1-1} - \binom{n-1-k}{1-1}.$$

EHRENFEUCHT & MYCIELSKI [16] conjectured that if the hypergraphs satisfy

$$|A_i| = k (A_i \in A), |B_i| = 1 (B_i \in B), |A| = |B| = m$$

and

$$A_i \cap B_j \neq \emptyset \quad \text{iff } i \neq j$$

then

$$(48) \quad m \leq \binom{k+1}{1}.$$

It is proved in [48]. T. TARJÁN [75] modified the proof yielding a stronger result:

Let (V, A) and (V, B) be two hypergraphs with $|A| = |B|$ and

$$(49) \quad A_i \cap B_j \neq \emptyset \quad \text{iff } i \neq j.$$

Lemma 1 will be applied for the following graph. W consists of all pairs (S, T) where $S, T \subseteq V$, $S \cap T = \emptyset$, and two distinct vertices $(S_1, T_1), (S_2, T_2)$ are connected iff one of the sets $S_1 \cap T_2, S_2 \cap T_1$ is empty. Fix an order on the elements in V . Let W_i consist of those vertices (S, T) in which all elements of S precede all elements of T according to the i -th permutation of the elements of V . Observe that W_i spans a complete graph. That means $f_i = 1$. We need the number

$$|\{i: (S, T) \in W_i\}| = \binom{n}{|S|+|T|} |S|! |T|! (n-|S|-|T|)! = \frac{n! |S|! |T|!}{(|S|+|T|)!}.$$

From inequality (5) we obtain

$$(50) \quad \sum_{i=1}^m \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

If $|A_i| = k$, $|B_i| = 1$, (48) trivially follows. Other variants follow, too. E.g. if $|A_i| + |B_i| \leq k$ then

$$m \leq \binom{k}{\lfloor k/2 \rfloor}.$$

9. n -DIMENSIONAL LATTICE-POINTS

SPERNER's question can be formulated in the following way. A square-free integer $N = p_1 p_2 \dots p_n$ is given; what is the maximal number of its divisors not dividing each other? After answering this question it is a must to answer the same for arbitrary $N = p_1^{\alpha_1} \dots p_n^{\alpha_n}$, too. The divisors of N have the form $p_1^{x_1} \dots p_n^{x_n}$, where $0 \leq x_i \leq \alpha_i$ ($1 \leq i \leq n$). Thus, with the divisors we can associate the lattice-points of an $(\alpha_1+1) \times \dots \times (\alpha_n+1)$ n -dimensional parallelotope. All questions can be extended to n -dimensional parallelotopes in this way. Some of these extensions are motivated by other applications.

If the character of the problem is such that in the parallelotopes there do not appear new phenomena (compared to the hypergraphs), then it is easier to start making a conjecture and proof for the 2- and 3-dimensional parallelotopes, since they are more graphic.

We briefly list the results which are generalizations of this type.

SPERNER's theorem was generalized in [28]. The bound for m is the maximal number of lattice-points with a fixed coordinate sum ($= [\sum \alpha_i / 2]$).

SCHÖNHEIM [73] generalized (46) and (39)-(40). In [44] the common generalization is given. (25) is generalized in [43].

ERDÖS & SCHÖNHEIM [26], further ERDÖS, HERZOG & SCHÖNHEIM [23] have investigated the generalization of (6). The max of m is not equal to the minimal m for which there exists an m -element set of divisors such that any other divisor is coprime to one of them. Both values are determined.

An analogue of (15)-(16) is generalized in [54].

The analogue of (44) is proved in [4]. Of course, there are no formulas, but it is proved that one of the optimal sets of lattice-points gives the first m in the lexicographic order. Other results concerning this theorem can be found in [7]. [5] gives the generalization of ERDÖS' problem of $(k-1)$ -representation of k -edges. [8] also concerns this generalization.

In [6] CLEMENTS shows, that the theorem of LINDSEY [65] (which maximizes the pairs of neighbouring lattice-points if their number is given) is an easy consequence of the generalized formula (44). Recently KLEITMAN, KRIEGER & ROTHSCHILD [61] determined the maximal number of such pairs which differ only in one coordinate.

LINDSTRÖM [66] solved an interesting question of KRUSKAL [64], which is an analogue of (44). A hypergraph can be imagined as a set of certain faces of an $(n-1)$ -dimensional simplex. Thus, if we fix the number of $(k-1)$ -dimensional faces, then (44) gives the minimal number of $(k-2)$ -dimensional subfaces. LINDSTRÖM solved the same question for more-dimensional cubes.

10. FURTHER ANALOGUES AND GENERALIZATIONS

There is an attempt to put these combinatorial theorems in a more general -algebraic- form. Most results concern SPERNER's theorem and close modifications. All these papers state the theorems for certain partial orders. We do not even give the list of these papers because KLEITMAN's paper in this tract contains it. The results contain all important combinatorial ana-

logues of SPERNER's theorem with one exception: the partitions of a finite set under refinement.

For generalizing other problems, there is only one result by HSIEH [38]. It solves an analogue of theorem 2: what is the maximal number of k -dimensional non-disjoint subspaces? And what is interesting, the harder problem, when the subspaces must have 1-dimensional common subspaces, is also solved for small n 's. Compare this with (11)-(12) which is true only for large n 's. The reason for the difference is, that the middle levels of the partial order of the subspaces are much larger than those of the subsets of a set.

It would be nice to have an algebraic generalization of (44). However, it seems to be hard, because besides the partial order we need an ordering in the levels of the elements of the same rank.

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APPLICATIONS OF RAMSEY STYLE THEOREMS
TO EIGENVALUES OF GRAPHS *)

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1. INTRODUCTION

Let G be a graph, $A(G)$ its adjacency matrix, i.e. $A = (a_{ij})$ is given by

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent vertices,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $A = A(G)$ is a symmetric matrix whose entries are 0 and 1, with every $a_{ii} = 0$. For any real symmetric A , we denote its eigenvalues by

$$\lambda_1(A) \geq \lambda_2(A) \geq \dots$$

or

$$\lambda^1(A) \leq \lambda^2(A) \leq \dots,$$

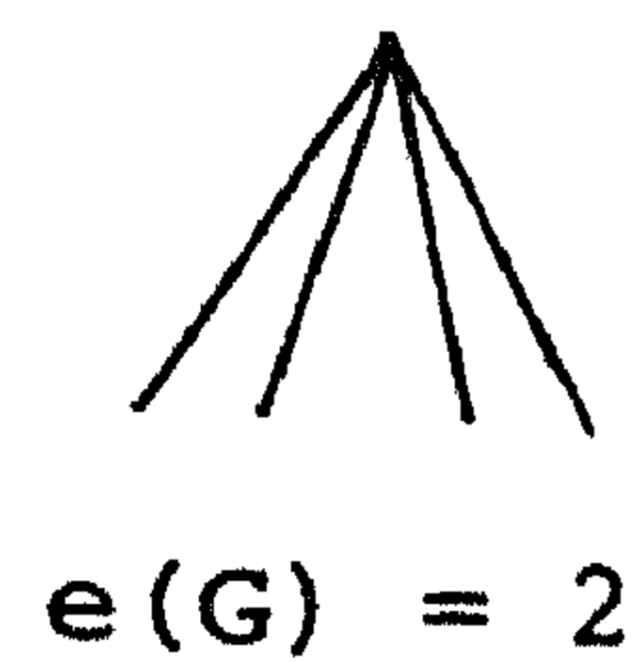
as is convenient. For $A = A(G)$, we sometimes write $\lambda_i(G)$ or $\lambda^i(G)$ for $\lambda_i(A(G))$ or $\lambda^i(A(G))$ respectively.

There have been many investigations in graph theory, experimental designs, group theory, etc. in which knowledge of properties of $\{\lambda_i(G)\}$ -which we shall henceforth call the *spectrum* of G - has been very useful even where the eigenvalues are not mentioned in either the hypotheses or conclusions of the theorems proved. These investigations furnish part of (the rest is natural curiosity) the motivation for study of questions where

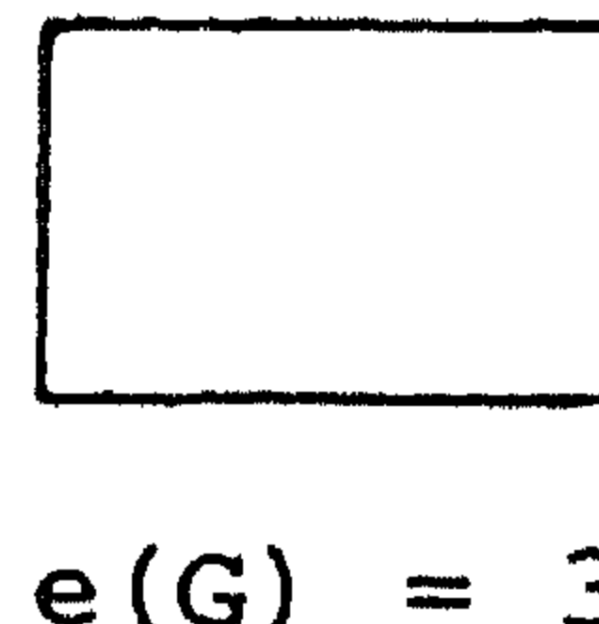
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the eigenvalues play an explicit role. More specifically, we can ask (and sometimes answer) questions of the following type:

- (1) What properties of a graph control the magnitude of $\lambda^i(G)$ or $\lambda_i(G)$?
We can answer this in a limited way for λ_1, λ_2 and λ^1 .
- (2) If G is an induced subgraph of H , written $G \subset H$ then (as we know from matrix theory), $\lambda_i(G) \leq \lambda_i(H)$, $\lambda^i(G) \geq \lambda^i(H)$. Suppose we specify that every vertex of H must have large valence. Then by how much must $\lambda_i(H)$ exceed $\lambda_i(G)$ or $\lambda^i(G)$ exceed $\lambda^i(H)$? We can answer this for λ_2 and λ^1 .
- (3) Define a relationship \sim on $V(G)$ to mean $i \sim j$ if for every $k \neq i, j$, $a_{ik} = a_{jk}$, and let $e(G)$ be the number of equivalence classes so defined. The examples



and



show that $e(G)$ is not uniquely determined by the spectrum of G (which is $(2, 0, 0, 0, -2)$ in both cases). But is the magnitude $e(G)$ roughly determined by the spectrum? The answer is yes in a sense to be made precise later (and this result will be completely proved in these notes, whereas other results will be sketched).

- (4) What real numbers can be limit points of the $\{\lambda_1(G)\}$, as G ranges over all graphs, or $\{\lambda_2(G)\}, \dots$, or $\{\lambda^1(G)\}$, etc.? We know the early limit points for $\{\lambda_1(G)\}$ and $\{\lambda^1(G)\}$.
- (5) Suppose (as is sometimes done) we represent G by the matrix $B(G) = J - 2A(G)$. What can be said about the spectrum of $B(G)$? To show that the study of $B(G)$ has its own surprises, we mention (and will indicate the proof of): for $i > 1$, $|\lambda_i(B)| \leq 2^{9(\lambda^1(B))^2}$.

Of course the study of these questions mixes ideas of graph theory and matrix theory, and the principal tools from graph theory are RAMSEY's theorem and some relatives.

2. THE RAMSEY STYLE THEOREMS

For theorems 2.1-2.3, S is a set of symbols, $|S| = s \geq 2$.

THEOREM 2.1. *There exists a function $R(n,s)$ such that every symmetric matrix of order $R(n,s)$ with entries in S contains a principal submatrix of order n , with all diagonal entries the same, all off diagonal entries the same.*

This is essentially RAMSEY's theorem, which needs no proof here.

THEOREM 2.2. *There exists a function $Z(n,s)$ such that every square matrix with entries in S of order n containing $Z(n,s)$ contains a square submatrix of order n , every entry of which is the same.*

This is easy to prove directly (see, e.g., the special case $s = 2$ given in [1], which easily generalizes) but we will give another proof soon.

THEOREM 2.3. *There exists a function $H(n,s)$ such that every matrix with entries in S containing $H(n,s)$ rows, no two the same, contains a square submatrix M of order n , which (after permutations of rows and columns) has the appearance*

$$(2.1) \quad \begin{bmatrix} a & & & c \\ & \cdot & & \\ & & \cdot & \\ b & & & a \end{bmatrix}$$

(all diagonal entries a , lower triangle entries b , upper triangle entries c), a, b, c not all the same.

PROOF OF THEOREM 2.2. (assuming theorem 2.3). Let $Z(n,s) = nH(2n,s)$, and let A be of order $Z(n,s)$. If at least $H(2n,s)$ rows are different, then the lower left (or upper right) part of (2.1) yields the desired submatrix of order n . Hence, we may assume that A has n rows the same. By symmetry, A has n columns the same. \square

PROOF OF THEOREM 2.3. (cf. [3]). We first prove:

$$(2.2) \left\{ \begin{array}{l} \text{If } f(r) = 2(s^r - 1)/(s - 1), \text{ if } A \text{ has all entries in } S \text{ and has } f(r) \\ \text{different rows, then } A \text{ contains a submatrix } B \text{ of } 2r \text{ rows and } r \\ \text{columns which, after permutation of rows and columns has the} \\ \text{form} \\ \\ b_{11} \neq b_{21}, b_{32} \neq b_{42}, \dots, b_{2r-1,r} \neq b_{2r,r} \\ \\ b_{2k-1,j} = b_{2k,j} \quad \text{for } j \neq k, k, j = 1, \dots, r. \end{array} \right.$$

When $r = 1$, $f(r) = 2$, verifying (2.2) in this case. Assume (2.2) has been shown for $r-1$, and let A have $f(r)$ different rows. We may also assume that each column of A is essential for the statement that any two rows are different (otherwise, such an inessential column can be discarded). Now suppose the first column of A were discarded. Then two rows of A (say 1 and 2) would be the same. Hence we may assume $a_{11} \neq a_{21}$, but $a_{ij} = a_{ij}$ for $j > 1$. Of the $f(r)-2$ remaining rows, at least $(f(r)-2)/s$ must be the same in column 1. Now

$$\frac{f(r)-2}{s} = \left(\frac{2(s^r - 1)}{s - 1} - 2 \right) / s = f(r-1).$$

The induction hypothesis applied to these $(f(r)-2)/s$ rows and the remaining columns of A completes the proof of (2.2).

Consider the matrix B given by (2.2). We use it to define a symmetric matrix C of order r on $\binom{s}{2} + s^2$ symbols as follows: c_{ii} is the unordered pair $\{b_{2i-1}, b_{2i}\}$ of distinct symbols in S ; c_{ij} is the ordered pair $(b_{2i,j}; b_{2j,i})$ if $i < j$, the reverse if $i > j$. Assume $r = R(n, \binom{s}{2} + s^2)$. By theorem 2.1, C contains a principal submatrix of order n in which all diagonal entries are the same, all off diagonal entries are the same. Referring back to B , this means B contains a matrix D with $2n$ rows and n columns such that (after possible row interchanges), we have

$$\begin{aligned} d_{11} &= d_{32} = \dots = d_{2n-1,n} = (\text{say}) a_1 \\ d_{21} &= d_{42} = \dots = d_{2n,n} = (\text{say}) a_2 \\ d_{2k-1,j} &= d_{2k,j} = (\text{say}) b, \quad \text{for all } j > k \\ d_{2k-1,j} &= d_{2k,j} = (\text{say}) c, \quad \text{for all } j < k. \end{aligned}$$

Clearly the odd or even rows of D (either unless $b = c$, and one of a_1, a_2 is $b=c$) produce the desired submatrix. Thus we have shown

$$H(n, s) = 2 \frac{s^{\binom{s}{2} + s^2 - 1}}{s-1} . \square$$

The last Ramsey style theorem we will use is

THEOREM 2.4. *There exists a function $S(n)$ such that, if G is a connected graph on $S(n)$ vertices, G contains a vertex of valence at least n , or a path of length n as an induced subgraph.*

The proof of this theorem is too easy to give here.

3. QUESTION (1)

Define H_n to be the graph on $2n+1$ vertices, in which one vertex is not adjacent to exactly n other vertices, but all other pairs of vertices are adjacent. For any graph G , let $\ell(G)$ be the smallest positive integer such that neither $K_{1, \ell}$ nor H_ℓ is an induced subgraph of G . The following theorem is proved in [2].

THEOREM 3.1. *The function $\ell(G)$ is bounded from above and below by a function of $|\lambda^1(G)|$.*

The proof of theorem 3.1 has three parts. The first part shows that $\ell(G)$ is bounded from above by a function of $|\lambda^1(G)|$. The second part proceeds as follows: The distance $d(G, H)$ between two graphs G and H with $V(G) = V(H)$ is defined by making $d(G, H)$ the maximum valence of the vertices in the graph $(G-H) \cup (H-G)$. Then $L(G)$ is defined to be the smallest integer L such that there exists a graph H with

$$d(G, H) \leq L$$

and H having a distinguished family of cliques K^1, K^2, \dots satisfying

- (i) every edge of H is in at least one K^i ,
- (ii) every vertex of H is in at most L of the K^i 's,
- (iii) $|V(K^i) \cap V(K^j)| \leq L$ for $i \neq j$.

Finally, it is proved that $L(G)$ is bounded from above by a function of $\ell(G)$. The third part of the proof shows that $|\lambda^1(G)|$ is bounded from above by a

function of $L(G)$.

The proofs in the first and third parts are arguments from matrix theory, and are in [2]. The proofs in the second part use RAMSEY's theorem (theorem 2.1 in these notes). For the remainder of this section, $S = \{0,1\}$, and we use $R(n)$ for $R(n,2)$.

The strategy is to look for "large" cliques in G ("large" depends on ℓ) and define an equivalence relation on large cliques. The equivalence classes of large cliques are themselves almost cliques, i.e., we can add a graph in which each vertex has valence bounded by a function of ℓ , so that they become cliques. These added edges are the edges of $H-G$. Further the edges of G not contained in any large clique form a graph in which each vertex has valence bounded by a function of ℓ , and these edges are the edges of $G-H$.

Let $N = N(\ell) = \ell^2 + \ell + 2$. Define W to be the set of all cliques $K \subset G$ such that $|V(K)| \geq N$. We shall prove the statements given in the preceding paragraph for the cliques in W (the full discussion, including (ii) and (iii), requires further conditions on N).

LEMMA 3.1. *If $K, K' \in W$, define*

$$K \sim K'$$

if each vertex of K is adjacent to all but at most $\ell-1$ vertices of K' . Then \sim is an equivalence relation.

PROOF. Reflexivity is clear, since $|V(K)| \geq \ell$. To prove symmetry, assume there is a vertex v in K' not adjacent to at least ℓ vertices in K , and let A denote that set of ℓ vertices in K . Each vertex in A is not adjacent to at most $\ell-2$ vertices in K' other than v , since $K \sim K'$. Hence, the set of vertices in K' each not adjacent to at least one vertex in A consists of v and at most $\ell(\ell-2)$ other vertices. Since $N > \ell + \ell(\ell-2) + 1$, it follows that K' contains at least ℓ vertices each of which is adjacent to each vertex in A . Call that set of ℓ vertices B . Then v, A, B generate an H_ℓ , contrary to the definition of ℓ . This contradiction proves that \sim is symmetric.

To prove transitivity, assume $K^1 \sim K^2$, $K^2 \sim K^3$, $K^1 \not\sim K^3$. Then K^3 contains a vertex v not adjacent to a set C of ℓ vertices in K^1 . Since $N > 2\ell + \ell(\ell-1) - 1$, and $K^1 \sim K^2$, it follows that K^2 contains a subset D of $2\ell-1$ vertices each of which is adjacent to all vertices in C . But since $K^3 \sim K^2$, D contains some subset F of ℓ vertices adjacent to v . Then

C, F, v generate an $H_\ell \subset G$, which is a contradiction. \square

Henceforth, the letter E will denote any equivalence class of cliques in W , and $V(E)$ will be the union of all vertices of all cliques in E .

LEMMA 3.2. *Let E be an equivalence class, $v \in V(E)$. Then v is adjacent to all but at most $R(\ell)-1$ other vertices in $V(E)$.*

PROOF. Let $K^v \in E$ be a clique containing v . By RAMSEY'S theorem, if $F \subset V(E)$, $|F| \geq R(\ell)$, and every vertex in F not adjacent to v , then F contains a K_ℓ or its complement \bar{K}_ℓ . If $\bar{K}_\ell \subset F$, then since $|V(K^v)| > \ell^2 - 2\ell + 1$, there exists a vertex $w \in K^v$ adjacent to all vertices in \bar{K}_ℓ . Thus $K_{1,\ell} \subset G$, a contradiction.

If $K_\ell \subset F$, then $|V(K^v)| > \ell + \ell(\ell-2) + 1$ implies that K^v contains a set of ℓ vertices each adjacent to all the vertices in K_ℓ , thus generating an H_ℓ . \square

LEMMA 3.3. *Let \tilde{H} be the graph formed by edges of G not in any clique in W . Then every vertex in \tilde{H} has valence at most $R(N)$.*

PROOF. If not, then by RAMSEY'S theorem we would have $K_{1,\ell} \subset G$, or the vertices adjacent to v in \tilde{H} would contain a clique in W , contradicting the definition of \tilde{H} . \square

Results analogous to theorem 3.1 have been established for $\lambda_2(G)$ by HOWES [6]. For $\lambda_1(G)$, it is easy to see that its size is controlled by the size of the smallest t such that $K_{1,t} \not\subset G$, $K_t \not\subset G$. Corresponding results for other $\lambda_i(G)$ or $\lambda^i(G)$ are not known.

4. QUESTION (2)

Let us define

$$\mu^i(G) \equiv \lim_{d \rightarrow \infty} \sup_{\substack{H \supset G \\ d(H) > d}} \lambda^i(H),$$

where $d(G)$ is the minimum valence of the vertices of G . Also

$$\mu_i(G) \equiv \lim_{d \rightarrow \infty} \inf_{\substack{H \supset G \\ d(H) > d}} \lambda_i(H).$$

It is easy to see that $\mu_1(G) = \infty$. Further, we shall give formulas for μ_2 and μ^1 showing that they are finite. By matrix theory arguments, this will prove the existence and finiteness of all other μ^i and μ_i , although we have no formulas for them.

THEOREM 4.1. (cf. [4]). Let $|V(G)| = m$ and let C^1 be the class of all $(0,1)$ matrices C with m rows such that every row sum of C is positive, but this property is lost if any column is deleted. Then

$$\mu^1(G) = \max_{C \in C^1} \lambda^1(A - CC^T).$$

THEOREM 4.2. Let $|V(G)| = m \geq 2$, and let C_2 be the class of all $(0,1)$ matrices C with m rows and at least two columns such that every row sum of C is positive, and, if C has more than two columns, no column can be deleted without destroying the property that C has positive row sums. Then

$$\mu_2(C) = \min_{C \in C_2} \lambda_1(A - C(J - I)^{-1}C^T).$$

We shall not make any remarks about the proof of theorem 4.2, except to state that, as far as the theme of this lecture is concerned, the use of Ramsey style theorems is analogous to the uses in the proof of theorem 4.1.

To prove theorem 4.1, one proceeds as follows: Let any $C \in C^1$ be given, and let C have k columns. Extend G to a graph $G_C(n)$ by adjoining k cliques K^1, \dots, K^k , each with n vertices, such that every vertex of clique K^j is adjacent to vertex i of G when $c_{ij} = 1$, and not adjacent when $c_{ij} = 0$. Additionally, no vertices of K^j and K^l , $j \neq l$, are adjacent. Then a little algebra shows that $\lim_{n \rightarrow \infty} \lambda^1(G_C(n)) = \lambda^1(A - CC^T)$, so

$$(4.1) \quad \mu^1(G) \geq \max_{C \in C^1} \lambda^1(A - CC^T).$$

Note that, if $d_1 > d_2$, $\sup_{\substack{H \supset G \\ d(H) > d_1}} \lambda_1(H) \leq \sup_{\substack{H \supset G \\ d(H) > d_2}} \lambda_1(H)$. Together with (4.1)

this implies $\mu^1(G)$ exists and is finite. To prove that

$$(4.2) \quad \mu^1(G) \leq \max_{C \in C^1} \lambda^1(A - CC^T),$$

let $\epsilon > 0$ be given. Choose an integer n such that, for all the finitely many graphs $G_C(n)$ discussed in the proof of inequality (4.1), we have

$$(4.3) \quad \lambda^1(G_C(n)) \leq \lambda^1(A-CC^T) - \epsilon.$$

From the definition of $\mu^1(G)$, we know that, for any d , there exists $H \supset G$ such that

$$(4.4) \quad d(H) > d$$

and

$$(4.5) \quad \mu^1(G) \leq \lambda^1(H).$$

We shall choose d (depending on n (hence on ϵ) and $\mu^1(G)$) such that, for any $H \supset G$ and satisfying (4.4) and (4.5), there is a $C \in \mathcal{C}^1$ such that

$$(4.6) \quad G_C(n) \subset H,$$

which, by matrix arguments, shows

$$(4.7) \quad \lambda^1(H) \leq \lambda^1(G_C(n)).$$

Combining (4.5), (4.7) and (4.3) yields (4.2).

So the critical thing is to prove (4.6). Besides RAMSEY's theorem, we need theorem 2.2, and we write $Z(n)$ for $Z(n,2)$ (our set S is $\{0,1\}$). $Z^{(k)}(n)$ will be the k -th iterate of $Z(n)$.

By section 3 we know that there is a function ℓ such that

$$(4.8) \quad K_{1,\ell(\mu^1)} \not\subset H$$

$$(4.9) \quad H_{\ell(\mu^1)} \not\subset H.$$

Let $D(G)$ be the maximum valence of the vertices of G , let

$$(4.10) \quad N = R(\max\{\ell(\mu^1), Z^{\binom{m}{2}}(\max\{n, \ell(\mu^1)\})\}),$$

$$(4.11) \quad d \geq 2^{m-1}N + D(G).$$

Since every vertex of H has valence at least d , every vertex of $G \subset H$ is adjacent to at least $2^{m-1}N$ vertices in $H-G$. For each $S \subset \{1, \dots, m\}$, $S \neq \emptyset$, let $H(S)$ be the set of vertices in $H-G$, each of which is adjacent to the vertices in S and to no other vertices in G . Hence, there exist sets S_1, S_2, \dots, S_t ($t \leq m$) of subsets of $\{1, \dots, m\}$ such that every vertex in G is in at least one S_i , discarding any S_i loses this property, the sets $H(S_i)$ are disjoint, and each $|H(S_i)| \geq N$. Let

$$\bar{Z} = Z \binom{m}{2} (\max \{n, \ell(u^1)\}).$$

We contend $H(S_i)$ contains a clique $K_{\bar{Z}}$. For, from (4.10) and RAMSEY's theorem, the only other possibility is that $H \supset H(S_i)$ contains $K_{1, \ell(u^1)}$, (note that $H(S_i)$ is attached to at least one vertex of G), violating (4.8).

Next, consider each of the t disjoint cliques $K_{\bar{Z}^1}, \dots, K_{\bar{Z}^t}$. Take any two of these cliques. By theorem 1.2, they either contain large subcliques (at least $\ell(u^1)$ in size) with all vertices adjacent (impossible by (4.9), since each is attached to a vertex of G not adjacent to the other), or large subcliques with no vertices of one adjacent to vertices of the other. Since $t \leq m$, we need only iterate this process at most $(m-1)+(m-2)+\dots+1 = \binom{m}{2}$ times. When we are done, we have the desired C and $G_C(n)$. (We have used here tacitly that $Z(a) \geq a$, of course).

5. QUESTION (3)

We shall show that the magnitude of $e(G)$, defined in the introduction (3), is controlled by the spectrum of G . For $a \leq b$, define $\Lambda(a, b)(G)$ to be the number of eigenvalues of G each of which is at most a or at least b . For the function $H(n, m)$ in theorem 2.3, write $H(n)$ for $H(n, 2)$ ($S=\{0, 1\}$).

THEOREM 5.1.

$$\Lambda((\sqrt{5}-1)/2, 1)(G) \leq e(G) < R(H(R([5\Lambda+1])),$$

(where Λ on the right is the same as Λ on the left).

To prove the left inequality is easy. Suppose $i \sim j$ and i and j are adjacent. Then it is easy to see that if $j \sim k$, i and k , and j and k are

adjacent. If i and j are not adjacent, and $j \sim k$, then i and k are not adjacent and j and k are not adjacent. Hence each equivalence class is a clique or an independent set. Now if there are e equivalence classes, the matrix $A(G)$ can be partitioned

$$A(G) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1e} \\ A_{21} & A_{22} & \cdots & A_{2e} \\ \vdots & \vdots & \ddots & \vdots \\ A_{e1} & A_{e2} & & A_{ee} \end{bmatrix}$$

such that each A_{ij} ($i \neq j$) is 0 or J , each A_{ii} is 0 or $J-I$. Thus the number of eigenvalues not 0 or -1 of G is at most e . But every eigenvalue in Λ is not 0 or -1 .

To prove the right inequality, assume it false and that $e(G) \geq R(H(R([5\Lambda+1])))$. It follows from theorem 2.1 that there exists a subset X of $H(R([5\Lambda+1]))$ vertices of G which are inequivalent, and form either a clique or an independent set. In any case, the rows of $A(G)$ corresponding to these vertices, together with the columns of $A(G)$ corresponding to the complementary set of vertices, form a submatrix of $A(G)$ in which all rows are different. By theorem 2.3, there exists a square submatrix of this matrix of order $R([5\Lambda+1])$, which, after row and column permutations, has the form $I, J-I$, or triangular. Let S be the set of vertices corresponding to the columns of this submatrix. By theorem 1.1, since $|S| = R([5\Lambda+1])$, S contains a subset Y of vertices, with $|Y| = [5\Lambda+1]$ where Y is a clique or independent set. The corresponding $[5\Lambda+1]$ vertices of X form a subset W such that the incidence matrix of W versus Y is $I, J-I$, or triangular (say V). Thus, setting $m = [5\Lambda+1]$, $A(G)$ contains a principal submatrix of order $(2m \times 2m)$ of one of the following forms

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & J-I \\ J-I & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & V \\ V^T & 0 \end{pmatrix}$$

$$\begin{pmatrix} J-I & I \\ I & 0 \end{pmatrix}, \quad \begin{pmatrix} J-I & J-I \\ J-I & 0 \end{pmatrix}, \quad \begin{pmatrix} J-I & V \\ V^T & 0 \end{pmatrix}$$

$$\begin{pmatrix} J-I & I \\ I & J-I \end{pmatrix}, \quad \begin{pmatrix} J-I & J-I \\ J-I & J-I \end{pmatrix}, \quad \begin{pmatrix} J-I & V \\ V^T & J-I \end{pmatrix}.$$

In each of these nine cases, the matrix has more than Λ eigenvalues $\leq (-\sqrt{5}-1)/2$ or ≥ 1 , which is impossible.

It is interesting that if 1 is replaced by $1 + \varepsilon$ or $(-\sqrt{5}-1)/2$ replaced by $((-\sqrt{5}-1)/2) - \varepsilon$, the right-hand inequality is no longer true.

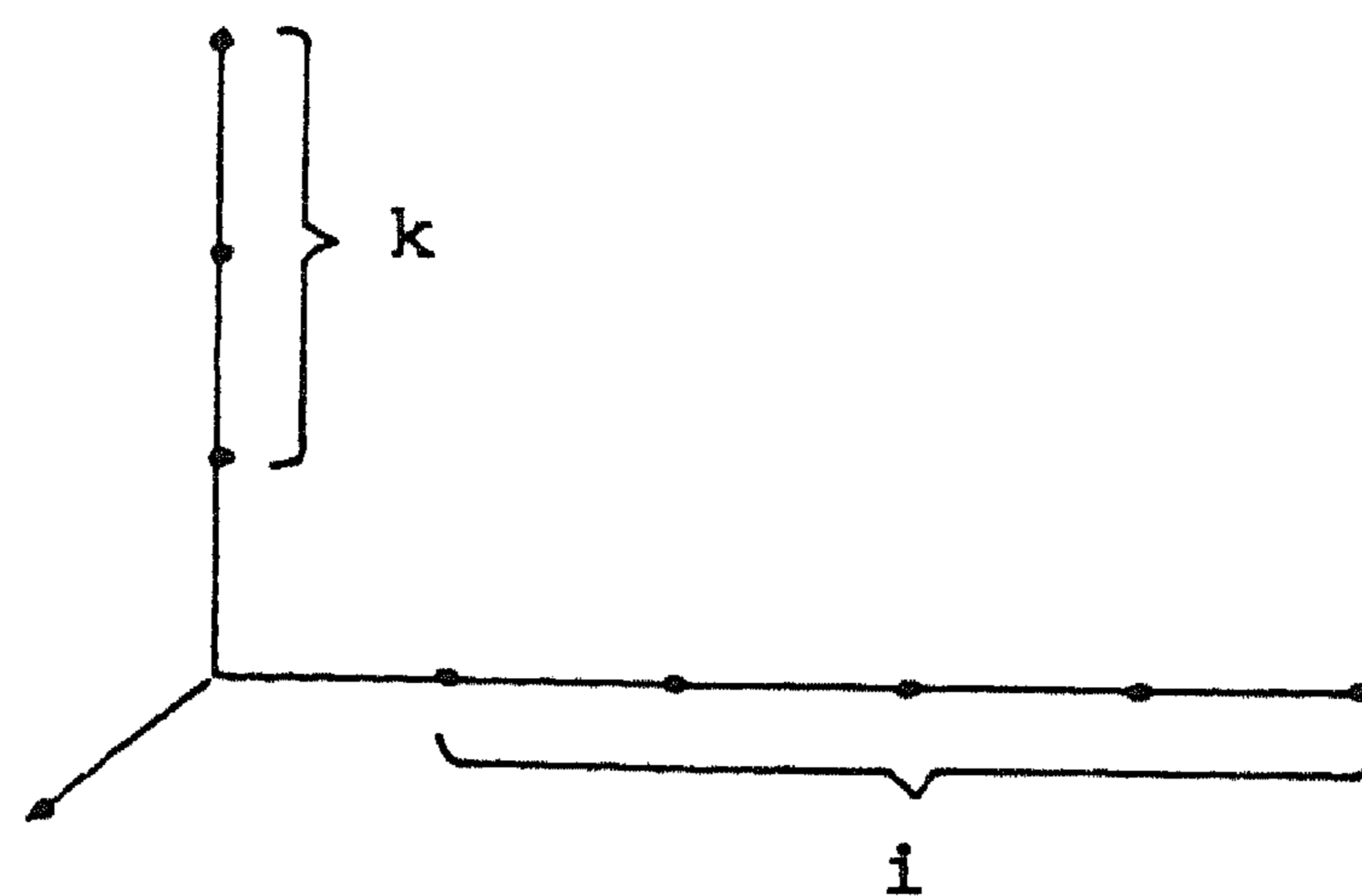
The same arguments will establish a theorem similar to theorem 5.1 if 1 is replaced by $(\sqrt{5}-1)/2$ and $(-\sqrt{5}-1)/2$ is replaced by -2 .

6. QUESTION (4)

We only know some small limit points of $\{\lambda_1(G)\}$ and of $\{\lambda^1(G)\}$ as G varies over all graphs. To be more specific, let Λ_1 be the set of distinct numbers each of which is $\lambda_1(G)$ for some G .

THEOREM 6.1. (cf. [5]). Let $P_n(x) = x^{n+1} - (1+x+\dots+x^{n-1})$, $n=1,2,\dots$. Let β_n be the unique positive root of P_n , and $\alpha_n = \beta_n^{1/2} + \beta_n^{-1/2}$. Then the numbers $2 = \alpha_1 < \alpha_2 < \dots$ are all limit points of Λ_1 less than $\tau^{1/2} + \tau^{-1/2}$, where $\tau = \frac{1}{2}(\sqrt{5}+1)$.

A very rough sketch of the proof proceeds as follows. If α is a limit point of Λ_1 , then there must exist a sequence of connected graphs G_1, G_2, \dots such that $\lambda_1(G_i)$ are all different, and $\lambda_1(G_i) \rightarrow \alpha$. But this means that the graphs G_i are all different, hence $|V(G_i)| \rightarrow \infty$, hence (theorem 2.4) for sufficiently large i , G_i contains a vertex of large valence (impossible, for this implies $\lambda_1(G_i) \rightarrow \infty$) or a long path. For a path S_n of length n , $\lambda_1(S_n) \rightarrow 2$, which must be the smallest limit point. Next, assume $2 < \alpha < \tau^{1/2} + \tau^{-1/2}$. Then G_i cannot be a simple circuit infinitely often, for $\lambda_1(\text{circuit}) = 2$. One can also show that if G_i contains a circuit, and at least one additional edge, $\lambda_1(G_i) > \tau^{1/2} + \tau^{-1/2}$. Hence, this cannot occur infinitely often, so we may assume each G_i is a tree. One can also prove that, if a tree G_i contains at least three vertices of valence at least three, $\lambda_1(G_i) > \tau^{1/2} + \tau^{-1/2}$. Further, if (infinitely often), G_i contains a vertex of valence at least four, then $\lim \lambda_1(G_i) > \tau^{1/2} + \tau^{-1/2}$. Continuing arguments and calculations in this vein, one finds that the desired limit points must be limit points of the biggest eigenvalues of the sequence of graphs $\{G_i^k\}$, where k is fixed, $i \rightarrow \infty$, and G_i^k is



and $\lim_{i \rightarrow \infty} \lambda_1(G_i^k) = \alpha_k$.

If we define Λ^1 to be the set of distinct numbers each of which is $\lambda^1(G)$ for some G , then

THEOREM 6.2. *Let T be a tree on at least two vertices, $L(T)$ its line graph, e an end of T , $\tilde{A}(L(T,e))$ be $A(L(T))$ modified so that there is -1 in the diagonal position corresponding to e . Then $\lambda^1(\tilde{A}(L(T,e))) > -2$, and every limit point of $\Lambda^1 > -2$ occurs in this way.*

Here again we use theorem 2.4. For if $\lim_{i \rightarrow \infty} \lambda^1(G_i) = \alpha > -2$, $\{G_i\}$ can not contain arbitrarily long paths S_n , because $\lim_{n \rightarrow \infty} \lambda^1(S_n) = -2$. So, for i large, G_i must contain at least one vertex v of large valence. By theorem 2.1, the vertices adjacent to v contain a large clique or large independent set; but the latter is impossible, because this would imply $K_{1,m} \subset G_i$ for m large, which contradicts $\lambda^1(G_i)$ of modest size (see section 3). Hence, we know that, for i large, G_i contains at least one large clique, say K_n . (This large clique eventually gets associated with the edge e of the tree which plays a special role.) The remainder of the proof uses theorems 2.2 and 2.1 to show that no vertex not in K_n can have large valence, in spirit similar to other applications already described.

We conjecture that $\lambda^1(\tilde{A}(L(T,e)))$ is always a limit point, but lack a proof at this time.

7. QUESTION (5)

To illustrate some of the fun in looking at symmetric matrices with entries ± 1 , (rather than $(0,1)$), we indicate the proof of

THEOREM 7.1. (cf. [1]). *There exists a function f such that, if A is any symmetric ± 1 matrix, $|\lambda_i(A)| \leq f(\lambda^1(A))$, for all $i > 1$.*

The argument consists of showing that there exists a matrix M of the form

$$M = \begin{pmatrix} + & - \\ - & + \end{pmatrix},$$

where the diagonal blocks are square (not necessarily of the same size; indeed, one may be empty), so that $A-M = P$ consists of a matrix in which only $\frac{1}{2}f(\lambda^1)$ entries per row are non-zero (i.e., ± 2), after which matrix arguments produce the desired inequality. To construct M , it turns out to be sufficient to prove that given a lower bound on $\lambda^1(A)$, A cannot have two rows (say 1 or 2) such that $a_{1j} = a_{2j}$ for many j , and also $a_{1j} = -a_{2j}$ for many j . Assume otherwise. Then we have a right to assume A has a principal submatrix

$$\tilde{A} = \begin{pmatrix} \pm 1 & \pm 1 & +1 \dots +1 & +1 \dots +1 \\ \pm 1 & \pm 1 & +1 \dots +1 & -1 \dots -1 \\ \hline +1 & +1 & & \\ \cdot & \cdot & A_{11} & A_{12} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ +1 & +1 & & \\ \hline +1 & -1 & & \\ \cdot & \cdot & A_{21} = A_{12} & A_{22} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ +1 & -1 & & \end{pmatrix} \begin{matrix} \left. \vphantom{\begin{matrix} \pm 1 \\ \pm 1 \\ +1 \\ \cdot \\ \cdot \\ +1 \end{matrix}} \right\} n \\ \left. \vphantom{\begin{matrix} +1 \\ \cdot \\ \cdot \\ +1 \end{matrix}} \right\} n \end{matrix}$$

By theorem 2.2, \tilde{A} (and hence A) has a principal submatrix (which we call $\tilde{\tilde{A}}$) of order n of the same form, where $A_{12} = J$ or $A_{12} = -J$. In either case, $\tilde{\tilde{A}}$ has a least eigenvalue which goes to $-\infty$ for large n .

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FOUNDATIONS, PARTITIONS AND COMBINATORIAL GEOMETRY

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SOME RECENT DEVELOPMENTS IN RAMSEY THEORY

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Recently a number of striking new results have been proved in an area becoming known as RAMSEY THEORY. It is our purpose here to describe some of these. Ramsey Theory is a part of combinatorial mathematics dealing with assertions of a certain type, which we will indicate below. Among the earliest theorems of this type are RAMSEY's theorem, of course, VANDER WAERDEN's theorem on arithmetic progressions and SCHUR's theorem on solutions of $x+y = z$.

To make our task easier, we will introduce the "arrow notation" of ERDŐS and RADO. This was originally used for generalizations of Ramsey's Theorem to infinite cardinals, but can be easily adapted to other cases as well. The meaning of the arrow notation will become clear by its use in the examples throughout this paper.

As our first example, consider:

$$n \xrightarrow{k} (l_1, \dots, l_r) .$$

This expression is just an abbreviation for the following assertion: if the k -element subsets of an n -element set are partitioned into r classes, then for some i there is an l_i -element subset L_i of the n -element set such that all the k -element subsets of L_i are in the i -th class.

THEOREM. (RAMSEY). *For all positive integers k, r, l_1, \dots, l_r , there exists an $N = N(k, r, l_1, \dots, l_r)$ such that if $n \geq N$, then $n \xrightarrow{k} (l_1, \dots, l_r)$.*

In fact, RAMSEY considered only the case where all the l_i are equal.

He also proved $\aleph_0 \xrightarrow[k]{\aleph_0, \dots, \aleph_0}$ which actually is stronger than the finite theorem above. The consideration of such statements with large cardinals or ordinals is subject in itself and will not be discussed here. For the large cardinals the subject is fairly complete and will be covered in a forthcoming book of ERDŐS, HAJNAL & RADO. For ordinals, the theory is developing rapidly, although there are still many open questions. To give the flavor of a result of this type, we mention one of the most interesting recent ones.

THEOREM. (CHANG, LARSEN, MILNER).

$$\omega^\omega \xrightarrow{2} (\omega^\omega, k)$$

This theorem asserts that if the pairs (i.e., 2-element subsets) of a set of order type ω^ω are partitioned into two classes, then either the first class contains all the pairs of a subset with induced order type ω^ω , or the second class contains all the pairs of some k -element subset.

This last example illustrates the arrow notation in a case where we deal with sets with structure (here the structure is that of order).

In general, in a Ramsey Theorem an assertion of the form $A \xrightarrow[B]{C_1, \dots, C_r}$, where the symbols A , B and C_i denote objects with certain structure. For example, as above, they could be sets or sets with order. Other examples include graphs, finite vector spaces, sets containing solutions to systems of linear equations, Boolean algebras and partitions of finite sets.

In the remainder of the paper, we will consider six examples of Ramsey theorems. The first two concern graphs and are due to W. DEUBER and to J. NEŠETŘIL & V. RÖDL. The next three concern systems of linear equations and their solution sets. These are results of N. HINDMAN, E. SZEMERÉDI and W. DEUBER. Finally, we will discuss some results of K. LEEB on abstract categories which are "*Ramsey*" graphs.

Recalling the previous statement of Ramsey's Theorem, we see that the first non-trivial case is

$$6 \xrightarrow{2} (3,3) .$$

This can be restated as follows: if the edges of the complete graph K_6 on six vertices are 2-colored arbitrarily, then some monochromatic triangle K_3 must be formed. This graphical form leads to several general considerations. The most natural of these, an immediate consequence of Ramsey's Theorem

(with $k = 2$), is simply:

For every finite graph H , there is a finite graph G such that $G \xrightarrow{2} (H, H)$.

Here, the arrow notation means that if the *edges* of G (represented by the 2 below the arrow) are 2-colored arbitrarily, then G will contain a monochromatic subgraph isomorphic to H .

It would be stronger to require that the monochromatic subgraph above be an *induced* subgraph of G . We could write $G \xrightarrow{2} (H, H)$ also in this case, provided we understand that we mean induced subgraphs here. Actually, to be rigorous, we should use a "different" kind of arrow for each different meaning. The proper setting for this is in terms of category theory as originally indicated by LEEB. We will elaborate on this when we discuss LEEB's recent results at the end of this paper.

We now turn our attention to the first result, which concerns *induced* subgraphs of graphs.

THEOREM. (DEUBER [2]). *For every finite graph H , there exists a finite graph G such that $G \xrightarrow{2} (H, H)$.*

SKETCH OF PROOF. What DEUBER actually proves is the equivalent but more convenient statement: for every choice of finite graphs G and H there exists a finite graph K such that $K \xrightarrow{2} (G, H)$. The proof is by induction on $|G| + |H|$ where $|G|$ denotes the number of vertices of G . The small cases are trivial. Let g be a vertex of G , $\bar{G} = G - \{g\}$, and let S be the subset of G to which g is connected. Also, let h in H , \bar{H} and T be defined similarly.

By induction we can find G^* and H^* such that $G^* \xrightarrow{2} (\bar{G}, H)$ and $H^* \xrightarrow{2} (G, \bar{H})$. We now form a large graph K as follows: Start with G^* . Let $\bar{G}_1, \dots, \bar{G}_m$ be all the occurrences of \bar{G} as an induced subgraph of G^* and let S_1, \dots, S_m be the corresponding subsets S (there may be more than one choice for an S_i ; any one is allowed). Now replace each vertex of $S = S_1 \cup \dots \cup S_m = \{x_1, \dots, x_\ell\}$ by a complete copy of H^* , with the copy of H^* replacing x_i denoted by H_i^* . Connect a vertex of H_i^* to a vertex of H_j^* iff x_i and x_j are connected in G^* . Also, if some vertex v is not in S , connect v to all the vertices of H_i^* iff v and x_i are connected in G^* . Thus, we have essentially "exploded" some of the vertices of G^* into H^* 's.

Suppose, in the simplest case, that all the S_i are disjoint. Let $\bar{H}_1, \dots, \bar{H}_n$ be the occurrences of \bar{H} in H^* and let T_1, \dots, T_n denote the corresponding subsets T . For each fixed S_i , consider the associated H^* 's and

choose one T_j from each H^* . For each i and such choice of T_j 's, we introduce a new vertex connected exactly to these T_j 's. Hence, if $|S_i| = k$, then for this i we have added n^k new vertices. Since we have m disjoint S_i , then there are altogether mn^k new vertices. This completes the definition of K .

Suppose now the edges of K are 2-colored, say, using the colors red and blue. By the construction of H^* , each H^* in K has either a red copy of G or a blue copy of \bar{H} . If the first alternative holds, then we are done. So assume each H^* in K contains a blue copy of H . Let y_1, \dots, y_m be the new vertices corresponding to the subsets T_j for these copies of \bar{H} (i.e., one y_i for each S_i). If any of the y_i are connected to any of the T_j by all blue edges, we are done since in this case we have a blue copy of H . Thus, we may assume that each y_i is connected by a red edge to some vertex $T \in T_j$ for each T_j to which it is connected. Let y_i be connected by red edges to t_{i1}, \dots, t_{iw} . Consider the graph \tilde{G} obtained from K by deleting all the vertices of all the copies of H^* except for the t_{ij} , and deleting all the new vertices except y_1, \dots, y_m . By construction, \tilde{G} is isomorphic to G^* together with the y_i . Also, it is an induced subgraph of K . Since each y_i is connected to the corresponding S_i by only red edges, we are done. For either $G^* \subseteq \tilde{G}$ contains a blue copy of H or, it contains a red copy of \bar{G} , say \bar{G}_i , which together with y_i forms a red copy of G . This completes the argument for the case that the S_i are disjoint.

The only obstruction preventing this from being completely general is that it usually happens that for some a and b , $S_a \cap S_b \neq \emptyset$ in G . This in turn would prevent us from choosing the same t_{ij} for both S_a and S_b when necessary. To get around this, we add another step to the construction. Namely, after replacing the vertices of S by copies of H^* , we take those in the $S_a \cap S_b$ and replace each vertex of the H^* itself by a copy of H^* , connecting it up in the same way as before. We can then be certain of obtaining a vertex connected by only red edges to some copy of H^* , and we can proceed essentially in the same way as before. \square

Of course, the graphs K resulting from this construction are usually much larger than are actually required. For example, the graph K constructed this way for the assertion $K \xrightarrow{2} (K_3, K_3)$ is $K = K_{81}$. Note also the high clique number K_{81} has relative to that of K_3 .

F. GALVIN had asked if for each finite graph H with clique number $\text{cl}(H) = k$ (where $\text{cl}(H) = \max\{n \mid K_n \text{ is a subgraph of } H\}$), there is a graph G also having $\text{cl}(G) = k$ such that $G \xrightarrow{2} (H, H)$. As above, we consider induced

subgraphs here. This question has been very recently answered in the affirmative by J. NEŠETŘIL & V. RÖDL. One sees easily that this implies DEUBER's result.

FOLKMAN, in response to a question of ERDÖS and HAJNAL, had earlier shown that there exists a graph G with $cl(G) = k$ such that $G \xrightarrow{2} (K_k, K_k)$. FOLKMAN also proved that for any G and H , there exists a K with $cl(K) = \max\{cl(G), cl(H)\}$ such that $K \xrightarrow{1} (G, H)$ (where the 1 below the arrow indicates that we are coloring the vertices of K instead of the edges). In fact, NEŠETŘIL & RÖDL also make use of this theorem.

The second result we discuss is the following:

THEOREM. (NEŠETŘIL & RÖDL). *For every finite graph H there exists a finite graph G such that $G \xrightarrow{2} (H, H)$ and $cl(G) = cl(H)$.*

SKETCH OF PROOF. The proof uses the ingenious idea of letting the vertices of G be subsets of a large set. By appropriately defining when edges occur between them, and applying Ramsey's Theorem to certain subsets, a large subset is obtained with the vertices and edges determined by it being very well behaved.

We will make some definitions first, and then indicate somewhat how the proof goes, especially for the case of $cl(H) = 2$, which is considerably simpler and more direct than the general case. We begin with the definition of the graphs (n, T, p) .

Let A, B be two p -subsets of $[1, n] = \{1, 2, \dots, n\}$. The *type* (or *p-type*) $t(A, B)$ of A and B is the pattern of their relative order, defined as follows: List the elements of $A \cup B$ in increasing order assuming $\min\{A-B\} < \min\{B-A\}$, say x_1, x_2, \dots, x_ℓ , $\ell \leq 2p$. If $x_i \in A \cap B$ replace it by two copies of itself. The new list thus obtained, say y_1, y_2, \dots, y_{2p} , is of length $2p$. The type $t(A, B)$ is then defined to be the sequence $(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{2p})$, where $\bar{y}_i = 2$ if $y_i \in A \cap B$, $\bar{y}_i = 0$ if $y_i \in A - B$, and $\bar{y}_i = 1$ if $y_i \in B - A$. We let $t(B, A) = t(A, B)$.

Let T be a set of p -types. The graph (n, T, p) is defined by having as vertices all $\binom{n}{p}$ p -subsets of $[1, n]$, and as edges, all pairs A, B of p -subsets with $t(A, B) \in T$. We define the clique number of T by $cl(T) = \sup_n cl((n, T, p))$. (Not all T have finite clique number, e.g., $\{(0, 1)\} = T$, although some do.)

The beautiful construction of (n, T, p) has the property that for large n it is extremely rich in induced subgraphs (m, T, p) , for $m < n$. This enables us to use Ramsey's Theorem ultimately to obtain very well behaved subgraphs.

The first result in the general case is to show that for each H there exist p and T , so that H is an induced subgraph of (n, T, p) for all large n , and with $\text{cl}(T) = \text{cl}(H)$. T and p are defined inductively, in general, and are quite complicated. However, for $\text{cl}(H) = 2$, we can describe T much more simply and, in fact, we can assert even more. Namely, for each H , let its vertices be ordered arbitrarily, say, x_1, x_2, \dots, x_k . Then there is a mapping $\phi: H \rightarrow (n, T, p)$ for a suitable n, T, p such that $\text{cl}(T) = 2$, and ϕ maps H isomorphically into an induced subgraph of (n, T, p) with $t(\phi(x_i), \phi(x_j))$ depending only on j , if $i < j$. In the general case $\text{cl}(H) = k$, a similar result holds, but the proof is much more complicated. For the remainder of the discussion, we restrict ourselves to $\text{cl}(H) = 2$. The mapping ϕ is defined inductively. T is the set of all types starting with some 0's, two 2's, then 0's and 1's only, e.g., $(0, 0, 0, 2, 2, 1, 1, 0, 1, 0, 1, 1)$. It is easy to see that $\text{cl}(T) = 2$.

Now suppose for large N that the edges of (N, T, p) are 2-colored. For each $(2p-1)$ -subset S of $[1, N]$ there are $\frac{(2p-1)(2p-2)}{2} \binom{2p-2}{p-1}$ pairs of p -subsets A, B with $A \cup B = S$. Of those pairs, some number m have their type in T . If we list these in some canonical order, say lexicographically, then we get for each $A \cup B$ a list of m types, corresponding to m edges, and thus m colors. But this produces a 2^m -coloring of the $(2p-1)$ -subsets of $[1, N]$. Thus, for any n , if N is large enough, Ramsey's Theorem implies that there is a subgraph (n, T, p) of (N, T, p) with all edges of a given type having the same color.

Let H be an arbitrary graph with $\text{cl}(H) = 2$, and let G^* be such that $G^* \xrightarrow{1} (H, H)$, which exists by FOLKMAN's result. Letting ϕ be as above, we have $\phi(G^*) \subseteq (n, T, p) \subseteq (N, T, p)$. Each vertex x_j of G^* is associated with a single type $t(\phi(x_i), \phi(x_j))$ for $i < j$, and thus with a single color. By choice of G^* , then, we obtain a subgraph H all of whose vertices have the same color. But by the definition of this coloring, all edges of H have the same color. This completes the case $\text{cl}(H) = 2$, since by letting $G = (N, T, p)$ we have $G \xrightarrow{2} (H, H)$. As previously remarked, the proof for the general case $\text{cl}(H) = k$ is similar in spirit but with somewhat more complicated details. \square

LINEAR EQUATIONS

Let $L = L(x_1, \dots, x_n)$ denote a finite system of homogeneous linear equations in the variables x_1, \dots, x_n with integer coefficients. For a set S of integers, we write $S \rightarrow \underline{(L, \dots, L)}$, if L always has a monochromatic solution

for any r -coloring of S . A system L is said to be *regular*, if, for all r , $\mathbb{P} \rightarrow (\underline{L}, \dots, \underline{L})$, where \mathbb{P} denotes the set of positive integers.

R. RADO has characterized all regular L by generalizing the properties of the two best known examples. These are, respectively, $L_2: x+y = z$ and $L(k): x_1 - x_2 = x_2 - x_3 = \dots = x_{k-1} - x_k$. That L_2 is regular is SCHUR's theorem. Of course, the regularity of $L(k)$ is trivial (by choosing all the x_i equal). However, if we rule out this possibility, then a solution of $L(k)$ determines an arithmetic progression of length k . This restricted regularity of $L(k)$ for all k is just VAN DER WAERDEN's well-known theorem.

Unfortunately, however, this surprising result still does not specify *which* color these progressions have. It was conjectured some 40 years ago by ERDÖS and TURAN that a solution must always occur in the most frequently occurring color. More precisely, they conjectured that if R is an infinite sequence of integers with positive upper density, i.e.,

$$(*) \quad \overline{\lim}_{n \rightarrow \infty} \frac{|R \cap [1, n]|}{n} > 0,$$

then R contains arbitrarily long arithmetic progressions. No progress was made on this problem until 1954 when K.F. ROTH showed that if R satisfies $(*)$, then R at least contains a *three*-term arithmetic progression. In fact, he showed more, namely, that for some $c > 0$, if $|R \cap [1, n]| > \frac{cn}{\log \log n}$ then R must contain a three-term arithmetic progression. The next significant step was not made until 1967 when SZEMERÉDI proved that $(*)$ implies that R contains a *four*-term progression. However, SZEMERÉDI's most recent result, which must be considered an achievement of the first magnitude, finally settles the original conjecture of ERDÖS and TURAN in the affirmative.

THEOREM. (SZEMERÉDI). $(*)$ implies R contains arbitrarily long arithmetic progressions.

SKETCH OF SKETCH OF PROOF. SZEMERÉDI's proof is completely combinatorial in nature and is based on a lemma on bipartite graphs which is of considerable importance in its own right. We shall give a very brief discussion of the flavor of the proof (which runs just under 100 pages in length), although we can only hint at the extreme ingenuity used in the proof itself.

Let G denote a bipartite graph with vertex sets A and B . We call G regular if all vertices in A have the same degree and all vertices in B have the same degree. We would like to assert that every sufficiently large

bipartite graph can be decomposed into a relatively small number of regular bipartite subgraphs, but unfortunately this is not true. However, it is true if the subgraphs are only required to be "approximately" regular and if we are allowed to ignore a small fraction of the vertices in A and B. More precisely, for $X \subseteq A$, $Y \subseteq B$, let $k(X,Y)$ denote the number of edges in the graph induced by the vertex sets X and Y and let $\beta(X,Y)$ denote $\frac{k(X,Y)}{|X||Y|}$, the density of edges in this induced subgraph. Then SZEMERÉDI proves the following:

LEMMA. For all $\varepsilon_1, \varepsilon_2, \delta, \rho, \sigma$ strictly between 0 and 1, there exist integers m_0, n_0, M, N such that for all bipartite graphs G with $|A| = m > M$, $|B| = n > N$ there exist disjoint $C_i \subseteq A$, $0 \leq i < m_0$, and for each $i < m_0$, disjoint $C_{i,j}$, $j < n_0$, such that:

- (a) $|A - \bigcup_{i < m_0} C_i| < \rho m$, $|B - \bigcup_{j < n_0} C_{i,j}| < \sigma n$ for any $i < m_0$;
- (b) for all $i < m_0$, $j < n_0$, $S \subseteq C_i$, $T \subseteq C_{i,j}$, with $|S| > \varepsilon_1 |C_i|$, $|T| > \varepsilon_2 |C_{i,j}|$, we have $\beta(S,T) \geq \beta(C_i, C_{i,j}) - \delta$;
- (c) for all $i < m_0$, $j < n_0$ and $x \in C_i$, $\beta(\{x\}, C_{i,j}) \leq \beta(C_i, C_{i,j}) + \delta$.

Condition (a) says that we have not omitted too many vertices in the decomposition. Conditions (b) and (c) express the approximate regularity of the subgraphs induced by the vertex sets C_i and $C_{i,j}$.

The basic objects dealt with in the proof are not just arithmetic progressions, but more general structures known as *configurations*. A 1-configuration is just a finite arithmetic progression; an m-configuration is a finite arithmetic progression of (m-1)-configurations.

Let R be an arbitrary fixed set of integers having positive upper density. The basic idea is to show inductively that there exist very long m-configurations which have an extremely restricted manner in which they intersect R. This is done by recursively defining certain special classes of higher order configurations in terms of rather well-behaved progressions of lower order configurations. Essentially, by showing that there exist extremely long configurations of some order which are moderately "regular", one can deduce the existence of configurations of a higher order which are even more "regular". This in turn is done by forming bipartite graphs based on the intersection patterns of the configurations with R and applying the decomposition lemma. Needless to say, the subtlety of the ideas used can only be appreciated by reading the actual proof. \square

Turning our attention back to SCHUR's system L_2 , we can generalize this to the system L_k defined as follows: for the variables x_s and y_s , L_k consists of all equations of the form $\sum_{s \in S} x_s = y_s$ where S ranges over all non-empty subsets of $[1, k]$. RADO's results imply that for all k and r ,

$$\mathbb{P} \rightarrow \underbrace{(L_k, \dots, L_k)}_r$$

It is natural to ask what happens for the system

$$L_\infty = \{ \sum_{s \in S} x_s = y_s \mid S \subseteq \mathbb{P}, 1 \leq |S| < \infty \}.$$

N. HINDMAN's remarkable theorem answers this question.

THEOREM. (HINDMAN). For all r , $\mathbb{P} \rightarrow \underbrace{(L_\infty, \dots, L_\infty)}_r$.

SKETCH OF PROOF. In the case L_k it is even true that for each r there is an $N = N(k, r)$ such that $N \rightarrow \underbrace{(L_k, \dots, L_k)}_r$. In other words, no matter which r -coloring we have, values x_1, \dots, x_k can be chosen from $[1, N]$ so that all the sums $\sum_{s \in S} x_s$ have the same color. For a fixed r -coloring of \mathbb{P} restricted to $[1, N(k, r)]$ it was not known whether upper bounds for the x_i existed independent of k . The existence of such bounds would allow HINDMAN's theorem to be obtained directly by a "compactness" argument.

What HINDMAN proves is that for each coloring π of \mathbb{P} with a finite number of colors, there is a function $f_\pi: \mathbb{P} \rightarrow \mathbb{P}$ such that for each m , $0 < m < \infty$, there is a set x_1, \dots, x_m with all its finite sums the same color, and in addition, such that $x_i \leq f_\pi(i)$ for $1 \leq i \leq m$. That is, we get monochromatic solutions to L_m for arbitrarily large m , where the sizes of the variables x_i are bounded above independently of k but depending on the coloring π (of all of \mathbb{P}).

We can illustrate several of the ideas of the proof, but we need some notation first. Let π be a finite coloring of \mathbb{P} , say $\mathbb{P} = A_1 \cup \dots \cup A_r$. For $1 \leq k \leq n$, we define

$$F_\pi(k, n) = \{ x \in \mathbb{P} \mid x \geq n \text{ and } \exists i \text{ such that } k, x, x+k \in A_i, \\ x \in F_\pi(j, n), j < k \}.$$

The $F_\pi(k, n)$ are sets which can be translated by k without changing color. If x_1, x_2, \dots is a sequence of integers, let $S(x_i)$ be the set of finite sums of the x_i .

The core of HINDMAN's proof is an "exceedingly technical" and quite

clever argument, which establishes that for each π there is an infinite sequence x_1, x_2, \dots and $n \in \mathbb{P}$ such that

$$S(x_i) \subseteq \bigcup_{k=1}^{n-1} F_{\pi}(k, n).$$

To manipulate sequences and sums conveniently, it would be nice to know that the numbers in the sequences were representable to base 2 in the following manner, e.g.,

$$\begin{aligned} x_1 &= && 10110111 \\ x_2 &= && 110000100000000 \\ x_3 &= && 110000000000000000 \\ &&& \vdots \end{aligned}$$

That is, the support of x_j should be all beyond the support of x_{j-1} for each j . Formally, if $2^{s-1} \leq x_{j-1}$, then $2^s \mid x_j$. Such a sequence will be called a *good sequence*. Now for every sequence x_1, x_2, \dots there is a good sequence (not necessarily a subsequence of the x_i 's) y_1, y_2, y_3, \dots with $S(y_i) \subseteq S(x_i)$. This follows from a compactness argument again.

Hence we basically need to deal only with good sequences. The nice property of these is that if $X = \{x_1, x_2, \dots\}$ is a good sequence, there is a bijection $\tau_X: S(x_i) \rightarrow \mathbb{P}$ which preserves sums, namely,

$$\tau_X\left(\sum_{s \in S} x_i\right) = \sum_{s \in S} 2^{s-1}.$$

That is, each block of support corresponds under τ to a single binary place.

We use this fact crucially in the following construction. Suppose π is a coloring, and $x_{\pi_1}, x_{\pi_2}, x_{\pi_3}, \dots$ is a good sequence with

$$S(x_{\pi_i}) \subseteq \bigcup_{k=1}^{n(\pi)-1} F_{\pi}(k, n(\pi)).$$

Then using the map τ_{π} determined by this sequence, we can get a new coloring π' of \mathbb{P} by letting two numbers have the same π' -color iff their images under τ_{π}^{-1} are in the same $F_{\pi}(k, n(\pi))$. This is an $(n(\pi)-1)$ -coloring.

Suppose for *all* π we have defined $f_{\pi}(i)$ for $i \leq \ell$ so that arbitrarily long finite good sequences have monochromatic sums and the i -th term is at most $f_{\pi}(i)$, $i \leq \ell$, where we take $f_{\pi}(1) = n(\pi)-1$ (which works for $\ell = 1$ by

the definition of $n(\pi)$). Then consider such a sequence for the coloring π' associated as above with π . Taking τ_π^{-1} of this sequence, we get a similar sequence which is constrained by the definition of π' to have its first ℓ terms respectively less than $\tau_\pi^{-1}(f_{\pi'}(i))$, $i \leq \ell$, and all greater than $n(\pi)-1$. Further, they must all have the same π -color, and for some common $k \leq n(\pi)-1$, adding k does not change this color. Then adjoining k as a first term gives us a new sequence with the first term not exceeding $f_\pi(1)$. Also, if we let $f_\pi(j) = \tau_\pi^{-1}(f_\pi(j-1))$, we have the j -th term not exceeding $f_\pi(i)$ for $j \leq \ell+1$.

We have thus constructed, simultaneously for all π , the bounds $f_\pi(i)$. What we have shown, then, is that for each $\pi = A_1 \cup A_2 \cup \dots \cup A_r$, and each k , there is a sequence x_1, \dots, x_k with all its sums in some $A_{i(k)}$ and $x_i \leq f_\pi(i)$, $1 \leq i \leq k$. As we noted above, a compactness argument now completes the proof. \square

We remark that because the supports of the x_i in a good sequence are disjoint, we can interpret the x_i as disjoint subsets of \mathbb{P} and their sums as disjoint unions. Thus, we obtain: *for every r -coloring of the finite subsets of \mathbb{P} , there exists an infinite sequence of finite disjoint sets A_1, A_2, A_3, \dots such that all the finite unions have the same color.*

The last of the results on equations is that of DEUBER, who settles a conjecture RADO raised in his original work. We recall that a system L of homogeneous linear equations is called regular if for any r , $\mathbb{P} \rightarrow (\underline{L}, \dots, \underline{L})_r$. RADO defined a set $S \subseteq \mathbb{P}$ to be *regular* if for every regular system L and any r , $S \rightarrow (\underline{L}, \dots, \underline{L})_r$. What RADO conjectured and what DEUBER proves is the following:

THEOREM. (DEUBER). *If $S \subseteq \mathbb{P}$ is regular and $S = A \cup B$, then either A or B is regular.*

SKETCH OF PROOF. The main idea of DEUBER's proof is to define certain sets, called (m,p,c) -sets, and to characterize regular sets in terms of (m,p,c) -sets. He then proves a finite RAMSEY theorem for these sets. Finally, by considering the nice structure of (m,p,c) -sets, he uses a compactness argument to establish the desired result.

We define (m,p,c) -sets below. However, we can describe them informally as a kind of ℓ -dimensional array of numbers (actually, certain subsets of these).

DEFINITION. For m, p, c positive integers, $p \geq c$, an (m, p, c) -set A is a set for which there exist m positive integers a_1, a_2, \dots, a_m , such that $A = \{ \sum_{i=1}^m \lambda_i a_i \mid |\lambda_i| \leq p, \text{ and the first non-zero coefficient } \lambda_i \text{ has the value } c \}$.

Now using RADO's characterization of regular systems of equations, we can show the following two facts:

- (a) for every regular system L there exist m, p, c such that every (m, p, c) -set contains a solution to L ;
- (b) for all m, p, c there is a regular system L such that every solution set for L contains an (m, p, c) -set.

As an example, consider the single equation $x+y = z$. Then a solution is any set of the form a_1, a_2, a_1+a_2 , which is certainly contained in the $(2, 1, 1)$ -set generated by a_1 and a_2 . On the other hand, the equations $x+y = z_1$, $x-y = z_2$, have solutions exactly of the form $x, y, z_1, z_2 = a_1, a_2, a_1+a_2, a_1-a_2$, a $(2, 1, 1)$ -set. These examples avoid $c \neq 1$, which can arise when the coefficients are more complicated.

By (a) and (b) we see that a regular set is any set containing (m, p, c) -sets for all m, p, c .

Suppose now that we know the following: for each (m, p, c) there is an (n, q, d) such that $(n, q, d) \rightarrow ((m, p, c), (m, p, c))$. That is, if the elements of any (n, q, d) -set are 2-colored, then there must be a monochromatic (m, p, c) -set. Thus for S regular, and $S = A \cup B$, either A or B must contain "arbitrarily large" (m, p, c) -sets and hence, by what we have noted, either A or B is regular. The main part of DEUBER's proof is concerned then with establishing the Ramsey property $(n, q, d) \rightarrow ((m, p, c), (m, p, c))$.

This result is similar to one of GALLAI concerning "n-dimensional arrays". For our purposes, we may consider an n-dimensional array as a set of the form

$$X_{n,p} = \{ a_0 + \sum_{i=1}^n \lambda_i a_i \mid |\lambda_i| \leq p \}.$$

For these we have that for n, p and r there is an N such that $X_{N,p} \rightarrow (X_{n,p}, \dots, X_{n,p})$.

However, this isn't quite good enough for our purposes, since an (m, p, c) -set will contain sums of the form $ca_i + \sum_{j>i} \lambda_j a_j$ along with certain differences as well (e.g., a_1 , a_1+a_2 and a_2 in the example above) while

$X_{N,p}$ may not contain any of its differences. To handle this problem we proceed iteratively.

First, we find a monochromatic

$$X_{N,p} = \{b_1 + \sum_{i=1}^N \lambda_i a_i \mid |\lambda_i| \leq p\} = b_1 + Z_{N,p}$$

where

$$Z_{N,p} = \{ \sum_{i=1}^N \lambda_i a_i \mid |\lambda_i| \leq p \}.$$

Then in $Z_{N,p}$ we find a monochromatic $b_2 + Z_{N,p}$, etc. Continuing in this manner we can find b_1, b_2, \dots, b_ℓ such that the color of the sum

$$b_i + \sum_{j>i} \lambda_j b_j$$

depends only on i . For large enough ℓ , we may select m of these b_j to generate a monochromatic $(m, p, 1)$ -set.

This completes the case $c = 1$. For $c > 1$, a similar argument can be applied where, however, at each step p must be adjusted to compensate for the effect of c . \square

CATEGORIES

The notion of a category having the Ramsey property was introduced by K. LEEB. It has been used to prove the Ramsey property for the category of finite vector spaces, among others. A category C is said to be *Ramsey* if for any objects A, B and number r , there is an object C such that for any r -coloring of the A -subobjects of C , all the A -subobjects of some B -subobject of C have the same color. Formally this says:

$$\forall A, B, r \exists C \ni \forall C \left(\begin{matrix} C \\ A \end{matrix} \right) \xrightarrow{f} [1, r],$$

\exists a monomorphism, $B \xrightarrow{\phi} C$ and i such that the following diagram commutes:

$$\begin{array}{ccc} C\binom{C}{A} & \xrightarrow{f} [1,r] \\ \bar{\phi} \uparrow & \uparrow \text{incl.} \\ C\binom{B}{A} & \rightarrow \{i\} . \end{array}$$

Here $C\binom{C}{A}$ denotes the set of subobjects of C of isomorphism type A , and $\bar{\phi}$ the function induced by ϕ .

We could also abbreviate this by using the arrow notation. A category C is Ramsey if for every r and objects A, B there is an object C such that

$$C \rightarrow \underbrace{(B, B, \dots, B)}_r .$$

To prove this property for a certain class of categories, including the category of sets (Ramsey's Theorem) and that of finite vector spaces, an elaborate induction is used. The induction is fundamentally determined by a generalization of the classical Pascal identity, $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$.

In his lecture notes on "Pascaltheorie", LEEB has developed more formally and generalized this kind of relationship and used it to prove some new Ramsey theorems, among other things. What we describe here is LEEB's generalization of the ordinary notion of labeled trees to that of trees labeled with objects from a category. A Ramsey theorem for these structures is then true if it was true in the original category.

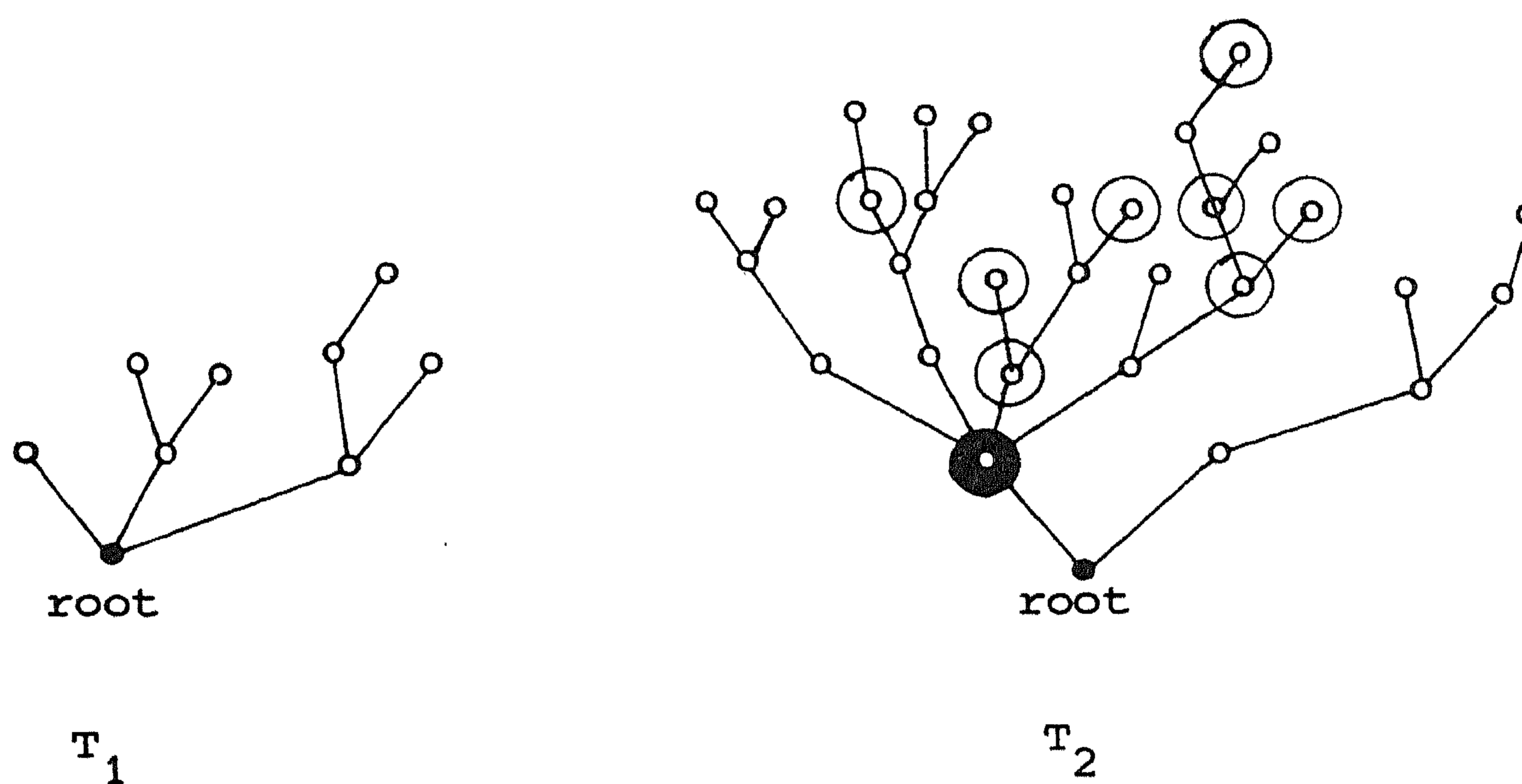
Consider a category C . Then the category $Ord(C)$ is defined to be the category of finite sequences of objects from C . That is, the objects of $Ord(C)$ are finite sequences of objects of C , and morphisms $(C_1, C_2, \dots, C_k) \rightarrow (D_1, D_2, \dots, D_\ell)$, $k \leq \ell$, are sequences $(\phi_1, \phi_2, \dots, \phi_k)$ of morphisms from C such that $\phi_i: C_i \rightarrow D_{j(i)}$ for some $j(i)$, and $1 \leq j(1) < j(2) < \dots < j(k) \leq \ell$.

We can define the category $Trees(C)$ similarly. We consider rooted, labeled trees with an orientation, or ordering, of the branches at each vertex. We take the labels from the objects of C . Morphisms are defined as follows. Let T_1, T_2 be two such objects, and let T_1, T_2 be their underlying rooted trees. First we "immerse" T_1 into T_2 . An immersion $\psi: T_1 \rightarrow T_2$ is a monomorphic mapping from the vertices of T_1 to those of T_2 such that:

- (a) For any two vertices x, y in T_1 , $\psi(x \wedge y) = \psi(x) \wedge \psi(y)$, where for two vertices u, v in a rooted tree T , $u \wedge v$ denotes the last common vertex in the paths from the root to u and from the root to v , respectively.
- (b) The order of the branches is preserved by ψ . That is, let B_1, B_2, \dots, B_k be the vertex sets of the branches at a vertex x in T_1 , given in order,

and let D_1, D_2, \dots, D_ℓ be the vertex sets of the branches at $\psi(x)$ in T_2 , given in order. Then for each i , $1 \leq i \leq k$, $\psi(B_i) \subseteq D_{j(i)}$ for some $j(i)$, and $1 \leq j(1) \leq \dots \leq j(k) \leq \ell$.

For example, the circled vertices in T_2 below indicate the immersion of T_1 into T_2 :



Once we have an immersion ψ of T_1 into T_2 , we then find a set of morphisms from C taking the labels from T_1 into the corresponding labels (by the immersion) of T_2 . Such sets of morphisms of C (with restrictions determined by (a) and (b)) are defined to be the morphisms of $Trees(C)$. If we denote a C -labeled tree by $[a, B]$, where a is the root label and B the sequence of branches at the root (with labels), we get the Pascal identity:

$$Trees(C) \left(\begin{array}{c} [a, B] \\ [c, D] \end{array} \right) = \coprod_{B_i \in B} Trees(C) \left(\begin{array}{c} B_i \\ [c, D] \end{array} \right) + C \left(\begin{array}{c} a \\ c \end{array} \right) \times Ord(Trees(C)) \left(\begin{array}{c} B \\ D \end{array} \right).$$

What this says is that every subtree (labeled) of type $[c, D]$ in a tree of type $[a, B]$ either has its root at the root of $[a, B]$, or lies entirely in one of the branches at the root, with labels mapped accordingly. If one considers the identity for trees with only one branch at each point, and C the category with only a single object, then this identity becomes the classical Pascal identity.

We say that a category C is *directed* if for any objects A and B , for

some object C there exist monomorphisms $A \rightarrow C$ and $B \rightarrow C$. What LEEB proves is the following:

THEOREM.(LEEB). *If C is Ramsey and directed, then $Trees(C)$ is Ramsey and directed.*

The proof uses the Ramsey property for C , together with the standard "product" argument, also used to prove (among other things) the result of GALLAI mentioned in the previous section.

A related and less complicated result, using the same basic techniques, is that if C is Ramsey and directed, then so is $Ord(C)$.

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ON AN EXTREMAL PROPERTY OF ANTICHAINS IN PARTIAL ORDERS.
THE LYM PROPERTY AND SOME OF ITS IMPLICATIONS
AND APPLICATIONS *)

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I.

Let F be a collection of subsets of an n element set S , such that no member of F contains another. We call such a collection an *antichain*, in contrast to a collection that is totally ordered by inclusion which one usually calls a chain.

A well-known theorem of SPERNER states that an antichain F can have no more than $\binom{n}{\lfloor n/2 \rfloor}$ members. YAMAMOTO and independently LUBELL and also MESHALKIN (and perhaps others) noticed that F satisfies a stronger property. If we denote the number of members of F of size k (having k elements) by f_k these numbers must satisfy

$$\sum_k f_k / \binom{n}{k} \leq 1 .$$

This result is stronger, in that the left-hand side is obviously greater than or equal to $\sum_k f_k / \binom{n}{\lfloor n/2 \rfloor}$ so that we may immediately deduce that $\sum_k f_k \leq \binom{n}{\lfloor n/2 \rfloor}$ which is SPERNER's theorem.

It is the purpose of this note first, to explore the class of partial orders (possessing a rank function) which satisfy this stronger property, called below the *LYM property*; that is in which an antichain under the partial order obeys a relation of the form $\sum_k f_k / N_k \leq 1$ where N_k represents the number of members of the order of rank k . Secondly, we show that this strong property is equivalent to a number of other properties of the partial order, which we shall shortly describe. Thirdly, we prove a number of general properties of such partial orders, and finally we give examples of application

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of these properties.

II.

The LUBELL (and YAMAMOTO) proof of this property is remarkably simple. If we examine all the maximum sized chains, it is clear from symmetry among the n elements of S that each k element subset of S appears in exactly the same proportion of these chains as does any other, and hence in a proportion $1/\binom{n}{k}$ of them. Since no two members of F can lie in any maximal chain, the average number of members of F per chain, which is $\sum_{A \in F} 1/\binom{n}{|A|}$ (each A in F contributes $1/\binom{n}{|A|}$ to this average), cannot exceed one. This relation $\sum_{A \in F} 1/\binom{n}{|A|} \leq 1$ is the LUBELL-YAMAMOTO inequality.

The form of the argument makes it immediately evident that an analogue holds for any partial order possessing a symmetry that is transitive on the members of each rank.

In any such order, one will again obtain that

$$\sum_{A \in F} 1/\binom{n}{|A|} \leq 1$$

by the same argument. This implies that, for example, the lattice of subspaces of a vector space of finite dimension over a finite field, will possess this property. One can also deduce from these remarks, that they hold equally well if we weight the members of our partial order by a weighting function w that depends only upon rank. That is, we can deduce that

$$\sum_{A \in F} \frac{w(|A|)}{N_{|A|}} \leq \max_k w(k),$$

or alternatively, for any weighting function w defined on the rank of the members of our order we must have

$$\sum_{A \in F} w(|A|) \leq \max_k N_k w(k).$$

The form of the LUBELL-YAMAMOTO argument is in fact so simple that it leads one to expect that there is a necessary connection between the symmetry of the partial order and the workings of the argument.

However, the argument does not really require this; symmetry is unnecessary; the argument does not require that each set of a given rank occurs in the same proportion of all maximal chains.

What is required is that there exists some list of maximal chains that contains every set of each rank the same number of times. The argument will go through if we apply it not to the set of all possible maximal chains, but to this list of such chains.

A partial order will thus satisfy the LYM property whenever there exists a list of maximal chains such that each member of the order of rank k occurs in a proportion $1/N_k$ of the chains.

When then does such a list exist?

GRAHAM & HARPER introduced the concept of *normalized matching* in a partial order. We define it as follows. A partial order is said to have the normalized matching property, if, for every k , given any collection F of rank k members of the order there is a collection G_{k-1} of rank $k-1$ members of the order, such that every member of G_{k-1} is ordered with respect to at least one member of F , and such that

$$|G_{k-1}|/N_{k-1} \geq |F|/N_k.$$

We shall now show that this normalized matching property is equivalent to the LYM property under discussion. We prove the following theorem.

THEOREM 1. *Let P be a partial order with a rank function and N_k members of rank k , then the following four conditions are equivalent:*

1. *every antichain F satisfies*

$$\sum_{A \in F} 1/N_k(A) \leq 1, \quad (\text{LYM property});$$

2. *every antichain F and real function w of k , the rank of A , satisfies*

$$\sum_{A \in F} w(|A|) \leq \max_k N_k w(k);$$

3. *there exists a list of maximal sized chains such that each member of P of rank k occurs in a proportion $1/N_k$ of the entries on the list;*

4. *P satisfies the normalized matching condition; that is, for every collection F of rank k members of P there is a collection G_{k-1} such that each member of G_{k-1} is ordered with respect to at least one member of F*

and

$$|G_{k-1}|/|F| \geq |N_{k-1}|/|N_k|.$$

PROOF. From the discussion previously, we have already noted that condition 3 implies conditions 1 and 2 and condition 2 implies condition 1 by choosing $w(k)$ to be $1/N_k$. We therefore need only prove that condition 4 implies condition 3, and that 1 implies 4.

We begin by proving the former. We assume that P satisfies the normalized matching condition. This condition can be seen to imply the ordinary PHILIP HALL matching condition if one takes a list of αN_{k-1} copies of the rank k members of P and a list of αN_k copies of the rank $k-1$ members for any integral α , and tries to match each member of the former list to one member of the latter ordered with respect to it in P . If we take any set of Q members of the former list, we will have at least $\lceil Q/\alpha N_{k-1} \rceil$ different members of rank k . By the normalized matching condition these will be ordered with respect to at least $\lceil Q/\alpha N_{k-1} \rceil N_{k-1}/N_k$ rank $k-1$ members of P and hence at least αN_k times as many or $\alpha N_{k-1} \lceil Q/\alpha N_{k-1} \rceil$ members (or at least Q members) of the second list. By the PHILIP HALL theorem there is a matching of the former to the latter.

We obtain a list of chains as required by condition 3 by starting at the highest rank r , starting with, for example, $(\prod_{k=1}^{r-1} N_k)$ copies of the rank r members of P and performing the matching of the last paragraph for $k=r, r-1, \dots, 2$. Each entry on the list of rank r will be matched into an entry of rank $r-1$ ordered with respect to it, which will be matched into one of rank $r-2$, etc. The orbits under the matchings will be the desired maximal chains of condition 3.

To prove that condition 1 implies condition 4 we prove the contrapositive, that not 4 implies that 1 cannot hold.

Suppose therefore for some k there is a collection F of rank k members of P such that the collection G_{k-1} of rank $k-1$ members of P that are ordered with respect to one or more members of F satisfies

$$\frac{|G_{k-1}|}{|F|} < \frac{N_{k-1}}{N_k}.$$

Then consider the antichain connecting F and the rank $k-1$ members of P not in G . These satisfy, by trivial manipulation of the last inequality,

$$\frac{|F|}{N_k} + \frac{N_{k-1} - |G_{k-1}|}{N_{k-1}} = 1 + \left(\frac{|F|}{N_k} - \frac{|G_{k-1}|}{N_{k-1}} \right) > 1 .$$

Thus condition 1 implies condition 4 and the theorem is proved. \square

III.

Granted all of this, what can we conclude about partial orders satisfying these conditions?

GRAHAM & HARPER, and also subsequently but independently HSIEH & KLEITMAN proved that direct products of partial orders each satisfying normalized matching and logarithmic convexity (for every k , $N_k^2 \geq N_{k-1}N_{k+1}$) also satisfy normalized matching and logarithmic convexity as well. Thus the lattice of divisors of an integer (with "divides" as order relation) satisfies this condition it being the direct product of chains.

In consequence of our theorem, this lattice satisfies the LYM property: antichains obey the $\sum_{A \in \mathcal{F}} 1/N_k(A) \leq 1$ inequality in the divisor-of- n lattice.

HSIEH & KLEITMAN proved further that if P_1 is a partial order satisfying normalized matching and P_2 is a chain, then the direct product $P_1 \otimes P_2$ satisfies normalized matching only if $P_1 \otimes P_2$ satisfies logarithmic convexity ($N_k^2 \geq N_{k-1}N_{k+1}$). If P_2 is an ordered pair, $P_1 \otimes P_2$ will satisfy normalized matching if and only if P_1 satisfies both normalized matching and logarithmic convexity.

On the other hand, the partial order consisting of partitions of an integer n ordered by refinement fails to satisfy normalized matching, at least for even n of magnitude at least 10. (For example, for $n = 10$, the partition 2222 is ordered with respect to only one partition into four integers (4222) while $N_4 = 9$ and $N_5 = 7$ where rank here is the number of blocks in the partition.) It is not known whether this partial order possesses the "Sperner property", that no antichain can have more than $\max_k N_k$ members.

GREENE & DILWORTH found an example of a geometric lattice which fails to satisfy either condition, Sperner or LYM. It is not known whether the lattice of partitions of a set satisfies either condition.

IV.

In a partial order satisfying any of the four conditions of theorem 1, a great many extremal properties of collections of order members can be deduced. We begin by enunciating a general theorem (which is itself a special case of the next one).

THEOREM 2. *Given a collection F of order members that obeys some restriction R and a weight function w of rank in P , with P a partial order satisfying the conditions of theorem 1, the sum of w over the collection is not greater than the maximum sum over chains satisfying R of the sum of $w(|A|)N_{|A|}$*

$$\sum_{A \in F} w(|A|) \leq \max_{\substack{C \\ C \text{ chain, } C \text{ obeys } R}} \sum_{A \in C} w(|A|)N_{|A|}.$$

Here and below is a restriction of the form "no set of members of F satisfies ...".

This theorem has many powerful implications including a number of theorems of KATONA and of ERDÖS that we shall describe in the next section. Such theorems have a wide range of validity here - they hold in partial orders satisfying the LYM property. The proof is immediate; by the existence of our list of maximal chains we know that the average value $\sum_{A \in F} w(|A|)N_{|A|}$ over maximal chains is less than its maximal value for any maximal chain. Since A occurs in $1/N_{|A|}$ of the maximal chains, it contributes $w(|A|)$ to the average.

A still more general theorem is the analogue of the KLEITMAN-KATONA theorem. It is the following

THEOREM 3. *Given two partial orders P_1 and P_2 satisfying the LYM property; let F be a family of members of the direct product partial order $P_1 \otimes P_2$, subject to some restriction R . Let w be a function of rank in P_1 and P_2 . Then the maximum value of the sum of $w(A)$ for A in F cannot exceed the maximum over a subfamily of the direct product of two maximal chains, of $w(r_1(A), r_2(A)) \frac{N_{r_1(A)}^{(1)}}{r_1(A)} \frac{N_{r_2(A)}^{(2)}}{r_2(A)}$. Here $r_i(A)$ is the rank of the i -th factor of A . That is*

$$\sum_{A \in F} w(r_1(A), r_2(A)) \leq \max_{\substack{F' \subset C_1 \otimes C_2 \\ F' \subset F, F' \text{ satisfying } R}} \sum_{A \in F'} w(A) \frac{N_{r_1(A)}^{(1)}}{r_1(A)} \frac{N_{r_2(A)}^{(2)}}{r_2(A)}.$$

This theorem has many important applications, as we shall discuss. It has an obvious generalization to direct products of k LYM-partial orders.

Its proof is also immediate, by exactly the argument previously applied to the lists of direct products of maximal chains in P_1 and P_2 whose existence follows from the LYM properties.

V.

We now examine some specific consequences of theorems 2 and 3. If in theorem 2 we let R be the restriction that if A, B are in F then $A \not\leq B$, we have a portion of theorem 1. If we let R be the restriction that no k members of F form a chain, and set $w = 1$, we obtain a generalization of the theorem of ERDÖS that the maximum size of F is the sum of the largest $(k-1)$ N_j 's. If we let R be the restriction that no two members of F are ordered and differ by rank $\geq k$, we obtain the sum of the k largest consecutive N_j 's. If we let R be the restriction that no two ordered members of F differ by less than $(<) k$ in rank we obtain, if P is unimodular (N_j has only one local maximum), $\max_t \sum_{j=0}^{r/k} N_{t+jk}$ as a bound on the size of F (this is a generalization of a result of KATONA for divisors of an integer, and generalized somewhat differently by KATONA); likewise, if R restricts F to have no $m+1$ members within any rank interval k , for unimodular P one obtains the largest m values of $\sum_{j=0}^{r/k} N_{t+jk}$ for distinct $t < j$. This result again generalizes a result of KATONA for the lattice of divisors of an integer. Many other similar results follow as we have freedom to choose R and w as we please, always finding that the maximum sum over F can be evaluated by looking at the appropriately reweighted sum over a single chain.

Theorem 3 has a number of consequences that generalize known theorems as well as some new ones. Thus if R restricts F to contain no ordered pair we can deduce immediately that $P_1 \otimes P_2$ will satisfy the SPERNER property as long as P_1 and P_2 are both unimodular, while, by some detailed argument one can show using theorem 3, that $P_1 \otimes P_2$ will obey condition (1) of theorem 1 if both are logarithmically convex, thus providing a proof of the HARPER-GRAHAM theorem.

If R restricts F to contain no ordered pair that are identical in one factor, and if weight functions w_1, w_2 are chosen for P_1 and P_2 such that when arranged in decreasing order the $w_1(k)N_k^{(1)}$ are

$$w_1(k_1)N_{k_1}^{(1)}, w_1(k_2)N_{k_2}^{(1)}, \dots, w_1(k_{r_1})N_{k_{r_1}}^{(1)},$$

and the $w_2(k)N_k^{(2)}$ arranged in decreasing order are

$$w_2(k'_1)N_{k'_1}^{(2)}, w_2(k'_2)N_{k'_2}^{(2)}, \dots, w_2(k'_{r_2})N_{k'_{r_2}}^{(2)},$$

then the sum of the product of the weight functions over members of F can't exceed

$$\sum_j w_1(k_j)N_{k_j}^{(1)} w_2(k'_j)N_{k'_j}^{(2)}.$$

If $k_j+k'_j$ is a constant then this sum becomes a sum over a single rank in $P_1 \otimes P_2$. This will occur for example if after weighting both P_1 and P_2 are symmetric in rank and unimodular (a theorem of KLEITMAN and KATONA for subsets of a set). It will occur also if P_2 is P_1 turned upside down, or if P_1 has only two ranks and the largest rank values of $w_2(k)N_k^{(2)}$ are consecutive and appropriately ordered, and in a wide variety of other circumstances. Notice that even for subsets or divisors of an integer the result remains true here for non-trivial weight functions obeying certain rules.

Similar results hold if R restricts F to contain no ordered pair identical in one factor and differing by more than ($>$) k in the other. For unimodular symmetric P_1 and P_2 (after weighting), one obtains either the maximum of the sum of $N^{(1)}N^{(2)}w_1w_2$ over k consecutive ranks in $P_1 \otimes P_2$ or a related sum. Similar results hold for P_2 being P_1 upside down (i.e. with order relation reversed) or if (after weighting) the relative sizes of the wN 's in P_1 and P_2 is as if P_2 was P_1 with order relations reversed.

One can obtain explicit best results when R is the restriction that no k members of F in $P_1 \otimes P_2$ that are identical in one factor form a chain. The results are not beautiful. For example if ranks in P_i run from 0 to r_i , and the population of rank k in P_i is $N_k^{(i)}$, we obtain for k = 3, and r_1 and r_2 even, that the size of F cannot exceed

$$\frac{2N^{(1+2)}_{r_1+r_2}}{2} - \left(\frac{N^{(1)}_{r_1}}{2} - \frac{N^{(1)}_{r_1+1}}{2} \right) \left(\frac{N^{(2)}_{r_2}}{2} - \frac{N^{(2)}_{r_2+1}}{2} \right) + 2\delta_{r_1 r_2} N_0^{(1)} N_0^{(2)}$$

when P_1 and P_2 are symmetric and unimodular.

Such results can be used to obtain results for direct products of three or more partial orders; thus for P_1, P_2 symmetric and unimodular and r_1, r_2 even and $r_3 = 1$, the maximal size of a family F having no ordered pair of members differing in only one factor is, if $N_0^{(3)} < N_1^{(3)}$,

$$\begin{aligned} & \frac{N_{r_1+r_2}^{(1+2)}}{2} (N_1^{(3)} + N_0^{(3)}) - N_0^{(3)} \left(\frac{N_{r_1}^{(1)}}{2} - \frac{N_{r_1+1}^{(1)}}{2} \right) \left(\frac{N_{r_2}^{(2)}}{2} - \frac{N_{r_2+1}^{(2)}}{2} \right) + \\ & + N_0^{(3)} \delta_{r_1 r_2} \left(N_0^{(1)} N_{r_2}^{(2)} + N_{r_1}^{(1)} N_0^{(2)} \right). \end{aligned}$$

It may be possible to juggle the restrictions on F to obtain SPERNER like conclusions in triple products, particularly through use of the "no differences $\geq k$ " result described above.

If R restricts F to contain no totally ordered members identical in one factor that differ by rank $< m$, then for symmetric unimodular orders P_1 and P_2 or for P_1 unimodular and P_2 being P_1 upside down, one obtains the maximum over j_0 of the sum of $N_i^{(1)} N_j^{(2)}$ over ranks whose sum is congruent to $j_0 \pmod m$. This is again a generalization of a divisor of n result of KATONA.

VI.

The basic method applied here to chains may be applied to other structures on orders defining appropriate classes of partial orders and developing properties of these. Similar ideas have been applied to partitions rather than chains. That is, given a list of partitions that contains members of identical rank the same number of times, one can draw similar conclusions about collections of subsets having disjointness (intersection) restrictions. Some examples of such results are described in [11].

If k divides n the ERDÖS-KO-RADO theorem follows immediately from this approach. It states that the number of non disjoint k element subsets of an n set cannot exceed a proportion k/n of them.

VII.

In this final section we give three examples of the application of some of the results in section V.

Consider sums of the form $\sum_{i=1}^n \epsilon_i a_i$ for a_i vectors in two-dimensional Euclidean space of magnitude at least one, and $\epsilon_i = \pm 1$. We could equally consider a wider coefficient set and remarks directly analogous to those that will follow can be made for such a set, and one in which the range of ϵ_i depends on i as well. For reasons of notational simplicity we shall confine ourselves to the present problem.

By elementary geometry, the sum of two or more vectors in any one quadrant each of magnitude at least one has magnitude at least $\sqrt{2}$. While the sum of three or more such vectors has magnitude at least $\sqrt{5}$, and the sum of k or more has magnitude at least $k\sqrt{2}$.

LITTLEWOOD & OFFORD raised the question, how many of the 2^n linear combinations considered above can lie inside a unit circle. KATONA raised (and solved) the same question for radius $\sqrt{2}$. If we arbitrarily divide the plane into quadrants, reverse the sign of enough a 's (this doesn't effect the 2^n linear combinations) such that they all lie in two quadrants; we may use the facts of the last paragraph to bring these problems into the language of sets.

We can imagine the indices corresponding to vectors in the two quadrants forming sets S_1 and S_2 . Each linear combination can be corresponded to a pair of subsets one of S_1 , one of S_2 namely those for which $\epsilon = +1$. If two linear combinations corresponding to the same subset in S_2 and to one subset containing the other in S_1 are to lie within a circle of radius r , they cannot differ by more than $\lceil r\sqrt{2} \rceil$ indices by those remarks, or for $r = \sqrt{5}$ by more than two indices.

We may conclude by one of the theorems of section V that the number of linear combinations lying in a circle of radius r cannot exceed the limits given in the following table. Details will be described in a subsequent paper.

$r = 1$	largest (KLEITMAN & KATONA) binomial coefficient,
$r = \sqrt{2}$	sum of largest two binomial coefficients (KATONA),
$r = \sqrt{5}$	sum of largest three binomial coefficients for $n \geq 5$,
$r = k$	sum of largest $2\lceil k\sqrt{2} \rceil$ binomial coefficients.

The results below $\sqrt{5}$ are best possible. The general k result, while not best possible, is interesting because it shows that for large n and reasonable

sized k the number of linear combinations in a circle can grow *at most linearly* with the radius, *not* quadratically as does the area of the circle. A similar result in n dimensions would be quite interesting.

The results up to $\sqrt{2}$ have been obtained in arbitrary dimension by the present author by a different method; a best result is known also for $r \leq \sqrt{3}$ in two dimensions, again proved by a different method.

A second application of the implications of theorem 3 is the proof of the HARPER-GRAHAM theorem itself. The theorem states that the direct product of two orders each satisfying LYM and each logarithmically convex will satisfy LYM and logarithmic convexity. Now LYM can be stated as the condition that for every chain $\sum_{A \in F} 1/N_a \leq 1$. By theorem 3 this reduces to the inequality

$$\sum_{A \in F} \frac{N_{r_1(A)}^{(1)} N_{r_2(A)}^{(2)}}{N_{r_1(A)+r_2(A)}^{(1+2)}} \leq 1$$

for F an antichain subfamily of the direct product of two chains. To prove the HARPER-GRAHAM theorem one must show that $P_1 \otimes P_2$ is logarithmically convex and that this inequality holds.

The former is a straightforward exercise (see HSIEH & KLEITMAN). We will outline an argument for the latter in the case that P_1 is itself a chain. In general, if P_1 has only two ranks, the inequality is easily seen to be equivalent to logarithmic convexity on P_2 . It is relatively easy to show by induction that if C_j is a chain of length j , that

$$C_j \otimes P_1 \text{ satisfying LYM implies } C_{j+1} \otimes P_1 \text{ satisfies LYM}$$

(see HSIEH & KLEITMAN). Thus logarithmic convexity and LYM for P_2 implies that $C_j \otimes P_2$ satisfies LYM.

By a slightly more detailed argument one can verify LYM for all P_1 and P_2 satisfying the given conditions.

A final application makes use of the HARPER-GRAHAM theorem, extending a result of LEVINE & LUBELL. The LYM property is independent of weighting that is constant over rank; by the HARPER-GRAHAM theorem if the weighting maintains logarithmic convexity in each factor, the direct product partial order will possess both logarithmic convexity and LYM. Since logarithmic

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SPERNER FAMILIES AND PARTITIONS OF A PARTIALLY ORDERED SET ^{*)}

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1. INTRODUCTION

This paper is a summary (without proofs) of the main results in a series of papers by the author and D.J. KLEITMAN [14] and the author [11, 12, 13] concerning subsets of a finite partially ordered set called *Sperner k-families*. If P is a finite partially ordered set, a subset $A \subseteq P$ is a *k-family* if A contains no chains of length $k+1$ (or, equivalently, if A can be expressed as the union of k 1-families in P). Maximum-sized k -families are called *Sperner k-families* of P .

The literature abounds with results about the maximum size of *Sperner k-families* for special classes of partially ordered sets. ^{**)} In this paper, however, we are not so much concerned with specific numbers as with structural properties. The results described here fall into one or more of the following categories:

- (1) Bounds on the size of a k -family induced by partitions of P into chains.
- (2) Relationships among numbers which arise for various values of k .
- (3) Intersection theorems for collections of k -families.
- (4) Complementary theorems obtained by interchanging the ideas of "chain" and "antichain".
- (5) Properties of a lattice-ordering defined on k -families.
- (6) Matching theorems and properties of certain submodular functions.

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^{**)} The name "Sperner k -family" comes from a generalization (due to ERDÖS [7]) of SPERNER's theorem on finite sets [17]. The generalization states that the maximum size of a k -family of subsets of an n -set is equal to the largest sum of k binomial coefficients $\binom{n}{j}$.

2. k -SATURATED PARTITIONS

If $k = 1$, k -families in P are also called *antichains*. Most of the results in this paper can be traced back to a deep theorem of DILWORTH [4], which states a basic relationship between chains and antichains of a partially ordered set:

THEOREM 2.1. *If every antichain of P has d or fewer elements, then P can be partitioned into d chains.*

The main object of [14] was to prove a similar statement about k -families. If $C = \{C_1, C_2, \dots, C_q\}$ is a partition of P into chains C_i , then, since chains meet k -families at most k times, it follows that no k -family can have more than

$$\sum_{i=1}^q \min \{k, |C_i|\}$$

members. Let $\beta_k(C)$ denote the bound induced by a partition C in this way, and let $d_k(P)$ denote the size of the largest k -family in P .

THEOREM 2.2. (cf. [14]). *For all k , $d_k(P) = \min_C \beta_k(C)$.*

A partition C which satisfies $d_k(P) = \beta_k(C)$ is called a *k -saturated partition* of P . The fact that k -saturated partitions always exist is apparently much more difficult to prove than DILWORTH's theorem (which is a special case), and there are many interesting consequences.

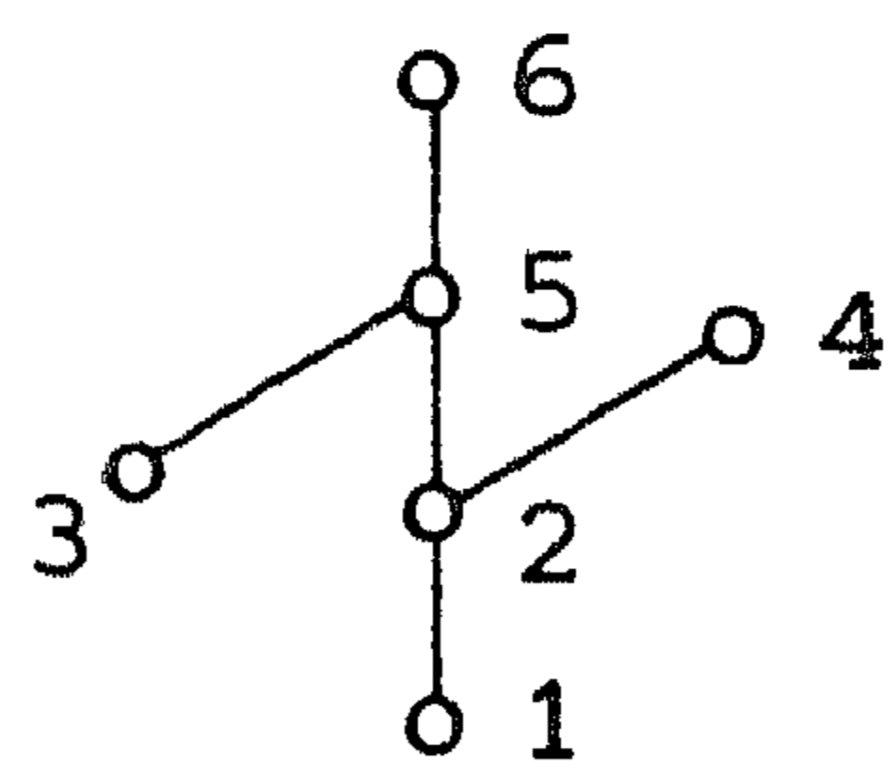
Another way of stating theorem 2.2 is as follows:

THEOREM 2.3. (cf. [14]). *For all k , $d_k(P) = \min_{S \subseteq P} \{|S| + k d_1(P-S)\}$.*

Theorem 2.3 follows from the fact that a k -saturated partition remains k -saturated if all of the chains of length $\leq k$ are broken up into singletons.

Many important examples (Boolean algebras, integer divisors, subspaces of a vector space) have the property that P can be simultaneously k -saturated for all k . That is, one can find a partition C such that $\beta_k(C) = d_k(P)$ for all k . However, this is not always possible if P is arbitrary.

EXAMPLE 2.1. Let $P =$



Then $d_1(P) = 2$, $d_2(P) = 4$, $d_3(P) = 5$ and $d_4(P) = 6$. $C = \{\overline{1\ 2\ 4}, \overline{3\ 5\ 6}\}$ is 1-saturated and 2-saturated but not 3-saturated, while $C' = \{\overline{3,4,1\ 2\ 5\ 6}\}$ is 2-saturated and 3-saturated but not 1-saturated. It is easy to see that no partition is simultaneously 1-, 2- and 3-saturated.

In view of this example, GREENE & KLEITMAN obtained the next best theorem on simultaneous k -saturation:

THEOREM 2.4. (cf. [14]). *For all k , there exists a partition which is simultaneously k -saturated and $(k+1)$ -saturated.*

In fact, it was only by proving this stronger result that a proof of theorem 2.2 was obtained in [14]. The proof is by induction on $|P|$, and most of it is easy except for one critical step: for each i , define $\Delta_i(P) = d_i(P) - d_{i-1}(P)$, with $\Delta_1(P) = d_1(P)$ by convention. Then

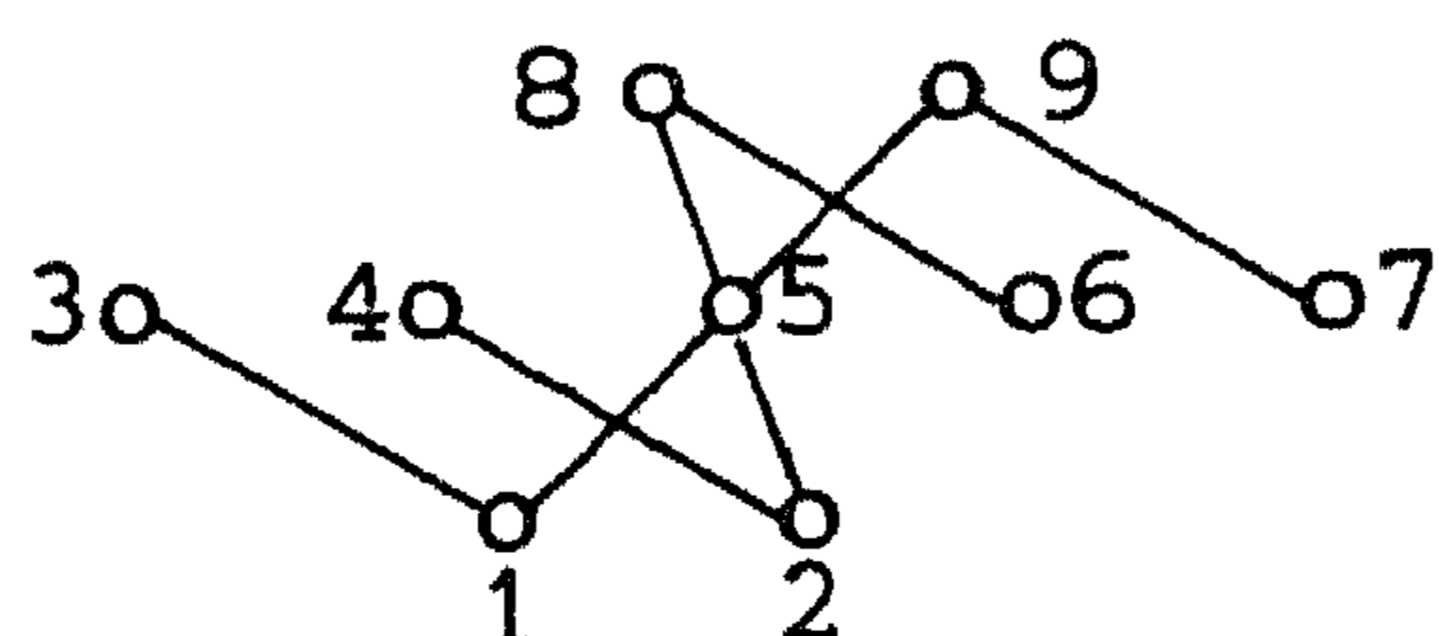
LEMMA 2.1. (cf. [14]). *If $\Delta_k(P) > \Delta_{k+1}(P)$, there exists an element $x \in P$ which is contained in every Sperner k -family and every Sperner $(k+1)$ -family of P .*

There is no reason to suppose that $\Delta_k(P) \geq \Delta_{k+1}(P)$ in general, but it turns out to be true. This result is more important (and less trivial) than it might seem at first glance:

THEOREM 2.5. (cf. [14]). $\Delta_k(P) \geq \Delta_{k+1}(P)$ for all k .

We know of no elementary proof of theorem 2.5, even when $k = 2$ (although it is trivial when $k = 1$). One difficulty is that there is apparently no combinatorial interpretation of $\Delta_k(P)$ if $k > 1$. It is not always true that a Sperner k -family can be obtained by adding $\Delta_k(P)$ elements to a Sperner $(k-1)$ -family, as the following example shows:

EXAMPLE 2.2. Let $P =$



Then $d_1(P) = 5$ and $d_2(P) = 8$, but $\{3,4,5,6,7\}$ is the only 1-family of size 5 and $\{1,2,6,7\} \cup \{3,4,8,9\}$ is the only 2-family of size 8. Thus no 2-family of size 8 can be obtained by adding 3 elements to a 1-family of size 5.

If it is known that k -saturated partitions exist, theorem 2.5 is trivial, by the following easy lemma:

LEMMA 2.2. *Suppose that $C = \{C_1, \dots, C_q\}$ is a k -saturated partition of P , and exactly h of the chains have length $\geq k$. Then $\Delta_k(P) \geq h \geq \Delta_{k+1}(P)$.*

Using theorem 2.4 and lemma 2.1, it is possible to completely characterize when the collection of all Sperner k -families of P has non-empty intersection.

THEOREM 2.6. (cf. [14]). *The following conditions are equivalent:*

- (1) $\Delta_1(P) > \Delta_{k+1}(P)$.
- (2) $d_{k+1}(P) > (k+1)d_1(P)$.
- (3) *Every set of $k+1$ Sperner k -families has non-empty intersection.*
- (4) *The collection of all Sperner k -families has non-empty intersection.*

The equivalence of the last two conditions suggests HELLY's theorem in k -dimensional euclidean space, which states that (3) and (4) are equivalent for any collection of convex sets. However, it is not true that Sperner k -families have the "Helly property" in a broad sense, since the property may not be inherited by subcollections.

3. COMPLEMENTARY PARTITIONS

DILWORTH's theorem (theorem 2.1) remains true if the words "chain" and "antichain" are interchanged. More surprising than the fact that this is true is its triviality by comparison with DILWORTH's theorem: define A_i to be the set of elements in P which have height i . (Define the *height* of an element x to be the length of the longest chain whose top is x .) If l is the length of the longest chain in P , then A_1, A_2, \dots, A_l is a partition of P into antichains.

Thus it is natural to ask whether a similar transformation can be applied to lemma 2.1 (as well as the other results in section 2). It turns out that almost everything remains true, although at the present time the

proofs do not seem to be trivial. These results were obtained by the author in [12].

We introduce the following terminology: if C is a subset of P which contains no antichains of size $h+1$, we call C an h -cofamily of P . By DILWORTH's theorem, $C = C_1 \cup \dots \cup C_h$ for some set of chains C_i . Let $\hat{d}_h(P)$ denote the size of the largest h -cofamily of P , and let $\hat{\Delta}_h(P) = \hat{d}_h(P) - \hat{d}_{h-1}(P)$. If $A = \{A_1, A_2, \dots, A_k\}$ is a partition of P into antichains, let

$$\hat{\beta}_h(A) = \sum_{i=1}^k \min \{h, |A_i|\}.$$

A partition A of P into antichains is h -saturated if $\hat{d}_h(P) = \hat{\beta}_h(A)$.

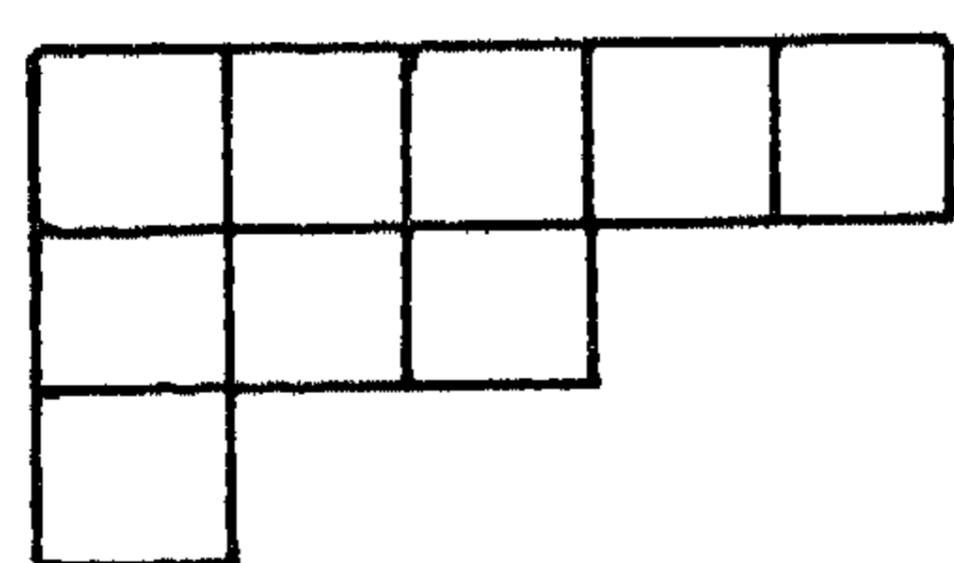
THEOREM 3.1. (cf. [12]). *For all h there exists a partition A of P into antichains which is both h -saturated and $(h+1)$ -saturated.*

THEOREM 3.2. (cf. [12]). *For all h , $\hat{\Delta}_h(P) \geq \hat{\Delta}_{h+1}(P)$.*

By virtue of theorems 2.5 and 3.2, we can think of the numbers $\hat{\Delta}_k(P)$ and $\hat{\Delta}_h(P)$ as forming the parts of a partition of the integer $|P|$, arranged in decreasing order. A remarkable relationship exists between these two sets of numbers: they are conjugate partitions.

THEOREM 3.3. (cf. [12]). *Define two partitions of $|P|$ as follows: $\Delta(P) = \{\Delta_1(P) \geq \Delta_2(P) \geq \dots \geq \Delta_1(P)\}$, where 1 is the length of the longest chain in P , and $\hat{\Delta}(P) = \{\hat{\Delta}_1(P) \geq \hat{\Delta}_2(P) \geq \dots \geq \hat{\Delta}_d(P)\}$, where d is the size of the largest antichain in P . Then $\Delta(P)$ and $\hat{\Delta}(P)$ are conjugate partitions. (That is, $\hat{\Delta}_h(P)$ equals the number of parts of $\Delta(P)$ of size $\geq h$, for all h .)*

As an illustration of theorem 3.3, consider the partially ordered set which appears in example 2.2. Since $d_1(P) = 5$, $d_2(P) = 8$ and $d_3(P) = 9$, and the partition $\Delta(P)$ has shape



it is easy to check that $\hat{d}_1(P) = 3$, $\hat{d}_2(P) = 5$, $\hat{d}_3(P) = 7$, $\hat{d}_4(P) = 8$, and $\hat{d}_5(P) = 9$. (The partition $A = \{1\ 2\ 6\ 7, 3\ 4\ 8\ 9, 5\}$ is 1-, 2- and 3-saturated, and the partition $A' = \{1\ 2, 3\ 4\ 5\ 6\ 7, 8\ 9\}$ is 3-saturated, 4-saturated and 5-saturated.)

A more interesting example is obtained from the theory of permutations. Suppose that $\sigma = \langle a_1, a_2, \dots, a_n \rangle$ is a sequence of distinct integers. Define P_σ to be the set of pairs (a_i, i) , with a partial order defined component-wise. It is easy to see that chains and antichains of P_σ correspond to increasing and decreasing subsequences of σ . Hence k -families (h -cofamilies) correspond to unions of k decreasing (h increasing) subsequences of σ . SCHENSTED [17] showed that the length of the longest increasing subsequence of σ could be computed by constructing a Young tableau (using what is now known as "Schensted's algorithm"), and counting the number of elements in the first row. Moreover, decreasing sequences can be considered by applying the same algorithm to σ in reverse order, in which case the tableau is transformed into its transpose.

In [11] the author extended SCHENSTED's theorem by giving a similar interpretation to the rest of the shape of the tableau associated with σ .

THEOREM 3.4. (cf. [11]). *Let σ be a sequence of distinct integers, and let P_σ be defined as above. If Schensted's algorithm maps σ onto a tableau of shape $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_h\}$, then $\hat{d}_h(P_\sigma) = \lambda_1 + \lambda_2 + \dots + \lambda_h$ for all h , and $\hat{d}_k(P_\sigma) = \lambda_1^* + \lambda_2^* + \dots + \lambda_k^*$ for all k . Hence $\hat{\Delta}_h(P_\sigma) = \lambda_h$ and $\Delta_k(P_\sigma) = \lambda_k^*$.*

Once it is proved that $\hat{\Delta}_h(P_\sigma) = \lambda_h$, it follows trivially that $\Delta_k(P_\sigma) = \lambda_k^*$, by reversing the order of σ . Hence theorem 3.3 is obvious in this case. It should be noted, however, that partially ordered sets of the form $P = P_\sigma$ are the *only* ones in which the relations of comparability and incomparability are interchangeable in this way.

Theorem 3.3 shows that we can associate a "shape" with every partially ordered set P , without actually constructing a tableau. When $P = P_\sigma$, this shape coincides with the one determined by Schensted's algorithm.

If $C = C_1 \cup \dots \cup C_h$ is an h -cofamily of P , we can define a partition of P into chains by taking C_1, C_2, \dots, C_h and each of the elements in $P-C$ as a singleton. Denote this partition by $\mathcal{C} = \{C_1, \dots, C_h; P-C\}$.

THEOREM 3.5. (cf. [12]).

- (i) *If $C = C_1 \cup \dots \cup C_h$ is an h -cofamily of size $\hat{d}_h(P)$, then $\mathcal{C} = \{C_1, \dots, C_h; P-C\}$ is a k -saturated partition for all k such that $\Delta_h^*(P) \geq k \geq \Delta_{h+1}^*(P)$.*
- (ii) *If $\mathcal{C} = \{C_1, \dots, C_h; T\}$ is a k -saturated partition, with each $|C_i| \geq k$,*

then $\Delta_k(P) \geq h \geq \Delta_{k+1}(P)$, and $C = C_1 \cup \dots \cup C_h$ is an h -cofamily of size $\hat{d}_h(P)$.

This result shows that, in a sense, h -cofamilies are k -saturated partitions provided that h and k are related properly. A similar statement holds for k -families and h -saturated partitions of P into antichains.

Next we mention a result which is the "complementary" analogue of theorem 2.6 (parts (3) and (4)):

THEOREM 3.6. (cf. [12]). *If every set of $h+1$ h -cofamilies of size $\hat{d}_h(P)$ has non-empty intersection, then there is an element $x \in P$ which is a member of every h -cofamily of size $\hat{d}_h(P)$.*

For example, if any two maximum-length chains have a common member, then they all have a common member.

The existence of "complementary" theorems makes one suspect that there might be a connection between these results and the theory of *perfect graphs*. A graph G is *perfect* if the analogue of DILWORTH's theorem holds for every subgraph of G . (We think of vertices connected by an edge as "comparable" and unconnected pairs as "incomparable". Hence chains correspond to complete subgraphs and antichains to independent sets.) If G is any graph, the complement G^* of G is obtained by interchanging the relations of "adjacent" and "non-adjacent". BERGE [1] conjectured and LOVÁSZ [16] proved (using ideas developed by FULKERSON [10]) that G^* is perfect whenever G is.

We have the following negative results:

- (1) *Theorem 2.2 need not hold for perfect graphs. That is, k -saturated partitions (into complete subgraphs) do not always exist.*
- (2) *If theorem 2.2 holds for all subgraphs of a graph G , it need not hold for G^* .*

An example which illustrates both observations is obtained by taking G and G^* to be the following graphs (both perfect):



One easily computes

$$\begin{array}{ll} d_1(G) = 3; & \hat{d}_1(G) = \bar{d}_1(G^*) = 3; \\ d_2(G) = 5; & \hat{d}_2(G) = \bar{d}_2(G^*) = 4; \\ d_3(G) = 6; & \hat{d}_3(G) = \bar{d}_3(G^*) = 6. \end{array}$$

It turns out that G^* has no 2-saturated partition, whereas G has 1-, 2- and 3-saturated partitions. Note that G^* also violates the condition $\Delta_1 \geq \Delta_2 \geq \Delta_3$.

On the other hand, G does not satisfy theorem 2.5. Although k -saturated partitions exist for $k = 1, 2, 3$, it is not possible to find a partition which is both 1-saturated and 2-saturated. The next theorem takes advantage of this loophole:

THEOREM 3.7. (cf. [12]). *Let G be a graph with the property that, for all k , there exists a partition of G into complete subgraphs which is both k -saturated and $(k+1)$ -saturated. Then G^* also has this property.*

4. THE LATTICE OF k -FAMILIES

The technical details of [14] were based on a careful study of a natural ordering which can be defined on the set of all Sperner k -families of P . In particular, properties of this ordering were used to prove lemma 2.1, after which most of the other results in section 2 follow by relatively simple arguments.

Let $F_k(P)$ denote the set of all k -families of P , and let $S_k(P)$ denote the set of all Sperner k -families of P . If $k = 1$, we define an ordering on $F_k(P)$ and $S_k(P)$ as follows: if A and B are antichains, we say that $A \leq B$ if every element of A is \leq some element of B . The following results are well-known:

THEOREM 4.1. (BIRKHOFF [2]). $F_1(P)$ is a distributive lattice.

THEOREM 4.2. (DILWORTH [5]). $S_1(P)$ is a sublattice of $F_1(P)$ (and hence is distributive).

It is easy to describe the lattice operations in $F_k(P)$. If U is any subset of P , define $\max[U]$ to be the set of maximal elements of U , and define \bar{U} to be the order ideal generated by U (that is, \bar{U} is the set of elements \leq some member of U .)

LEMMA 4.1. For any antichains A and $B \in F_1(P)$,

- (1) $A \leq B$ if and only if $\bar{A} \subseteq \bar{B}$;
- (2) $A \vee B = \max [\bar{A} \cup \bar{B}] = \max [A \cup B]$;
- (3) $A \wedge B = \max [\bar{A} \cap \bar{B}]$;
- (4) $|A \vee B| + |A \wedge B| \geq |A| + |B|$.

Theorem 4.2 above is an immediate corollary of inequality (4). To prove (4), it is convenient to introduce an auxiliary operation $A \Delta B = ((A \cup B) - (A \cap B)) \cup (A \cap B)$. That is, $A \Delta B$ is the set of "non-maximal" elements of $A \cup B$ (plus all elements which occur twice). It is immediate that $A \Delta B \subseteq A \wedge B$ and $|A \vee B| + |A \Delta B| = |A| + |B|$, from which (4) follows. Using these properties alone, it is possible to obtain some interesting facts about antichains:

COROLLARY 4.1. (KLEITMAN, EDELBERG & LUBELL [15]). *There exists an antichain A of maximum size in P which is invariant under every automorphism of P .*

The proof is easy: take A to be the top element of $S_1(P)$. This argument (due to FREESE [8]) can be used to give a one-line proof of SPERNER's theorem for Boolean algebras.

Another application is the following: if A^+ and A^- denote the top and bottom elements of $S_1(P)$, then every member of $S_1(P)$ lies between them. Hence A^+ and A^- have non-empty intersection if and only if all of the members of $S_1(P)$ have non-empty intersection, and we have a special case of theorem 2.6:

COROLLARY 4.2. *The antichains of maximum size in P have non-empty intersection if and only if any two of them have non-empty intersection.*

Next we extend the ordering defined on antichains to $F_k(P)$ and $S_k(P)$. If A is a k -family, then A can always be partitioned into antichains A_1, A_2, \dots, A_k by taking $A_1 = \max [A]$, $A_2 = \max [A - A_1]$, $A_3 = \max [A - A_1 - A_2]$, and so forth. That is, A_i is the set of elements of "depth" i in A . We call this partition the *canonical partition* of A .

If A and B are k -families, define $A \leq B$ if $A_i \leq B_i$ ($1 \leq i \leq k$), where A_i and B_i denote antichains in the canonical partitions of A and B . It is clear that this definition makes both $F_k(P)$ and $S_k(P)$ into partially ordered sets. To show that both are lattices, we must define new operations:

$$A \vee B = \bigcup_{i=1}^k A_i \vee B_i,$$

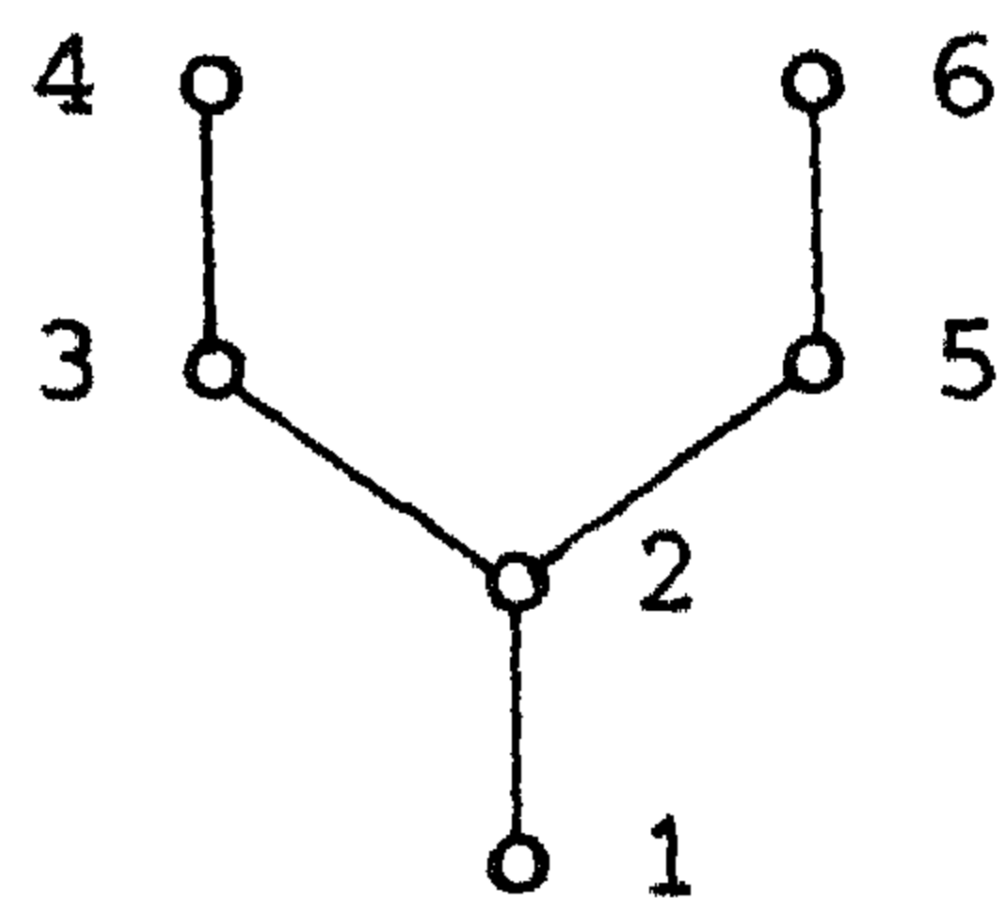
$$A \Delta B = \bigcup_{i=1}^k A_i \Delta B_i,$$

$$A \hat{*} B = \bigcup_{i=1}^k A_i \wedge B_i.$$

THEOREM 4.3. (cf. [14]). $F_k(P)$ is a lattice, in which the join of two k -families is given by $A \vee B$.

However, it may not be true that either $A \Delta B$ or $A \hat{*} B$ coincides with $A \wedge B$ (the true g.l.b. of A and B in $F_k(P)$).

EXAMPLE 4.1. Let $P =$



and $A = \{4, 3\}$, $B = \{6, 5\}$. Then $A \Delta B = \emptyset$ and $A \hat{*} B = \{2\}$. Yet $A \wedge B = \{2, 1\}$.

A procedure for computing $A \wedge B$ was described in [14].

LEMMA 4.2. For any $A, B \in F_k(P)$

- (1) $A \Delta B \subseteq A \hat{*} B \subseteq A \wedge B$;
- (2) $|A \vee B| + |A \Delta B| = |A| + |B|$;
- (3) $|A \vee B| + |A \wedge B| \geq |A| + |B|$.

It follows from inequality (3) that $S_k(P)$ is closed under \vee and \wedge , and hence forms a sublattice of $F_k(P)$. If $k > 1$, it is no longer true in general that $F_k(P)$ is distributive (although it can be shown to be "locally distributive"). Hence one cannot conclude from (3) that $S_k(P)$ is a distributive lattice. Surprisingly, this turns out to be true anyway.

THEOREM 4.4. (cf. [14]). $S_k(P)$ is a distributive sublattice of $F_k(P)$. Moreover, if $A, B \in S_k(P)$, then $A \wedge B = A \Delta B = A \hat{*} B$.

The structure of $F_k(P)$ and $S_k(P)$ is discussed more carefully in [14]. We conclude this section by giving an application of theorem 4.4:

THEOREM 4.5. (cf. [14]). *If P is any partially ordered set, then for each $k \geq 1$ there exists a Sperner k -family $A \in S_k(P)$ which is invariant under every automorphism of P .*

This extends corollary 4.1 to k -families. The proof is exactly the same.

5. GRADED MULTIPARTITE GRAPHS

A different approach to the study of k -families was taken by the author in [13]. Essentially, the idea was to extend FULKERSON's method of obtaining DILWORTH's theorem from HALL's matching theorem [9].

If P is a partially ordered set, define $\Gamma_k(P)$ to be the *graded multipartite graph* obtained by taking $k+1$ copies of P (denoted by P_1, P_2, \dots, P_{k+1}) and connecting $x \in P_i$ to $y \in P_{i+1}$ if $x < y$ in P . A *partial matching* in $\Gamma_k(P)$ is a collection of disjoint paths of length $(k+1)$ which link some element of P_1 to some element of P_{k+1} . If $k = 1$, FULKERSON observed that the edges of a maximum partial matching can always be joined to form a minimum partition of P into chains. A proper interpretation of the HALL-ORE matching condition gives DILWORTH's theorem immediately. If $k > 1$, the situation is somewhat more complicated, since the first part of FULKERSON's argument is difficult to duplicate. Nevertheless, the second part carries over easily, and we obtain the following:

THEOREM 5.1. (cf. [13]). *The maximum number of paths in a partial matching in $\Gamma_k(P)$ is equal to $|P| - d_k(P)$.*

Theorem 5.1 is proved by showing that every minimal separating set in $\Gamma_k(P)$ is obtained by partitioning a set of the form $P-A$, where A is a Sperner k -family. Minimal separating sets can be found using a flow algorithm (or other methods in this case) and hence there is an effective procedure for constructing Sperner k -families.

If l is the length of the longest chain in P , and $k = l-1$, it is easy to see that disjoint paths in $\Gamma_k(P)$ correspond to disjoint maximum-length chains in P . Moreover A is a Sperner k -family if and only if $P-A$ meets every chain of length l .

COROLLARY 5.1. *The maximum number of disjoint 1-chains in P is equal to the minimum number of elements in P which meet every 1-chain.*

This result is just another way of stating that (1-1)-saturated partitions exist. Hence we obtain an easier proof of theorem 2.2 in this case. It is interesting to note that corollary 5.1 remains true if chains are replaced by antichains (theorem 3.1), although there is apparently no analogous proof using flows.

If we attempt to extend FULKERSON's proof of DILWORTH's theorem, the difficulty for $k > 1$ is that partial matchings in $\Gamma_k(P)$ cannot be readily transformed into collections of chains in P . However, the converse problem is trivial: if $C = \{C_1, C_2, \dots, C_q\}$ is a partition of P into chains a partial matching in $\Gamma_k(P)$ is obtained by taking all consecutive segments of length $(k+1)$ appearing in C . Moreover, it is easy to see that the number of paths in the matching is exactly $|P| - \beta_k(C)$. Hence another (more constructive) proof of theorem 2.2 follows if we show that every maximum partial matching in $\Gamma_k(P)$ can be "straightened out", so that it corresponds to one obtained from a partition C by taking consecutive segments of length $(k+1)$. This is easy for small values of k but becomes more difficult as k increases. A general algorithm was described by the author in [13].

6. SUBMODULAR FUNCTIONS

In this section we describe a class of combinatorial geometries which can be associated with a partially ordered set by means of submodular functions related to k -families. The basic tool is the identity

$$|A \vee B| + |A \Delta B| = |A| + |B| \quad (\text{lemma 4.2}).$$

LEMMA 6.1. (cf. [14]). *For any k , the function d_k is a super-modular function on order ideals of P . That is, if \bar{M} and \bar{N} are order ideals of P , then $d_k(\bar{M} \cup \bar{N}) + d_k(\bar{M} \cap \bar{N}) \geq d_k(\bar{M}) + d_k(\bar{N})$.*

If P is any partially ordered set, let P_0 be the set of maximal elements of P , and let $P^* = P - P_0$. If $X \subseteq P_0$ define

$$\delta_k(X) = d_k(P^* \cup X) - d_k(X), \quad \text{and} \quad r_k(X) = |X| - \delta_k(X).$$

THEOREM 6.1. (cf. [14]). With the above notation, r_k is the rank function of a combinatorial geometry on P_0 . That is

- (1) $r_k(\emptyset) = 0$;
- (2) $r_k(X) \leq r_k(X \cup p) \leq r_k(X) + 1$, $X \subseteq P_0, p \in P_0$;
- (3) $r_k(X \cup Y) + r_k(X \cap Y) \leq r_k(X) + r_k(Y)$, $X, Y \subseteq P_0$.

If $k = 1$ and P has height two (so that P_0 and P^* are both antichains) r_k coincides with the usual rank function on bipartite graphs associated with the HALL-ORE matching theorem. In this case, a set X is independent if it forms the set of initial vertices of a matching. In general, one can give the following interpretation of what it means for a set to be independent:

THEOREM 6.2. (cf. [14]). A subset $X \subseteq P_0$ satisfies $r_k(X) = |X|$ if and only if there exists a k -saturated partition of P^* and a matching of X into the set of tops of chains of length $\geq k$.

We conjecture that this geometry is actually induced (in the sense of [3]) by another geometry on P^* defined by taking bases to be the sets of tops of chains of length $\geq k$ formed by k -saturated partitions of P^* . We have not been able to prove or disprove this.

If $k = 1$, however, it is true. Form the bipartite graph $\Gamma_1(P^*)$ (defined in section 5), and consider the standard transversal geometry which it determines on P^* . A subset $B \subseteq P^*$ is a basis if and only if for some 1-saturated partition of P^* , B is the set of elements which are not tops of chains. If we take the dual of this geometry, then the sets which are tops of chains become bases and we have proved the following:

THEOREM 6.3. If Q is any partially ordered set, let $B(Q)$ denote the collection of subsets of Q which are the tops of chains in some minimal partition of Q into chains. Then $B(Q)$ is the set of bases of a combinatorial geometry.

We conclude this paper with an application of theorems 6.1 and 6.2, giving a completely different proof of theorem 2.4 in the special case $k = 1$.

THEOREM 6.4. For any partially ordered set P , there exists a partition of P into chains which is 1-saturated and 2-saturated.

PROOF. Let A be an antichain of maximum size in P , and let P^+ and P^- denote the parts of P which lie above A and below A , respectively. Using the partially ordered set $P^+ \cup A$, define a geometry $G^+(A)$ on A , whose rank function

r^+ is obtained by taking $k = 1$ in the definition preceding theorem 6.1. Similarly, define a geometry $G^-(A)$ with rank function r^- , by using $P^- \cup A$ in the same way. Then $r^+(A) = |A| - (d_1(P^+ \cup A) - d_1(P^+)) = d_1(P^+)$, and $r^-(A) = |A| - (d_1(P^- \cup A) - d_1(P^-)) = d_1(P^-)$. Let X be a basis of $G^+(A)$ and let Y be a basis of $G^-(A)$. By theorem 6.2 we can partition $X \cup P^+$ into $|X|$ chains, and $Y \cup P^-$ into $|Y|$ chains. By linking these chains together and adding the remaining singletons of A , we obtain a partition of P into $d_1(P)$ chains which has exactly $|X \cup Y| = d_1(P^+) + d_1(P^-) - |X \cap Y|$ chains of length two or more. This partition (which is trivially 1-saturated) will be 2-saturated if the number of chains of length two or more is $d_2(P) - d_1(P)$. Hence we must show that X and Y can be chosen so that

$$|X \cap Y| = d_1(P) - d_2(P) + d_1(P^+) + d_1(P^-).$$

But this turns out to be a direct consequence of EDMONDS' matroid intersection theorem [6]. According to EDMONDS' theorem, there exists a set of size q which is independent in both $G^+(A)$ and $G^-(A)$ if and only if $q \leq r^+(U) + r^-(A-U)$ for all $U \subseteq A$. But

$$\begin{aligned} \min_{U \subseteq A} (r^+(U) + r^-(A-U)) &= \min_{U \subseteq A} \{ (|U| - d_1(P^+ \cup U) + d_1(P^+)) + \\ &\quad + (|A-U| - d_1(P^- \cup (A-U)) + d_1(P^-)) \} = \\ &= d_1(P) + d_1(P^+) + d_1(P^-) - \max_{U \subseteq A} \{ d_1(P^+ \cup U) + \\ &\quad + d_1(P^- \cup (A-U)) \} \geq \\ &\geq d_1(P) + d_1(P^+) + d_1(P^-) - d_2(P) \end{aligned}$$

as desired. (The last step follows if we observe that the sum in brackets is the size of some 2-family.) \square

It seems likely that an extension of this argument could be used to prove theorem 2.4 for arbitrary k , but so far we have not been able to find such a proof.

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COMBINATORIAL RECIPROCITY THEOREMS ^{*})

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A *combinatorial reciprocity theorem* is a result which establishes a kind of duality between two related enumeration problems. This rather vague concept will become clearer as more and more examples of such theorems are given. We shall be content in this paper with explaining the meaning of various reciprocity theorems *via* mere statements of results, together with clarifying examples. A rigorous treatment with detailed proofs appears in [11].

1. POLYNOMIALS

A *polynomial reciprocity theorem* takes the following form. Two combinatorially defined sequences S_1, S_2, \dots and $\bar{S}_1, \bar{S}_2, \dots$ of finite sets are given, such that the functions $f(n) = |S_n|$ and $\bar{f}(n) = |\bar{S}_n|$ are polynomials in n for all integers $n \geq 1$. One then concludes that $\bar{f}(n) = (-1)^d f(-n)$, where $d = \deg f$. Frequently the numbers $f(0)$ and $\bar{f}(0)$ will have a special significance.

EXAMPLE 1.1. Fix $p > 0$. Let $f(n)$ be the number of combinations *with repetitions* of n things taken p at a time. Let $\bar{f}(n)$ be the number of such combinations *without repetitions*. Thus $f(n) = \binom{n+p-1}{p}$ and $\bar{f}(n) = \binom{n}{p}$. Hence it can be verified by inspection that $f(n)$ and $\bar{f}(n)$ are polynomials in n of degree p , related by $\bar{f}(n) = (-1)^p f(-n)$.

EXAMPLE 1.2. (THE ORDER POLYNOMIAL). Let P be a finite partially ordered set of cardinality $p > 0$. Let $\omega: P \rightarrow [p]$ be a fixed bijection, where we use the "French notation" $[p] = \{1, 2, \dots, p\}$. Let $\Omega(n)$ denote the number of maps

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$\sigma: P \rightarrow [n]$ such that (i) $x \leq y$ in P implies $\sigma(x) \leq \sigma(y)$, and (ii) $x < y$ in P and $\omega(x) > \omega(y)$ implies $\sigma(x) < \sigma(y)$. Let $\bar{\Omega}(n)$ denote the number of maps $\tau: P \rightarrow [n]$ such that (i) $x \leq y$ in P implies $\tau(x) \leq \tau(y)$, and (ii) $x < y$ in P and $\omega(x) < \omega(y)$ implies $\tau(x) < \tau(y)$. Then it can be shown [8, Proposition 13.2(i)] that Ω and $\bar{\Omega}$ are polynomial functions in n of degree p related by $\bar{\Omega}(n) = (-1)^P \Omega(-n)$. We call Ω the *order polynomial* of (P, ω) . There are several ways to prove this reciprocity relationship between Ω and $\bar{\Omega}$, perhaps the simplest by a judicious use of the Principle of Inclusion-Exclusion which we leave to the reader. Note that if P is a p -element chain and ω is order-preserving, then $\Omega(n) = \binom{n+p-1}{p}$ and $\bar{\Omega}(n) = \binom{n}{p}$, so example 1.1 is a special case.

Several interesting consequences of the reciprocity between Ω and $\bar{\Omega}$ are derived in [8, §19]. For instance, if ω is order-preserving then for some integer ℓ we have $\Omega(n) = (-1)^P \bar{\Omega}(-\ell-n)$ for all n if and only if every maximal chain of P has length ℓ .

EXAMPLE 1.3. (CHROMATIC POLYNOMIALS). Let G be a finite graph without loops or multiple edges, with vertex set V of cardinality p . Let $\chi(n)$ denote the number of pairs (θ, σ) , where (i) θ is an acyclic orientation of the edges of G , and (ii) $\sigma: V \rightarrow [n]$ is any map $V \rightarrow [n]$ such that if $u \rightarrow v$ in θ (so $u, v \in V$ and uv is an edge of G) then $\sigma(u) > \sigma(v)$. Let $\bar{\chi}(n)$ be the number of such maps with the condition $\sigma(u) > \sigma(v)$ replaced with $\sigma(u) \geq \sigma(v)$. It is easily seen that $\chi(n)$ is the chromatic polynomial of G . In [9] two proofs are given of the reciprocity theorem $\bar{\chi}(n) = (-1)^P \chi(-n)$. In particular, $(-1)^P \chi(-1)$ is the number of acyclic orientations of G .

EXAMPLE 1.4. (ABSTRACT MANIFOLDS). Let Δ be a finite simplicial complex with vertex set V , with $|V| = p$. Thus Δ is a collection of subsets S of V such that $\{v\} \in \Delta$ for all $v \in V$, and if $S \in \Delta$ and $T \subset S$, then $T \in \Delta$. Let $f_i = f_i(\Delta)$ be the number of $(i+1)$ -sets contained in Δ . Hence $f_{-1} = 1$ and $f_0 = p$. Define the polynomial $\Lambda(\Delta, n)$ by

$$\Lambda(\Delta, n) = \sum_{i \geq 0} f_i \binom{n-1}{i}.$$

Note that $\Lambda(\Delta, 0) = f_0 - f_1 + f_2 - \dots = \chi(\Delta)$, the Euler characteristic of Δ .

Now suppose that the underlying topological space $|\Delta|$ of Δ is homeomorphic to a d -dimensional manifold with boundary. Hence $\deg \Lambda(\Delta, n) = d$. Denote by $\partial\Delta$ those elements of Δ such that $|\partial\Delta| = \partial|\Delta|$, in the obvious

sense. Hence $\partial\Delta$ is itself a simplicial complex, with vertex set contained in V . It follows from a result of MACDONALD [5, Proposition 1.1] that

$$(1.1) \quad (-1)^d \Lambda(\Delta, -n) = \Lambda(\Delta, n) - \Lambda(\partial\Delta, n) .$$

For instance, let Δ consist of ABCD, BCDE, and all their subsets (ABCD is short for $\{A, B, C, D\}$, etc.) Then $d = 3$, $|\Delta|$ is a 3-ball, and $\partial\Delta$ consists of ABC, ABD, ACD, BCE, CDE, BDE, and all their subsets. Moreover,

$$\Lambda(\Delta, n) = 5 + 9 \binom{n-1}{1} + 7 \binom{n-1}{2} + 2 \binom{n-1}{3}$$

and

$$\Lambda(\partial\Delta, n) = 5 + 9 \binom{n-1}{1} + 6 \binom{n-1}{2} .$$

It follows from (1.1) that

$$-\Lambda(\Delta, -n) = \binom{n-1}{2} + 2 \binom{n-1}{3} .$$

A special case of particular interest occurs when $\partial\Delta = \emptyset$, i.e., when $|\Delta|$ is a manifold. We then have from (1.1) that

$$(1.2) \quad (-1)^d \Lambda(\Delta, -n) = \Lambda(\Delta, n) .$$

Now (1.2) imposes certain constraints on the numbers f_i which define Λ . When $|\Delta|$ is a sphere, these constraints are simply the well-known DEHN-SOMMERVILLE equations [4, Chapter 9] [6, Chapter 2.4].

EXAMPLE 1.5. (CONCRETE MANIFOLDS). Let M be a subset of the s -dimensional euclidean space with the following properties: (i) M is a union of finitely many convex polytopes, any two of which intersect in a common face of both, (ii) the vertices of these convex polytopes have integer coordinates, and (iii) M is homeomorphic to a d -dimensional manifold with boundary. If $n > 0$, then let $j(n)$ be the number of points $\alpha \in M$ such that $n\alpha$ has integer coordinates, and let $i(n)$ be the number of such points not belonging to ∂M . Then a result due essentially to E. EHRHART [2] (for the generality considered here, one also needs [5, Proposition 1.1]) states that $j(n)$ and $i(n)$ are polynomial functions of n of degree d satisfying

$$(1.3) \quad j(0) = \chi(M), \quad i(n) = (-1)^d j(-n) .$$

We remark that condition (ii) can be replaced by the requirement (ii') the vertices have *rational* coordinates. In this case i and j need no longer be polynomials, but instead there is some $N > 0$ and polynomials j_0, j_1, \dots, j_{N-1} and i_0, i_1, \dots, i_{N-1} such that $j(n) = j_a(n)$ and $i(n) = i_a(n)$ whenever $n \equiv a \pmod{N}$. We then have in place of (1.3) that $j_0(0) = \chi(M)$ and $i_a(n) = (-1)^d j_{-a}(-n)$, where the subscripts are taken modulo N .

An interesting application of (1.3) is to the problem of finding the volume $V(M)$ of a subset M satisfying conditions (i), (ii), (iii), and the additional condition that $s = d$. It is easy to see that then the leading coefficient of $j(n)$ is $V(M)$. Hence from (1.3) we see that if we know any $d+1$ of the numbers $\chi(M), j(n), i(n), n \geq 1$, then we can compute $V(M)$. For a further discussion of this result (including references), see [11].

EXAMPLE 1.6. (MAGIC SQUARES). As a special case of example 1.5, take M to be the set of all doubly stochastic $N \times N$ matrices, so $s = N^2$ and $d = (N-1)^2$. It is well-known that M is a convex polytope whose vertices have integer coordinates, so $j(n)$ and $i(n)$ are polynomials in n of degree $(N-1)^2$. It is easy to see that $j(n)$ is the number of $N \times N$ matrices of non-negative integers with every row and column sum equal to n , while $i(n)$ is the number of such matrices with positive entries. Clearly $i(0) = i(1) = \dots = i(N-1) = 0$ and $i(N+n) = j(n)$ for $n \geq 0$. There follows from (1.3),

$$j(-1) = j(-2) = \dots = j(-N+1) = 0 ,$$

$$j(n) = (-1)^{N-1} j(-N-n) .$$

These results were first obtained in [10]. Another proof is given in [3].

2. HOMOGENEOUS LINEAR EQUATIONS

Consider the homogeneous linear equation $x = y$. Let $F(X, Y) = \sum X^\alpha Y^\beta$, where the sum is over all solutions $(x, y) = (\alpha, \beta)$ to $x = y$ in non-negative integers α, β . Let $\bar{F}(X, Y)$ be the corresponding sum over all solutions in positive integers. Clearly $F(X, Y) = 1/(1-XY)$ and $\bar{F}(X, Y) = XY/(1-XY)$. Hence as rational functions we have $\bar{F}(X, Y) = -F(1/X, 1/Y)$. It is this result we

wish to extend to more general systems of equations.

THEOREM 2.1. [10, Theorem 4.1]. *Let E be a system of finitely many linear homogeneous equations with integer coefficients, in the variables x_1, x_2, \dots, x_s . Define*

$$F(x_1, x_2, \dots, x_s) = \sum x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s}, \quad (2.1)$$

$$\bar{F}(x_1, x_2, \dots, x_s) = \sum x_1^{\beta_1} x_2^{\beta_2} \dots x_s^{\beta_s},$$

where $(\alpha_1, \alpha_2, \dots, \alpha_s)$ ranges over all solutions $x_i = \alpha_i$ of E in non-negative integers α_i , while $(\beta_1, \beta_2, \dots, \beta_s)$ ranges over all solutions in positive integers. Then F and \bar{F} are rational functions of the x_i 's (in the algebra of formal power series, or for $|x_i| < 1$). A necessary and sufficient condition that

$$\bar{F}(x_1, x_2, \dots, x_s) = \pm F(1/x_1, 1/x_2, \dots, 1/x_s),$$

as rational functions, is for E to possess a solution in positive integers. In this case the correct sign is $(-1)^{\kappa}$, where κ is the corank (= s-rank E) of E.

Many of the results in section 1 can be deduced from the above theorem. We require a connection between evaluating polynomials at $+n$ and $-n$, and substituting $1/x_i$ for x_i in a rational function. Such a connection is provided by the next result, which EHRHART [1] attributes to POPOVICIU [7].

PROPOSITION 2.1. *Let H(n) be a function from the integers Z to the complex numbers C of the form*

$$H(n) = \sum_{i=1}^r P_i(n) \alpha_i^n,$$

where the α_i 's are fixed non-zero complex numbers and each P_i is a polynomial in n. Define

$$F(X) = \sum_{n=0}^{\infty} H(n) X^n, \quad \bar{F}(X) = \sum_{n=1}^{\infty} H(-n) X^n.$$

Then F and \bar{F} are rational functions of X , related by $\bar{F}(X) = -F(1/X)$.

Theorem 2.1 suggests that we try to find "rational function analogues" of examples 1.4 and 1.5.

PROPOSITION 2.2. *Let Δ be a finite simplicial complex with vertices v_1, v_2, \dots, v_p . Suppose $|\Delta|$ is homeomorphic to a d -manifold with boundary. Define the generating functions*

$$F(v_1, v_2, \dots, v_p) = \sum v_1^{\delta_1} v_2^{\delta_2} \dots v_p^{\delta_p} + \chi(\Delta) - 1,$$

$$\bar{F}(v_1, v_2, \dots, v_p) = \sum v_1^{\epsilon_1} v_2^{\epsilon_2} \dots v_p^{\epsilon_p},$$

where $(\delta_1, \delta_2, \dots, \delta_p)$ ranges over all p -tuples of non-negative integers such that $\{v_i \mid \delta_i > 0\} \in \Delta$, while $(\epsilon_1, \epsilon_2, \dots, \epsilon_p)$ ranges over all p -tuples of non-negative integers such that $\emptyset \neq \{v_i \mid \epsilon_i > 0\} \in \Delta - \partial\Delta$. Then F and \bar{F} are rational functions of the v_i 's related by

$$\bar{F}(v_1, v_2, \dots, v_p) = (-1)^{d+1} F(1/v_1, 1/v_2, \dots, 1/v_p).$$

Proposition 2.2 is a consequence of MACDONALD's result [5, Proposition 1.1] mentioned earlier. It is easily seen that

$$F(X, X, \dots, X) = \sum_{n=0}^{\infty} \Lambda(\Delta, n) X^n,$$

$$\bar{F}(X, X, \dots, X) = \sum_{n=1}^{\infty} [\Lambda(\Delta, n) - \Lambda(\partial\Delta, n)] X^n,$$

in the notation of example 1.4. Thus (1.1) follows from propositions 2.1 and 2.2.

PROPOSITION 2.3. *Let M satisfy properties (i), (ii') and (iii) of example 1.5. Define*

$$F(x_1, x_2, \dots, x_s, y) = \chi(M) + \sum x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} y^n,$$

$$\bar{F}(x_1, x_2, \dots, x_s, y) = \sum x_1^{\beta_1} x_2^{\beta_2} \dots x_s^{\beta_s} y^n,$$

where $(\alpha_1, \alpha_2, \dots, \alpha_s, n)$ ranges over all $(s+1)$ -tuples of non-negative integers

α_i and positive integers n such that $(\alpha_1/n, \alpha_2/n, \dots, \alpha_s/n) \in M$, while $(\beta_1, \beta_2, \dots, \beta_s, n)$ ranges over all such $(s+1)$ -tuples with $(\beta_1/n, \beta_2/n, \dots, \beta_s/n) \in M - \partial M$. Then F and \bar{F} are rational functions related by

$$\bar{F}(X_1, X_2, \dots, X_s, Y) = (-1)^{d+1} F(1/X_1, 1/X_2, \dots, 1/X_s, 1/Y) .$$

If we put each $X_i = 1$ and apply proposition 2.1, then we get (1.3).

3. RECIPROCAL DOMAINS

In theorem 2.1, we considered solutions $\alpha_i \geq 0$ ($i=1,2,\dots,s$) and $\beta_j > 0$ ($j=1,2,\dots,s$) to a system of homogeneous linear equations. It is natural to consider the following generalization. Let E be a system of finitely many linear homogeneous equations with integer coefficients, in the variables x_1, x_2, \dots, x_s (as in theorem 2.1). Let $S \subset [s]$. Define

$$(3.1) \quad F_S(X_1, X_2, \dots, X_s) = \sum X_1^{\alpha_1} X_2^{\alpha_2} \dots X_s^{\alpha_s} ,$$

$$\bar{F}_S(X_1, X_2, \dots, X_s) = \sum X_1^{\beta_1} X_2^{\beta_2} \dots X_s^{\beta_s} ,$$

where $(\alpha_1, \alpha_2, \dots, \alpha_s)$ ranges over all solutions to E in non-negative integers such that $\alpha_i > 0$ if $i \in S$, while $(\beta_1, \beta_2, \dots, \beta_s)$ ranges over all solutions to E in non-negative integers with $\beta_i > 0$ if $i \notin S$. Thus $\bar{F}_S = F_{[s]-S}$. Note that $F_\emptyset = F$ and $\bar{F}_\emptyset = \bar{F}$, where F and \bar{F} are given by (2.1).

We now ask under what conditions do we have

$$(3.2) \quad \bar{F}(X_1, X_2, \dots, X_s) = (-1)^\kappa F(1/X_1, 1/X_2, \dots, 1/X_s) ,$$

where κ is the corank of E . It seems plausible that (3.2) will hold whenever E has a solution in positive integers, as in theorem 2.1. In [11], however, we show that this is not the case; and we show why it is likely that there are no simple necessary and sufficient conditions for (3.2) to hold. There is, however, an elegant and surprising sufficient condition.

THEOREM 3.1. [11, Proposition 8.3]. *A sufficient condition for (3.2) to hold is that there exists a solution $(\gamma_1, \gamma_2, \dots, \gamma_s)$ to E in integers γ_i such that*

$\gamma_i > 0$ if $i \in S$ and $\gamma_i < 0$ if $i \notin S$.

The proof of theorem 3.1 depends on a rather complicated geometric argument suggested by a result of EHRHART [1, p.22] on "reciprocal domains". It is much easier, on the other hand, to give a necessary condition for (3.2) to hold.

PROPOSITION 3.1. *If (3.2) holds, then either $F = \bar{F} = 0$, or else E has a solution in positive integers.*

PROOF. Assume (3.2) holds but not $F = \bar{F} = 0$. Then $F \neq 0$ and $\bar{F} \neq 0$, so E has solutions $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_s)$ as given in (3.1). Then $\alpha + \beta$ is a solution to E in positive integers. \square

4. INHOMOGENEOUS EQUATIONS

Another way of extending theorem 2.1 besides theorem 3.1 is to consider *inhomogeneous* linear equations. Suppose we have a system

$$(4.1) \quad \sum_{i=1}^s a_{ij} x_i = b_j, \quad j \in [p],$$

of p inhomogeneous linear equations with integer coefficients a_{ij} and integer constants b_j , in the variables x_1, x_2, \dots, x_s . It turns out that the correct reciprocal notions to consider in this context are (i) solutions to (4.1) in non-negative integers, and (ii) solutions in positive integers to the "reciprocal system"

$$(4.2) \quad \sum_{i=1}^s a_{ij} x_i = -b_j, \quad j \in [p].$$

Suppose, for example, that $S \subset [s]$ and that

$$b_j = - \sum_{i \in S} a_{ij}, \quad j \in [p].$$

Hence a solution $(\alpha_1, \dots, \alpha_s)$ to (4.1) in non-negative integers corresponds to a solution $(\beta_1, \dots, \beta_s)$ of the system $\sum a_{ij} x_i = 0$ in integers β_i satisfying $\beta_i \geq 0$ if $i \notin S$, $\beta_i > 0$ if $i \in S$ (set $\beta_i = \alpha_i$ if $i \notin S$, $\beta_i = \alpha_i + 1$ if $i \in S$). Moreover, a solution $(\alpha_1, \dots, \alpha_s)$ to (4.2) in positive integers corresponds to a solution $(\beta_1, \dots, \beta_s)$ of the system $\sum a_{ij} x_i = 0$ in integers β_i satisfying

$\beta_i > 0$ if $i \notin S$, $\beta_i \geq 0$ if $i \in S$ (set $\beta_i = \alpha_i$ if $i \notin S$; $\beta_i = \alpha_i - 1$ if $i \in S$). Hence our notion of reciprocity for inhomogeneous systems includes the reciprocity of section 3 as a special case.

We therefore define

$$F(x_1, x_2, \dots, x_s) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s},$$

(4.3)

$$\bar{F}(x_1, x_2, \dots, x_s) = x_1^{\beta_1} x_2^{\beta_2} \dots x_s^{\beta_s},$$

where $(\alpha_1, \alpha_2, \dots, \alpha_s)$ ranges over all solutions to (4.1) in non-negative integers, while $(\beta_1, \beta_2, \dots, \beta_s)$ ranges over all solutions to (4.2) in positive integers. As usual, we seek conditions when $\bar{F}(x_1, x_2, \dots, x_s) = (-1)^\kappa F(1/x_1, 1/x_2, \dots, 1/x_s)$, where κ is the corank of (4.1) or (4.2). We shall say that (4.1) has the *R-property* if $\bar{F}(x_1, x_2, \dots, x_s) = (-1)^\kappa F(1/x_1, 1/x_2, \dots, 1/x_s)$. The possibility of obtaining reasonable necessary and sufficient conditions for E to have the *R-property* appears hopeless, and even reasonably general sufficient conditions are rather complex and not very edifying. We shall now discuss the nature of the sufficient conditions obtained in [11].

Let $\{i_1, i_2, \dots, i_k\}$ be a set of $k < p$ elements from $[s]$ such that the determinant of coefficients taken from the first k rows and from columns i_1, i_2, \dots, i_k of (4.1) is non-zero. Hence we can solve the first k equations (i.e., $j \in [k]$) of (4.1) for $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ in terms of the remaining x_i 's and substitute these values in the remaining $p-k$ equations, obtaining $p-k$ equations in $s-k$ unknowns. Let $E(i_1, i_2, \dots, i_k)$ denote the first of these $p-k$ equations (i.e., the equation resulting from making the above substitution into the $(k+1)$ -st equation of (4.1)). Thus in particular $E(\emptyset)$ is just the first equation $\sum a_{i1} x_i = b_1$ of (4.1). Note that the equations $E(i_1, i_2, \dots, i_k)$ are really determined only up to a non-zero multiplicative constant. This need not concern us since we will be interested only in solutions to these equations.

EXAMPLE 4.1. Consider the system

$$\begin{aligned} x_1 - x_2 + 3x_3 &= b_1 \\ 2x_2 - x_3 - x_4 &= b_2. \end{aligned}$$

Then we obtain the equations

$$\begin{aligned}
 E(\emptyset): & \quad x_1 - x_2 + 3x_3 = b_1 \\
 E(1): & \quad 2x_2 - x_3 - x_4 = b_2 \\
 E(2): & \quad 2x_1 + 5x_3 - x_4 = 2b_1 + b_2 \\
 E(3): & \quad x_1 + 5x_2 - 3x_4 = -b_1 + 3b_2 .
 \end{aligned}$$

THEOREM 4.1. *A sufficient condition that the system (4.1) has the R-property is the following. For every set $\{i_1, i_2, \dots, i_k\} \in [s-1]$ for which $E(i_1, i_2, \dots, i_k)$ is defined, the single equation $E(i_1, i_2, \dots, i_k)$ should possess the R-property.*

It should be mentioned that in [11] theorem 4.1 is strengthened so that only a special subset of the equations $E(i_1, i_2, \dots, i_k)$ need be considered. However, the definition of this subset is rather complicated and will be omitted here. Theorem 4.1 is proved in [11] using iterated contour integration. Contour integration may seem like an unwarranted artifice for a result like theorem 4.1. While it is undoubtedly possible to dispense with contour integration, the next results show that it is not too unnatural in the present context. We would like to complement theorem 4.1 by obtaining conditions for a single equation to possess the R-property.

THEOREM 4.2. *Let $a_1x_1 + a_2x_2 + \dots + a_sx_s = b$ be a single linear equation E with integer coefficients a_i and integer constant term b. Then the following three conditions are equivalent.*

(i) *The rational functions*

$$(4.4) \quad \lambda^{b-1} / (1-\lambda^{-a_1})(1-\lambda^{-a_2}) \dots (1-\lambda^{-a_s})$$

and

$$(4.5) \quad \lambda^{\bar{b}-1} / (1-\lambda^{-a_1})(1-\lambda^{-a_2}) \dots (1-\lambda^{-a_s})$$

have zero residues at $\lambda = 0$. Here $\bar{b} = -b - a_1 - a_2 - \dots - a_s$.

(ii) *The following two conditions are both satisfied.*

(a) *There does not exist a solution $(\alpha_1, \alpha_2, \dots, \alpha_s)$ to E in integers such that*

$$\begin{aligned}
 \alpha_t &< 0 \quad \text{if} \quad a_t > 0, \quad \text{and} \\
 \alpha_t &\geq 0 \quad \text{if} \quad a_t < 0.
 \end{aligned}$$

(b) There does not exist a solution $(\beta_1, \beta_2, \dots, \beta_s)$ to E in integers such that

$$\begin{aligned} \beta_t &\geq 0 \quad \text{if } a_t > 0, \text{ and} \\ \beta_t &< 0 \quad \text{if } a_t < 0. \end{aligned}$$

(Note: It is clear that at least one of (a) or (b) always holds.)

(iii) E has the R-property.

THEOREM 4.3. With the hypotheses of theorem 4.2, the following two conditions are equivalent.

- (i) The rational functions of (4.4) and (4.5) have no poles at $\lambda = 0$.
- (ii) $\sum_{t^-} a_t < -b < \sum_{t^+} a_t$, where $\sum_{t^-} a_t$ (resp. $\sum_{t^+} a_t$) denotes the sum of all a_t satisfying $a_t < 0$ (resp. $a_t > 0$).

If, moreover, either of the two (equivalent) conditions (i) or (ii) is satisfied, then E has the R-property.

EXAMPLE 4.2. Consider the system E of example 4.1. By theorems 4.1 and 4.3, we see that E has the R-property if

$$\begin{aligned} -1 &< -b_1 &< 4 \\ -2 &< &-b_2 < 2 \\ -1 &< -2b_1 - b_2 &< 7 \\ -3 &< b_1 - 3b_2 &< 6. \end{aligned}$$

These conditions hold if and only if $(b_1, b_2) = (0, -1), (0, 0), (-1, -1), (-1, 0), (-2, -1)$ or $(-2, 0)$.

Analogously to proposition 3.1, we have a simple necessary condition for a system (4.1) to have the R-property. The proof is essentially the same as the proof of proposition 3.1.

PROPOSITION 4.1. Suppose the system (4.1) has the R-property. Then either $F = \bar{F} = 0$, or else the homogeneous system $\sum_{i=1}^s a_{ij} x_i = 0$, $j \in [p]$, has a solution in positive integers.

We have given a sampling of what we believe to be the most interesting examples of combinatorial reciprocity theorems. Some additional types of reciprocity theorems are given in [11]. There are many other combinatorial relationships which can be viewed as reciprocity theorems and which we have

not touched on. Examples include the inverse relationship between the Stirling numbers of the first and second kinds, and the MacWilliams identities of coding theory. We believe that many new interesting results and unifying principles are awaiting discovery in the field of combinatorial reciprocity.

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