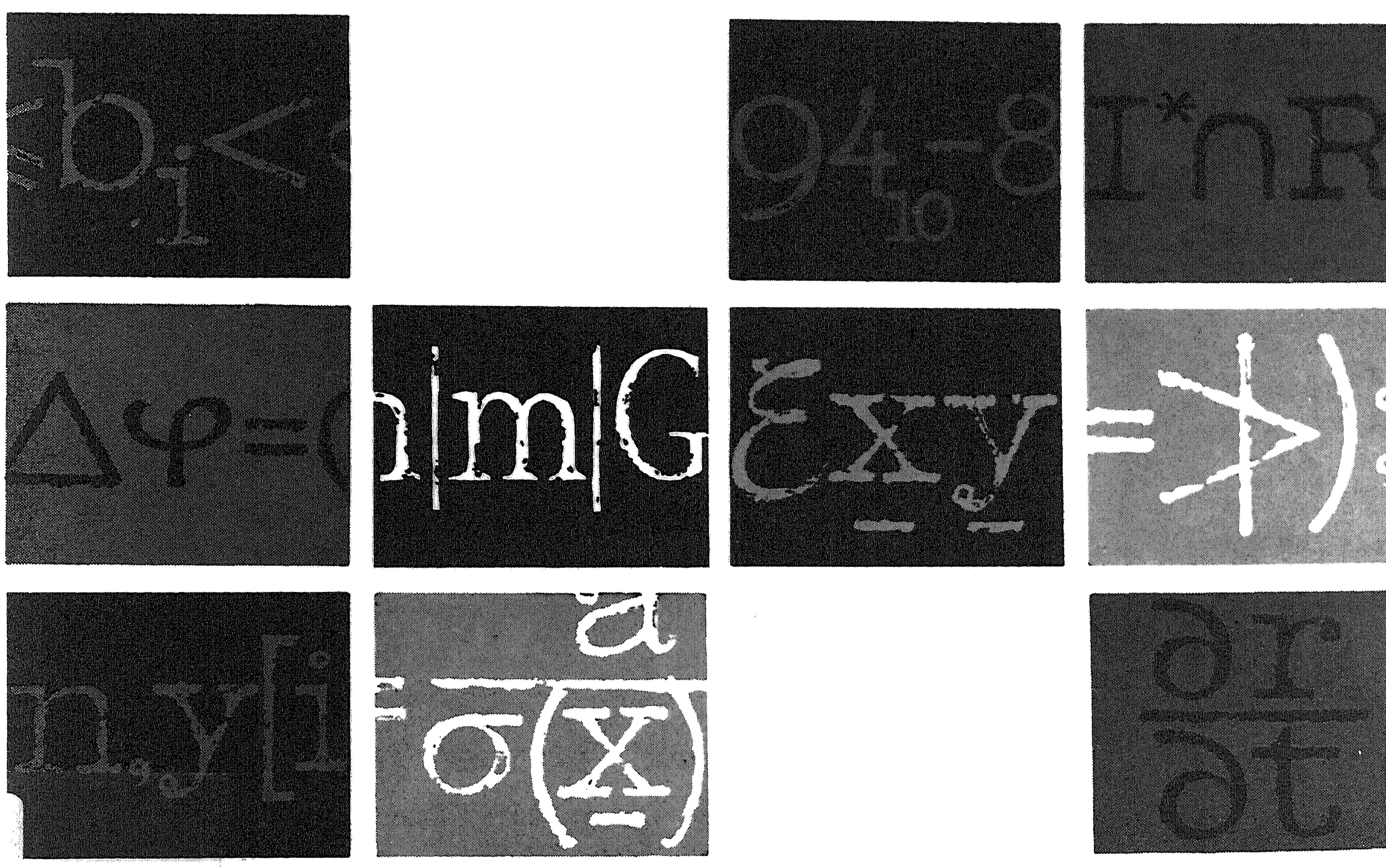
TOPOLOGY AND ORDER STRUCTURES

PART 2

edited by H.R. BENNETT
D.J. LUTZER



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PREFACE

In August, 1981, NATO and Free University of Amsterdam jointly sponsored the second meeting of a workshop on topology and linear orderings. This volume includes contributions from most of the participants in that workshop plus papers by several others who were not able to attend the workshop. We wish to express our gratitude to NATO and Free University for their financial support, to the Mathematical Centre for agreeing to publish this volume, and to our colleagues J. van MILL and E. WATTEL for their invaluable assistance in preparing this manuscript. In addition, we wish to thank the National Science Foundation for support during the editing of this volume.

H.R. Bennett and D.J. Lutzer

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TWENTY QUESTIONS ON ORDERED SPACES

D.J. LUTZER

INTRODUCTION

In August 1980, NATO and Texas Tech University sponsored a workshop on topology and orderings in Lubbock, Texas. During a two week period, a dozen specialists met to collaborate on a wide spectrum of problems associated primarily with the interplay of topology and linear orderings. A number of questions were raised, many of which were published in the proceedings of that workshop [11]. In this paper I will discuss the status of several of those questions, some of which were solved during the year between August, 1980 and the second meeting of the workshop in Amsterdam. I will concentrate on twenty questions which were open in August 1981 (some of which have since been solved; see "Added in Proof").

For basic information about linearly ordered topological spaces (LOTS) and generalized ordered spaces (GO-spaces), see [20], [19], [42]. Recently the term "suborderable space" has become widely used for a topological space which can be given an ordering that makes it a GO-space. I will use the terms interchangably. At many points in this paper, the phrase "if and only if" will be abbreviated to "iff".

My goal in this paper is not to write yet another survey of GO-spaces but rather to present some of the most interesting problems in hope that some of my readers will attack and solve them.

ORDERABILITY PROBLEMS AND ORDERABLE VS. SUBORDERABLE SPACES

The (sub) orderability problem asks for characterizations of those topological spaces which are (sub) orderable. The first of the posed problems in [11] concerned orderability via selections. Van Mill and Wattel have since solved that problem, proving

<u>THEOREM 2.1</u>. [28]: *A completely regular space* X *is suborderable if and only if there is a continuous mapping* $s: X \times X \to X$ *which satisfies*

(a) $s(x,y) = s(y,x) \in \{x,y\}$

(b) *if* U *is open and* $x \in U$*, there is an open* V *with* $x \in V \subset U$ *such that whenever* $y \in V$ *and* $z \in X - U$, $s(y,z) = y$ *if and only if* $s(x,z) = x$.

An immediate corollary of (2.1) is an earlier result by van Mill and Wattel which solves a problem posed by E. Michael [26].

<u>COROLLARY 2.2</u>. [27]: *For a compact* T_2*-space, the following are equivalent:*

(a) X *is orderable;*

(b) *if* 2^X *denotes the Vietoris hyperspace of all closed, nonvoid subsets of* X*, then there is a continuous mapping* $s: 2^X \to X$ *having* $s(F) \in F$ *for each* $F \in 2^X$*;*

(c) *there is a continuous mapping* $s: X \times X \to X$ *such that for* $(x,y) \in X^2$, $s(y,x) = s(x,y) \in \{x,y\}$.

A third orderability theorem was proved by Purisch, answering a question posed by Galvin.

<u>THEOREM 2.3</u>. [31]: *Any scattered suborderable space is orderable.*

It is easy to mis-read a result such as (2.3). Theorem 2.3 does *not* say that the given ordering of a scattered GO-space X yields the topology of X as its usual open-interval topology. Usually some other linear ordering of X must be constructed. For example, if $X = \{\alpha < \omega_1 \mid \alpha \text{ is not a limit ordinal}\}$ then X is a scattered GO-space whose given ordering yields a copy of $[0,\omega_1)$ if we use its open-interval topology on X.

Still open is an old orderability problem for compact spaces, originally posed by Purisch [33]. I will list it as my first open question.

<u>QUESTION 1</u>. *Suppose* X *is separable, zerodimensional, monotonically normal and compact. Is* X *orderable?*

One approach to Question (1) is to attempt to decide whether such an X has properties which it would have if it were orderable. In this direction, Ostaszewski proved that the space X in Question (1) must be hereditarily separable and hereditarily Lindelöf.

Asking whether a given GO-space is orderable is a very natural orderability problem. The "classical" approach to obtaining a negative answer is to discover a theorem which holds for every orderable space and yet which fails to hold for the GO-space in question. For example, I proved

<u>THEOREM 2.4</u>. [18]: *Any orderable space with a* G_δ*-diagonal is metrizable.*

It follows immediately that neither the Michael line nor the Sorgenfrey line can be orderable. However, this "classical" approach is not always successful. For example, it does not decide the orderability problem for the set Q of rational numbers topologized as a subspace of the Sorgenfrey line (the space *is* orderable). In August, 1980, I asked for other approaches. Purisch and Wattel responded by proving a non-orderability theorem based on counting pseudogaps. (We say that a *pseudogap* occurs at a point p of a GO-space X if either $[p,\rightarrow)$ or $(\leftarrow,p]$ is open in the topology of X but not open in the usual open interval topology defined by the ordering of X.)

<u>THEOREM 2.5</u>. [32]: *Suppose* X *is a* GO*-space and that the number of pseudogaps of* X *exceeds each of:*

(a) *the density of the derived set,* X^d*, of* X*;*

(b) sup {card (S) : S *is a closed discrete subspace of* X}*;*

(c) *the cardinality of the set*

$\cup\{C : C$ *is a convex component of* $X - X^d$ *and* $\overline{C}$ *is not compact*$\}$.

Then X *is <u>not</u> orderable. In particular, if the number of pseudogaps of* X *exceeds the density of* X*, then* X *is not orderable.*

Purisch and Wattel apply (2.5) to show that the space $X = [0,1] \times \{0,1\}$, with the lexicographic ordering and with each point (t,1) isolated, is a GO-space which is not orderable.

Van Douwen and I raised another orderability problem at the first NATO workshop. We asked "Is it true that each GO-space has a dense, orderable subspace?" That question has been completely answered by S. Williams who has been able to characterize spaces with dense orderable subspaces. He begins by introducing the notion of a butterfly space: X is a *butterfly space* provided that each $x \in X$ is associated with two nests $\mathcal{U}_0(x)$ and $\mathcal{U}_1(x)$ of open sets such that

(a) $\{U_0 \cup \{x\} \cup U_1 : U_i \in \mathcal{U}_i(x)$ for $i \in \{0,1\}\}$ is neigborhood base at x;

(b) if $U_i \in \mathcal{U}_i(x)$, then $U_0 \cap U_1 = \emptyset$.

Then Williams proves

THEOREM 2.6. [40]: *Let* X *be a* $T_{3.5}$ *space. Then the following are equivalent:*
(a) X *has a dense, orderable subspace;*
(b) X *has a* π*-base which is a tree under inclusion and has a dense subspace which is a butterfly space;*
(c) βX*, the Čech-Stone compactification of* X*, is co-absolute with some* LOTS *and* X *has a dense butterfly subspace.*

Since every GO-space is itself a butterfly space and since it is easy to (inductively) construct a tree π-base for X, he obtains

COROLLARY 2.7. *Any* GO*-space has a dense orderable subspace.*

That corollary has an interesting (but already known) application. Recall that a *Blumberg space* is a space X such that for every real valued function f on X, there is a dense subset D = D(f) of X such that the function $f|_D : D \to R$ is continuous. A classical result of Bradford and Goffman [13] asserts that a metrizable space is a Blumberg space iff it satisfies the Baire Category Theorem, and for many years it was unknown whether every compact T_2 space is a Blumberg space. W. Weiss [38] provided the required counterexample using a very clever argument; an earlier consistency result (obtained independently by Weiss and myself) is now easy to prove. Recall that a *Souslin line* is a non-separable LOTS in which every collection of pairwise disjoint open sets is at most countable. It is consistent with (and independent of) ZFC that such spaces exist.

COROLLARY 2.8. *No Souslin line is a Blumberg space.*

PROOF: Let X be a Souslin line. Then card(X) ≤ $\mathfrak{c}$. Fix any 1-1 function f: X → R where R is the usual space of real numbers and suppose there is a dense set D ⊂ X such that the restriction of f to D is continuous. Since D is a GO-space, there is a dense subset E of D which is orderable. Then E is also dense in X, so that E has the property that any pairwise disjoint collection of relatively open sets in E is at most countable. Let g be the restriction of f to E. Since g: E → R is 1-1 and continuous, and since R has a G_δ-diagonal, so does E. But then, by Theorem 2.4, E is metrizable. It is well known that a metrizable space in which pairwise disjoint open collections are countable is actually separable. But then so is X, and that is impossible. □

Williams also obtains two other curious results on the existence of dense orderable subspaces [40].

THEOREM 2.9. *Let* X *be completely regular. If* $X \times [0,1]$ *has a dense orderable subspace, then* X *has a dense metrizable subspace.*

THEOREM 2.10. *Suppose that* $Y = \Pi\{X_\alpha : \alpha \in A\}$ *is an infinite product of infinite completely regular spaces. If* Y *has a dense orderable subspace, then* $\operatorname{card}(A) = \omega$ *and* Y *also has a dense metrizable subspace.*

Since each metrizable space has a dense orderable subspace, we remark that the converses of Theorems 2.9 and 2.10 are also true.

Williams' results led him to pose three questions [40].

QUESTION (II). *Suppose* X_1 *and* X_2 *are coabsolute* LOTS. *Does it follow that* X_i *must contain a dense subspace* D_i *such that* D_1 *and* D_2 *are homeomorphic?*

QUESTION (III). *Assume the Continuum Hypothesis and suppose that* X *is a non-compact, paracompact, locally compact* LOTS. *Must* $\beta X - X$ *have a dense, orderable subspace?*

QUESTION (IV). *Assume that* X^ω*, the product of countably many copies of a completely regular space* X*, has a dense orderable subspace. Must* X *also have a dense orderable subspace?* (cf. "Added in proof".)

I raised another question about the relation between GO-spaces and the linearly orderable spaces in which they embed. It is known, for example, that a GO-space X is paracompact (resp., metrizable) iff X embeds as a closed subspace of a LOTS which is paracompact (resp., metrizable). On the other hand, it is known that the Sorgenfrey line is a perfect (= closed sets are G_δ's) GO-space which cannot be embedded as a closed subset of any perfect LOTS. While working on another problem, I discovered that it would be useful to have an affirmative answer to the following question.

QUESTION (V). *Suppose* X *is a perfect* GO-*space. Does there exist a perfect orderable space* Y *such that* X *densely embeds in* Y? (The ordering of Y is not required to extend the ordering of X. This question has an easy affirmative answer if X has countable cellularity.)

In closing this subsection, let me note that Williams' Theorem 2.7, above, shows that there is an orderable counterexample to (VII) if and only if there is a suborderable counterexample. One can show that, in suborderable spaces, the property "There is a σ-discrete dense subset" is a hereditary property. By way of contrast, the weaker property "There is a dense, metrizable subspace" is *not* hereditary in suborderable spaces. For example, let X be a connected LOTS having no dense, metrizable subspace; such a space is constructed in [35]. According to [10, Lemma 4.2] or [17], there are disjoint dense subsets D and E of X. Let $Y = \{(x,n) \mid \text{if } x \in D \text{ then } n \text{ is an integer with } n \leq 0 \text{ and if } x \in X-D,\ n = 0\}$ and let Y have the open interval topology of the lexicographic ordering. Then $\{(x,n) \in Y \mid x \in D \text{ and } n < 0\}$ is a dense metrizable subspace of the LOTS Y while the subspace $E \times \{0\}$ of Y has no dense metrizable subspace.

B. Perfect spaces and point-countable bases.

Bennett [3] and Ponomarev [30] have proved that if there is a Souslin line then there is a Souslin line with a point-countable base. Such a space would be orderable, perfect, and have a point-countable base but would be non-metrizable. Real examples (i.e., in ZFC) of such a space still elude us and our next question, originally posed by R.W. Heath, seems to be hard.

QUESTION (IX). *In ZFC, is there an orderable, perfect space* X *which has a point-countable base and yet is non-metrizable?*

General theory (cf. (3.1), below) shows that the point-countable base in a potential counterexample for (IX) could not be σ-point finite or σ-disjoint. Bennett [6] has constructed (in ZFC) a LOTS with a point-countable base but not a σ-point-finite base, but his space is not perfect. Finally, let us point out that any counterexample for (IX) is also a solution of (VII) and (VIII) because any GO-space with a point-countable base and a σ-discrete dense subspace must be metrizable [12].

C. σ-minimal bases and quasi-developments.

In his dissertation, Bennett studied the class of quasi-developable spaces [5] and later Bennett and I proved that the quasi-developable spaces are exactly the same as the spaces having θ-bases in the sense of Worrell and Wicke [41] [8]. Subsequently, C. Aull isolated the notion of a σ-minimal base: a *σ-minimal base* for a space X is a base

$\mathcal{B} = \cup\{\mathcal{B}(n) \mid n \geq 1\}$ where each $\mathcal{B}(n)$ has the property that if $\mathcal{C} \subsetneqq \mathcal{B}(n)$, then $\cup\, \mathcal{C} \subsetneqq \cup\, \mathcal{B}(n)$. Aull observed that every quasi-developable space has a σ-minimal base and asked whether the converse might be true [1]. Bennett and Berney [7] gave a negative answer by showing that the lexicographic square has a σ-minimal base while its subspace $[0,1] \times \{0,1\}$ does not. That contrasts with the fact that any subspace of a quasi-developable space is again quasi-developable.

All known examples of GO-spaces which have σ-minimal bases but are not quasi-developable have subspaces, usually closed subspaces, which do not have σ-minimal bases. That leads to the following question, originally (mis-) posed in [9].

<u>QUESTION (X)</u>. *Suppose* X *is a (sub) orderable space such that each subspace of* X *has a* σ*-minimal base. Must* X *be quasi-developable? If we assume that* X *is compact, must* X *be metrizable?*

It is easy to see that, for the second half of (X) to have an affirmative solution, it would be sufficient (and necessary) to show that the compact space X is perfect.

Very little progress has been made on (X). It is not hard to see that, while the existence of a σ-minimal base for a GO-space X does not force X to be first countable [9], the space X will be first-countable if each closed subset of X has a σ-minimal base. Burke observed that if a first countable GO-space has a σ-minimal base then it has a dense metrizable subspace [39]. And that is essentially all that we know about (X), except for a result asserting that any suborderable space with a σ-minimal base is hereditarily paracompact [9].

That result about hereditary paracompactness is a surprising one in the light of a theorem due to D. Burke [14] asserting that any topological space embeds into a space with a σ-minimal base. Thus, the existence of a σ-minimal base for a space has *no* hereditary consequences among general spaces. Van Douwen was led to ask whether each hereditarily paracompact GO-space can be embedded in a GO-space having a σ-minimal base. The answer is negative. Todočevič [35], using only ZFC, constructed a compact, connected, first-countable LOTS which has no dense metrizable subspace. (Unfortunately, the space does not provide an answer for (VII) since it is not perfect: if it were perfect then it would be a Souslin line and such things don't exist in ZFC.) If Todočevič's space X were embedded in a suborderable space Y having a σ-minimal base, then (being connected) it would have to be a convex set in Y. Since the existence of a σ-minimal base is hereditary to open subspaces. $Int_Y(X)$ would have a σ-minimal base. But, as noted above, a first-countable space with a σ-minimal base has a dense metrizable subspace, so X would have such a subspace, contrary to Todočević's construction.

In closing this subsection, it might be worthwhile to say a few words about quasi-developable spaces. The class of quasi-developable spaces is a particularly useful tool in the study of metrizability in GO-spaces because of the following results [19], [8].

THEOREM 3.1. *Let* X *be a suborderable space. The following are equivalent:*
(a) X *is quasi-developable;*
(b) X *has a* σ*-point finite base;*
(c) X *has a* σ*-disjoint base whose members are convex sets.*

THEOREM 3.2. *Let* X *be a first countable suborderable space and suppose* $X = \cup\{X(n) : n \geq 1\}$ *where each* X(n) *is a quasi-developable subspace of* X. *Then* X *is also quasi-developable.*

The general metrization theory for quasi-developable spaces also lends itself to the study of GO-spaces.

THEOREM 3.3. (a) *A space is metrizable if it is quasi-developable, perfect, and collectionwise normal* [5]; (b) *A space is metrizable if it is a quasi-developable paracompact* p*-space* [4]; (c) *A space is metrizable if it is a quasi-developable* M*-space* [4].

D. Hereditary properties and metrizability of GO-spaces.

Some years ago, Bennett and I proved the following rather curious theorem [10].

THEOREM 3.4. *Let* X *be suborderable. Then the following are equivalent:*
(a) X *is metrizable*;
(b) *each subspace of* X *is a* p*-space;*
(c) *each subspace of* X *is an* M*-space;*
(d) *each subspace of* X *is a* wΔ*-space;*
(e) *each subspace of* X *is quasi-complete in the sense of Creede.*

Recent years have provided several new and easier proofs ([42], [17]) and the equivalence of (a) and (b) is now quite simple, given Balogh's structure theorems in [2]. Theorem 3.4 provides the motivation for the next open question, but first we must introduce Nagami's Σ-spaces.

Nagami introduced the class of Σ-spaces in order to describe a very large class of spaces which is well-behaved with respect to paracompactness in products [29]. Every M-space in the sense of Morita and every paracompact p-space is a Σ-space, and any countable product of paracompact Σ-spaces is again a paracompact Σ-space. Recall that X is a *Σ-space* if there are locally finite closed covers $F(1), F(2), \ldots$ of X such that if we write $C(x,n) = \cap\{F \in F(n) \mid x \in F\}$ and $C(x) = \cap\{C(x,n) \mid n \geq 1\}$ for each $x \in X$, then

(a) C(x) is countably compact;

(b) if $C(x) \subset U$, where U is open, then for some n, $C(x,n) \subset U$.

In his dissertation [42], van Wouwe began the study of suborderable, *hereditary Σ-spaces*, i.e., suborderable spaces whose every subspace is a Σ-space. He showed that such spaces must be paracompact and first countable and he investigated the following question (which is still open):

QUESTION (XI). *Suppose* X *is a suborderable, hereditary Σ-space. Must* X *be metrizable? What if* X *is assumed to be compact?*

Van Wouwe proved [42, Thm. 4.1.3] that a perfect suborderable Σ-space is an M-space. Just as with (X), therefore, it would suffice to show that each suborderable, hereditary Σ-space is perfect, for then Theorem 3.4, above, would yield an affirmative answer for (XI). Van Wouwe [42] also gave what appears to be a significant reduction of (XI) proving that (XI) is equivalent to

QUESTION (XII). *Let* X *be a Lindelöf suborderable space which is a hereditary Σ-space. Must* X *be perfect?*

One possible attack on (XII) uses products of subspaces of suborderable, hereditarily Σ-spaces, and suggests the following question, to which I expect a negative answer.

QUESTION (XIII). *Suppose* X *is a compact* LOTS *and that for any subspace* Y *of* X, *the product space* Y^{ω} *is paracompact. Must* X *be metrizable? What if we assume the Continuum Hypothesis?*

Both (X) and (XI) deal with GO-spaces which are first-countable and paracompact. In [10] we give a structure theorem for such spaces which seems to say that they are somewhat like the lexicographic square.

THEOREM 3.5. *Let* X *be a first-countable, paracompact* GO-*space. Then there are subsets* G *and* H *of* X *such that:*

(a) G *is open and metrizable and* H = X-G*;*

(b) H *is dense-in-intself;*

(c) H *can be written as* H = D ∪ E *where* D *and* E *are disjoint dense subsets of* H *such that if* $d_1 < d_2$ *are in* D *(resp., if* $e_1 < e_2$ *are in* E*) then* $[d_1,d_2] \cap D$ *is not compact (resp.,* $[e_1,e_2] \cap E$ *is not compact).*

Theorem 3.5 led van Douwen to pose the following question.

QUESTION (XIV). *Suppose* X *is a compact* LOTS *without isolated points. Does there exist a subset* B ⊂ X *such that both* B *and* (X-B) *intersect every nonempty closed* C ⊂ X *having no (relatively) isolated points?*

4. THE HAHN-MAZURKIEWICZ PROBLEM

Since 1914 we have known that a space X is a continuous image of [0,1] if X is a connected, locally connected, compact metric space; that is the classical Hahn-Mazurkiewicz theorem. Since 1916 we have known that every continuous image of [0,1] is arcwise connected; that is Moore's theorem. Modern Hahn-Mazurkiewicz theory asks what happens if the space [0,1] is replaced by an arbitrary compact, connected ordered space, and seeks analogues of the classical results of Hahn, Mazurkiewicz and Moore. Perhaps the most basic question in the area was posed by Mardešić and Papič [23].

QUESTION (XV). *Suppose* X *is a compact, connected, locally connected Hausdorff space which is known to be the continuous image of some compact ordered space. Must* X *be the continuous image of some compact connected* LOTS*?*

Readers may wonder about including the phrase "which is known to be the continuous image of some compact ordered space" in the statement of (XV). That qualifying phrase must be included because if Y is any compact, connected, locally connected infinite Hausdorff space, then X = Y × [0,1]

is another space of the same type and the new space X cannot be the continuous image of any compact LOTS since it is not monotonically normal ([15]). (The first examples of this type are due to Mardešič [21]).

Today, perhaps the best result which guarantees that a compact, connected space X is the image of some compact, connected LOTS is due to Ward and involves the notion of approximability by finite trees. We say that a tree is finite if it has only finitely many endpoints and that a compact connected space X can be *approximated by finite trees* if there is a family $\mathcal{T}$ of finite trees in the space X such that

(a) $\mathcal{T}$ is directed by inclusion;

(b) $\cup\mathcal{T}$ is dense in X;

(c) given any open cover $\mathcal{U}$ of X there is a tree $T(\mathcal{U}) \in \mathcal{T}$ such that for any tree T having $T(\mathcal{U}) \subset T \in \mathcal{T}$ and any component C of $T - T(\mathcal{U})$, some member of $\mathcal{U}$ contains C.

Ward has proved that if a compact, connected Hausdorff space X can be approximated by finite trees, then X is a continuous image of a compact, connected LOTS [37], and Ward and Treybig [36] ask

<u>QUESTION (XVI)</u>. *Is it true that a connected, compact Hausdorff space* X *is the continuous image of a connected, compact* LOTS *if and only if* X *can be approximated by finite trees?*

Next we mention a question asking about anaogues of Moore's theorem. The question is due to Mardešič [22].

<u>QUESTION (XVII)</u>. *Let* X *be a connected, locally connected, compact Hausdorff space and suppose* X *is known to be the continuous image of some compact* LOTS. *Must* X *be "arcwise connected" in the sense that any two points of* X *are contained in some compact, connected, orderable subspace of* X?

Another question asks about certain continuous images of [0,1]; it may be surprising that such a question is still open. A mapping $f\colon X \to Y$ is *irreducible* if $f[C] \subsetneq f[X]$ whenever C is a proper closed subset of X. (Continua theorists prefer the term "strongly irreducible".)

<u>QUESTION (XVIII)</u>. *Characterize those spaces which are continuous images of* [0,1] *under irreducible mappings*. (cf. "Added in Proof".)

Readers are referred to the excellent survey paper written by Treybig and Ward [36] for the first volume of these preceedings where these last four questions and related work are discussed in detail, and to the earlier survey of Mardešić [22].

The two final questions were posed at the first session of the workshop by L.B. Treybig.

QUESTION (XIV). *Supposed* X *is a compact, connected* LOTS *and* X *is homeomorphic to each of its nondegenerate closed intervals. In* ZFC, *does it follow that there is an order-reversing homeomorphism* $h: X \to X$? (cf. "Added in Proof".)

QUESTION (XV). *Suppose* $f: X \to Y$ *is a continuous mapping from a compact, connected* LOTS *onto a compact, connected Hausdorff space. We say that* f *has finite oscillation at local separating points of* Y *provided that if* U *is open in* Y *and* $p \in U$ *has the property that* $U - \{p\} = R \cup S$ *is the union of two mutually separated sets, then there is a finite open cover* G *of* $f^{-1}[R \cup S]$ *by subintervals of* X *such that no member of* G *meets both* $f^{-1}[R]$ *and* $f^{-1}[S]$. *Is it true that if the connected, compact Hausdorff space* Y *is the continuous image of some compact, connected* LOTS, *and if no point separates* Y, *then* Y *is the image of some (other) compact, connected* LOTS *under a mapping which has finite oscillation at local separating points of* Y ?

That final rather technical question is related to one possible attack on (XV).

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Added in proof. After first drafts of this paper had been circulated, I received letters from colleagues about several of my questions.

(1) P. Simon (Prague) wrote me that he has a negative answer to Williams' question (IV).

(2) S. Todočević wrote to me announcing a negative solution in ZFC to Treybig's question (XIX). Details will appear in Section 5 of his paper on trees in the forthcoming *Handbook of Set Theoretic Topology* edited by Kunen and Vaughan.

(3) W. Bula (Katowice) wrote to me announcing that he and M. Turzanski have obtained:

(a) a characterization of continuous images of compact ordered spaces;

(b) a characterization of continuous irreducible images of compact connected orderable spaces.

Both characterizations, Bula writes, involve special closed subbases.

A NOTE ON PERFECT NORMALITY IN GENERALIZED ORDERED SPACES

H. BENNET & D.J. LUTZER

Recall that a subset S of a space X is *regularly closed* if
S = Cl (Int(S)) and is *regularly open*
[4], Šč̌epin defined a space X to be *perfectly* κ-*normal* if

(1) given disjoint regularly closed sets H_1 and H_2, there are disjoint open sets U_1 and U_2 having $H_i \subset U_i$, and

(2) every regularly closed subset of X is the intersection of countably many regularly open sets.

It is clear that each *perfectly normal* space (= a normal space in which every closed set is a G_δ) must be perfectly κ-normal. According to a theorem of Šč̌epin [4, Thm. 1] any product of metrizable spaces is perfectly κ-normal so it is clear that the perfectly normal spaces are a proper subfamily of the perfectly κ-normal spaces. Our goal in this note is to show that for GO-spaces, the two classes coincide. We prove a bit more, namely:

<u>THEOREM</u>. *The following properties of a generalized ordered space* X *are equivalent:*

(a) X *is perfectly normal;*

(b) X *is perfectly* κ-*normal;*

(c) *each regularly closed subset of* X *is a* G_δ.

It is obvious that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). We show that (c) implies (a), using a sequence of lemmas. Clearly (c) is equivalent to the assertion that every regularly open set is an F_σ.

<u>LEMMA 1</u>. *Suppose C is a pairwise disjoint collection of convex* F_σ-*subsets of a* GO-*space* X. *Suppose there is a* F_σ-*subset* D *of* X *having*

(a) $D \subset \cup\mathcal{C}$ *and*

(b) $D \cap C \neq \emptyset$ *for each* $C \in \mathcal{C}$.

Then $\cup\mathcal{C}$ *is also an* F_σ*-subset of* X.

LEMMA 2. *Suppose* X *is a* GO*-space in which each regularly closed set is a* G_δ. *Then*

(a) X *is first countable, and*

(b) *each convex subset of* X *is an* F_σ*-set.*

LEMMA 3. [1, Lemma 4.2], [2]. *Let* X *be a* GO*-space and let* $Y \subset X$. *Then there are sets* D *and* E *such that:*

(a) $D \cup E = Y$, $D \cap E = \emptyset$;

(b) *if* $p \in X$ *has the property that whenever* $b > p$ *the set* $[p,b) \cap Y$ *is infinite, then for each* $b > p$, *both* $[p,b) \cap D$ *and* $[p,b) \cap E$ *are infinite;*

(c) *if* $p \in X$ *has the property that whenever* $a < p$ *the set* $(a,p] \cap Y$ *is infinite, then for each* $a < p$, *both* $(a,p] \cap D$ *and* $(a,p] \cap E$ *are infinite.*

COROLLARY. *Let* X *be a* GO*-space and let* $Y \subset X$ *be a set of isolated points of* X. *Then there are sets* D *and* E *such that:*

(a) $D \cup E = Y$ *and* $D \cap E = \emptyset$;

(b) $Cl(D) - D = Cl(Y) - Y = Cl(E) - E$.

LEMMA 5. *For any space* X, *each closed subset of* X *is a* G_δ *if and only if each nowhere dense closed subset of* X *is a* G_δ.

We may now prove that (c) implies (a) in our Theorem as follows. Let C be a closed nowhere dense subset of X and let $\mathcal{V}$ be the family of convex components of X–C. Let

$$\mathcal{V}_0 = \{V \in \mathcal{V} : \text{card}(V) \text{ is finite}\} \text{ and}$$
$$\mathcal{V}_1 = \mathcal{V} - \mathcal{V}_0.$$

Since C is nowhere dense in X, $X = Cl(X-C) = Cl(\cup\mathcal{V}_0) \cup Cl(\cup\mathcal{V}_1)$. Therefore $C = C_0 \cup C_1$ where $C_i = C \cap Cl(\cup\mathcal{V}_i)$.

For each $V \in \mathcal{V}_1$ we may choose a set U(V) and points a(V) and b(V) satisfying:

(i) U(V) is a nonempty, convex, open set;

(ii) $U(V) = Int(Cl(U(V)))$; and

(iii) $Cl(U(V)) \subset (a(V),b(V)) \subset [a(V),b(V)] \subset V$.

Let $W = \cup\{U(V) : V \in \mathcal{V}_1\}$. We will show that W is a regular open set. First observe that if we write $\hat{C}_1 = C_1 \cap Cl(W)$, then

(iv) $Cl(W) = \hat{C}_1 \cup (\cup\{Cl(U(V)) : V \in \mathcal{V}_1\})$.

Let $p \in Cl(W) - W$. According to (ii), if $p \in Cl(U(V)) - U(V)$ for some $V \in \mathcal{V}_1$, then $p \notin Int(Cl(W))$, so assume $p \in \hat{C}_1$. Suppose there is a convex neighborhood J of p which meets only a finite number of members of $\mathcal{V}_1$. Let $\mathcal{V}' = \{V \in \mathcal{V}_1 : J \cap V \neq \emptyset \text{ and } V \subset (\leftarrow,p)\}$ and $\mathcal{V}'' = \{V \in \mathcal{V}_1 : J \cap V \neq \emptyset$ and $V \subset (p,\rightarrow)\}$. Since $p \in Cl(W)$ and $W \subset \cup\mathcal{V}_1$, at least one of $\mathcal{V}'$ and $\mathcal{V}''$ is non-void. Assume $\mathcal{V}' \neq \emptyset$. Let V_1 be the member of V' closest to the point p and compute $b(V_1)$ as in (iii). Then $b(V_1) < p$ so that if $\mathcal{V}'' = \emptyset$, then $J \cap (b(V_1),\rightarrow)$ would be a neighborhood of p that is disjoint from W. Hence $\mathcal{V}'' \neq \emptyset$. Let V_2 be the member of $\mathcal{V}''$ closest to the point p, and compute $a(V_2)$ as in (iii). But then $J \cap (b(V_1),a(V_2))$ is a neighborhood of p which is disjoint from W, contrary to $p \in Cl(W)$. Therefore, each convex neighborhood J of p meets infinitely many members of $\mathcal{V}_1$ so that convexity of J forces $V \subset J$ for some $V \in \mathcal{V}_1$. But then $a(V) \in J$ even though $a(V) \notin Cl(W)$, showing that $J \not\subset Cl(W)$. Therefore we have $W = Int(Cl(W))$ so that W is an F_σ-subset of X. According to Lemmas 1 and 2, we obtain

(v) the set $\cup\mathcal{V}_1$ is an F_σ-subset of X.

Now consider the set $Y = \cup\mathcal{V}_0$. Each point of Y is isolated in X so that from Corollary 4 we obtain disjoint sets D and E having $Cl(D) - D = Cl(Y) - Y = Cl(E) - E$. Consider the set $H = C \cup (\cup\mathcal{V}_1) \cup D$. Since $X - H = E$ is open, H is closed. Since $C_1 \subset Cl(\cup\mathcal{V}_1)$ and since $C_0 = Cl(Y) \cap C = Cl(Y) - Y = Cl(D) - D$, we see that $C \subset Cl(\cup\mathcal{V}_1 \cup D)$ so that $H \subset Cl(\cup\mathcal{V}_1 \cup D) \subset Cl(Int(H)) \subset H$. Therefore H is a regular closed set and, hence, is a G_δ-subset of X. Therefore

(vi) the set $E = X - H$ is an F_σ-subset of X.

Analogously we obtain

(vii) the set D is an F_σ-subset of X.

But then the set $X - C$ is the union of the three F_σ-subsets $\cup\mathcal{V}_1$, D and E so that the set C must be a G_δ-set. Since C was an arbitrary closed nowhere dense subset of X, it now follows from Lemma 5 that every closed subset of X is a G_δ. Hence X is perfectly normal.

In closing, let us remark that perfect κ-normallity is not a productive property, even if one uses only orderable spaces as factors. Consider any perfectly normal, nonmetrizable LOTS (such as the lexicographic product $[0,1] \times \{0,1\}$). One of the regularly closed sets $A = \{(x,y) \in X^2 : x \leq y\}$ and $B = \{(x,y) \in X^2 : x \geq y\}$ must fail to be a G_δ-set in X^2 since their intersection is the diagonal of the LOTS X which cannot be a G_δ unless X is metrizable. This argument shows that for a LOTS X, X is metrizable if and only if X^2 is perfectly κ-normal.

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METRIZATION THEOREMS FOR CONTINUOUS IMAGES OF COMPACT ORDRED SPACES

W. BULA

It was shown by L.B. Treybig [8] that if a product of two infinite Hausdorff spaces can be obtained as a continuous image of a compact ordered space, then it is metrizable. This theorem generalizes previous results of Mardešić and Papić [5] for continuous images of ordered continua.

If both factors of the product are infinite then each of them can be considered as an image of the product space under an open continuous map (namely, projection) all fibers of which are infinite. This leads to a natural question, whether an open infinite-to-one continuous image of a space which is a continuous image of a compact ordered space must be metrizable. In the present paper, we give an affirmative answer to this question.

A map $f : X \to Y$ is said to be *local infinitely covering* if there exists a countable family $\mathcal{R}$ consisting of pairs $\langle F,U\rangle$, where F is a closed subset of X and U is an open neighborhood of F, such that

(*) for each point $y \in Y$ there exists a sequence $\langle F_1,U_1\rangle,\langle F_2,U_2\rangle,\dots$ of elements of $\mathcal{R}$ such that
(a) $y \in f(F_k)$, for $k = 1,2,\dots$, and
(b) $U_1,U_2,\dots$ are disjoint.

Let us note that infinitely covering maps introduced in [1] are local infinitely covering. The following lemma is proved in [1].

LEMMA 1. *Let* f *be a continuous map from a compact ordered space* K *into a* T_1*-space* X. *If* $\mathcal{P}$ *is a family of convex disjoint subsets of* K, *then the set* $\{P \in \mathcal{P} : x,y \in f(P)\}$ *is finite for all pairs* $x,y \in X$ *such that* $x \neq y$.

THEOREM 1. *If a Hausdorff space is a continuous image of a compact ordered space under a local infinitely covering map, then it is metrizable.*

PROOF. Let f be a local infinitely covering map from a compact ordered space K onto a Hausdorff space X. Let $\mathcal{R}$ be a family satisfying condition (*) from the definition of a local infinitely covering map. For each $\langle F,U\rangle \in \mathcal{R}$, the set F is covered by finitely many convex components of U, so the family

$$\mathcal{S} = \{C \cap F : C \text{ is a convex component of } U \text{ which meets } F, \text{ where } \langle F,U\rangle \in \mathcal{R}\},$$

which clearly consists of closed sets, is countable.

Fix $x,y \in X$, $x \neq y$. Choose a sequence $\langle F_1,U_1\rangle, \langle F_2,U_2\rangle, \ldots$ of elements of $\mathcal{R}$ such that $x \in f(F_k)$ for $k = 1,2,\ldots$, and the sets $U_1, U_2, \ldots$ are disjoint. For each k choose C_k to be a convex component of U_k which meets F_k and such that $x \in f(C_k)$. The family $\{C_k : k = 1,2,\ldots\}$ consists of convex disjoint subsets of K, so, in view of Lemma 1, there is at least one k such that $y \notin f(C_k)$. Hence the family $\{f(S_1) \cap \ldots \cap f(S_n) : S_1, \ldots S_n \in \mathcal{S}, n = 1,2,\ldots\}$ is a countable network for X, so X is metrizable in view of Arhangel'skiĭ's Theorem (see R. Engelking [2], Theorem 3.1.19).

Let $\mathcal{U}$ be a family of open subsets of a space X and H be a subset of X. The family $\mathcal{U}$ is called an *irreducible open cover of* H if $H \subset \bigcup \mathcal{U}$ and $H - \mathrm{cl} \bigcup (\mathcal{U} - \{U\}) \neq \emptyset$ for each $U \in \mathcal{U}$.

Hereafter, we will use the following notation: $\mathrm{st}(x,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}$, whenever $\mathcal{U}$ is a family of subsets of a space X and $x \in X$.

Let $\mathcal{U}$ and $\mathcal{V}$ be families of open subsets of X. The family $\mathcal{V}$ is called a (*closed*) [*star*] *refinement of* $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$ ($\mathrm{cl}\, V \subset U$) [$\mathrm{st}(x,\mathcal{V}) \subset U$ for each $x \in V$]. If the family $\mathcal{V}$ is both star and closed refinement of $\mathcal{U}$ then it is called a *star-and-closed refinement of* $\mathcal{U}$.

A family $\mathcal{W} = \{W(U) : U \in \mathcal{U}\}$ is called a *closed shrinking of* $\mathcal{U}$ if $\mathrm{cl}\, W(U) \subset U$ for each $U \in \mathcal{U}$.

LEMMA 2. *Let* X *be a compact Hausdorff space and* H *be a closed infinite subset. For every sequence* $\{\mathcal{U}_n : n = 1,2,\ldots\}$ *such that*

(1) $\mathcal{U}_n$ *is an irreducible finite open cover of* H,

(2) $\mathcal{U}_n$ *contains at least* n *disjoint elements, and*

(3) $\mathcal{U}_{n+1}$ *is a star-and-closed refinement of* $\mathcal{U}_n$,

for $n = 1,2,\ldots$, *the family* $\bigcup\{\mathcal{U}_n : n = 1,2,\ldots\}$ *contains an infinite subfamily consisting of disjoint elements.*

PROOF. Put $F = \bigcap\{\bigcup\mathcal{U}_n : n = 1,2,\ldots\}$. Let us note that in view of (1) and (3), F is a closed infinite subset of X containing H. For each $x \in F$, denote $[x] = \bigcap\{st(x,\mathcal{U}_n) : n = 1,2,\ldots\}$. Put $\mathcal{R} = \{[x] : x \in F\}$. The family $\mathcal{R}$ is an upper semi-continuous partition of the set F (see W. Kulpa [4]) and for each $x \in F$, the family $\{st(x,\mathcal{U}_n) : n = 1,2,\ldots\}$ is a neighborhood base around [x]. By assumptions (2) and (3), the family $\mathcal{R}$ is infinite. Hence, there exists $x \in F$ such that for every open set U containing [x], the family $\{[y] \in \mathcal{R} : [y] \cap U \neq \emptyset\}$ is infinite. The partition $\mathcal{R}$ is upper semi-continuous, so the family $\{[y] \in \mathcal{R} : [y] \subset U\}$ is infinite as well. So, by induction, one can construct a sequence $\{[x_k] : k = 1,2,\ldots\}$ and a subsequence $\{\mathcal{U}_{n(k)} : k = 1,2,\ldots\}$ of the sequence $\{\mathcal{U}_n : n = 1,2,\ldots\}$ such that $[x_k] \subset st(x,\mathcal{U}_{n(k)})$, and $[x_k] \cap \mathrm{cl}\,st(x,\mathcal{U}_{n(k+1)}) = \emptyset$. The family $\{st(x_k,\mathcal{U}_m) : m = 1,2,\ldots\}$ is a neighborhood base around $[x_k]$, so one can choose a sequence $\{m(k) : k = 1,2,\ldots\}$ such that $[x_k] \subset st(x_k,\mathcal{U}_{m(k)}) \subset$ $\subset st(x,\mathcal{U}_{n(k)}) - \mathrm{cl}\,st(x,\mathcal{U}_{n(k+1)})$. One can see that the elements of the sequence $\{st(x_k,\mathcal{U}_{m(k)}) : k = 1,2,\ldots\}$ are disjoint.

Let $f : X \xrightarrow{\text{onto}} Y$ be an open continuous map and $\mathcal{U}$ be a finite family of open subsets of X. Hereafter, we will use the following notation

$$I(U) = U - \mathrm{cl}\bigcup(\mathcal{U} - \{U\}),\ \text{for } U \in \mathcal{U},$$
$$C(\mathcal{U}) = \bigcap\{f(I(U)) : U \in \mathcal{U}\},\ \text{and}$$
$$\mathcal{T}(\mathcal{U}) = \{U \cap f^{-1}(C(\mathcal{U})) : U \in \mathcal{U}\}.$$

The elementary proof of the following lemma is omitted.

LEMMA 3. *Let* $f : X \xrightarrow{\text{onto}} Y$ *be an open continuous map. If a family* $\mathcal{U}$ *is an irreducible finite open cover of the set* $f^{-1}(y)$, *where* $y \in Y$, *then*

(1) $C(\mathcal{U})$ *is an open neighborhood of* y, *and*

(2) *if* $z \in C(\mathcal{U})$, *then* $\mathcal{T}(\mathcal{U})$ *is an irreducible open cover of* $f^{-1}(z)$ *refining* $\mathcal{U}$.

THEOREM 2. *Every open infinite-to-one continuous map from a compact Hausdorff space onto a Hausdorff space is a local infinitely covering map.*

PROOF. Let f be an open infinite-to-one continuous map from a compact Hausdorff space X onto a Hausdorff space Y. For each $p \in Y$, let $\mathcal{U}(p)$ be a finite irreducible open cover of $f^{-1}(p)$. In view of Lemma 3(1), the family $\{C(\mathcal{U}(p)) : p \in Y\}$ is an open cover of Y, so there exist points $p_1,\ldots,p_{n(1)} \in Y$ such that the family $\mathcal{W}_1 = \{C(\mathcal{U}(p_k)) : k = 1,\ldots,n(1)\}$ is an open cover of Y. Denote $W(1,k) = C(\mathcal{U}(p_k))$. By normality of Y, we can find a family $\mathcal{Z}_1 = \{Z(1,k) : k = 1,\ldots,n(1)\}$ which is a closed shrinking of $\mathcal{W}_1$ and such that $\cup\mathcal{Z}_1 = Y$ (see R. Engelking [2], Theorem 1.5.18).

Fix $k \in \{1,\ldots,n(1)\}$. Let us note that in view of Lemma 3(2), the family $\mathcal{U}(1,k) = \mathcal{T}(\mathcal{U}(p_k))$ is an irreducible open cover of $f^{-1}(z)$ for each $z \in clZ(1,k)$. By normality of X, we can find a family $\mathcal{V}(1,k)$ which is a closed shrinking of $\mathcal{U}(1,k)$ covering $f^{-1}(clZ(1,k))$.

Denote $\mathcal{M}_1 = \{\langle\mathcal{U}(1,k),\mathcal{V}(1,k),W(1,k),Z(1,k)\rangle : k = 1,\ldots,n(1)\}$. Assume that we have already defined families

$$\mathcal{M}_i = \{\langle\mathcal{U}(i,j),\mathcal{V}(i,j),W(i,j),Z(i,j)\rangle : j = 1,\ldots,n(i)\},$$

for $i = 1,\ldots,m$, such that the following inductive assumptions are satisfied:

(1) $\{W(i,j) : j = 1,\ldots,n(i)\}$ is an open cover of Y, and $\{Z(i,j) : j = 1,\ldots,n(i)\}$ is its closed shrinking which also covers Y, for $i = 1,\ldots,m$,

(2) $\mathcal{U}(i,j)$ is a finite family of open subsets of X which is an irreducible open cover of $f^{-1}(z)$, for each $z \in clZ(i,j)$, where $j = 1,\ldots,n(i)$, and $i = 1,\ldots,m$,

(3) $\mathcal{V}(i,j)$ is a closed shrinking of $\mathcal{U}(i,j)$ covering $f^{-1}(clZ(i,j))$, for $j = 1,\ldots,n(i)$, and $i = 1,\ldots,m$,

(4) $\mathcal{U}(i,j)$ contains at least i disjoint elements, for $j = 1,\ldots,n(i)$ and $i = 1,\ldots,m$,

(5) $\mathcal{U}(i,j) = \mathcal{T}(\mathcal{U}(i,j))$ and $W(i,j) = C(\mathcal{U}(i,j))$, for $j = 1,\ldots,n(i)$ and $i = 1,\ldots,m$,

(6) if $1 \le k < i \le m$ then $\{W(i,l) : l = 1,\ldots,n(i)\}$ is a closed refinement of $\{W(k,j) : j = 1,\ldots,n(k)\}$, and

(7) if $1 \le k < m$ then for each $l \in \{1,\ldots,n(k)\}$ and $j \in \{1,\ldots,n(k+1)\}$ such that $clW(k+1,l) \subset W(k,j)$, the family $\mathcal{U}(k+1,l)$ is a star-and-closed refinement of $\mathcal{U}(k,j)$.

Fix $k \in \{1,\ldots,n(m)\}$. Let us note that for each point $p \in clZ(m,k)$ the set $W_p = \cap\{W(m,j) : p \in W(m,j),\ j = 1,\ldots,n(m)\}$ is an open neighborhood of p contained in $W(m,k)$. By (2), the family $\mathcal{R}_p = \{U \cap f^{-1}(W_p) : U \in \mathcal{U}(m,k)\}$ is an irreducible open cover of the set $f^{-1}(p)$ and it is a refinement of $\mathcal{U}(m,k)$, so there exists a finite family $\mathcal{S}_p$ which is an irreducible open cover of $f^{-1}(p)$ and is a star-and-closed refinement of $\mathcal{R}_p$. In addition, we can assume that $\mathcal{S}_p$ contains at least m+1 disjoint elements, because the set $f^{-1}(p)$ is infinite. The family $\mathcal{C} = \{C(S_p) : p \in \mathrm{cl}Z(m,k)\}$ covers the set $\mathrm{cl}Z(m,k)$, so there exists a finite subfamily $\mathcal{W}_k \subset \mathcal{C}$ which also covers $\mathrm{cl}Z(m,k)$.

Let us note that $\mathcal{W} = \cup\{\mathcal{W}_k : k = 1,\ldots,n(m)\}$ is a finite open cover of Y and $\mathcal{W}$ is a closed refinement of $\{W(m,i) : i = 1,\ldots,n(m)\}$. Let $n(m+1)$ be the number of elements of $\mathcal{W}$. Let us enumerate elements of $\mathcal{W}$ is the following way: $\mathcal{W} = \{W(m+1,j) : j = 1,\ldots,n(m+1)\}$.

For each $j \in \{1,\ldots,n(m+1)\}$ there is $k \in \{1,\ldots,n(m)\}$ and a point $p \in \mathrm{cl}Z(m,k)$ such that $W(m+1,j) = C(S_p)$. Put $\mathcal{U}(m+1,j) = \mathcal{T}(S_p)$.

The family $\mathcal{W} = \{W(m+1,j) : j = 1,\ldots,n(m+1)\}$ covers Y, so we can find a family $\{Z(m+1,j) : j = 1,\ldots,n(m+1)\}$, a closed shrinking of $\mathcal{W}$ which also covers Y.

For each $k \in \{1,\ldots,n(m+1)\}$, the family $\mathcal{U}(m+1,k)$ covers $f^{-1}(\mathrm{cl}Z(m+1,k))$, so we can find a family $\mathcal{V}(m+1,k)$ which is a closed shrinking of $\mathcal{U}(m+1,k)$ and also covers $f^{-1}(\mathrm{cl}Z(m+1,k))$.

Hence, we obtain a family

$$\mathcal{M}_{m+1} = \{\langle\mathcal{U}(m+1,k),\mathcal{V}(m+1,k),W(m+1,k),Z(m+1,k)\rangle : k = 1,\ldots,n(m+1)\}$$

such that the sequence $\mathcal{M}_1,\mathcal{M}_2,\ldots,\mathcal{M}_{m+1}$ satisfies the inductive assumptions (1) - (7) (with m being replaced by m+1).

Denote

$$\mathcal{R} = \{\langle\mathrm{cl}V,U\rangle : U \in \mathcal{U}(m,k),\ V \in \mathcal{V}(m,k),\ \mathrm{cl}V \subset U,\ k = 1,\ldots,n(m),\ m = 1,2,\ldots\}.$$

Observe that the family $\mathcal{R}$ is countable.

Fix $p \in Y$. By conditions (1), (6) and (7), there exists a sequence $\{k(m) : m = 1,2,\ldots\}$ such that $p \in \mathrm{cl}Z(m,k(m)), \mathrm{cl}W(m+1,k(m+1)) \subset W(m,k(m))$. By (2), the family $\mathcal{U}(m,k(m))$ is an irreducible open cover of $f^{-1}(p)$, and, by (3), $\mathcal{V}(m,k(m))$ is as well. Thus in view of Lemma 2, the family $\cup\{\mathcal{U}(m,k(m)) : m = 1,2,\ldots\}$ contains an infinite subfamily $\mathcal{U}$ of dis-
family $\cup\{\mathcal{U}(m,k(m)) : m = 1,2,\ldots\}$ contains an infinite subfamily $\mathcal{U}$ of disjoint sets. The family $\mathcal{V}(m,k(m))$ is a closed shrinking of $\mathcal{U}(m,k(m))$, so there exists a family $\{V(U) : U \in \mathcal{U}\} \subset \cup\{\mathcal{V}(m,k(m)) : m = 1,2,\ldots\}$ such that $\mathrm{cl}V(U) \subset U$, for each $U \in \mathcal{U}$. By condition (2), $p \in f(\mathrm{cl}V(U))$ for each $U \in \mathcal{U}$. Hence the family $\{\langle\mathrm{cl}V(U),U\rangle : U \in \mathcal{U}\}$ is an infinite subfamily of $\mathcal{R}$ such that $p \in f(\mathrm{cl}V(U))$ for each $U \in \mathcal{U}$ and the elements of $\mathcal{U}$ are disjoint. This proves that f is a local infinitely covering map.

The following lemma follows from Ward's theorem [9] quoted by Simone [6].

LEMMA 4. *Every closed* G_δ*-subset of a continuous image of a compact ordered space has a separable boundary.*

The easy proof of the following lemma is omitted.

LEMMA 5. *Let* $g : X \xrightarrow{\text{onto}} Y$ *and* $f : Y \xrightarrow{\text{onto}} Z$ *be continuous maps. If* f *is a local infinitely covering map then so is* $f \circ g$.

THEOREM 3. *Let a Hausdorff space* X *be a continuous image of a compact ordered space. If* f *is an open infinite-to-one continuous map from* X *onto a Hausdorff space* Y, *then*

(1) Y *is metrizable,*

(2) *if* Y *is dense in itself then* X *is seperable, and*

(3) *if* X *is connected then it is metrizable.*

PROOF. (1) Let g be a continuous map from a compact ordered space onto X. In view of Theorem 2, f is a local infinitely covering map; by Lemma 5, so is $f \circ g$. Hence, in view of Theorem 1, the space Y is metrizable.
(2) The space Y is metrizable, so it is first countable. Hence, each fiber of f is a closed G_δ-subset of X. If, in addition, Y has no isolated point then each fiber of f is nowhere dense, so, in view of Lemma 4, each fiber of f is separable. Let D be a countable dense subset of Y and E(d) be a countable dense subset of $f^{-1}(d)$, for each $d \in D$. One can see that the set $E = \bigcup\{E(d) : d \in D\}$ is a countbale dense subset of X.
(3) If X is connected, then so is Y. Hence, by (2), X is separable, so it is metrizable in view of a Theorem of Treybig [7], being a connected separable image of a compact ordered space.

REMARK. Theorem 1 generalizes the result of W. Bula, W. Debski and W. Kulpa [1], and also the Theorem of Treybig [8]. Another approach to this is given in [3].

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A PATHOLOGICAL HOMOGENEOUS SUBSPACE OF THE REAL LINE

A.J.M. van ENGELEN & J. van MILL

1. INTRODUCTION

All spaces under discussion are separable metric.

A zero-dimensional space is called *strongly homogeneous* provided that all nonempty clopen subspaces are hemeomorphic. Strongly homogeneous spaces behave very well; for example, they have the pleasant property that any homeomorphism between closed and nowhere dense sets can be extended to a homeomorphism of the whole space, [4]. As a consequence, all strongly homogeneous spaces are homogeneous (we encourage the reader to find a direct elementary proof of this corollary). Observe that a strongly homogeneous nowhere locally compact space has the property that all nonempty open subspaces are homeomorphic.

From the above observations it is clear that a strongly homogeneous space has "many" autohomeomorphisms. Many familiar subspaces of the real line are strongly homogeneous, for example, the rationals, the irrationals and the Cantor set. R.D. Anderson [1] has shown that, in particular, the autohomeomorphism group of a srongly homogeneous space is algebraically simple.

The aim of this note is to construct a very pathological exmple of a strongly homogeneous subspace X of the real line $\mathbb{R}$. The space X is pathological since it contains a countable dense subset $D \subset X$ such that $Y = X \setminus D$ is rigid, i.e. has only one autohomeomorphism, namely the identity. In fact, we prove a little bit more, i.e., if $h: Y \to Y$ is an embedding (not necessarily surjective), then h = identity.

A space X is called *strongly locally homogeneous* if it has an open base $\mathcal{U}$ such that for each $U \in \mathcal{U}$ and points $x,y \in U$, there exists a homeomorphism $h: X \to X$ with $h(x) = y$ and $h \mid X \setminus U$ equal to the identity. The most obvious examples of strongly locally homogeneous spaces are locally euclidean spaces and zero-dimensional homogeneous spaces. Clearly, every connected strongly locally homogenous space is homogeneous.

By a result of Anderson, Curtis and van Mill [2], if X is strongly locally homogeneous and topologically complete, and if $D \subset X$ is countable and dense, then $Y = X \setminus D$ is strongly locally homogeneous and Y has the property that for every countable subset E of Y we have that $Y \approx Y \setminus E$ (i.e., Y is homeomorphic to $Y \setminus E$). Our example shows that in this theorem the assumption of topological completeness is essential. In fact, both conclusions are false in general since our example has the property that for some countable dense set D the complement of D admits no non-trivial embeddings. As we will show, our example is even Baire which shows that the above cited result of Anderson, Curtis and van Mill, in a sense, is the best possible.

2. PRELIMINARIES

If X is a space then Auth(X) denotes the group of autohomeomorphisms of X. The domain and range of a function f will be denoted by dom(f) and range(f), respectively.

Let Y be a fixed dense in itself, topologically complete space and let $\Phi \subset \mathrm{Auth}(Y)$ be a countable subgroup. For all $x \in Y$ let $V(x) = \{h(x) : h \in \Phi\}$. Define

$$\mathcal{F} = \{f\colon \mathrm{dom}(f) \text{ and } \mathrm{range}(f) \text{ are } G_\delta\text{-subsets of } Y \text{ and } f\colon \mathrm{dom}(f) \to \mathrm{range}(f) \text{ is a homeomorphism}\},$$

$$\mathcal{G} = \{f \in \mathcal{F}\colon |\{x \in \mathrm{dom}(f) : f(x) \notin V(x)\}| = 2^{\aleph_0}\},$$

respectively.

The following result was proven in detail for the special case $Y = \mathbb{R}^2$ in van Mill [5, section 3]. The reader can easily check that the only thing used in the proof was that $\mathbb{R}^2$ is topologically complete and dense in itself. We therefore state Theorem 2.1 without proof.

<u>THEOREM 2.1</u>. *For each* $f \in \mathcal{F}$ *there is a point* $x_f \in \mathrm{dom}(f)$ *such that if* $X = \bigcup_{f\in\mathcal{F}} V(x_f)$, *then*

(3) $f(x_f) \notin X$,

(2) *if* $f,g \in \mathcal{F}$ *are distinct, then* $V(x_f) \cap V(x_g) = \emptyset$,

(3) *if* $\mathcal{D}$ *is a family of countably many nowhere dense subsets of* X *and if* $U \subset X$ *is open and nonempty, then* $|U \setminus \bigcup\mathcal{D}| = 2^{\aleph_0}$. □

3. THE EXAMPLE

Let $Q \subset \mathbb{R}$ be the set of rational numbers and put $Q' = Q \setminus \{0\}$. Put

$$\Phi = \{h \in \mathrm{Auth}(\mathbb{R}) : \exists q \in Q \, \exists s \in Q' \, \forall x \in \mathbb{R} : h(x) = s \cdot x + q\}.$$

Observe that Φ is a countable subgroup of Auth($\mathbb{R}$) and that for all $x \in \mathbb{R}$ the set $V(x)$, defined in section 2, is equal to $Q' \cdot x + Q$. Let $Y = \mathbb{R}$ and let X be as in Theorem 2.1. We claim that X is as required.

LEMMA 3.1.
(1) *If* $K \subset \mathbb{R}$ *is a Cantor set, then both* $K \cap X \neq \emptyset$ *and* $K \cap (\mathbb{R} \setminus X) \neq \emptyset$,
(2) *if* $x \in \mathbb{R} \setminus X$ *then* $V(x) \subset \mathbb{R} \setminus X$.

PROOF. Let $K \subset \mathbb{R}$ be a Cantor set and let $L \subset \mathbb{R}$ be a Cantor set disjoint from $Q' \cdot K + Q$. If $h: K \to L$ is any homeomorphism, then $h \in G$ and consequently, by 2.1 (1), $X \cap K \neq \emptyset$. Since also $h^{-1} \in G$, again by 2.1 (1), $K \cap (\mathbb{R} \setminus X) \neq \emptyset$. This proves (1) and the trivial proof of (2) is left to the reader. □

Fix a point $x \in \mathbb{R} \setminus X$.

LEMMA 3.2.
(1) X *is zero-dimensional,*
(2) *if* $a,b,c,d \in V(x)$ *and if* $a < b$ *and* $c < d$ *then there is a homeomorphism* $h: X \to X$ *such that* $h([a,b] \cap X) = [c,d] \cap X$.

PROOF. (1) follows immediately since $V(x)$ is dense. For (2), find a point $s \in Q'$ and a point $q \in Q$ such that the homeomorphism $\tilde{h}: \mathbb{R} \to \mathbb{R}$ defined by $\tilde{h}(t) = s \cdot t + q$ has the property that $\tilde{h}([a,b]) = [c,d]$. Then $\tilde{h} \in \Phi$ and consequently, by the definition of X, we see that $\tilde{h} = h \mid X$ is as required. □

COROLLARY 3.3. X *has the property that all of its nonempty open subsets are homeomorphic; in particular,* X *is strongly homogeneous.*

PROOF. Let $U \subset X$ be nonempty and open. By 3.1 (1), U is not compact. Consequently, we can find a countably infinite collection $\mathcal{L}$ of nonempty intervals such that
(1) If $L \in \mathcal{L}$ then $\{\min L, \max L\} \subset V(x)$,
(2) if $L,M \in \mathcal{L}$ are distinct, then $(L \cap X) \cap (M \cap X) = \emptyset$, and

(3) $U = \bigcup_{L \in \mathcal{L}} L \cap X$.

By 3.2 (2), if $L, M \in \mathcal{L}$ are distinct, then $(L \cap X) \approx (M \cap X)$. In addition, for each $L \in \mathcal{L}$ we have that $L \cap X$ is a clopen subset of X. Therefore, again by 3.2 (2), we find that

$$U \approx ([x,x+1] \cap X) \times \mathbb{N}.$$

We therefore conclude that all nonempty open subspaces of X are homeomorphic. □

By 2.1 (2), we can find a countable dense set $D \subset X$ such that

(1) if $d, e \in D$ are distinct, then $d \notin V(e)$ (and $e \notin V(d)$).

(2) $D \cap \{x_f : f \in \mathcal{G}\} = \emptyset$.

We claim that $Y = X \setminus D$ admits no nontrivial embeddings.

<u>LEMMA 3.4</u>. *If* $h: Y \to Y$ *is an embedding, then* h = identity.

<u>PROOF</u>. The technique used in this proof is similar to the one in [5, 3.4 and 3.5].

<u>CLAIM</u>. $|\{y \in Y : h(y) \notin V(y)\}| < 2^{\aleph_0}$.

Suppose that this is not true. By the classical Lavrentieff Theorem [3], there are G_δ-subsets $S, T \subset \mathbb{R}$ with $Y \subset S$ and range$(h) \subset T$ such that h can be extended to a homeomorphism $f: S \to T$. Then $f \in \mathcal{G}$ and consequently, $x_f \in Y$. However, since f extends h, $h(x_f) = f(x_f) \notin X$, by 2.1 (1). This obviously is a contradiction.

For each $s \in Q'$ and $q \in Q$ put

$$A_q^s = \{y \in Y : h(y) = s \cdot y + q\}.$$

Observe that each A_q^s is closed in Y. Let B_q^s be the closure of A_q^s in X. Since D is countable, by the claim and by 2.1 (3), the union of the B_q^s's having nonempty interior in X is dense in X.

Take $s \in Q'$ and $q \in Q$ such that B_q^s has nonempty interior in X. Since Y is dense in X and since $B_q^s \cap Y = A_q^s$, this implies that A_q^s is not nowhere dense in Y. Since D is dense in X, the set

$$E = \{\tfrac{1}{s}(d-q) : d \in D\}$$

is also dense in X. Suppose that either $s \neq 1$ or $q \neq 0$. Then obviously $E \subset Y$ and consequently we can find a point $d \in D$ such that $\frac{1}{s}(d-q) \in A_q^s$. Since

$$h(\frac{1}{s}(d-q)) = d \notin Y,$$

this is a contradiction. We conclude that $s = 1$ and $q = 0$.

This argument shows that there is only one B_q^s with nonempty interior, namely B_0^1. Since B_0^1 is dense, h = identity. □

REMARK 3.5. In [6] it was shown that there is a homogenenous subset $A \subset \mathbb{R}$ such that

(1) $A \approx \mathbb{R} \setminus A$, and

(2) A does not admit the structure of a topological group.

It can be shown that there is also a countable dense subset $D \subset A$ such that $A \setminus D$ is rigid. However, A is not strongly homogenenous. So if one is willing to sacrifice strong homogeneity, it is possible to construct X as above with the additional curious property that $X \approx \mathbb{R} \setminus X$. With similar arguments as in [6] it can be shown that our space X constructed above also has the property that it is not a topological group.

REMARK 3.6. If one performs the above construction in the plane, one gets an example of a one dimensional, connected and locally connected space Z such that

(1) Z is strongly locally homogeneous,

(2) if $K \subset Z$ is compact, then $Z \approx K \setminus Z$,

(3) there is a countable dense set $D \subset Z$ such that $Z \setminus D$ does not admit a nontrivial embedding.

For details, see [5]. The argument given there that a space such as Z is strongly locally homogeneous can also be used to show that $Z \setminus \{pt\} \approx Z$ and with a little bit more work that $Z \setminus$ compact $\approx Z$ (use that Z contains no Cantor sets (Lemma 3.1 (1)). The details of working this out are left to the reader.

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A NOTE ON TOPOLOGICAL GROUPS WITH DENSE ORDERABLE SUBSPACES

R. FOX

This note was written to answer a question raised by Scott Williams at the workshop on Topology and Order in Amsterdam, concerning the result of Theorem 2.3 of [2]. Williams asked whether a topological group which had a dense linearly orderable subspace (not necessarily a subgroup) was either metrizable or topologically orderable. The answer to this question is "yes", as we shall now show.

Let G be a topological group, and $\mathcal{D}$ a dense linearly orderable subspace of G. Nyikos and Reichel have shown [1, Theorem 6] that if the identity of G has a linearly ordered neighborhood base, then G is either metrizable or topologically orderable. Thus it will suffice to show that the identity (or any other point a of G) has a linearly ordered neighborhood base (because if one point does, then all do).

Since $\mathcal{D}$ is dense and G (like all topological groups) is regular, it will suffice just to show that a point $a \in \mathcal{D}$ has a linearly ordered neighborhood base $\mathcal{B}$ in $\mathcal{D}$. For then, $\{cl_G B : B \in \mathcal{B}\}$ will be a linearly ordered neighborhood base for a in G (here cl_G denotes closure in G).

In order that a point a in a linearly ordered topological space, such as $\mathcal{D}$, have a linearly ordered neighborhood base, it is necessary and sufficient that one of the following hold:

(i) a is not a limit from the left;

(ii) a is not a limit from the right; or

(iii) a is a limit from both the left and the right and has the same cofinality on the left as on the right.

The cofinality of a on the left (right) is the least ordinal ω_μ for which there is an ω_μ-sequence converging to a from the left(right); ω_μ is necessarily a regular ordinal.

Thus to complete the proof all we need is the following lemma, which improves on Lemma 3 of [1].

LEMMA. *Let $\mathcal{D}$ be a dense orderable subspace of a topological group G, and $a \in \mathcal{D}$. If a is a limit in $\mathcal{D}$ from both the left and the right, then a has the same cofinality on the left as on the right.*

PROOF. Suppose the left and right cofinality are unequal. Consider the case where the right-cofinality is less than the left cofinality, the other case being analogous. Let A be the closure in G of the interval $(a,\rightarrow)$ in $\mathcal{D}$. Observe that A contains no points x of $\mathcal{D}$ with $x < a$ and that A is a neighborhood (in G) of each point $y \in \mathcal{D}$ having $y > a$. Since $xa^{-1}y$ converges (in G) to y as x converges to a, there must be a point $f(y) \in \mathcal{D}$ such that $f(y) < a$ and such that $xa^{-1}y \in A$ for each $x \in (f(y),a) \subset \mathcal{D}$. Since the left cofinality of a in $\mathcal{D}$ is larger than the right cofinality of a, we may find a set $C \subset (a,\rightarrow)$ and a point x_0 in $\mathcal{D}$ such that $\inf(C) = a$ and $f(y) < x_0 < a$ for each $y \in C$. Then $x_0a^{-1}y \in A$ for each $y \in C$. Since $x_0a^{-1}y$ approaches x_0 as y approached a, it follows that $x_0 \in A$ which is impossible since $x_0 < a$.

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THE K_n-PROPERTY IN ω_1-TREES

K.P. HART

0. INTRODUCTION

In [2] Eric van Douwen introduced the classes of K_n-spaces ($n \in \omega$). He showed that retractable spaces are K_0, that K_n-spaces are K_{n+1} for all n and that K_1-spaces are hereditarily collectionwise normal (CWN).

None of the implications can be reversed as examples in [2] and [4] show. In this note we study the K_n-properties in the class of ω_1-trees. We show that K_1- (and hence K_0-)trees are retractable and that for every $n \geq 2$ the K_n-property is equivalent to collectionwise Hausdorffness. In [2] an example was given of a normal non-collectionwise Hausdorff K_2-space.

Our results provide us with three more examples to show that the gap between K_2 and K_1 is very wide:

(a) Devlin and Shelah [1] gave an example of a collectionwise Hausdorff non-normal Aronszajn tree. This tree is K_2.

(b) In [3] it was shown that Souslin trees are not retractable and hence not K_1, but Souslin trees are collectionwise normal and hence certainly K_2.

(c) In §3 we show that the one-point compactification of a locally compact K_n-space is again K_n; hence the one-point compactification of a Souslin tree is a compact K_2-space which is not K_1. I am indebted to Jan van Mill for the latter observation.

1. DEFINITIONS AND PREPARATORY REMARKS

1.0. ω_1-TREES.

A tree is a partially ordered set $\langle T,<_T\rangle$ such that $\hat{x} = \{y \in T : y <_T x\}$ is well-ordered by $<_T$. For $x \in T$ we denote by ht(x) (height) the ordertype of $\hat{x}$. If α is an ordinal we let $T_\alpha = \{x \in T : ht(x) = \alpha\}$. If C is a set of ordinals then $T \restriction C = \bigcup_{\alpha\in C} T_\alpha$; in particular $T \restriction \alpha = \{x \in T : ht(x) < \alpha\}$.

An ω_1-tree is a tree $\langle T, < \rangle$ with the following properties:

(i) $T_{\omega_1} = \emptyset$, T_0 has one point: $\underline{0}$,

(ii) if $\alpha \in \omega_1$ then $0 < |T_\alpha| \le \omega_0$,

(iii) if $x \in T$ and $\alpha > ht(x)$ then there are (at least) two different successors of x in T_α ,

(iv) if α is a limit and $x,y \in T_\alpha$ then $x = y \leftrightarrow \hat{x} = \hat{y}$.

The tree topology of an ω_1-tree T is generated by the following base:

$$\{\{0\}\} \cup \{(x,y] : x,y \in T \wedge x <_T y\},$$

where $(x,y] = \{z \in T : x <_T z \le_T y\}$.

With this topology T is first-countable, locally compact, Hausdorff (clause (iv)) and zero-dimensional.

A Souslin tree is an ω_1-tree in which every antichain (= set of pairwise incomparable elements) is countable.

We let Λ denote the set of limit ordinals in ω_1.

1.1. RETRACTABILITY AND K_n-SPACES

Let X be a topological space and $A \subseteq X$. A is said to be a retract of X if there is a continuous map $r : X \to A$ such that for all $a \in A$, $r(a) = a$. A is said to be K_n-embedded ($n \in \omega$) if there exists a function $\kappa : \tau A \to \tau X$ (τY denotes the topology of Y) such that:

(i) $\kappa(U) \cap A = U$ for all $U \in \tau A$ and

(ii) $n = 0$: $\kappa(\emptyset) = \emptyset$ and $\kappa(U \cap V) = \kappa(U) \cap \kappa(V)$ for all $U,V \in \tau A$;
$n > 0$: If $\{U_0,\dots,U_n\} \subseteq \tau A$ and $U_i \cap U_j = \emptyset$ for $i \neq j$ then $\bigcap_{i=0}^n \kappa(U_i) = \emptyset$.

In this situation κ is said to be a K_n-function.

<u>DEFINITION</u>. Let X be topological space. Then X is said to be retractable if every closed subset of X is a retract of X, equivalently if every closed nowhere dense subset of X is a retract of X (if $A \subseteq X$ is closed, use a retraction of X onto the boundary of A to construct a retraction onto A).

X is said to be a K_n-space ($n \in \omega$) [2] iff every subspace of X is K_n-embedded if X, iff every closed subspace is K_n-embedded, iff every closed nowhere dense subspace is K_n-embedded.

2. $n \leq 1$.

In this section we show that for ω_1-trees the properties of being retractable, K_0 and K_1 are all equivalent.

<u>LEMMA 2.0</u>. *Let* T *be an* ω_1*-tree which is collectionwise normal. Then* T *is strongly zero-dimensional.*

<u>PROOF</u>. Let $A,B \subseteq T$ be closed and disjoint.
For $n \in \omega$, define $C_n \subseteq A \cup B$ as follows:

$$C_0 = \{x \in A \cup B : x \text{ is minimal}\}.$$

For $a \in A[B]$, define

$$K(a) = \{b \in B[A] : a < b \text{ and } b \text{ is minimal}\}.$$

If $n \in \omega$ and C_n is defined let

$$C_{n+1} = \bigcup_{a \in C_n} K(a).$$

$C = \bigcup_{n\in\omega} C_n$ is a closed and discrete subspace of T (a point in C' would have to be in $A \cap B$). So since T is CWN for all $c \in C$ choose $x_c < c$ such that:

$$\{(x_c, c]\}_{c \in C}$$

is a discrete collection and $c \in A[B] \to (x_c, c] \cap B[A] = \emptyset$.
Next for $c \in C$ let $T^c = \{z \in T : z \geq c\}$ and

$$X(c) = (x_c, c] \cup T^c \setminus \bigcup\{(x_d, d] \cup T^d : d \in K(c)\}.$$

The following facts are easy to verify:
- each $X(c)$ is clopen;
- $\{X(c)\}_{c \in C}$ is a discrete collection;
- $A \subseteq \bigcup_{a\in A\cap C} X(a)$, $B \subseteq \bigcup_{b \in B\cap C} X(b)$;
- $a \in A \cap C$, $b \in B \cap C \to X(a) \cap X(b) = \emptyset$.

But then we see that

$$0_A = \bigcup_{a \in A\cap C} X(a)$$

is a clopen set which contains A and is disjoint from B. □

We now turn to the main result of this section.

<u>THEOREM 2.1</u>. *The following are equivalent for an* ω_1*-tree* T:

(i) T *is retractable.*

(ii) T *is a* K_0*-space.*

(iii) T *is a* K_1*-space.*

<u>PROOF</u>. (i) $\to$ (ii) $\to$ (iii) always holds, so we turn to proving:
(iii)$\to$(i): Let $A \subseteq T$ be closed and nowhere dense, so $A \subseteq T \restriction \Lambda$. We have to find a retraction of T onto A. We first show that A is a retract of a suitable neighborhood U_A of A, then using Lemma 2.0 we find a clopen set 0_A such that $A \subseteq 0_A \subseteq U_A$. But then, since A is a retract of the clopen set 0_A, it immediately follows that A is a retract of T (map $T \setminus 0_A$ onto some isolated point of A). Now, let $\kappa : \tau A \to \tau T$ be a K_1-function.
Let $A^0 = \{a \in A : a \text{ is isolated in } A\}$, and $A_0 = \{a \in A : a \text{ is minimal}\}$.
For $a \in A$ define x_a as follows:

$$\text{if } a \in A^0, \quad x_a = \min\{x \in \hat{a} : [x,a] \subseteq \kappa(\{a\})\},$$
$$\text{if } a \in A \setminus A^0, x_a = \min\{x \in \hat{a} : [x,a] \subseteq \kappa([0,a] \cap A \setminus A_0)\}.$$

We shall show that A is a retract of $U_A = \bigcup_{a \in A} [x_a,a]$.
Define $r : U_A \to A$ as follows:

$$\text{if } x \in [x_a,a] \text{ for some } a \in A^0, \text{ let } r(x) = a.$$
$$\text{if } x \in U_A \setminus \bigcup_{a \in A^0} [x_a,a], \qquad \text{let } r(x) = \max([0,x] \cap A).$$

Clearly r maps U_A onto A and satisfies $r(a) = a$ for all $a \in A$.
We show that r is continuous.
Let $x \in U_A$.
If x is isolated in T or in $\bigcup_{a \in A^0} [x_a,a]$, then r is constant on a neighborhood of x and hence continuous at x.
Assume x is not isolated and not in $\bigcup_{a \in A^0} [x_a,a]$.
a) $x \notin A$: pick $a \in A \setminus A^0$ such that $x \in [x_a,a]$. Take $y < x$ such that $(y,x] \subset [x_a,a] \setminus A$. If for some $b \in A^0$ we have $(y,x] \cap [x_b,b] \neq \emptyset$ then since $[x_a,a] \cap [x_b,b] \neq \emptyset$ it would follow that $b \in [x_a,a]$ since κ is a K_1-function, but then either $b \leq x$ or $x \leq b$. In the first case we would have $b \in (y,x]$, in the second $x \in [x_b,b]$, both of which are impossible. So $(y,x] \cap \bigcup_{b \in A^0} [x_b,b] = \emptyset$ and we see that r is constant on $(y,x]$ with value $\max([0,x] \cap A)$.

b) $x \in A$: for notation's sake, write $x = a$.
Let $y < a$ be arbitrary.
Pick $b \in A^0 \cap (x_a, a]$ such that $y < x_b$. Take $z \in [x_b, a]$.
If $z \notin \bigcup_{c \in A^0} [x_c, c]$ then $b \leq r(z) < z \leq a$, so $r(z) \in (y, a]$.
If $z \in [x_c, c]$ for some $c \in A^0$, then
as in case a) it follows that $c \in [x_a, a]$ and so $x_b \leq z < c \leq a$.
So again $r(z) \in (y, a]$.
Since y was arbitrary, we conclude that r is continuous at a. □

REMARK 2.2.

In Lemma 2.0 it was assumed that T was CWN. This was sufficient for our purpose since K_1-spaces are hereditarily CWN. The question naturally arises whether we can do with less:
Is every normal ω_1-tree strongly zero-dimensional?
Or if we consider separation of zero-sets by clopen sets:
Is every ω_1-tree strongly zero-dimensional?
Let us note that the answer is an unsatisfactory "yes" if we assume $\neg$CH:
If A and B are disjoint zero-sets and $f : T \to \mathbb{R}$ is a continuous function which is zero on A and one on B then $f^{\leftarrow}[(\leftarrow, r)]$ is a clopen set separating A from B where $r \in (0,1) \setminus f[T]$ is arbitrary.

3. $n \geq 2$.

In this section we show that the K_n-property is equivalent to collectionwise Hausdorffness in the class of ω_1-trees.

THEOREM 3.1. *The following are equivalent for an ω_1-tree* T:
(i) T *is a* K_2*-space*,
(ii) T *is a* K_n*-space for all* $n \geq 2$,
(iii) T *is a* K_n*-space for some* $n \geq 2$,
(iv) T *is collectionwise Hausdorff*

PROOF. (i) $\leftrightarrow$ (ii) $\to$ (iii) always holds.
(iii) $\to$ (iv): By [1] it suffices to check that no antichain of T meets stationary many levels. Suppose $A \subseteq T$ is an antichain which meets stationary many levels; w.l.o.g. $A \subseteq T \upharpoonright \Lambda$. Let $\kappa : \tau A \to \tau T$ be a K_n-function. Define $f : A \to T$ by $f(a) = \min \{x \in \hat{a} : [x,a] \subseteq \kappa(\{a\})\}$. Since f is regressive, f is constant on a subset A' of A which also meets stationary

many levels.
However, if we have $\{a_0,a_1,\ldots,a_{n+1}\} \subseteq A'$, then $\bigcap_{i=0}^{n+1} \kappa(\{a_i\}) = \emptyset$, showing that f cannot even be constant on $\{a_0,a_1,\ldots,a_{n+1}\}$. So A cannot meet stationary many levels.
(iv) $\rightarrow$ (i): Assume T is collectionwise Hausdorff. Let $A \subseteq T$ be closed. For $\eta \in \omega_1$, let $A_\eta = \{a \in A : tp(\hat{a} \cap A) = \eta\}$. A_η is an antichain and $\bigcup_{\xi\leq\eta} A_\xi$ is closed for all $\eta \in \omega_1$.
So if $\eta \in \omega_1 \setminus \Lambda$, for all $a \in A_\eta$ pick $x_a < a$ such that $\{(x_a,a]\}_{a\in A_\eta}$ is disjoint and $(x_a,a] \cap \bigcup_{\xi<\eta} A_\xi = \emptyset$ for all $a \in A_\eta$.
Let $A^0 = \{a \in A : a \text{ is isolated in } A\} = \bigcup_{\eta\in\omega_1\setminus\Lambda} A_\eta$. It follows that $\{(x_a,a]\}_{a\in A^0}$ is a disjoint collection.
For each $a \in A$ define a local base for T at a as follows:

$$\text{if } a \in A^0 \quad \text{let } \mathcal{B}(a) = \{(y,a] : x_a \leq y < a\},$$
$$\text{if } a \in A\setminus A^0 \quad \text{let } \mathcal{B}(a) = \{(x_b,a] : b \in \hat{a} \cap A^0\}.$$

Let $\mathcal{B} = \bigcup_{a\in A} \mathcal{B}(a)$.
$\mathcal{B}$ has the following two properties:

$\{B \cap A : B \in \mathcal{B}\}$ is a base for A, and
if $B_i \cap B_j \cap A = \emptyset$ for $0 \leq i < j \leq 2$ then $\bigcap_{i=0}^{2} B_i = \emptyset$
($\mathcal{B}$ is a K_2-base for A as defined in [2]).

It is easy to see that by $\kappa(U) = \bigcup\{B \in \mathcal{B} : B \cap A \subseteq U\}$, a K_2-function from A to T is defined. By definition $\{B \cap A : B \in \mathcal{B}\}$ is a base for A. Now take $(y_0,a_0],(y_1,a_1],(y_2,a_2] \in \mathcal{B}$ such that $(y_i,a_i] \cap (y_j,a_j] \cap A = \emptyset$ if $i \neq j$. For $i = 0,1,2$ define

$$H_i = \bigcup\{(x_p,p] : p \in A^0 \cap (y_i,a_i]\} \text{ and}$$
$$K_i = (y_i,a_i] \setminus H_i.$$

Then $H_i \cap H_j = K_i \cap K_j = \emptyset$ for $0 \leq i < j \leq 2$.
For $H_i \cap H_j$, this follows from the fact that $\{(x_a,a]\}_{a\in A^0}$ is a disjoint collection.
For $K_i \cap K_j$, suppose we have $x \in K_i \cap K_j$. Pick p_i and p_j from A^0 such that $y_i = x_{p_i}$ and $y_j = x_{p_j}$.
It follows that $y_i < p_i < x$ and $y_j < p_j < x$ so that

$$\max(p_i,p_j) \in (y_i,x] \cap (y_j,x] \cap A \subseteq (y_i,a_i] \cap (y_j,a_j] \cap A,$$

which is a contradiction.
Now if $(y_0,a_0] \cap (y_1,a_1] = \emptyset$ then we are done, so suppose $(y_0,a_0] \cap (y_1,a_1] \neq \emptyset$. But then

$$\begin{aligned}(y_0,a_0] \cap (y_1,a_1] \cap (y_2,a_2] &\subseteq [(H_0 \cap K_1) \cup (K_0 \cap H_1)] \cap (H_2 \cup K_2) \\ &\subseteq [(H_0 \cup H_1) \cap H_2] \cup [(K_0 \cup K_1) \cap K_2] \\ &= \emptyset. \qquad \square\end{aligned}$$

REMARK 3.1. In [3] is was shown that Souslin trees are not retractable and hence by Theorem 2.1 not even K_1. By Theorem 3.0 they are K_2-spaces. So in connection with the results from [2] the following question arises: What kind of extension properties do Souslin trees have?
From results in [2] it follows that they do not have proerty D_c^* for any $c < 3$ (see [2] for definitions).

We now show that the one-point compactification of a Souslin tree is a K_2-space (which is not K_1 since the K_n-property is hereditary).

LEMMA 3.2. *Let* X *be a locally compact* K_n*-space. Then* αX *is also a* K_n*-space.*

PROOF. Denote the point at infinity by ∞ (we assume of course that X is not compact). First note that $\lambda : \tau X \to \tau(\alpha X)$ defined by

$$\lambda(U) = U \text{ if } X \setminus U \text{ is not compact}$$
$$\lambda(U) = U \cup \{\infty\} \text{ if } X \setminus U \text{ is compact}$$

is a K_0-function.
Now let $A \subseteq \alpha X$ and let $\kappa : \tau(A \cap X) \to \tau X$ be a K_n-function. It is straightforward to check that $\theta : \tau A \to \tau(\alpha X)$ defined by $\theta(U) = \lambda(\kappa(U \cap X))$ is a K_n-function. □

EXAMPLE 3.3. Let T be a Souslin tree and αT its one-point compactification. Then αT is K_2 (by the lemma) but not K_1. Furthermore αT is hereditarily collectionwise normal.
So we have an example of a compact hereditarily collectionwise normal K_2-space which is not K_1. □

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ON COMPACTIFICATIONS OF GO-SPACES

J. PELANT

E.K. van DOUWEN raised the question of whether there is any connected compactification of the Sorgenfrey line. This question was answered in the negative [2]. The method of [2] works not only for compactifications of the Sorgenfrey line but, for example, also for connected extensions of other GO-spaces; however some completeness-type condition is needed (see [1] for details). We suggest a different method which seems to work only for compactifications, but nevertheless proves some new results: e.g., that no uncountable subspace of the Sorgenfrey line has a connected compactification.

If S is an ordered space, $\mathcal{T}$ is its topology, $M \subset S$ and $M^- \cup M^+ \subset M$ then $\mathcal{T}(M^+,M^-)$ denotes a GO-topology on M obtained as the coarsest topology on M containing $\mathcal{T} \upharpoonright M$ and containing each $(\leftarrow,x]$, for $x \in M^-$, and each $[x,\rightarrow)$, for $x \in M^+$, as clopen sets.
If K is a compactification of M and $V \subset M$ is an open set in M then let $\tilde{V} = \bigcup\{G \subset K\colon G \text{ is open in } K \text{ and } G \cap M \subset V\}$. In what follows, α is a cardinal number.

<u>THEOREM</u>. *Let* R *be a compact, connected linearly ordered space. Suppose that there is a* π*-base* $\mathcal{B}$ *in* R *such that* card $\mathcal{B} \le \alpha$ *(i.e., each* $P \in \mathcal{B}$ *is a nonempty open set and for each open* $G \subset R$*, there is* $P \in \mathcal{B}$ *such that* $P \subset G$*). Let* M *be a subset of* R*. If* $M^+ \cup M^- \subset M$ *and* card$(M^+ \cup M^-) > \alpha$ *then* $(M,\mathcal{T}(M^+,M^-))$ *has no connected compactification. (Hence* $(M,\mathcal{T}(M^+,M^-))$ *is not orderable by a dense order.*

Notation: We let:

(a) $0 = \min R$, $1 = \max R$.

(b) For $x \in R$, $y \in R$:

$$[x,y] = \{z \in R : x \le z \le y\}$$
$$(x,y) = \{z \in R : x < z < y\}$$

The sets $[x,y)$ and $(x,y]$ are defined analogously.

(c) As a topological space M will be always supposed to be equipped with $T(M^+,M^-)$.

(d) If T is a topological space then the terms T-neighborhood, T-isolated, etc., are used to emphasize what topology is under consideration.

PROOF OF THEOREM. If M contains some M-isolated points then obviously the topological space M has no connected compactification. Hence suppose that there are no M-isolated points in M. We shall suppose that card $M^- > \alpha$ (the case card $M^+ > \alpha$ is quite similar). Suppose that K is a connected compactification of M.

We see immediately that for each M-clopen set D there is $p \in K - M$ such that $p \in cl_K(D) \cap cl_K(M - D)$. We shall need a more detailed assertion:

OBSERVATION 1.

1. For each nonempty M-clopen $D \subset M$ there exist $x_0 \in cl_R(D)$, $x_1 \in cl_R(M - D)$, $p \in K - M$ such that for each R-neighborhood V_i of x_i, $i = 0,1$, we have that

(*) $$p \in cl_K(V_0 \cap D) \cap cl_K(V_1 \cap (M - D))$$

2. If $x_0 \in M^-$, then $x_0 \neq \sup D$.
If $x_1 \in M^-$, then $x_1 \neq \sup (M - D)$.

PROOF OF OBSERVATION 1.

1. There is $p \in cl_K(D) \cap cl_K(M - D)$.
Put $t_0 = \sup \{r \in R : p \notin cl_K([0,r] \cap D)\}$.
Put $t_0 = x_0$. For any R-neighborhood V of x_0, we have $p \in cl_K(V \cap D)$.
To see that this x_0 satisfies (1), we use the definition of t_0 if $t_0 = 0$ and we use the fact that $p \in cl_K D$ if $t_0 = 1$.
If $0 < t_0 < 1$ then for each $y,z \in R$ such that $y < t_0 < z$ we have $p \in cl_K((y,z) \cap D)$; otherwise t_0 is not the supremum (we use the fact that the order of R is dense).
Similarly, one finds x_1 for $M - D$.

2. Suppose the contrary, i.e., $x_0 \in M^-$ and $x_0 = \sup D$. Hence $x_0 \in D$. Then $\{D \cap V_0 : V_0$ is an R-neighborhood of $x_0\}$ forms a local base of M-neighborhoods at x_0, so that $x_0 \in cl_K(\{p\})$, a contradiction. An analogous argument applies for x_1.

NOTATION. The set of all two-point sets $\{a,b\}$ which occur in (*) for some non-empty M-clopen D and $p \in K - M$ will be denoted by $\mathcal{D}$.

OBSERVATION 2. For each $x \in M^-$ and each $y \in [0,x)$ there is $t \in (y,x)$ such that whenever $\{a,b\} \in \mathcal{D}$ and $a \in (t,x)$ then $b \in [y,x)$.

PROOF. By the regularity of K, for each $y \in [0,x)$ there is $t \in (y,x)$ such that $cl_K((t,x] \cap M) \subset \widetilde{[y,x]} = \cup\{G \subset K : G \text{ is open and } G \cap M \subset [y,x] \cap M\}$. If $\{a,b\} \in \mathcal{D}$ and $a \in (t,x)$ then the corresponding p assigned to $\{a,b\}$ by (*) is an element of $cl_K((t,x] \cap M)$. Hence $\widetilde{[y,x]}$ is a K-neighborhood of p. It follows that $b \in [y,x]$. But the case $b = x$ cannot occur (since $[y,x]$ is a K-neighborhood of p so that $b = x$ would imply that $p \in cl_K((z,x] \cap M)$ for each $z < x$ which is impossible).

DEFINITION.

$$A^- = \{x \in M^- : \text{there is } z_x \in (x,1] \text{ such that for each } d \in M^- \cap (x,z_x) \text{ there is } \{a,b\} \in \mathcal{D} \text{ such that } x < a < d \le b\}.$$

OBSERVATION 3. Let $x \in M^- - \cup\mathcal{D}$.
If x is not right R-isolated in M^-, then $x \in A^-$.

PROOF. Suppose that x is not right R-isolated in M^- and $x \notin A^-$, i.e. there is a transfinite sequence $\{d_\gamma\}_{\gamma\in\beta}$ such that:

(i) β is a regular initial ordinal

(ii) if $\beta > \gamma > \gamma'$, then $d_\gamma < d_{\gamma'}$,

(iii) $d_\gamma \in M^-$ for each $\gamma \in \beta$

(iv) $x = \inf \{d_\gamma\}_{\gamma\in\beta}$

(v) for each $\{a,b\} \in \mathcal{D}$ and each $\gamma \in \beta$ it holds that
either $(x,d_\gamma) \cap \{a,b\} = \emptyset$
or $[d_\gamma,1) \cap \{a,b\} = \emptyset$.

(Observe that $\beta \le \alpha$ since $\chi(R) \le \text{card } (\beta) \le \alpha$.)
The set $M \cap (x,d_\gamma]$ is M-clopen for each $\gamma \in \beta$. Hence for each $\gamma \in \beta$ there are points

$$x_\gamma \in [x,d_\gamma],$$
$$y_\gamma \in R - (x,d_\gamma),\ \{x_\gamma,y_\gamma\} \in \mathcal{D},$$
$$\text{and } p_\gamma \in K - M$$

satisfying (*). Recall that by Observation 2 and the fact that $x \notin \cup\mathcal{D}$, necessarily $x_\gamma \in (x,d_\gamma)$. Hence $y_\gamma \in [0,x]$ for each $\gamma \in \beta$. By Observation 2 and the assumption that $x \notin \cup\mathcal{D}$, there is $\eta \in R$ such that $[\eta,x] \cap \{y_\gamma : \gamma \in \beta\} = \emptyset$.

Let $\mathcal{F}$ be a uniform ultrafilter on β (i.e., card $F = \beta$ for each $F \in \mathcal{F}$). Denote by $\mathcal{G}$ (resp. $\mathcal{H}$) an ultrafilter on R (resp. on K) the base of which is $\{\{y_\gamma : \gamma \in F\} : F \in \mathcal{F}\}$ ($\{\{p_\gamma : \gamma \in F\} : F \in \mathcal{F}\}$ resp.). Since R and K are compact there are $t \in [0,\eta]$ and $p \in K$ such that t is an R-limit of $\mathcal{G}$ and p is an K-limit of $\mathcal{H}$. Take some R-closed R-neighborhood V of t such that $V \subset [0,x]$ and some R-closed R-neighborhood W of x so that $V \cap W = \emptyset$. There is $F \in \mathcal{F}$ such that $\{y_\gamma : \gamma \in F\} \subset V$ and $\{x_\gamma : \gamma \in F\} \subset W$ (recall that $\mathcal{F}$ is uniform). Hence

$$\{p_\gamma : \gamma \in F\} \cup \{p\} \subset cl_K(V \cap (M - (x,d_0])) \cap cl_K(W \cap M \cap (x,d_0]).$$

Therefore $p \in K - M$ and $\{t,x\} \in \mathcal{D}$ which is contradiction.

<u>OBSERVATION 4</u>. card $A^- \leq \alpha$.

<u>PROOF</u>. Suppose card $A^- > \alpha$.
There is $H \subset A^-$, card $H > \alpha$ and $P \in \mathcal{B}$ such that $(x,z_x) \supset P$ for each $x \in H$. Since M^- is α-Lindelöf (see [1]) and card $H > \alpha$ there is $h \in M^-$ such that $h = \sup (H \cap [0,h))$ (hence in particular $h \leq \inf (P)$). Take $y \in [0,h)$. Suppose $t \in (y,x)$ is the element, the existence of which is given by Observation 2. There is $g \in H \cap [0,h)$ such that $t < g < h \leq \inf P < \sup P \leq z_g$. Hence there is $\{a,b\} \in \mathcal{D}$ such that $g < a < h \leq b$ which contradicts Observation 2.

<u>OBSERVATION 5</u>. card $(M^- \cap (\cup\mathcal{D})) \leq \alpha$.

<u>PROOF</u>. Suppose the contrary. Then there are $H \subset M^-$, $\mathcal{E} \subset \mathcal{D}$ and $P \in \mathcal{B}$ such that:

(i) card H = card $\mathcal{E} > \alpha$,

and, with $\mathcal{E} = \{\{h,b_h\} : h \in H\}$,

(ii) $P \subset (h,b_h) \cup (b_h,h)$ for each $h \in H$.

Again, M^- is α-Lindelöf so there is $x \in M^-$ such that $x = \sup (H \cap [0,x))$. Take $y \in [0,x)$ such that $P - [y,x) \neq \emptyset$ and take $t \in (y,x)$, the existence of which is given by Observation 2. Take $h \in (t,x) \cap H$. By Observation 2,

$b_h \in [y,x)$. Hence $P \subset [y,x)$ which is a contradiction.

Now the Theorem follows since $M^- = A^- \cup ((\cup\mathcal{D}) \cap M^-) \cup \{x \in M^- : x$ is right R-isolated in $M^-\}$ and the cardinality of the last set is at most α.

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THE ORDERABILITY OF COMPACT SCATTERED SPACES

S. PURISCH

1. INTRODUCTION

A space is *scattered* if each of its nonempty subsets has an isolated point.

Recently G. MORAN [4] gave a complicated proof that a Hausdorff space is homeomorphic to a compact, scattered (totally) orderable space iff it is the 2-to-1 continuous image of a compact ordinal space. It would be desirable to find a simple proof of Moran's theorem and, further, to find an intrinsic characterization of such spaces.

In 1972 J.W. BAKER [1] gave such an intrinsic characterization of the compact ordinal spaces in terms of a neighborhood base. Each point of an ordinal space has a neighborhood base which is linearly ordered by reverse inclusion. In general this is not true for orderable, compact, scattered spaces. For example consider the point ω_1 in $\omega_1 + 1 + \omega_0^*$ (where α^* denotes the ordinal α with its order reversed).

Modifying Baker's result in terms of a neighborhood subbase, we give here a characterization of orderable compact scattered spaces. The proof is not long, and with a little extra effort Moran's result follows.

There are two kinds of examples to avoid. One is a "T-shaped" space, e.g., the quotient of the discrete sum of three copies of $\omega_1 + 1$ obtained by identifying the three copies of the point ω_1. The theorem given here eliminates such a possibility by assuming that each point has a neighborhood subbase which essentially allows at most one (possibly transfinite) sequence converging to the point form the "left" and at most one sequence converging to it from the "right". The other type of example given below is more complicated and subtle, in that although orderability may be satisfied locally, there is a breakdown globally. This leads to considering a type of stationary set whose existence is independent of ZFC.

2. EXAMPLES.

2.1. Let X be the set $\omega_1 \times (\omega_0 + 1 + \omega_1^*)$ and $Y = X \cup \{p\}$. For each $<\alpha,\beta> \in X$ define a basic neighborhood as follows:

if $\beta < \omega_0$, $\{<\alpha,\beta>\}$, i.e. $<\alpha,\beta>$ is isolated;
if $\beta > \omega_0$, $\{<\alpha,\gamma> \mid \beta \le \gamma < \tau\}$ for some $\tau > \beta$;
if $\beta = \omega_0$, $\{<\alpha,\gamma> \mid \tau_1 < \gamma < \tau_2\} \cup \{<\lambda,\gamma> \mid \tau < \lambda < \alpha,\ \gamma \in \omega_0 + 1 + \omega_1^*\}$
for some $\tau_1 < \omega_0$, $\tau_2 < \omega_0$ and $\tau < \alpha$.

(note that the order type on the second factor space of X is $\omega_0 + 1 + \omega_1^*$; it is not ω_1.)

A basic neighborhood of p is

$$\{p\} \cup \{<\lambda,\gamma> \mid \tau < \lambda, \gamma \in \omega_0 + 1 + \omega_1^*\} \text{ for some } \tau < \omega_1.$$

The set Y is an ω_1-sequence converging to p. Each member of the sequence is a copy of $\omega_0 + 1 + \omega_1^*$. For $\alpha < \omega_1$ a basic neighborhood of $<\alpha,\omega_0>$ is the union of a cofinal segment of the set of predecessors of the α^{th} member of this sequence plus a neighborhood of the point ω_1 in the α^{th} copy of $\omega_0 + 1 + \omega_1^*$. Note, however, points such as $<\omega_0,0^*>$ are isolated where 0^* is the maximum of ω_1^*; the sequence $\{<n,0^*> \mid n < \omega_0\}$ converges to $<\omega_0,\omega_0>$ and not to $<\omega_0,0^*>$.

Note that Y is a chandelier space as defined by RAJAGOPALAN, SOUNDARARAJAN and JAKEL [6].

Y is scattered and compact. (The only reason for adding the point p is to compactify the example.)

PROPOSITION. Y *is not orderable.*

The proof, given after some preliminaries, is needed in the proof of Theorem 2.

Let κ be an ordinal. A subset A of κ is *stationary* in κ if A meets every closed unbounded (*cub*) subset of κ.

PRESSING DOWN LEMMA (PDL). *Suppose* S *is a stationary subset of* κ, *where* κ *is a regular uncountable cardinal. If* $f : S \to \kappa$ *has the property that* $f(\alpha) < \alpha$ *for each* $\alpha \in S - \{0\}$, *then there is a* $\beta < \kappa$ *such that* $f^{-1}\{\beta\}$ *is stationary in* κ.

PROOF OF PROPOSITION. Suppose Y were orderable, and let $\leq$ be an admissible order on Y. Note $W = \{<\beta,\omega_0> \mid \beta \in \omega_1\} \cup \{p\}$ is homeomorphic to $\omega_1 + 1$. So with respect to $\leq$, p is the limit of an ω_1 or ω_1^*, say ω_1, type sequence W_1 which is closed in $W - \{p\}$ (i.e. an isomorphic copy of ω_1). Then by induction find a cub subset S_1 of ω_1 such that with respect to $\leq$, the sequence $\{<\beta,\omega_0> \mid \beta \in S_1\}$ is increasing (and is a subset of W_1). Let S be the set of limit points of S_1. Then S is also a cub of ω_1.

Fix $\alpha \in S$. Then $\{<\alpha,n> \mid n < \omega_0\}$ is eventually less than $<\alpha,\omega_0>$ and $\{<\alpha,\gamma> \mid \omega_0 < \gamma\}$ is eventually greater than $<\alpha,\omega_0>$ since the former sequence and $\{<\beta,\omega_0> \mid \beta \in S_1 \cap \alpha\}$ both have cofinality ω_0 while the latter has cofinality ω_1. Let n_α be the first natural number such that $<\alpha,n_\alpha> < <\alpha,\omega_0>$. Define $f(\alpha) = \min\{\beta \mid \beta \in S_1 \cap \alpha \text{ and } <\alpha,n_\alpha> < <\beta,\omega_0>\}$.

Then $f : S \to \omega_1$ has the property that $f(\alpha) < \alpha$ for all $\alpha \in S$ (unfixing α). So by PDL there exists $\beta < \omega_1$ such that $f^{-1}\{\beta\}$ is stationary in ω_1. Let α_0 be a limit point of $f^{-1}\{\beta\}$ greater than β. Then with respect to the original topology on Y, $\{<\alpha,n_\alpha> \mid \alpha \in \alpha_0 \cap f^{-1}\{\beta\}\}$ converges to $<\alpha_0,\omega_0>$. But since $<\beta,\omega_0> < <\alpha_0,\omega_0>$ and for all $\alpha \in \alpha_0 \cap f^{-1}\{\beta\}$, $<\alpha,n_\alpha> < <\beta,\omega_0>$, $\{<\alpha,n_\alpha> \mid \alpha \in \alpha_0 \cap f^{-1}\{\beta\}\}$ does not converge to $<\alpha_0,\omega_0>$ in the order topology generated by $\leq$. Hence Y is not orderable.

2.2. With the previous example in mind the author believed that by a proof analogous to the one in 2.1, a necessary condition (and one of two suffucient conditions) for a compact, scattered space X to be orderable would be: there is no subset Y of X with $Y = \bigcup_{s\in S} X_s$, where the X_s's are pairwise disjoint, S is a stationary subset of a regular, uncountable ordinal, and for each $s \in S$, X_s is homeomorphic to $(\omega_0 + 1) + \alpha_s^*$, where α_s is an uncountable, regular ordinal, $x_s \in X_s$ is the point corresponding to ω_0 under the homeomorphism, and $\{x_s\}_{s\in S}$ is homeomorphic to S.

However, let X be the lexicographic product $(\omega_1 + 1) \times (\omega_1 \cdot 2+1)^*$. For each $\alpha \in \omega_1 \cdot 2+1$ let α^* denote the corresponding element in $(\omega_1 \cdot 2+1)^*$. Let S be the set of limit ordinals in ω_1. For each $s \in S$ let $x_s = <s,(\omega_1 \cdot 2)^*>$, let $\{s_n\}_{n\in\omega_0}$ be an increasing sequence in ω_1 converging to s, and let $X_s = \{<s_n,(s+1)^*> \mid n \in \omega_0\} \cup \{x_s\} \cup \{<s,(\omega_1 + \alpha)^*> \mid 0 < \alpha < \omega_1\}$. (Here $(s+1)^*$ is used instead of s^* and $\alpha \neq 0$ to insure that $X_s - \{x_s\}$ is open. This is not required but makes the example nicer.) Then $Y = \bigcup_{s\in S} X_s$ satisfies the conditions of the last paragraph.

3. RESULTS.

LEMMA. *Let* A *be a compact, scattered, orderable space, and let* $\ell \in A$ *not be the limit of two disjoint, closed sequences, one of which has uncountable cofinality. Then there is an admissible order on* A *with* ℓ *as its first point.*

PROOF. Let $\le$ be an admissible order on A. If ℓ is the first point of $\langle A,\le\rangle$, we are done. Otherwise if $\ell \notin \mathrm{cl}(-\infty,\ell)$, then ℓ has an immediate predecessor ℓ'. Let $A_1 = (-\infty,\ell']$ and $A_2 = [\ell,\infty)$. Define $\le'$ on A as follows: for $x,y \in A$ $x \le' y$ if $x \in A_2$ and $y \le A_1$, or $x \le y$ and x and y are both elements of A_i, $i = 1,2$. Since A is compact, A_1 and A_2 both have two endpoints. Then $\le'$ satisfies the lemma.

If ℓ is the last point of $\langle A,\le\rangle$, or $\ell \notin \mathrm{cl}(\ell,\infty)$, then, by first inverting the order $\le$, an analogous argument produces an order satisfying the lemma.

Finally, suppose that $\{\ell\} = \mathrm{cl}(-\infty,\ell) \cap \mathrm{cl}(\ell,\infty)$. Then there are monotone increasing $\{a_n\}_{n<\omega_0}$ and monotone decreasing $\{b_n\}_{n<\omega_0}$, sequences of isolated points converging to ℓ where a_0 is the first and b_0 the last point of $\langle A,\le\rangle$. For each $n < \omega_0$ let $A_n = [a_n,a_{n+1})$ and $B_n = (b_{n+1},b_n]$. Define $\le'$ on A so that $\le$ and $\le'$ have identical restrictions to each A_n and B_n, and $\ell \le' A_{n+1} \le' B_n \le' A_n$. (Define $B_n \le' A_n$ if $b \le' a$ for every $a \in A_n$ and every $b \in B_n$.) Then $\le'$ satisfies the Lemma.

In the following theorem, compactness is necessary since the Cartesian product $\omega_1 \times (\omega_0 + 1)$ is locally compact, scattered, and satisfies the condition in the theorem but is not orderable.

THEOREM 1. *A compact, scattered space* X *is orderable iff for each non-isolated* $x \in X$ *there is a neighborhood subbase* $\{L_\alpha\}_{\alpha<\tau} \cup \{R_\alpha\}_{\alpha<\gamma}$ *consisting of two decreasing nests of clopen sets (these nests may be identical) such that for every limit ordinal* $\beta \le \tau$ ($\beta \le \gamma$ *respectively*), $\bigcap_{\alpha<\beta} L_\alpha$ ($\bigcap_{\alpha<\beta} R_\alpha$ *respectively*) *has one boundary point* ℓ_β (r_β *respectively*), *where* ℓ_β (r_β *respectively*) *is not the limit of two disjoint closed sequences in* $\bigcap_{\alpha<\beta} L_\alpha$ ($\bigcap_{\alpha<\beta} R_\alpha$ *respectively*), *one of which has uncountable cofinality.*

PROOF. For necessity let $\le$ be an admissible order on X. Let $x \in X$ be nonisolated. If x is in the closure of the open ray $(-\infty,x)$, there is an increasing sequence $\{y_\alpha\}_{\alpha<\tau}$ of isolated points converging to x. In this case for each $\alpha < \tau$ let L_α be the ray (y_α,∞); otherwise let $L_\alpha = [x,\infty)$. If $x \in cl(x,\infty)$ there is a decreasing sequence $\{z_\alpha\}_{\alpha<\gamma}$ of isolated points converging to x. In this case for each $\alpha < \gamma$ let R_α be the ray $(-\infty,z_\alpha)$; otherwise let $R_\alpha = (-\infty,x]$. This clearly satisfies the condition of the theorem.

The sufficiency proof is by induction on the *length* of X, i.e., the least ordinal α such that the α^{th} derived set $X^{(\alpha)}$ is empty. So assume that λ is the length of X and that for all $\alpha < \gamma$ the theorem holds for all compact, scattered spaces of length α.

Since X is compact, λ is a nonlimit ordinal. Then X can be finitely partitioned into clopen sets $\{0_x \mid x \in X^{(\lambda-1)}\}$ such that $0_x \cap X^{(\lambda-1)} = \{x\}$. Thus, if each 0_x satisfies the theorem, so does X. Then without loss of generality let $X^{(\lambda-1)}$ be a singleton $\{q\}$. There is a neighborhood subbase $\{L_\alpha\}_{\alpha<\tau} \cup \{R_\alpha\}_{\alpha<\gamma}$ of q satisfying the condition in the theorem with $L_0 = R_0 = X$.

By the induction hypothesis, for each $\beta < \tau$, there is an admissible order $\le_{L_{\beta+1}}$ on the clopen set $L'_{\beta+1} = L_\beta - L_{\beta+1}$; and if β is a limit ordinal there is an admissible order $\le_{L_\beta}$ on $L'_\beta = \bigcap_{\alpha<\beta} L_\alpha - L_\beta$ whose first point, by the lemma, is its boundary point. Similarity for each $\beta < \gamma$ there is an admissible order $\le_{R_{\beta+1}}$ on the clopen set $R'_{\beta+1} = (R_\beta - R_{\beta+1}) \cap \bigcap_{\alpha<\beta} L_\alpha$ and, for β a limit ordinal, an admissible order $\le_{R_\beta}$ on $R'_\beta = (\bigcap_{\alpha<\beta} R_\alpha - R_\beta) \cap \bigcap_{\alpha<\beta} L_\alpha$ whose last point (by the lemma and inverting the order) is its boundary point, if one exists (iff $r_\beta \in \bigcap_{\alpha<\tau} L_\alpha$).

Define $\le$ on X so that for each $\alpha < \alpha_1 < \tau$ and $\beta < \beta_1 < \gamma$, the order $\le_{L_\alpha}$ (respectively $\le_{R_\beta}$) is identical to $\le$ restricted to L'_α (respectively R'_β), and $L'_\alpha \le L'_{\alpha_1} \le q \le R'_{\beta_1} \le R'_\beta$.

We now show $\le$ is an admissible order on X. The only possible problems are at q and, for each limit ordinal β, at ℓ_β and r_β. So fix β and concentrate on ℓ_β; the proofs for q and r_β are analogous. First note that for each $\alpha < \beta$, if $x_\alpha \in L'_\alpha$ then $\{x_\alpha\}_{\alpha<\beta}$ converges to ℓ_β. To see this, observe that compactness implies that $\{x_\alpha\}_{\alpha<\beta}$ has a cluster point; if $y \in X - L_\alpha$ for some $\alpha < \beta$, then y is not a cluster point of $\{x_\alpha\}_{\alpha<\beta}$ and ℓ_β is the only boundary point of $\bigcap_{\alpha<\beta} L_\alpha$.

So let V be a neighborhood of ℓ_β. Then there is a point b such that $\ell_\beta < b \le$ (the first point of $L'_{\beta+1}$) and $[\ell_\beta,b) \subseteq V$, since $\le_{L_\beta}$ is an admissible order on L'_β. By what was shown in the last paragraph there is an

$\alpha_0 < \beta$ such that $L'_\alpha \subseteq V$ for all α such that $\alpha_0 \le \alpha < \beta$. So for $a \in L_{\alpha_0}$, $(a,b) \subseteq V$.

Finally let $c < \ell_\beta < d$. For some $\alpha < \beta$, $c \in L'_\alpha$. Then $\alpha+1 < \beta$ and $(L_{\alpha+1} - L_\beta) \cap (-\infty,d)$ is a neighborhood of ℓ_β contained in (c,d). So $\le$ is an admissible order on X. □

COROLLARY 1. *A compact, scattered space* X *is orderable if for each non-isolated* $x \in X$ *there is a neighborhood subbase* $\{L_\alpha\}_{\alpha<\tau} \cup \{R_\alpha\}_{\alpha<\beta}$ *consisting of two decreasing nests of clopen sets (these nests may be identical) such that for every limit ordinal* β, $\bigcap_{\alpha<\beta} L_\alpha - L_\beta$ *contains one point whenever* $\beta < \tau$, *and* $\bigcap_{\alpha<\beta} R_\alpha - R_\beta$ *contains one point whenever* $\beta < \gamma$. □

Note that the lexicographic product $(\omega_1 + 1) \times (\omega_1 + 1)^*$ is ordered, compact, and scattered but doesn't satisfy the above condition.

COROLLARY 2. *A compact, scattered space is orderable iff it is the* 2-*to*-1 *continuous image of a compact ordinal space.*

PROOF. If X is ordered, compact, and scattered, consider the projective cover $\pi_X : aX \to X$ (see [5]), where aX is obtained by identifying points $(y,0)$ and $(y,1)$ in the lexicographic product $X \times \{0,1\}$ for those $y \notin cl(-\infty,y) \cap cl(y,\infty)$, and π_X is the canonical map. Then π_X is 2-to-1, and X satisfies Baker's property (D) (see [1]). So by [1], aX is homeomorphic to a compact ordinal space.

Now let $f : \lambda+1 \to X$ be a 2-to-1 map from a compact ordinal space $\lambda+1$. Then X is compact and scattered, being the continuous image of a compact, scattered space. Let $x \in X$. If $f^{-1}(x)$ is a singleton $\{\delta\}$, we show that for all $\xi < \delta$, $f((\xi,\delta])$ is a neighborhood (not necessarily open) of x. So let $\{x_a\}_{a\in A}$ be a net in X converging to x. For each $a \in A$ pick $s_a \in f^{-1}(x_a)$. Since $\lambda+1$ is compact and $\{\delta\} = f^{-1}(x)$, $\{s_a\}_{a\in A} \to \delta$. Then $\{s_a\}_{a\in A}$ is eventually in $(\xi,\delta]$. So $\{x_\alpha\}_{a\in A}$ is eventually in $f((\xi,\delta])$, and $f((\xi,\delta])$ is a neighborhood of x.

Now we construct the desired neighborhood base of x. Since X is 0-dimensional, for all $\xi < \delta$ there is a clopen neighborhood L of x such that (by the compactness of $\lambda+1$ and the fact that $\{\delta\} = f^{-1}(x))f^{-1}(L) \subseteq (\xi,\delta]$. So there is an ordinal τ, a closed increasing sequence $\{\xi_\alpha\}_{\alpha<\tau} \to \delta$, and for all $\alpha < \alpha' < \tau$ a clopen neighborhood L_α of x such that $(\xi_\alpha,\delta] \supset f^{-1}(L_\alpha) \supset (\xi_{\alpha'},\delta]$.

Then $\bigcap_{\alpha<\tau} L_\alpha = \{x\}$, and so by the compactness of X, $\{L_\alpha\}_{\alpha<\tau}$ is a nested neighborhood base of x.

For a limit ordinal $\beta < \tau$ note that $f(\xi_\alpha) \notin L_\alpha$ for each $\alpha < \beta$. But since $\{\xi_\alpha\}_{\alpha<\beta} \to \xi_\beta$, $\{f(\xi_\alpha)\}_{\alpha<\beta} \to f(\xi_\beta)$. So $f(\xi_\beta)$ is a boundary point of $\bigcap_{\alpha<\beta} L_\alpha$. Moreover, $f(\xi_\beta)$ is the only boundary point of $\bigcap_{\alpha<\beta} L_\alpha$, since any net outside of $\bigcap_{\alpha<\beta} L_\alpha$ converging to a point of $\bigcap_{\alpha<\beta} L_\alpha$ must eventually, for each $\alpha < \beta$, be in the open set L_α. Hence a net of inverse images of the members of the given net is eventually in $f^{-1}(L_\alpha) \subset (\xi_\alpha,\delta]$ but not in $[\xi_\beta,\delta]$ and so converges to ξ_β.

If $\{\xi_\beta\} = f^{-1}f(\xi_\beta)$, then as before, $f((\xi,\xi_\beta])$ is a neighborhood of $f(\xi_\beta)$ for all $\xi < \xi_\beta$. But $f((\xi,\xi_\beta]) \cap \bigcap_{\alpha<\beta} L_\alpha = \{f(\xi_\beta)\}$. Therefore, since $f(\xi_\beta)$ is isolated in $\bigcap_{\alpha<\beta} L_\alpha$, it cannot be the limit of any nontrivial sequence in $\bigcap_{\alpha<\beta} L_\alpha$.

If $\{\xi_\beta,\xi\} = f^{-1}f(\xi_\beta)$, then by construction $\xi_\beta < \xi \le \xi_{\beta+1}$. Note that $f^{-1}(\bigcap_{\alpha<\beta} L_\alpha) = \bigcap_{\alpha<\beta} f^{-1}(L_\alpha) = [\xi_\beta,\delta]$, and so $f([\xi_\beta,\delta]) = \bigcap_{\alpha<\beta} L_\alpha$. If $f(\xi_\beta)$ is the limit in $\bigcap_{\alpha<\beta} L_\alpha$ of a closed sequence of uncountable cofinality, then the inverse image of this sequence is closed, uncountable, and unbounded in ξ. So no closed set cofinal in ξ can be disjoint from this set of inverse images. Thus there is no other closed sequence in $\bigcap_{\alpha<\beta} L_\alpha$ converging to $f(\xi_\beta)$ and disjoint from the given sequence.

If $f^{-1}(x) = \{\delta,\delta'\}$, where $\delta < \delta'$, then by slightly modifying the argument above we have that $f((\xi,\lambda])$ and $f([0,\delta] \cup (\xi',\delta'])$ (and in fact $f((\xi,\delta] \cup (\xi',\delta']))$ are neighborhoods of x for all ξ and ξ' such that $\xi < \delta \le \xi' < \delta'$, and there are clopen neighborhoods L and R of x such that $f^{-1}(L) - (\xi,\lambda]$ and $f^{-1}(R) \subseteq [0,\delta] \cup (\xi',\delta']$. So there are ordinals τ and γ, closed, increasing sequences $\{\xi_\alpha\}_{\alpha<\tau} \to \delta$ and $\{\xi'_\alpha\}_{\alpha<\gamma} \to \delta'$ with $\delta < \xi'_0$, and for all $\alpha < \alpha_1 < \tau$ and $\alpha' < \alpha'_1 < \gamma$ clopen neighborhoods L_α and R_α of x such that $(\xi_\alpha,\lambda] \supset f^{-1}(L_\alpha) \supset (\xi_{\alpha_1},\lambda]$, and $[0,\delta] \cup (\xi'_{\alpha'},\delta'] \supset f^{-1}(R_{\alpha'}) \supset [0,\delta] \cup (\xi'_{\alpha'_1},\delta']$. Then, much as before, $\{L_\alpha\}_{\alpha<\tau} \cup \{R_\alpha\}_{\alpha<\tau}$ is a neighborhood subbase of x satisfying the conditions of the theorem. Hence X is orderable. □

<u>THEOREM 2</u>. *A compact, scattered space* X *is orderable iff*

(1) *for each nonisolated* $x \in X$ *there is a neighborhood subbase* $\{L_\alpha\}_{\alpha<\tau} \cup \{R_\alpha\}_{\alpha<\tau}$ *consisting of two decreasing nests of clopen sets (these nests may be identical) such that for every limit ordinal* β, $\bigcap_{\alpha<\beta} L_\alpha$ *has one boundary point whenever* $\beta \le \tau$ *and* $\bigcap_{\alpha<\beta} R_\alpha$ *has one boundary point whenever* $\beta \le \gamma$*; and*

(2) *there is no subset* $Y \cup \{p\}$ *of* X *such that* $Y = \bigcup_{s\in S} X_s$*, where the*

X_s's *are pairwise disjoint,* S *is a stationary set of some uncountable, regular ordinal, a basic neighborhood of* p *is* $\{p\} \cup \bigcup\{X_s \mid s \in S - \alpha\}$ *for some* $\alpha \in S$, *and for each* $s \in S$, X_s *is homeomorphic to* $(\omega_0 + 1) + \alpha_s^*$, *where* α_s *is an uncountable, regular ordinal,* $x_s \in X_s$ *is the point corresponding to* ω_0 *under the homeomorphism, and* $\{x_s\}_{s\in S}$ *is homeomorphic to* S.

PROOF. For necessity we need only consider condition (2). The proof is analogous to the proof of the proposition in 2.1 except that p is substituted for α_0.

For sufficiency let condition (1) hold and let X not be orderable. Then by Theorem 1 there is a nonisolated $p \in X$ and a neighborhood subbase $\{L_\alpha\}_{\alpha<\tau} \cup \{R_\alpha\}_{\alpha<\gamma}$ of p satisfying condition (1) above and such that for some stationary set S of limit ordinals in τ and every point s of S the boundary point ℓ_s of $\bigcap_{\alpha<s} L_\alpha$ is the limit of two disjoint closed sequences, $\{y_\alpha\}_{\alpha<s_1}$ and $\{z_\alpha\}_{\alpha<s_2}$, in $\bigcap_{\alpha<s} L_\alpha - L_s$, where s_2 has uncountable cofinality, for otherwise $\{\ell_\alpha \mid \alpha \le s, \alpha \text{ a limit ordinal}\} \cup \{y_\alpha\}_{\alpha<s_1} \cup \{z_\alpha\}_{\alpha<s_2}$ would be the type of "T-shaped" space clustering at ℓ_s which condition (1) prevents. Then for some countable cofinal subset s_1' of s_1 and cofinal subset s_2' of s_2, $X_s = \{y_\alpha\}_{\alpha\in s_1'} \cup \{\ell_s\} \cup \{z_\alpha\}_{\alpha\in s_2'}$ is homeomorphic to $(\omega_0 + 1) + \alpha_s^*$ where α_s is some uncountable, regular ordinal. Hence $\{p\} \cup (\bigcup_{s\in S} X_s)$ is the required space. So condition 2 does not hold, and sufficiency has been proven. □

Note that in condition (2) of Theorem 2, every element of S that is a cluster point of S must have countable cofinality, since if there exists a cluster point $\alpha \in S$ of uncountable cofinality, then $X_\alpha \cup \{x_s \mid s < \alpha\}$ would be the type of "T-shaped" space clustering at x_α which condition (1) prevents. In condition (2) the equivalence of replacing "S is a stationary subset of a regular, uncountable ordinal κ" by "S is a stationary subset of ω_1" is independent of ZFC. Let S be a stationary subset of κ, each element of which has countable cofinality, such that S contains no subset homeomorphic to a stationary subset of ω_1; such sets are precisely the κ-stationary supersets of E *sets*. The independence of the existence of an E set is discussed in [3] and [2], as was pointed out to the author by F. TALL.

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SUBMETRIZABLE TOPOLOGIES ON THE ORDINALS

G.M. REED

In this paper, the author considers topological spaces in the form $(X,\mathcal{T})$ where $X = \{x_\alpha \mid \alpha < \lambda\}$, $\mathcal{T}_1$ is a given topology on X, $\mathcal{T}_2$ is the order topology on X w.r.t. λ, and $\{U_1 \cap U_2 \mid U_1 \in \mathcal{T}_1 \text{ and } U_2 \in \mathcal{T}_2\}$ is a basis for the topology $\mathcal{T}$. Whereas particular applications of this construction technique have been made previously in the literature (notably by R. POL to construct a perfectly normal, locally metrizable, non-paracompact space and by E. POL and R. POL to construct a hereditarily normal, strongly zero-dimensional space with a subspace of positive dimension), there has evidently been no general study of the technique. The following surprising new applications were recently discovered by the author in the preliminary stages of such a study.

I. Let $\mathcal{C}$ denote the class of spaces such that $(X,\mathcal{T}) \in \mathcal{C}$ iff $X = \{x_\alpha \mid \alpha < \omega_1\} \subseteq \mathbb{R}$ and $\mathcal{T}$ is the intersection topology as above.

(1) If $X \in \mathcal{C}$, then X is regular, submetrizable, pseudo-normal, collectionwise Hausdorff, first countable, locally countable, countably metacompact space that is not subparacompact.

(2) (MA + $\neg$CH) If $X \in \mathcal{C}$, X is perfect.

(3) There exists $X \in \mathcal{C}$ such that X is not normal.

(4) (CH) There exists $X \in \mathcal{C}$ such that X is collectionwise normal.

II. Let (SP^*) denote the statement that there exists a collection $\mathcal{S}$ of almost disjoint, infinite sequences in ω_1 such that

(i) any collection of subsequences which contains a final part from each sequence in $\mathcal{S}$ is not pairwise disjoint and

(ii) for any coloring of $\mathcal{S}$ by two colors, there is a coloring of ω_1 agreeing with each of the sequences in $\mathcal{S}$ on a final part.

(SP^*) is a direct consequence of the set-theoretic principle recently introduced by S. SHELAH to show the consistency w.r.t. (GCH) of a normal, collectionwise Hausdorff Moore space.

(1) (SP^*) is true iff there exists a metric Q-set (i.e. a space which is not σ-discrete but in which each subset is an F_σ-set) of weight $\leq\omega_1$.

(2) It is consistent with (GCH) that there exists a metric Q-set. Recall that the author has previously observed (Fund. Math., to appear) that normal first countable Q-sets do not exist under ($V = L$).

WHEN IS A GO-SPACE JO(2)?

B.M. SCOTT

The most familiar generalizations of orderable spaces are the suborderable spaces. However, another possible direction of generalization was considered by MEYER in [2] and by MEYER, NEUMANN-LARA, and WILSON in [3].

DEFINITION. A space $\langle X,\tau\rangle$ is $JO(\kappa)$ iff τ can be represented as the join of at most κ linear order topologies on the set X, where κ is some cardinal number.

Obviously the orderable spaces are precisely the JO(1) spaces, so the natural question is: How are the classes of suborderable and JO(2) spaces related? It is rather easy to find a JO(2) space that is not suborderable; see Example 1. It is, however, not known whether every suborderable space is JO(2) (or even JO(n) for some $n < \omega$). The best 'universal' result is an observation from [3]: if $\langle X,\leq,\tau\rangle$ is a GO-space (= suborderable space with suborder specified), and κ is the number of pseudogaps of $\langle X,\leq,\tau\rangle$, then $\langle X,\tau\rangle$ is $JO(\kappa)$. All other results are of the form: if $\langle X,\leq,\tau\rangle$ is a GO-space satisfying some additional condition, then $\langle X,\tau\rangle$ is JO(2). Sometimes, indeed, the additional condition is that X be a particular space: in [2], for instance, it is shown that the Sorgenfrey Line is JO(2). Our results also take this form. The first slightly generalizes the main theorem of [3], while the second is completely new.

Let $\langle X,\leq,\tau\rangle$ be a GO-space. Ψ and Γ denote, respectively, the sets of pseudogaps and of gaps of X, and $X^* = X \cup \Psi \cup \Gamma$ ordered by the obvious extension of $\leq$. Ψ^+ is the set of right pseudogaps: $\Psi^+ = \{p \in \Psi : X \cap (\leftarrow,p)$ has a greatest element$\}$. (That is, $p \in \Psi^+$ means that p has an immediate predecessor, x, in X^*, where $x \in X$, $(\leftarrow,x] \in \tau$, and $(\leftarrow,x] \notin \tau_{\leq}$, the order topology induced by $\leq$.) Similarly, Ψ^- is the set of left pseudogaps of X. We write X^*-inf and X^*-sup to emphasize that inf's and sup's are taken in X^*.

Essentially the following lemma can be extracted from the proof of the main theorem in [3].

LEMMA 1. *Suppose that there is a family* $\mathcal{S}$ *of open intervals of* X^* *such that:*

(i) $\langle \mathcal{S}, \supseteq \rangle$ *is a tree of height* $\le \omega$;

(ii) *for each* $S \in \mathcal{S}$, $\{\lambda(S), \rho(S)\} \subseteq \Gamma \cup \Psi$, *where* $S = (\lambda(S), \rho(S))$;

(iii) $\Psi^+ \subseteq \{\lambda(S) : S \in \mathcal{S}\}$;

(iv) $\Psi^- \subseteq \{\rho(S) : S \in \mathcal{S}\}$; *and*

(v) *if* $\mathcal{B}$ *is an infinite branch through* $\mathcal{S}$, *then* $|\cap \mathcal{B}| \le 1$.

Then τ *is the join of two order topologies. (Specifically,* $\tau = \tau_{\le} \vee \tau_{\preccurlyeq}$, *where the order* $\preccurlyeq$ *is defined as follows: if* $x, y \in X$ *with* $x < y$, *then* $x \preccurlyeq y$ *iff* $|\{S \in \mathcal{S} : \{x,y\} \subseteq S\}|$ *is even; otherwise* $y \preccurlyeq x$.) □

THEOREM 1. *Let* $\mathcal{C}$ *be the set of non-degenerate order-components of* X *in* $X \cup \Psi$. *If* $\mathcal{C}$ *and* Ψ *are* σ*-discrete in* $X \cup \Psi$, *then* τ *is the join of* $\tau_{\le}$ *and another order topology.*

(The result in [3] is the same with 'countable' replacing 'σ-discrete', so our improvement is not surprising.)

PROOF. We shall of course simply produce a tree, $\mathcal{S}$, as in Lemma 1. Let $\mathcal{C} = \cup\{\mathcal{C}_n : n \in \omega\}$ and $\Psi = \cup\{\Psi_n : n \in \omega\}$, where each $\mathcal{C}_n$ and Ψ_n is discrete, $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$, and $\Psi_n \subseteq \Psi_{n+1}$. Also, let $\Psi_n^{\pm} = \Psi_n \cap \Psi^{\pm}$. We construct $\mathcal{S}$ recursively as a union of families $\mathcal{S}_n$ $(n \in \omega)$ of open intervals of X^*; at stage n the following conditions are to be met.

$(a)_n$ $\langle \mathcal{T}_n, \supseteq \rangle$ is a tree of height $\le n$, where $\mathcal{T}_n = \cup\{\mathcal{S}_i : i < n\}$.

$(b)_n$ For each $S \in \mathcal{S}_n$, $\{\lambda(S), \rho(S)\} \subseteq \Psi \cup \Gamma$.

$(c)_n$ $\Psi^+ \subseteq \{\lambda(S) : S \in \mathcal{T}_n \cup \mathcal{S}_n \ (= \mathcal{T}_{n+1})\}$.

$(d)_n$ $\Psi_n^- \subseteq \{\rho(S) : S \in \mathcal{T}_{n+1}\}$.

$(e)_n$ If $S \in \mathcal{S}_n$, $C \in \mathcal{C}_n$, and $S \cap C \neq \emptyset$, then $C \subseteq S$, and either

$$\lambda(C) = \lambda(S) \in \Psi_n^+ \setminus \Psi_{n-1}, \text{ or } \rho(C) = \rho(S) \in \Psi_n^- \setminus \Psi_{n-1}$$

(where $\Psi_{-1} = \emptyset$).

Suppose that for some $n \in \omega$ we have already constructed $\mathcal{T}_n = \cup\{\mathcal{S}_i : i < n\}$ in accordance with $(a)_i - (e)_i$, $i < n$. Let $D^+ = \{p \in \Psi_n^+ : \neg \exists\, S \in \mathcal{T}_n\ (p = \lambda(S))\}$, $D^- = \{p \in \Psi_n^- : \neg \exists\, S \in \mathcal{T}_n\ (p = \rho(S))\}$, and $D = D^+ \cup D^-$. For each $p \in D^+$ let $u(p) = X^*\text{-}\inf(\Psi_n \cap (p, \rightarrow))$, and set

$I(p) = (p,u(p))$. For each $p \in D^-$ let $u(p) = X^*\text{-sup}\,[(\Psi_n \cup \{u(q) : q \in D^+\}) \cap (\leftarrow,p)]$, and set $I(p) = (u(p),p)$. Then $F = \{I(p) : p \in D\}$ is a disjoint family of open intervals with clopen traces on X. Moreover, for any $I \in F$ and $C \in \mathcal{C}$ either $I \cap C = \emptyset$, or $C \subseteq I$. We now shrink (some of) the members of F.

Consider a point $p \in D^+$. If $I(p)$ already contains no member of $\mathcal{C}_n$, let $I'(p) = I(p)$. Otherwise, $\mathcal{C}_n$ being discrete, there is a point $q \in (\Psi \cup \Gamma) \cap I(p)$ such that if $C \in \mathcal{C}_n$ and $C \subseteq (p,q)$, then $p = \lambda(C)$; in this case we let $I'(p) = (p,q)$. Similarly, if $p \in D^-$, then either $I(p)$ contains no member of $\mathcal{C}_n$, and we let $I'(p) = I(p)$; or there is a point $q \in (\Psi \cup \Gamma) \cap I(p)$ such that if $C \in \mathcal{C}_n$ and $C \subseteq (q,p)$, then $p = \rho(C)$, and moreover such that if $u(p) \in D^+$, then $I'(u(p)) \cap (q,p) = \emptyset$, in which case we let $I'(p) = (q,p)$.

Let $\mathcal{S}_n = \{I'(p) : p \in D\}$; clearly $(a)_n - (e)_n$ are satisfied, so the construction can proceed to ω.

Finally, let $\mathcal{S} = \bigcup\{\mathcal{S}_n : n \in \omega\}$; clearly (i) - (iv) of Lemma 1 follow at once from (a) - (d) of the inductive hypothesis. To check (v), suppose that $\mathcal{B}$ is an infinite branch through $\mathcal{S}$. If $|\bigcap \mathcal{B}| > 1$, then by (e) there are $n \in \omega$ and $C \in \mathcal{C}_n$ such that $C \subseteq \bigcap\mathcal{B}$. Choose $m \geq n$ so that $\{\lambda(C),\rho(C)\} \cap \Psi \subseteq \Psi_m$; if $k > m$, and $S \in \mathcal{S}_k$, then by $(e)_k$ $C \cap S = \emptyset$. But then $\mathcal{B} \subseteq \mathcal{T}_{m+1}$, which contradicts $(a)_{m+1}$. Thus, $|\bigcap \mathcal{B}| \leq 1$, and $\langle \mathcal{S},\supseteq\rangle$ satisfies (i) - (v) of Lemma 1. □

<u>REMARK</u>. Every disjoint family of non-degenerate, convex open sets in a perfect GO-space is σ-discrete. Thus, if $X \cup \Psi$ is perfect we need only ask that Ψ be σ-discrete.

We now show that even if Theorem 1 is inapplicable, X will be JO(2) if it has enough isolated points. (This result was inspired by the technique used in [2] to show that the Sorgenfrey Line is JO(2).)

Let I be the set of τ-isolated points of X, and let $J = \{x \in I : x$ is adjacent in X^* to two points of $\Psi\}$, $R = \{x \in X \setminus J : x$ is adjacent in X^* to a point of $\Psi^-\}$, and $L = \{x \in X \setminus J : x$ is adjacent in X^* to a point of $\Psi^+\}$.

<u>THEOREM 2</u>. *If there is a* τ*-closed set* $C \subseteq I$ *such that* $\omega \leq |C| = |L \cup R \cup J \setminus C|$, *then* X *is* JO(2).

<u>PROOF</u>. Let $Y = X \setminus C$, an open subspace of X, and let $\kappa = |C|$. We view Y as

a GO-space with order and topology inherited from X. Let $Y_0 = \{y \in Y :$ y is adjacent to a pseudogap of $Y\}$. Fix $y \in Y_0$, and suppose that y is immediately preceded by a pseudogap of Y: $(\leftarrow,y) \cap Y$ has no last element, but $[y,\rightarrow) \cap Y$ is open in Y. Then either y is immediately preceded by a pseudogap of X (i.e., $y \in J \cup R$), or in X, y is immediately preceded by an order-component of C. And $y \notin C$, so $|Y_0| \leq |L \cup R \cup J \setminus C| + |C| = \kappa$, and hence Y has at most κ pseudogaps. Let λ be the number of pseudogaps of Y.

Partition C into three sets, C_0, C_1 and C_2, such that $|C_1| = \lambda$, and C_0 and C_2 are infinite. Let $Z = C_0 \cup Y \cup C_2$, and extend $\leq$ to Z so that:

(1) $C_0 < Y < C_2$;

(2) Z has no endpoints;

(3) C_0 has a last element iff Y has a first element;

(4) C_2 has a first element iff Y has a last element; and

(5) C_0 and C_2 are discretely ordered.

As in [1], let Z^+ be the set of $\langle z,n\rangle \in Z \times \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers) such that one of the following conditions holds:

(α) $z \in Z \setminus Y_0$ and $n = 0$;

(β) $z \in Y_0$ is immediately preceded by a pseudogap of Y and $n \leq 0$; or

(γ) $z \in Y_0$ is immediately followed by a pseudogap of Y and $n \geq 0$.

Let $\preccurlyeq$ be the lexicographic order on Z^+, and let $\trianglelefteq$ be the 'reverse' lexicographic order: $\langle y,n\rangle \trianglelefteq \langle z,m\rangle$ iff either $n < m$, or $n = m$ and $y \leq z$. Let $\tau' = \tau_{\preccurlyeq} \vee \tau_{\trianglelefteq}$; we shall complete the proof by constructing a homeomorphism from $\langle Z^+,\tau'\rangle$ onto $\langle X,\tau\rangle$.

Let $\tilde{Y} = \{\langle y,0\rangle \in Z^+ : y \in Y\}$, $\tilde{C}_0 = \{\langle z,0\rangle \in Z^+ : z \in C_0\}$, $\tilde{C}_2 = \{\langle z,0\rangle \in Z^+ : z \in C_2\}$, and $\tilde{C}_1 = \{\langle z,n\rangle \in Z^+ : n \neq 0\}$. Let $f : \tilde{C}_1 \leftrightarrow C_1$ be any bijection, and define a bijection $h : Z^+ \leftrightarrow X$ by

$$h(\langle z,n\rangle) = \begin{cases} z, & \text{if } n = 0 \\ f(\langle z,n\rangle), & \text{if } n \neq 0. \end{cases}$$

We show that h is a homeomorphism.

Each point of $Z^+ \setminus \tilde{Y}$ is $\tau_{\preccurlyeq}$-isolated and hence τ'-isolated, and each point of $h[Z^+ \setminus \tilde{Y}]$ is τ-isolated. $\tilde{Y}$ is a $\tau_{\trianglelefteq}$-open and hence τ'-open subset of Z^+, and $h[\tilde{Y}]$ is a τ-open subset of X. In the subspace $\tilde{Y}$ of Z^+, every relatively $\tau_{\trianglelefteq}$-open set is $\tau_{\preccurlyeq}$-open, so τ' agrees with $\tau_{\preccurlyeq}$ on $\tilde{Y}$. But by construction $h \restriction \tilde{Y}$ is a homeomorphism of $\tilde{Y}$ (in its relative topology from $\tau_{\preccurlyeq}$ and hence from τ' also) onto Y (as a subspace of $\langle X,\tau\rangle$; putting all the pieces together, we see that h is indeed a homeomorphism. □

MEYER remarks in [3] that he has a very complicated proof that the Michael Line (obtained from the real line by isolating each irrational number) is JO(3), but that he does not know whether it is JO(2). Theorem 2 settles this question.

COROLLARY 1. *The Michael Line,* M, *and the 'Reverse Michael Line',* R *(obtained by isolating each rational number), are* JO(2).

PROOF. In M take C to be any Cantor set contained in the set of irrational numbers; in R, take $C = \mathbb{Z}$. In each case C is a large enough closed set of isolated points. □

The question still remains: Is every suborderable space JO(2)? I doubt it.

CONJECTURE. There is a suborderable space that is not JO(2) (and probably not JO(n) for any $n < \omega$).

In fact, I believe that one of the following spaces (or some simple modification thereof) is such a space, but so far I have been unable to prove it.

1. Let X be a 'very rigid' 2^{ω}-dense subset of the real line. That is, X meets every non-empty open set in $\mathbb{R}$ is a set of power 2^{ω}, and no two bounded intervals of X are order-isomorphic or order-anti-isomorphic unless they are identical. (X can be constructed recursively, one point at a time.) More rigidly can be added if necessary. For instance, I believe that X can be chosen so that if $V,W \in \tau$, the Sorgenfrey topology on X, and $V \neq W$, then there is no 1-1, τ-continuous map $f : V \to W$, though I haven't checked the details.

2. Let B be a Bernstein set in $\mathbb{R}$ such that $\mathbb{R} \setminus B$ is also a Bernstein set. (That is, neither B nor $\mathbb{R} \setminus B$ contains an uncountable, closed subset of $\mathbb{R}$.) Let Y be the Michael-like line formed by isolating each point of B.

My hope for X is based on the intuition that the technique of [2] is in some sense the only way to handle a Sorgenfrey topology on a dense linear order. Y is more questionable; it merely fails to meet the requirements of Theorem 2.

EXAMPLE 1. Let $X = (\omega \times \omega_1) \cup \{\langle\omega,\omega_1\rangle\}$, topologized as follows. Points of $\omega \times \omega_1$ are isolated. For $n \in \omega$ let $H_n = [n,\omega) \times \omega_1$, and for $\alpha \in \omega_1$ let $V_\alpha = \omega \times [\alpha,\omega_1)$. A nbhd base at $p = \langle\omega,\omega_1\rangle$ consists of all sets $B(n,\alpha) = \{p\} \cup (H_n \cap V_\alpha)$, $\langle n,\alpha\rangle \in \omega \times \omega_1$.

To see that X is JO(2), define orders $\leq$ and $\preccurlyeq$ on X as follows. The order $\leq$ is to order each set $\{n\} \times \omega_1$ $(n \in \omega)$ discretely. Moreover, $\langle n,\alpha\rangle < \langle m,\beta\rangle$ whenever $n < m \leq \omega$. The order $\preccurlyeq$ is order each set $\omega \times \{\alpha\}$ $(\alpha \in \omega_1)$ discretely. Moreover, $\langle n,\alpha\rangle \prec \langle m,\beta\rangle$ whenever $\alpha < \beta \leq \omega_1$. If $\tau = \tau_\leq \vee \tau_\prec$, and W is a τ-nbhd of p, then $W \supseteq W_\leq \cap W_\preccurlyeq$ for some $W_\leq \in \tau_\leq$ and $W_\prec \in \tau_\prec$, each a nbhd of p. But then clearly $W \supset B(n,\alpha)$ for some $\langle n,\alpha\rangle \in \omega \times \omega_1$. Conversely, it is clear that each $B(n,\alpha) \in \tau$, so X is JO(2).

Now suppose that $\trianglelefteq$ is a compatible suborder on X. What $\trianglelefteq$ looks like at points of $\omega \times \omega_1$ is immaterial, since they are isolated; the problem is p. Obviously p is not isolated. Moreover, $\langle H_n \cup \{p\} : n \in \omega\rangle$ and $\langle V_\alpha \cup \{p\} : \alpha \in \omega_1\rangle$ are strictly decreasing sequences of nbhds of p, each intersecting down to $\{p\}$, and they have different cofinalities. From this it is easy to deduce that p must be a two-sided limit point in the order $\trianglelefteq$, the limit of an ω-sequence from one side and an ω_1-sequence from the other. It then follows that the topology at p must be the order topology at p induced by $\trianglelefteq$, so that in fact there must be sequences $\langle x_n : n \in \omega\rangle$ and $\langle y_\alpha : \alpha \in \omega_1\rangle$ converging to p *in* X. Such sequences, however, are easily seen not to exist, so X is not suborderable. □

(Example 1 is also an example of a globular space of glob-dimension 2 that is not suborderable; see [4]. An example of this kind is referenced in [3].)

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LOCAL BASES AND PRODUCT PARTIAL ORDER II

B.M. SCOTT

This note is a continuation of [5], and all conventions, notation, and terminology from [5] will be assumed henceforth. Moreover, numbering of section, theorems, etc., will be continued in sequence, and results from [5] will be referred to simply by number.

5. REGULAR GLOBS.

We have not thus far said anything about the possible effect of separation properties on the structure of globs. In this section we show that if X is T_3 and globular, its globs can be chosen to demonstrate the regularity of X in a very nice way. As a by-product we obtain a result (5.2) about partial orders which may be of some independent interest. As usual, however, the only surprising feature of these results is the labor required to obtain them.

Recall that a partial order $\langle P,\leq\rangle$ is well-founded iff it contains no infinite, descending chain. If P is well-founded there is a natural decomposition of P into levels: define an ordinal-valued function ℓ on P recursively by $\ell(p) = \sup\{\ell(q) + 1 : q < p\}$, and for any ordinal α define the α-th level of P to be $P_\alpha = \{p \in P : \ell(p) = \alpha\}$. Clearly each level of P is an antichain; i.e., if $p,q \in P_\alpha$, and $p \neq q$, then $p \nleq q$ and $q \nleq p$. Thus, by well-ordering each level of P (independently) we can define an order-preserving injection $\phi : P \to \alpha$ for some ordinal α. (Note: ϕ is *not* in general an order-isomorphism.) Conversely, if such an injection ϕ exists, it is clear that P is well-founded.

<u>PROPOSITION 5.0</u>. *A partial order $\langle P,\leq\rangle$ is well-founded iff there is an order-preserving injection $\phi : P \to \alpha$ for some ordinal α.* □

We shall call a partial order $\langle P,\leq\rangle$ *uniform* if: (i) P is well-founded;

and (ii) for any cofinal set $C \subseteq P$ there is an isomorphism $\phi : P \to C$ such that ran ϕ is cofinal in C. (That is, every cofinal subset of P contains cofinally a copy of P.) Any regular cardinal $\kappa \geq \omega$ (with the usual order) is obviously uniform.

THEOREM 5.1. *Let* $\langle P,\leq\rangle$ *be a uniform partial order. If* $\kappa \geq \omega$ *is a regular cardinal, and* $\kappa > |P|$, *then the product partial order* $P \times \kappa$ *is uniform.*

PROOF. Certainly $P \times \kappa$ is well-founded. Suppose that C is a cofinal subset of $P \times \kappa$, and let $K = \{p \in P : |\{\xi \in \kappa : \langle p,\xi\rangle \in C\}| = \kappa\}$; plainly K must be cofinal in P. (We use here the assumption that $\kappa > |P|$.) For each $p \in K$ let $\Gamma_p = \{\xi \in \kappa : \langle p,\xi\rangle \in C\}$.

Now fix an order-preserving bijection $\phi : P \to \alpha$ for some ordinal $\alpha < \kappa$, let h be an isomorphism of P into K, and let Q = ran h. Let $\trianglelefteq$ be the reverse lexicographic order on $\alpha \times \kappa$: for $\langle\xi,\rho\rangle,\langle\zeta,\sigma\rangle \in \alpha \times \kappa$, $\langle\xi,\rho\rangle \trianglelefteq \langle\zeta,\sigma\rangle$ iff either $\rho < \sigma$, or $\rho = \sigma$ and $\xi \leq \zeta$. Clearly $\trianglelefteq$ well-orders $\alpha \times \kappa$ in type κ (since $\alpha < \kappa$). Thus, by recursion along $\trianglelefteq$ we may define a function $f : \alpha \times \kappa \to \kappa$ such that:

(1) if $\langle\xi,\rho\rangle \trianglelefteq \langle\zeta,\sigma\rangle$, then $f(\langle\xi,\rho\rangle) \leq f(\langle\zeta,\sigma\rangle)$ (and conversely);

(2) f is 1-1; and

(3) for each $\langle\xi,\rho\rangle \in \alpha \times \kappa$, $f(\langle\xi,\rho\rangle) \in \Gamma_{h(\phi^{-1}(\xi))}$.

Finally, let $e : P \times \kappa \to C$: $\langle p,\rho\rangle \mapsto \langle h(p),f(\langle\phi(p),\rho\rangle)\rangle$. (That ran $e \subseteq C$ follows from the observation that $f(\langle\phi(p),\rho\rangle) \in \Gamma_{h(\phi^{-1}(\phi(p)))} = \Gamma_{h(p)}$.) Since f is 1-1, so is e. Suppose $\langle p,\rho\rangle \leq \langle q,\sigma\rangle$ in $P \times \kappa$. Then $p \leq q$, so $h(p) \leq h(q)$ and $\phi(p) \leq \phi(q)$. And $\rho \leq \sigma$, so $\langle\phi(p),\rho\rangle \trianglelefteq \langle\phi(q),\sigma\rangle$, whence $f(\langle\phi(p),\rho\rangle) \leq f(\langle\phi(q),\sigma\rangle)$. Thus, $e(\langle p,\rho\rangle) \leq e(\langle q,\sigma\rangle)$, and e is order-preserving. Conversely, $e(\langle p,\rho\rangle) \leq e(\langle q,\sigma\rangle)$ implies that $h(p) \leq h(q)$ (whence also $p \leq q$) and $f(\langle\phi(p),\rho\rangle) \leq f(\langle\phi(q),\sigma\rangle)$, so that $\langle\phi(p),\rho\rangle \trianglelefteq \langle\phi(q),\sigma\rangle$, $\rho \leq \sigma$, and, finally, $\langle p,\rho\rangle \leq \langle q,\sigma\rangle$. Thus, e is an isomorphism. And for any $\langle p,\rho\rangle \in C$ there are a $q \in P$ and a $\sigma \in \kappa$ such that $h(q) \geq p$ and $f(\langle\phi(q),\sigma\rangle) \geq \rho$ (so that $e(\langle q,\sigma\rangle) \geq \langle p,\rho\rangle$), showing that ran e is cofinal in $P \times \kappa$. □

COROLLARY 5.2. *Let* Ω *be a finite set of distinct regular infinite cardinals. Then the partial order* $\langle\Pi\Omega,\leq\rangle$ *is uniform.*

REMARK 5.3. By using a result of Pouzet [4] (see also [3]) we can give a short proof of 5.2, but Pouzet's result is deeper (and harder) than 5.1.

He shows that if $\langle P,\leq\rangle$ is a directed set with no infinite antichain, then P has a cofinal set of the form $\lambda_0 \times \lambda_1 \times \ldots \times \lambda_{m-1}$ where $\lambda_0 < \lambda_1 < \ldots < \lambda_{m-1}$ and each λ_i is a regular cardinal. Let $P = \Pi\Omega$, where $\Omega = \{\kappa_i : i < n\}$ is as in 5.2, and $\kappa_0 < \ldots < \kappa_{n-1}$. If $C \subseteq P$ is cofinal, $\langle C,\leq\rangle$ satisfies the hypothesis of Pouzet's theorem, and C contains a cofinal copy, Q, of $\Pi\Omega'$, where $\Omega' = \{\lambda_i : i < m\}$. Now let the space X, the point p, and the Ω-glob $\mathcal{B} = \{B(\bar{\alpha}) : \bar{\alpha} \in P\}$ be as in Example 4.2. Then $\{B(\bar{\alpha}) \in \mathcal{B} : \bar{\alpha} \in Q\}$ is an Ω'-glob at p, so by the Uniqueness Theorem (2.2) $\Omega' = \Omega$.

If P and Ω are as in 5.3, and $\bar{\alpha},\bar{\beta} \in P$, we write $\bar{\alpha} \ll \bar{\beta}$ iff $\alpha_i < \beta_i$ for each $i < n$.

<u>LEMMA 5.4</u>. *Let* P *and* Ω *be as above. Then there is a isomorphism* $\phi : P \to P$ *such that*: (1) ran ϕ *is cofinal in* P; *and* (2) *for any* $\bar{\alpha},\bar{\beta} \in P$, $\bar{\alpha} < \bar{\beta}$ *iff* $\phi(\bar{\alpha}) < \phi(\bar{\beta})$ *iff* $\phi(\bar{\alpha}) \ll \phi(\beta)$.

<u>PROOF</u>. Define ϕ by setting $\phi(\bar{\alpha})_i = \Sigma_{j\leq i}\,(\kappa_{i-j-1}\cdot\alpha_{i-j})$, where all arithmetic is ordinal arithmetic, and $\kappa_{-1} = 1$. Written out, $\phi(\bar{\alpha}) = \langle\alpha_0, \kappa_0\cdot\alpha_1 + \alpha_0, \kappa_1\cdot\alpha_2 + \kappa_0\cdot\alpha_1 + \alpha_0, \ldots\rangle$. Verification that ϕ has the desired properties is routine. □

Recall that if X is a space, and $p \in X$, we say that X is *regular at* p iff p has a local base of closed nbhds.

<u>THEOREM 5.5</u>. *Let* X *be regular at* p, *and suppose that* $\gamma\chi(p,X) = \Omega = \{\kappa_i : i < n\}$, *where* $\kappa_0 < \ldots < \kappa_{n-1}$. *Let* $P = \Pi\Omega$. *Then there is an* Ω*-glob* $\mathcal{G} = \{G(\bar{\alpha}) : \bar{\alpha} \in P\}$ *at* p *such that* $G(\bar{\alpha}) \supseteq \mathrm{cl}\, G(\bar{\beta})$ *whenever*, $\bar{\alpha},\bar{\beta} \in P$ *with* $\bar{\alpha} < \bar{\beta}$.

<u>PROOF</u>. Let $\mathcal{B} = \{B(\bar{\alpha}) : \bar{\alpha} \in P\}$ be any Ω-glob at p. For each $i < n$ let $P_i^- = \Pi\{\kappa_j : j < i\}$ and $P_i^+ = \Pi\{\kappa_j : i < j < n\}$. ($\Pi\emptyset = 1$, as usual.) Fix $i < n$.

For each $\xi \in \kappa_i$ there are $\bar{\alpha}(i,\xi) \in P_i^-$, $\eta(i,\xi) \in \kappa_i$, and $\bar{\beta}(i,\xi) \in P_i^+$ such that $\mathrm{cl}\, B(\bar{\alpha}(i,\xi)^\frown\eta(i,\xi)^\frown\bar{\beta}(i,\xi)) \subseteq B(\bar{0}[i \to \xi])$. And $|P_i^-| < \kappa_i < |P_i^+|$ (with trivial modifications in case $i = 0$ or $i = n-1$), so there are $\bar{\alpha}(i) \in P_i^-$, $\bar{\beta}(i) \in P_i^+$, and a cofinal $\Gamma_i \subseteq \kappa_i$ such that $\bar{\alpha}(i) = \bar{\alpha}(i,\xi)$ for each $\xi \in \Gamma_i$ and $\bar{\beta}(i) \geq \bar{\beta}(i,\xi)$ for each $\xi \in \kappa_i$. Thus, $\mathrm{cl}\, B(\bar{\alpha}(i)^\frown\eta(i,\xi)^\frown\bar{\beta}(i)) \subseteq B(0[i \to \xi])$ for each $\xi \in \Gamma_i$, and it follows that

$$(*) \qquad \bigcap \{\mathrm{cl}\ B(\bar\alpha(i)^\frown 0^\frown \bar\beta(i)[i \to \xi]) : \xi \in \kappa_i\} \subseteq \bigcap \{B(\bar 0[i \to \xi]) : \xi \in \kappa_i\} = E^{\mathcal{B}}_{\{i\}}(\bar 0).$$

Let $\bar\gamma \in P$ be the least upper bound of $\{\bar\alpha(i)^\frown 0^\frown \bar\beta(i) : i < n\}$, and let $Q = \{\bar\alpha \in P : \bar\alpha \geq \bar\gamma\}$. Let $\phi : P \to Q$ be the obvious surjective isomorphism, and for $\bar\alpha \in P$ let $C(\bar\alpha) = \mathrm{cl}\ B(\phi(\bar\alpha))$. Let $\mathcal{C} = \{C(\bar\alpha) : \bar\alpha \in P\}$; by (*), $E^{\mathcal{C}}_{\{i\}}(\bar 0) \subseteq E^{\mathcal{B}}_{\{i\}}(\bar 0)$ for each $i < n$, so $p \notin \mathrm{int}\ E^{\mathcal{C}}_{\{i\}}(\bar 0)$ for any $i < n$. It follows that $\mathcal{C}$ is a strict weak nbhd Ω-glob at p. By the Equivalence Theorem (2.0), therefore, $\mathcal{C}$ contains a nbhd Ω-glob $\mathcal{D} = \{D(\bar\alpha) : \bar\alpha \in P\}$. Let $\mathcal{V} = \{V(\bar\alpha) : \bar\alpha \in P\}$, where $V(\bar\alpha) = \mathrm{int}\ D(\bar\alpha)$; $\mathcal{D}$ consists of regular closed sets, and $\mathcal{V}$ consists of their regular open interiors, so $\mathcal{V}$ is an Ω-glob at p. (For $V(\bar\alpha) \subseteq B(\bar\beta)$ iff $D(\bar\alpha) \subseteq D(\bar\beta)$ iff $\bar\alpha \leq \bar\beta$.) By the Theorem of Cofinal Similarity (3.4) there are cofinal sets $K_i \subseteq \kappa_i$ $(i < n)$ such that if $P^* = \Pi\{K_i : i < n\}$, $\bar\alpha, \bar\beta \in P^*$, and $\bar\alpha \ll \bar\beta$, then $V(\bar\alpha) \supseteq D(\bar\beta)$.

Thus, p has an Ω-glob $\mathcal{W} = \{W(\bar\alpha) : \bar\alpha \in P\}$ $(= \{V(\bar\alpha) : \bar\alpha \in P^*\})$ such that for any $\bar\alpha, \bar\beta \in P$ with $\bar\alpha \ll \bar\beta$, $W(\bar\alpha) \supseteq \mathrm{cl}\ W(\bar\beta)$. Let $\psi : P \to P$ be the function whose existence is guaranteed by Lemma 5.4. For each $\bar\alpha \in P$ let $G(\bar\alpha) = W(\psi(\bar\alpha))$, and let $\mathcal{G} = \{G(\bar\alpha) : \bar\alpha \in P\}$. Clearly $\mathcal{G}$ has the desired properties. □

6. AN INTERESTING CLASS OF GLOBULAR SPACES

Let X be a space and $\mathcal{B}$ a base for X. Recall that $\mathcal{B}$ is of *subinfinite rank* [2] iff for each $p \in X$ the partial order $\langle \mathcal{B}_p, \supseteq \rangle$ has no infinite antichain, where $\mathcal{B}_p = \{B \in \mathcal{B} : p \in B\}$. Spaces with bases of subinfinite rank are very nice: FORSTER and GRABNER [1] have shown that they are hereditarily metacompact. They are also globular.

To see this, consider the local base $\mathcal{B}_p$ at p: $\langle \mathcal{B}_p, \supseteq \rangle$ is obviously a directed set with no infinite antichain, so the result of Pouzet menmentioned in 5.3 applies, and we conclude that $\langle \mathcal{B}_p, \supseteq \rangle$ contains a cofinal glob.

<u>THEOREM 6.0</u>. *If the space* X *has a base of subinfinite rank, then* X *is globular.* □

The converse is of course false: the LOTS ω_1 is certainly globular - even an lob-space - but, not being even meta-Lindelöf, it cannot bave a base of subinfinite rank. A more reasonable question is whether a hereditarily metacompact globular space must have a base of subinfinite rank. The following example shows that the answer is emphatically 'no'.

EXAMPLE 6.1. *A first-countable, hereditarily Lindelöf, hereditarily separable, 0-dimensional Tikhonov space with no base of subinfinite rank.* Let $Q = \{\langle p,q\rangle \in \mathbb{R}^2 : q > 0$ and p and q are rational$\}$ and $Y = \{\langle x,0\rangle \in \mathbb{R}^2 :$ x is irrational$\}$. Our example has as underlying set $X = Q \cup Y$. Points of Q are isolated. Basic nbhds of a point $\langle x,0\rangle \in Y$ are bowties: given $\varepsilon > 0$ we define $B(x,\varepsilon) = \{\langle y,p\rangle \in X : 0 \le p \le |y-x| < \varepsilon\}$. X is obviously first-countable, 0-dimensional, and Tikhonov. And since Q is countable, while Y is homeomorphic to the space of irrationals, X is hereditarily separable and hereditarily Lindelöf.

Now suppose that $\mathcal{B}$ is a base for X. For each irrational x there are a $B_x \in \mathcal{B}$ and an $n_x \in \omega$ such that $B(x,2^{-n_x}) \subseteq B_x \subseteq B(x,1)$. By the Baire Category Theorem there are an $n \in \omega$ and an interval J of irrationals such that $D = \{x \in J : n_x = n\}$ is dense in J. Fix $p \in J$; then $\{B_x : x \in D$ and $|p-x| < 2^{-n}\}$ is an infinite antichain in $\mathcal{B}_{\langle p,0\rangle}$, so $\mathcal{B}$ cannot have subinfinite rank. □

7. SOME REMARKS ON QUESTION 4.8.

This section is essentially an attempt to stimulate interest in the main unresolved structural problem about globs. As usual we have a space X, a point $p \in X$ with $\gamma\chi(p,X) = \Omega = \{\kappa_i : i < n\}$, where $\kappa_0 < \ldots < \kappa_{n-1}$, and an Ω-glob $\mathcal{B} = \{B(\bar{\alpha}) : \bar{\alpha} \in P\}$ at p, where $P = \Pi\Omega$. Let $K = K(p,X)$, the type of p. Recall that $\mathcal{B}$ is said to be ectomorphic if there are families $A_i = \{A_i(\alpha) : \alpha \in \kappa_i\}$ (for $i < n$) such that $B(\bar{\alpha}) = \bigcup\{A_i(\alpha_i) : i < n\}$ for each $\bar{\alpha} \in P$. It is very easy to show that if $\mathcal{B}$ is ectomorphic, then $K = [n]^1$, and if $n = 2$ it can be shown (using results and techniques from Section 3) that $\mathcal{B}$ is ectomorphic iff $K = [n]^1$. The question is whether $K = [n]^1$ implies that $\mathcal{B}$ is ectomorphic even when $n > 2$. Though I am still unable to answer the question, I can prove some structural results, moderately interesting in themselves, that seem to point the way toward a solution.

For the moment we assume only that $n \notin K$, not that $K = [n]^1$. Let $K^* = \{I \in K : I$ is $\subseteq$-maximal in $K\}$. For any $I \in P^*(n)$ and $\bar{\alpha} \in P$ let $F_I(\bar{\alpha}) = \bigcap\{E_{\{i\}}(\bar{\alpha}) : i \in n \setminus I\}$. (Considering the prototypical Example 4.2, we might think of the $F_I(\bar{\alpha})$ as lying *in* the 'I-face' of $\mathcal{B}$, while the

$E_I(\bar{\alpha})$ lie, so to speak, around its edge.)

LEMMA 7.0. *For any* $I \in K^*$ *and* $\bar{\alpha} \in P$, $p \in \mathrm{int}(E_I(\bar{\alpha}) \cup F_I(\bar{\alpha}))$.

PROOF. Fix $I \in K^*$ and $\bar{\alpha} \in P$. For each $i \in n \setminus I$ there is a $\bar{\beta}^i \in P$ such that $\bar{\beta}^i \geq \bar{\alpha}$ and $B(\bar{\beta}^i) \subseteq E_I(\bar{\alpha}) \cup E_{\{i\}}(\bar{\alpha})$ (since $I \cup \{i\} \notin K$). Let $\bar{\beta} = \max\{\bar{\beta}^i : i \in n \setminus I\}$. Then $\bar{\beta} \geq \bar{\alpha}$, and $B(\bar{\beta}) \subseteq E_I(\alpha) \cup F_I(\bar{\alpha})$. □

LEMMA 7.1. *Let* $I, J \in K^*$ *with* $I \neq J$. *Then for any* $\bar{\alpha} \in P$, $F_I(\bar{\alpha}) \subseteq E_J(\bar{\alpha})$.

PROOF. Since I and J are maximal and distinct, we can pick a $j \in J \setminus I$, and clearly $F_I(\bar{\alpha}) \subseteq E_{\{j\}}(\bar{\alpha}) \subseteq E_J(\bar{\alpha})$. □

LEMMA 7.2. *If* $\emptyset \neq A \subseteq K^*$, *then* $\bigcap\{E_I(\bar{\alpha}) \cup F_I(\bar{\alpha}) : I \in A\} = \bigcup\{F_I(\bar{\alpha}) : I \in A\} \cup \bigcap\{E_I(\bar{\alpha}) : I \in A\}$ *for any* $\bar{\alpha} \in P$.

PROOF. Fix $\bar{\alpha} \in P$. We prove the lemma by induction on $|A|$. Assume that $|A| > 1$ (and that the result has been proved for smaller families), fix $J \in A$, and let $L = A \setminus \{J\}$. Then:

$$\begin{aligned}
\bigcap_{I\in A} (E_I(\bar{\alpha}) \cup F_I(\bar{\alpha})) &= (E_J(\bar{\alpha}) \cup F_J(\bar{\alpha})) \cap \bigcap_{I\in L} (E_I(\bar{\alpha}) \cup F_J(\bar{\alpha})) \\
&= (E_J(\bar{\alpha}) \cup F_J(\bar{\alpha})) \cap \left(\bigcup_{I\in L} F_I(\bar{\alpha}) \cap \bigcap_{I\in L} E_I(\bar{\alpha})\right) \\
&= (E_J(\bar{\alpha}) \cap \bigcup_{I\in L} F_I(\bar{\alpha})) \cup (E_J(\bar{\alpha}) \cap \bigcap_{I\in L} E_I(\bar{\alpha})) \cup \\
&\qquad \cup (F_J(\bar{\alpha}) \cap \bigcup_{I\in L} F_I(\bar{\alpha})) \cup (F_J(\bar{\alpha}) \cap \bigcap_{I\in L} E_I(\bar{\alpha})) \\
&= \bigcup_{I\in L} (E_J(\bar{\alpha}) \cap F_I(\bar{\alpha})) \cup \bigcap_{I\in A} E_I(\bar{\alpha}) \cup \\
&\qquad \cup \bigcup_{I\in L} (F_J(\bar{\alpha}) \cap F_I(\bar{\alpha})) \cup \bigcap_{I\in L} F_J(\bar{\alpha}) \cap E_I(\bar{\alpha})) \\
&= \bigcup_{I\in L} F_I(\bar{\alpha}) \cup \bigcap_{I\in A} E_I(\bar{\alpha}) \cup \bigcup_{I\in L} F_{J\cap I}(\bar{\alpha}) \cup \bigcap_{I\in L} F_J(\bar{\alpha}) \\
&= \bigcup_{I\in A} F_I(\bar{\alpha}) \cup \bigcap_{I\in A} E_I(\bar{\alpha}),
\end{aligned}$$

since $F_{J\cap I}(\bar{\alpha}) \subseteq F_I(\bar{\alpha})$. □

COROLLARY 7.3. *For each* $\bar{\alpha} \in P$ *there is a* $\bar{\beta} \in P$ *such that:*

(i) $\bar{\beta} \geq \bar{\alpha}$;

(ii) $B(\bar{\beta}) \subseteq \bigcup_{I\in K^*} F_I(\bar{\alpha}) \cup \bigcap_{I\in K^*} E_I(\bar{\alpha})$; *and*

(iii) $\forall\, I \in P^*(n) \setminus K (B(\bar{\beta}) \subseteq E_I(\bar{\alpha}))$.

PROOF. By the definition of K, (iii) is true for all sufficiently large $\bar{\beta}$, as of course is (i). And since $P^*(n) \setminus K$ is finite, Lemmas 7.0 and 7.2 imply that (ii) is also satisfied for all sufficiently large $\bar{\beta}$. □

LEMMA 7.4. *For any* $\bar{\alpha} \in P$, $p \in \operatorname{int} \bigcup_{I \in K^*} F_I(\bar{\alpha})$.

PROOF. Fix $\bar{\alpha} \in P$, and find $\bar{\beta}$ as in Corollary 7.3. Fix $x \in B(\bar{\beta}) \cap \bigcap\{E_I(\bar{\alpha}) : I \in K^*\}$, and let $A = \{i < n : x \in E_{\{i\}}(\bar{\alpha})\}$, so that $x \notin E_{n\setminus A}(\bar{\alpha})$.

If $n \setminus A \notin K$, then by (iii) $x \in B(\bar{\beta}) \subseteq E_{n\setminus A}(\bar{\alpha})$, which is absurd, so $n \setminus A \in K$, and there is a $J \in K^*$ such that $n \setminus A \subseteq J$. But then $x \in F_{n\setminus A}(\bar{\alpha}) \subseteq F_J(\bar{\alpha})$, so

$$(*) \qquad B(\bar{\beta}) = B(\bar{\beta}) \cap \left(\bigcup_{I\in K^*} F_I(\bar{\alpha}) \cup \bigcap_{I\in K^*} E_I(\bar{\alpha})\right) = B(\bar{\beta}) \cap \bigcup_{I\in K^*} F_I(\bar{\alpha}),$$

whence $B(\bar{\beta}) \subseteq \bigcup_{I\in K^*} F_I(\bar{\alpha})$, and the result follows immediately.

(A minor technical difficulty arises if $A = n$, as in that case (iii) of 7.3 does not apply. But if $A = n$, then $x \in \bigcap\{E_{\{i\}}(\bar{\alpha}) : i < n\} \subseteq F_I(\bar{\alpha})$ for *any* $I \in K^*$, so the computation (*) is still valid.) □

For each $\bar{\alpha} \in P$ define $F(\bar{\alpha}) = \bigcup\{F_I(\bar{\alpha}) : I \in K^*\}$, and let $F = \{F(\bar{\alpha}) : \bar{\alpha} \in P\}$. (Note that as usual the notation suppresses the dependence of F on B.)

THEOREM 7.5. *F is a strict weak Ω-glob at* p.

PROOF. Only the strictness of F might be in question, and it follows from the observation that $F(\bar{\alpha}) \subseteq B(\bar{\alpha})$ for each $\bar{\alpha} \in P$. □

Now suppose that $K = [n]^1 = K^*$, so that $F(\bar{\alpha}) = \bigcup\{F_{\{i\}}(\bar{\alpha}) : i < n\}$. Here $F_{\{i\}}(\bar{\alpha})$ seems to play something like the role of $A_i(\alpha_i)$ in the definition of ectomorphism. Unfortunately, there are still a couple of difficulties in the way of getting an ectomorphic Ω-glob at p. The crucial problem is that $F_{\{i\}}(\bar{\alpha})$ does not depend only on i and α_i: the other components of $\bar{\alpha}$ affect it also. This phenomenon can be seen for instance, in Example 4.10, which is in fact extomorphic: take $A_i(\xi)$ to be $X \setminus \{\bar{\alpha} \in P : \alpha_i < \xi\}$. But then $A_i(\xi)$ turns out to be $\bigcap\{F_{\{i\}}(\bar{\alpha}) : \alpha_i = \xi\}$, which suggests defining $A_i(\xi) = \bigcap\{F_{\{i\}}(\bar{\alpha}) : \alpha_i = \xi\}$ in general and then trying to prove at least that $\bigcup\{A_i(\alpha_i) : i < n\}$ is a nbhd of p for each $\bar{\alpha} \in P$ – and this I have been unable to do. (Of course, this is the same as setting $A_i(\xi) = \cap\{B(\bar{\alpha}) : \alpha_i = \xi\}$; perhaps it is encouraging that the two most obvious ways to define the $A_i(\xi)$'s are in fact the same.)

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MAPPINGS OF FINITE OSCILLATION AT LOCAL SEPARATING POINTS

L.B. TREYBIG

Some of the motivation for this paper can be seen in a simple example. Let G denote the upper semi-continuous decomposition of [0,1] whose only nondegenerate element is the interval $[\frac{1}{3},\frac{2}{3}]$. Let $g : [0,1] \to [0,1]/G$ be a continuous onto map, where each $g^{-1}(x)$ is totally disconnected. Now if $g^{-1}([\frac{1}{3},\frac{2}{3}])$ is finite, for example, and $t_0 \in g^{-1}([\frac{1}{3},\frac{2}{3}])$, then if g(t) moves across $[\frac{1}{3},\frac{2}{3}]$ in $[0,1]/G$ as t moves across t_0, then we could replace t_0 by a copy of [0,1] and define a path that takes a trip across the longer bridge $[\frac{1}{3},\frac{2}{3}]$ in [0,1]. We could then use a finite set of such replacements and the map g in order to define a continuous map $f : A \to [0,1]$, where A is an arc.

This is an essential problem that arose in the author's work [7] on a problem of Mardešić and Papić [5]: Is every locally connected continuum which is the continuous image of a compact ordered space the continuous image of an ordered continuum? In the actual application $[0,1]/G$ is replaced by a Peano continuum M/G_2, where G_2 may have uncountably many nondegenerate elements and M is a locally connected (Hausdorff) continuum.

If M is a locally connected continuum, then a point P of M will be called a local separating point of M provided there is a connected open set U containing P such that $U - P = R \cup S$ mutually separated (see also [3]). If in addition, R or S is not connected, then P will be called a multiple local separating point of M. If $f : [0,1] \to M$ is a continuous onto map and the point P of M is a local separating point, then f will be said to be of finite oscillation at P provided that if U, R, and S are as above, then there is a finite collection G of open intervals (half open at 0 or 1) covering $f^{-1}(R \cup S)$ so that no interval of G intersects both $f^{-1}(R)$ and $f^{-1}(S)$. In general, f will be said to be of finite oscillation at local separating points if f is of finite oscillation at each local separating point. The main result of this paper is

THEOREM 3. *If* M *is a locally connected metric continuum such that no point separates* M, *then there is a continuous onto map* $f : [0,1] \to M$ *such that* f *is of finite oscillation at local separating points.*

EXAMPLE. If the hypothesis that no point separates the space is omitted, then in the plane the sum of the intervals $[(0,0),(\frac{1}{n},\frac{1}{n^2})]$ $(n = 1,2,3,\ldots)$ does not satisfy the conclusion of the theorem.

In Theorem 4 we show that under certain conditions Theorem 3 may be strengthened to yield a mapping that is also strongly irreducible. In [3] HARROLD proves a stronger result for a mapping that is only required to be strongly irreducible. [A continuous onto map $f : X \to Y$ is said to be strongly irreducible if f maps no closed proper subset of X onto Y.]

A continuum is a compact connected Hausdorff space. A tree or dendrite is a metric continuum in which each pair of points can be separated by a point. A finite tree is a tree with only finitely many endpoints [6]. If d is a metric on M, $P \in M$, $r > 0$, and $A \subset M$, then $N(P,r) = \{Q : Q \in M$ and $d(P,Q) < r\}$ and diam. $A = \text{l.u.b.}\ \{d(x,y) : x,y \in A\}$. Following WARD [8], we say that a metric continuum M can be approximated by a sequence of finite dendrites if there exists a sequence $D_1, D_2, \ldots$ of finite dendrites such that

(1) $D_1 \subset D_2 \subset D_3 \subset \ldots \subset M$,

(2) $\cup\{D_n,\ n = 1,2,\ldots\}$ is dense in M, and

(3) if C is a component of $D_{n+1} - D_n$, then diam. $C < 2^{-n}$.

Other definitions and theorems related to this paper may be found in papers by BING [1], CAPEL [2], HAROLD [3], WARD [8], [9], and books by HOCKING and YOUNG [4], MOORE [6], and WHYBURN [10].

THEOREM 1. *If* M *is a locally connected metric continuum, and* G *is the set of all multiple local separating points of* M, *then* G *is countable.*

PROOF. (See Theorem 9.2 of Chapter III of WHYBURN [10]) □

THEOREM 2. *If* M *is a locally connected metric continuum such that no point separates* M, *then* M *can be approximated by a sequence of finite dendrites* $D_1, D_2, D_3, \ldots$ *such that if* $P \in \cup_1^\infty D_i$, *then there exist at most finitely many integers* i *such that* P *is the boundary point of a component* C *of* $D_{i+1} - D_i$.

PROOF. We note that if P is a point of a connected open set U, then

$U' = U - P$ has at most finitely many components. For if not, let V be an open set so that $P \in V \in \overline{V} \subset U$. At most finitely many components of U' intersect Bd.V, and any component of U' not intersecting Bd.V is a subset of V, and is thus a component of $M - P$.

As in [8] there is a sequence $S_1^1, S_2^1, \ldots$ of partitions of M such that (*) for each n, S_n^1 is a 2^{-1} partition, S_{n+1}^1 refines S_n^1, and each member of S_n^1 has property S. Thus, by the first paragraph, if $\{P_1, P_2, \ldots\}$ denotes the set of multiple local separating points of M, then the sequence $S_1, S_2, \ldots$ defined so that each S_n is the set of components of $\cup S_n^1 - \{P_1, P_2, \ldots, P_n\}$, $n = 1,2,3,\ldots$, satisfies (*).

LEMMA 2A. *Suppose* D *is a finite dendrite,* G *and* G' *are elements of* S_n, W *is a connected open set containing a point of* $Bd.G \cap Bd.G'$ (*respectively,* Bd.G) *such that* $W \cup G \cup G'$ ($W \cup G$) *is a subset of an element* H *of* S_{n-1} *and* $D \cap G$ *is void, but* $D \cap G'$ ($D \cap W \cap Bd.G$) *is not void. Then, there is an arc* A *contained in* $W \cup G \cup G'$ ($W \cup G$) *such that* $G \cap A$ *is not void and* $A \cap D$ *is a point* P *such that* $D - P$ *has at most two components, unless either* (1) P *is a multiple local separating point of* M, (2) *there is a connected open set* U *containing* P *such that* $U - P = R \cup S$ *mutually separated, where* R *and* S *are connected,* $R \cap D$ *is void, and* $A \cap (R \cup P)$ *contains a subarc* QP *of* A.

PROOF. There is an arc XP from a point X of G to a point P of D such that $XP \cap D = P$ and XP is a subset of $G \cup W \cup G'$ ($G \cup W$). Suppose P is a local separating point of M which is not a multiple local separating point. There is a connected open set U containing P such that (1) U is a subset of $G \cup W \cup G'$ ($G \cup W$), (2) $U - P = R \cup S$ mutually separated, where R and S are connected, and (3) no point Q of $(D - P) \cap U$ has the property that $D - Q$ has more that two components.

One of $R \cup P$ and $S \cup P$ contains a subarc YP of XP, so suppose $R \cup P$ does. If $R \cap D$ is not void, there is an arc YP' from Y to a point P' of D where $YP' \subset R$ and $YP' \cap D = P'$. Thus, $(XP - P) \cup YP'$ contains an arc XP' with the desired properties. If $R \cap D$ is void, then XP is the desired arc.

Now suppose P is not a local separating point of M and that U is a connected open set containing P such that (1) $U \subset W \cup G \cup G'$ ($U \subset W \cup G$) and (2) no point Q of $(D - P) \cap U$ has the property that $D - Q$ has more than two components. Since $U - P$ is connected, there is an arc YP' from a point Y of XP to a point P' of D such that $YP' \subset U - P$ and $YP' \cap D = P'$. The set $(XP - P) \cup YP'$ contains an arc XP' with the desired properties. This completes the proof of Lemma 2A. □

As in Lemma 2.2 of [9] there is a finite sequence of arcs whose union is a finite dendrite D_1 which intersects each element of S_1. Let the elements of S_1 be labeled $s_1^1, s_2^1, \ldots, s_{n_1}^1$ and let the elements of S_2 in s_1^1 be labeled $w_1, w_2, \ldots, w_x$.

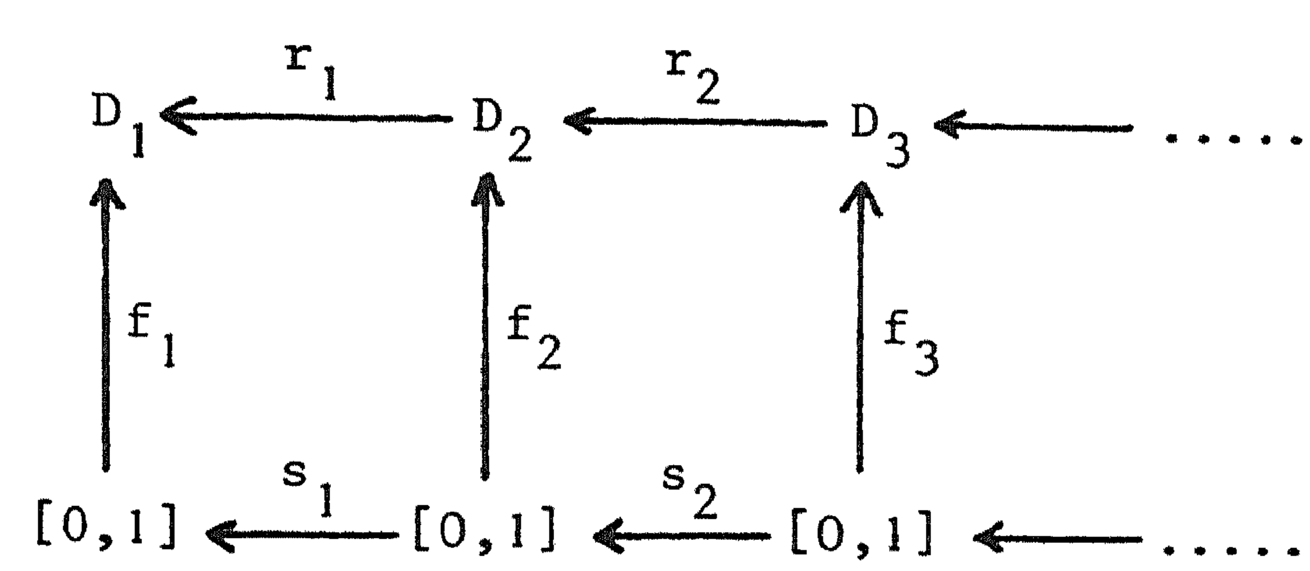
If there is a w_t so that $w_t \cap D_1$ is void, then there is such a w_t such that either (1) there is a w_{s_1} such that $w_s \cap D_1$ is not void and $Bd.w_x \cap Bd.w_t$ contains a point x of s_1^1, or (2) there is a point x of $D_1 \cap s_1^1 \cap Bd.w_t$. In either case we apply Lemma 2A with $D = D_1$, $G = w_t$, $G' = w_s$ or void, and $W = s_1^1$ to find an arc A'_t. This process is continued to find a finite dendrite D_1^1 which meets each one of $w_1, \ldots, w_x$, and where each component of $D_1^1 - D_1$ is a subset of s_1^1.

We now continue the applications of Lemma 2A to find finite dendrites $D_2^1, \ldots, D_{n_1}^1 = D_2$ such that (1) $D_1^1 \subset D_2^1 \subset \ldots \subset D_{n_1}^1$, (2) if $1 \leq i \leq j \leq n_1$ then each element of S_2 in S_i^1 meets D_j^1 and (3) each component of $D_2 - D_1$ is a subset of an element of S_1.

In general an analogous process is used to find D_{n+1} given D_n. We note that if n is a positive integer and $j > n$ then (1) no component of $D_{j+1} - D_j$ contains any one of $P_1, \ldots, P_n$, (2) D_n meets each element of S_n and (3) each component of $D_j - D_n$ is a subset of an element of S_n. Also since new arcs have always been added using Lemma 2A at each stage, if P is a local separating point of M, there is a positive integer n so that if $j < n$, then no component of $D_{j+1} - D_j$ has P in its boundary.

<u>THEOREM 3</u>. *If* M *is a locally connected metric continuum such that no point separates* M*, then there is a continuous onto map* $f : [0,1] \to M$ *such that* f *is of finite oscillation at local separating points.*

<u>PROOF</u>. We let M be approximated by a sequence of finite dendrites $D_1, D_2, \ldots$ satisfying the conditions of Theorem 2. For each n let $r_n : D_{n+1} \to D_n$ be the retraction such that $r(x) = x$ if $x \in D_n$, and $r(x) = p$ if x belongs to a component C of $D_{n+1} - D_n$ and p is the unique boundary point of C. Using Lemmas 5 and 6 and Theorem 2 of [8], for each positive integer n there is a continuous surjection f_n and a continuous monotone surjection s_n so that the ladder



is commutative and each $f_n^{-1}(x)$ is finite for $x \in M$. Then $D_\infty = \operatorname{inv\,lim}\{D_n, r_n\}$ is a dendrite, the limit of the inverse sequence $\{[0,1], s_n\}$ is $[0,1]$, and there is induced a continuous surjection $f' : [0,1] \to D_\infty$. By Lemmas 3 and 4 of [8], there is a continuous surjection $g : D_\infty \to M$. Let $f = gf' : [0,1] \to M$.

The sets $D = D_1 \times D_2 \times \ldots$ and $E = [0,1] \times [0,1] \times \ldots$, respectively, can be given metrics d', e defined by

$$d'(x_1, x_2, \ldots; y_1, y_2, \ldots) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i},$$

$$e(u_1, u_2, \ldots; v_1, v_2, \ldots) = \sum_{i=1}^{\infty} \frac{|u_i - v_1|}{2^i},$$

where d',e generate the product topology on D, E, respectively, and d is a metric on M. Suppose there is a local separating point P of M such that f is not of finite oscillation at P. The method of construction of f yields that there is a positive integer N_1 so that if $n > N_1$, then card. $f_n^{-1}(P) \le j$. Since f is not of finite oscillation at P, there is a connected open set U containing P such that (1) $U - P = R \cup S$ mutually separated and, (2) there is an increasing sequence $P_1, P_2, \ldots, P_{2j+3}$ of elements of $f^{-1}(P)$ in $[0,1]$ so that if $C_i = [P_i, P_{i+1}]$, $1 \le i \le 2j+2$, then $f(C_i)$ intersects both R, S and is a subset of U.

There exists $\varepsilon > 0$ such that $f(C_i) \cap R'$ and $f(C_i) \cap S'$ are not void for $i = 1, \ldots, 2j+3$, where $R' = R \cap (M - \overline{N(P,\varepsilon)})$, $S' = S \cap (M - \overline{N(P,\varepsilon)})$. Let $P_n = (x_1^n, x_2^n, \ldots)$, $1 \le n \le 2j+3$, and let $Q_i = [P_i, P_{i+2}]$, $i = 1, 3, \ldots, 2j+1$. For each integer $i = 1, 3, 5, \ldots, 2j+1$ let w_i denote an open interval containing P_i and contained in $f^{-1}(\{P\} \cup (R - \overline{R'}) \cup (S - \overline{S'}))$.

There is a positive integer N_1 so that $\sum_{p=1}^{N_1} \frac{1}{2^p} <$ (the minimum distance α between any of the sets $Q'_z = Q_z - \cup w_i$). There is a positive number ε' so that there exists $r_z \in f(Q'_z) \cap R'$ and $s_z \in f(Q'_z) \cap S'$, $z = 1, 3, \ldots, 2j+1$, where $d(r_z, P) > \varepsilon + \varepsilon'$ and $d(s_z, P) > \varepsilon + \varepsilon'$. There is a positive integer N_2 so that $\Sigma_{i=N_2}^{\infty} \frac{\text{diam } M}{2^i} < \varepsilon'$. Thus, if $(x_1,x_2,\ldots) \in E$ where $r_i(x_{i+1}) = x_i$, $(i = 1,2,3,\ldots)$ then the fact $f(x_1,x_2,\ldots) = \lim_{n\to\infty} f_n(x_n)$ means that $d(f_n(x_n),f(x_1,x_2,\ldots)) < \varepsilon'$ if $n \geq N_2$.

Let $N = N_1 + N_2$. Now if $(x_1,x_2,\ldots,x_N,\ldots) \in Q'_i$, $(y_1,y_2,\ldots,y_N,\ldots) \in Q'_j$, $i \neq j$, then $\Sigma_{N+1}^{\infty} \frac{|x_i - y_i|}{2^i} < \alpha$, so $(x_1,\ldots,x_N) \neq (y_1,\ldots,y_N)$. Thus $x_N \neq y_N$ because $x_{p-1} = s_{p-1}(x_p)$, $y_{p-1} = s_{p-1}(y_p)$ for $p = 2,3,\ldots$. We now let $R_z = \{x_N : (x_1,x_2,\ldots,x_N,\ldots) \in Q'_z\}$ $z = 1,3,\ldots,2j+1$, and note that $\{R_z : z = 1,3,\ldots,2j+1\}$ is a set of $j+1$ disjoint open intervals in S_N. However, $f_N(R_z)$ is a connected subset of U which intersects R' and S', so $P \in f_N(R_z)$. Therefore, card. $(f_N^{-1}(P)) \geq j+1$, a contradiction. □

THEOREM 4. *Suppose that* M *is a locally connected metric continuum such that* (1) *no point separates* M, *and* (2) *the set* N_1 *of all non-local separating points of* M *contains an open set* N' *dense in* M. *Then, there is a continuous onto map* $f : [0,1] \to M$ *such that* (1) f *is of finite oscillation at local separating points, and* (2) *there is a countable set* C *dense in* N_1 *such that* (a) $f^{-1}(C)$ *is dense in* [0,1] *and* (b) *if* $c \in C$, *then* $f^{-1}(c)$ *is a single point* (*thus* f *is strongly irreducible*).

PROOF. Let $S_1,S_2,\ldots$ be a sequence of partitions of M as in paragraph two of the proof of Theorem 2. We shall define a sequence $D_1,D_2,\ldots$ of dendrites as in Theorem 2, but with some added conditions. Thus, if D_n is given and D_{n+1} is to be constructed, we will always add new arcs using Lemma 2A. Also given a finite dendrite D a map $f : [0,1] \to D$ will be in C(D) provided there is a partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of [0,1] such that (1) if $0 \leq i < n$ there exists $h \in A(D)$ (= set of all maximal open arcs which are open in D) such that $f|[x_i,x_{i+1}]$ is a homeomorphism onto $\bar{h}$, (2) at most two $[x_i,x_{i+1}]$'s are mapped onto a given $\bar{h}$, and (3) f is onto and continuous.

Now let D_1 be as in the proof of Theorem 2 and let $f_1 \in C(D)$. Let D_2 be a finite dendron intersecting each element of S_2 such that (1) $D_1 \subset D_2$, (2) each component of $D_2 - D_1$ is a subset of an element of S_1 and intersects the set N', and (3) if I is a subinterval of [0,1] of length $\geq \frac{1}{2^2}$, then $f_1(I)$ intersects the closure of a component of $D_2 - D_1$. We let $f_2 : [0,1] \to D_2$ be an element of $C(D_2)$ such that (1) if h is a component of $H_2 = f_1^{-1}(D_1 - \overline{D_2 - D_1})$ there is a closed interval $g_h \subset h$ such that (a) $f_2|_{g_h} = f_1|_{g_h}$ and (b) if $f_1(h) \cap N'$ is not void, then $f_1(g_h) \subset N'$, (2) if $x \in [0,1]$ then $d(f_1(x),f_2(x)) < 1$, and (3) as in the proof of

Theorem 3 there exists continuous monotone $s_1 : [0,1] \to [0,1]$ and retraction $r_1 : D_2 \to D_1$ so that $r_1 f_2 = f_1 s_1$. We let $H_2' = \{g_h : h \in H_2\}$.

Now suppose $f_1,\ldots,f_n$; $D_1,\ldots,D_n$; $H_2,\ldots,h_n$; $H_2',\ldots,H_n'$; $r_1,\ldots,r_{n-1}$; $s_1,\ldots,s_{n-1}$ have been chosen. We let D_{n+1} be a finite dendron intersecting each element of S_{n+1} such that (1) $D_n \subset D_{n+1}$, (2) each component of $D_{n+1} - D_n$ is a subset of an element of S_n and intersects the set N', and (3) if I is a subinterval of $[0,1]$ of length $\geq \frac{1}{2^{n+1}}$, then $f_n(I)$ intersects the closure of a component of $D_{n+1} - D_n$. We let $f_{n+1} : [0,1] \to D_{n+1}$ be an element of $C(D_{n+1})$ such that (1) if h is a component of $H_{n+1} = H_n \cap f_n^{-1}(D_n - \overline{D_{n+1} - D_n})$, there is a closed interval $g_h \subset h$ such that (a) $f_{n+1}|g_h = f_n|g_h$, (b) if $f_n(h) \cap N'$ is not void, then $f_{n+1}(g_h) \subset N'$, and (c) if there exists $g_{h'}$ such that $h' \in H_n$ and $g_{h'} \cap h$ is not void, then $g_h \subset g_{h'}$, (2) if $x \in [0,1]$ then $d(f_{n+1}(x),f_n(x)) < \frac{1}{2^{n-1}}$, and (3) as in the proof of Theorem 3 there exist continuous monotone $s_n : [0,1] \to [0,1]$ and retraction $r_n : D_{n+1} \to D_n$ so that $r_n f_{n+1} = f_n s_n$. We let $H_{n+1}' = \{g_h : h \in H_{n+1}\}$.

We continue the process to find D_∞, f', g, f as in Theorem 3. Thus f is a continuous onto map $[0,1] \to M$ such that f is of finite oscillation at local separating points. We now show that if (a,b) is a subinterval of $[0,1]$, then $f((a,b)) \cap N'$ is not void.

There is a positive integer n such that $\frac{1}{2^n} < \frac{b-a}{8}$, and thus an integer i_0 such that $\frac{i_0 + i}{2^n} \in (a,b)$ for $i = 0,1,\ldots,5$. There exist components C_i, $i = 0,2,4$, of $D_n - D_{n-1}$ whose closures intersect

$$f_{n-1}\left(\left[\frac{i_0+i}{2^n}, \frac{i_0+i+1}{2^n}\right]\right),$$

$i = 0,2,4$, respectively. Therefore, there exist closed subintervals $g_n, g_{n+1}, \ldots$ of

$$\left[\frac{i_0}{2^n}, \frac{i_0+5}{2^n}\right]$$

such that (1) $g_n \supset g_{n+1} \supset g_{n+2} \supset \ldots$, (2) $C_2 \cap N' \supset f_i(g_i)$, $i = n,n+1,\ldots$, and (3) $f_{i+1}|g_i = f_i|g_i$, $i = n,n+1,\ldots$. Therefore, if $x \in \bigcap_{i=1}^{\infty} g_i \subset (a,b)$, then $f(x) \in N'$.

We now adjust f so as to find conclusion (2) of Theorem 4. Let $\{B_1,B_2,\ldots\}$ denote a countable basis for $[0,1]$, and also assume without loss of generality that f is light [4].

Let $d_1 \in B_1 \cap f^{-1}(N')$ and let $c_1 = f(d_1)$. There is a positive number $\varepsilon_1 < \frac{1}{2}$ such that $\overline{N(c_1,\varepsilon_1)} \subset N'$. Let $(a_1,b_1),\ldots,(a_n,b_n)$ be an irreducible open cover of $f^{-1}(c_1)$ so that (1) $\cup_{i=1}^{n} (a_i,b_i) \subset f^{-1}(N(c_1,\varepsilon_1))$, (2) no two of the (a_i,b_i) intersect, and (3) assume without loss of generality that $d_1 \in (a_1,b_1)$. The set $Y_1 = \cup_{i=1}^{n} f([a_i,b_i])$ is a Peano continuum containing c_1 such that c_1 is not a local separating point of Y_1. By Lemma 2 of HARROLD [3], there is a continuous map g_1 of $[a_1,b_1]$ onto Y_1 such that $g_1(a_1) = f(a_1), g_1(b_1) = f(b_1)$, and $g_1^{-1}(c_1)$ is the single point d_1. For each i in $\{2,\ldots,n\}$ let g_i map $[a_i,b_i]$ continuously into $Y_1 - c_1$ so that $g_i(a_i) = f(a_i), g_i(b_i) = f(b_i)$. Now define a map $h_1 : [0,1] \to M$ such that

$$h_1(x) = \begin{cases} f(x), & x \in [0,1] - \cup_1^n (a_i,b_i) \\ g_i(x), & x \in [a_i,b_i] \end{cases}$$

Now suppose $h_1,\ldots,h_n$; $\varepsilon_1,\ldots,\varepsilon_n$; $d_1,\ldots,d_n$; $c_1,\ldots,c_n$; have been defined. Let $d_{n+1} \in h_n^{-1}(N') \cap (B_{n+1} - \cup_1^n h_n^{-1}(c_p))$ and let $c_{n+1} = h_n(d_{n+1})$. There is a positive number ε_{n+1} such that $\varepsilon_{n+1} < 1/2^{n+1}$ and $\overline{N(c_{n+1},\varepsilon_{n+1})} \subset N' - \{c_1,\ldots,c_n\}$. Now let $(a_1,b_1),\ldots,(a_j,b_j)$ be an irreducible open cover of $h_n^{-1}(c_{n+1})$ such that (1) $\cup_{i=1}^{i} (a_i,b_i) \subset h_n^{-1}(N(c_{n+1},\varepsilon_{n+1}))$, (2) no two of the (a_i,b_i) intersect, and (3) assume without loss of generality that $d_{n+1} \in (a_1,b_1)$. From above we may find a continuous onto map

$$k : \bigcup_{i=1}^{j} [a_i,b_i] \to \bigcup_{i=1}^{j} h_n([a_i,b_i])$$

so that k agrees with h_n on the a_i's and b_i's, and $k^{-1}(c_{n+1}) = d_{n+1}$. The map h_{n+1} is defined by

$$h_{n+1}(x) = \begin{cases} k(x), & x \in \cup_1^j [a_i,b_i] \\ h_n(x), & x \in [0,1] - \cup_1^j (a_i,b_i). \end{cases}$$

The map $h : [0,1] \to M$ defined by $h(x) = \lim_{i\to\infty} h_i(x)$ satisfies the conclusion of the theorem. □

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ORDERABLE SUBSPACES OF COMPACT F-SPACES

S.W. WILLIAMS

0. INTRODUCTION

A space is *orderable* if its topology is induced by a linear ordering of its ground set. Further recall that a space is *almost*-P [10] if each non-empty G_δ-set has infinite interior. We consider the question:

0.1. *Is there a compact almost*-P, F-*space of weight* c *possessing a dense orderable subspace*?

Suppose that Y is a locally compact, real-compact, non-compact space. Then we know that $\beta Y - Y$ is a compact almost-P ([5]) F-space ([6]). If, in addition, Y has weight c (and is metrizable), and if CH (MA) is assumed, then $\beta Y - Y$ has a dense orderable subspace [2] ([7]). In a parallel paper [13] we established:

(1). if CH is false, then $\beta Y - Y$ has no dense orderable subspace for some Y which is also zero-dimensional separable and σ-compact, and

(2). it is consistent with the axioms of ZFC that $\beta Y - Y$ never has a dense orderable subspace.

From (2) we find motivation for 0.1.

A space is *Parovičenko* [1] if it is a compact, almost-P, zero-dimensional, F-space. We build here, using direct limits of trees and Boolean algebras, a machine for producing Parovičenko spaces. As an application of our machine, we obtain an affirmative answer to 0.1 with

<u>THEOREM 5.2</u>. *For each regular uncountable cardinal* $\Delta \leq c$ *there is a Parovičenko space* X_Δ *of weight* c *possessing a dense orderable subspace* L_Δ *such that for each regular uncountable cardinal* Γ, $\Delta < \Gamma \leq c$, L_Γ *is not densely embeddable into* X_Δ.

In section 5 we discuss other methods for answering 0.1 in the affirmative.

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Throughout this paper c is the cardinal exp (ω).

1. PRELIMINARIES

All cardinals are ordinals and all ordinals are von Neuman ordinals. The first three infinite cardinals are $\omega = \omega_0$, $\omega_1 = \omega^+$, and $\omega_2 = \omega^{++}$; c is the cardinality of the continuum. If A is a set, $|A|$ is the cardinality of A. If A is a set and if κ is a cardinal, then

$$[A]^{\leq\kappa} = \{I \subseteq A : \omega \leq |I| \leq \kappa\}.$$

If A and B are sets, then ${}^{A}B$ is the set of functions from A to B. If κ and λ are cardinals, then $\lambda^{\kappa} = |{}^{\kappa}\lambda|$ and

$$\lambda^{\kappa} = \sup \{\lambda^{\mu} : \mu < \kappa, \mu \text{ is a cardinal}\}.$$

If $f \in {}^{A}B$ and $C \subseteq A$, then $f \upharpoonright C$ denotes the restriction of f to C. The end of a proof will be denoted by $\square$.

Suppose B is a Boolean algebra. We will ust St(B) for its Stone-space [3, p. 40]. We will use + and Σ for addition, and we use $\cdot$ and Π for product. If $P \subseteq B$, then <<P>> is the *subalgebra* of B *generated by* P. The set $P \subseteq B$ is *dense* in B if

$$0 < b \in B \Rightarrow \exists\, p \in P,\ 0 < p \leq b.$$

A POSET (partially ordered set) P is said to be *separative* provided

$$p,q \in P,\ p \nleq q \Rightarrow \exists\, r \in P,\ r \leq p,\ \nexists\, s \in P,\ s \leq r,\ s \leq q.$$

PROPOSITION 1.1. (1) *A dense subset of a Boolean algebra is separative.* (2) [8, 29B] *Each separative* POSET P *can be embedded as a dense subset of a unique (up to an isomorphism) complete Boolean algebra (which we denote by* cp<<P>>). $\square$

Part (2) of (1.1) gives us another algebra to consider. Suppose κ is an infinite cardinal. Then the κ-completion, $\kappa cp\langle\langle P\rangle\rangle$, of a separative POSET P is the subalgebra of $cp\langle\langle P\rangle\rangle$

$$\langle\langle\{\Sigma\, I : I \subseteq P,\ |I| \le \kappa\}\rangle\rangle.$$

A POSET P is *κ-closed* if each decreasing sequence of order type at most κ is bounded below by a non-minimal element when no member of the sequence is minimal.

Suppose P is a POSET. If $p \in P$, then

$$p\!\downarrow\ = \{q \in P : q < p\} \text{ and } p\!\uparrow\ = \{q \in P : p < q\}.$$

P is said to be a *tree* whenever

(i) P has a greatest element 1_P,

(ii) for each $p \in P$, $p\!\uparrow$ is well-ordered, and

(iii) for each $p \in P$, $\exists\ r, s \in P$ such that p is the smallest element of $r\!\uparrow \cup\ s\!\uparrow$.

(Warning - a tree, as we have defined it, has its order the reverse of the usual one and is defined to make it a separative POSET.)

Suppose T is a tree. Then

ord[t] is the order type of $t\!\uparrow$ when $t \in T$;

$T \uparrow \alpha = \{t \in T : \text{ord}[t] < \alpha\}$ is the α'th truncation of T;

$\ell v(T,\alpha) = \{t \in T : \text{ord}[t] = \alpha\}$ is the α'th *level* of T;

$\Delta(T) = \inf\{\alpha : \ell v(T,\alpha) = \emptyset\}$ is the *depth* of T.

A branch of T is a maximal (with respect to inclusion) linearly ordered set;

$$\text{ord}[M] = \bigcup\{\text{ord}[t] : t \in M\} \text{ when M is a branch of T.}$$

PROPOSITION 1.2. (1) *A tree is a separative* POSET;

(2) [11] *Suppose* P *is a separative ω-closed* POSET *of cardinality* c. *If* P *has no minimal elements, then* P *has a dense tree.* □

2. THE TREE MACHINE

In [1] a machine is constructed whose input takes any Boolean algebra of cardinality $\not{c}$, and whose output is always the $\not{c}$-homogeneous-universal Boolean algebra of cardinality $\not{c}$ (which exists if, and only if, $\not{c} = 2^{\pounds}$ [3, 6.12]). For our purposes we will need to discern a little more about the dense subsets of complete Boolean algebras. Proposition 1.2 suggested to us that trees might be used to control the cardinality of algebras.

Suppose that T is a tree and κ is an infinite cardinal. We will say that T satisfies *the input conditions for* κ whenever

2.1 (1) T is κ-closed;

(2) $|T| = ¢$ and $\Delta(T)$ is a regular cardinal $\leq c$;

(3) if $\alpha \in \Delta(T)$ and if M is a branch of $T \uparrow \alpha$ bounded below in T, then $|\{t \in T : M = t\uparrow\}| = ¢$; and

(4) if $t \in T$, then $\Delta(\{t\} \cup t\downarrow) = \Delta(T)$.

One quickly observes that if T is a tree satisfying 2.1 (1), (2) and (4), then

$$S = \{t \in T : \mathrm{ord}[t] = 0 \text{ or } \mathrm{cl}(\mathrm{ord}[t]) = \omega\}$$

is a dense tree in T, $\Delta(S) = \Delta(T)$, and S satisfies the input conditions for κ.

2.1. THE TREE MACHINE

Suppose κ is a cardinal and T is a tree satisfying the input conditions for κ. Then the machine's job is to construct, recursively, according to restrictions given below, a Boolean algebra $p<<T>>$ and a tree $T^{\#}$ as an increasing union of, respectively, Boolean algebras $\{B(\delta) : \delta \in \Delta(T)\}$ and trees $\{T(\delta) : \delta \in \Delta(T)\}$.

First we give $W = \{(0,0)\} \cup ((\Delta(T) - \{0\}) \times c)$ the lexicographic order. As W is well-ordered we can define our objects recursively over W. Let

$$T(0,0) = T \text{ and } B(0,0) = \kappa cp<<T>>.$$

From 2.1 (1) every branch of $T(0,0) \uparrow \kappa$ is bounded below in T. So

$$¢ \leq 2^{\kappa} \leq |T| \leq |T|^{\kappa} = |B(0,0)| = ¢^{\kappa}.$$

From 2.2 $|T| = ¢$, and so $¢^{\kappa} = ¢$.

Suppose that for a given $w \in W$ we have found for each $v < w$ a tree $T(v)$, a Boolean algebra $B(v)$, and a map $\Phi(u,v)$ $\forall\, u < v$ subject to the restrictions:

(1) $T(v)$ satisfies the input conditions for κ and $\Delta(T(v)) = \Delta(T)$;

(2) $B(v)$ is κ-complete and $|B(v)| = ¢$;

(3) $T(v)$ is dense in $B(v)$;

(4) $\Phi(u,v) : B(u) \to B(v)$ is an embedding of algebras;

(5) $(\Phi(u,v) \restriction T(u)) : T(u) \to T(v)$ is an embedding of trees;

(6) if $x < u$, then $\Phi(x,v) = \Phi(u,v) \circ \Phi(x,u)$;

(7) if $t \in T(u)$, then $\mathrm{ord}[\Phi(u,v)(t)] = \mathrm{ord}[t]$;

(8) if $u = (\beta,0) < v$, then there is a listing $\{I(\eta) : \eta < \cent\}$ of $[\ell v(T(u),\beta)]^{\leq\kappa}$ such that if $(\beta,\rho) < v$ (or if $(\beta+1,0) \leq v$), $\Sigma_{B(v)}\, \Phi(u,v)(I(\eta)) < \Phi(u,v)(\Sigma_{B(u)}\, I(\eta))\ \forall\ \eta < \rho$ (resp. $\eta < \cent$).

CASE 1. $w = (\delta,\theta+1)$ *for some* θ, $0 \leq \theta \leq c$.

Set $x = (\delta,0)$ and let $\{I(\eta) : \eta < \cent\}$ be the listing of $[\ell v(T(x),\beta)]^{\leq\kappa}$ given by (8). Set $y = (\delta,\theta)$ and choose a copy of T disjoint from $B(y)$. For convenience we set

$$b = \Sigma\, \Phi(x,y)(I(\theta)) \text{ and } P = S \cup (B(y) - \{0\}),$$

where $\Phi(x,x)$ is the identity map. We define a partial order $<$ on P so that $p < q$ iff

(i) $p,q \in S$ and $p <_S q$, or

(ii) $p,q \in B(y)$ and $p <_{B(y)} q$, or

(iii) $p \in S$, $q \in B(y)$, and $b \leq_{B(y)} q$.

Now $S \cup T(y)$ is a dense tree in P satisfying the input condition for κ. So we let $T(w) = S \cup T(y)$. From 1.1 (1), $T(y) \cup \{b\}$ is separative so P is separative. Set $B(w) = \kappa cp\langle\langle P\rangle\rangle$. From 1.1 (2) it is clear that the identity injection from $T(y) \cup \{b\}$ into P extends to an embedding Φ from $B(y)$ into $B(w)$. For v, $x \leq v \leq y$, let $\Phi(v,w) = \Phi \circ \Phi(v,y)$. From (ii) above, (7) is satisfied. From (iii) above

$$\sum_{B(w)} \Phi I(\theta) = \sum_{B(w)} I(\theta) < b = \Phi(b).$$

So (8) is satisfied. This completes case 1.

CASE 2. $w = (\delta,0)$ *and* $w \leq cf(\delta) \leq \kappa$.

Let $S = \varinjlim \{T(v) : v < w\}$ and $B = \varinjlim \{B(v) : v < w\}$ where $\varinjlim$ is taken over the connecting maps $\Phi(u,v)$ for $v < w$. From (7) it is clear that S is a tree dense in the Boolean algebra B; however, S is probably not κ-closed. Set

$$\mathcal{M} = \{M : M \text{ is a branch of } S \text{ and } \mathrm{ord}[M] \leq \kappa\}.$$

For $M \in \mathcal{M}$, let $S(M)$ be a copy of $T - \{1_T\}$. We define a partial order $<$ on

$$P = B \cup (\cup\{S(M) : M \in \mathcal{M}\}).$$

We say $p < q$ iff

(i) $p,q \in S(M)$ for $M \in \mathcal{M}$ and $p <_{S(M)} q$, or
(ii) $p,q \in B$ and $p <_B q$, or
(iii) $p \in S(M)$ for $M \in \mathcal{M}$, $q \in B$, and $\exists\, m \in M$ with $m <_B q$.

Let $T(w) = S \cup (\cup\{S(M) : M \in \mathcal{M}\})$. Then $T(w)$ is κ-closed, and by 2.1 ((1) and (2)), $|T(w)| = ¢^\kappa = ¢$. Let $B(w) = \kappa cp\langle\langle P\rangle\rangle$ and for $u < w$, let

$$\Phi(u,w) = \varinjlim \{\Phi(u,v) : u < v < w\}.$$

Since $|P| = ¢$ and $¢^\kappa = ¢$, $|B(w)| = ¢$. The rest of the restrictions (1) through (8) are satisfied. This completes case 2.

<u>CASE 3</u>. *Not [Case 1 or Case 2].*

Let $T(w) = \varinjlim \{T(v) : v < w\}$ and $B(w) = \kappa cp\langle\langle \varinjlim \{B(v) : v < w\}\rangle\rangle$, where $\varinjlim$ is taken over the connecting maps $\Phi(u,v)$ for $v < w$. Now suppose $w = (\delta,\theta)$. If δ is a successor and $\theta = 0$, then in W we have $cf(w) = ¢$. If δ is not a successor and $\theta = 0$, then $cf(w) > \kappa$ in W. If $\theta \neq 0$, then $cf(\theta) > \kappa$. In any of these cases, $T(w)$ is easily seen to be κ-closed, and hence, to satisfy the input conditions of κ. Restrictions (2) through (8) are satisfied if we let

$$\Phi(u,w) = \varinjlim \Phi\{(u,v) : u < v < w\}, \text{ when } u < w.$$

This completes Case 3.

We complete the construction of the Tree Machine with the following definitions. Suppose $\delta \in \Delta(T)$, we set

$$T(\delta) = \Phi((\delta,0),\Delta)(T((\delta,0))) \text{ and}$$
$$B(\delta) = \Phi((\delta,0),\Delta)(B((\delta,0))), \text{ where}$$
$$\Phi((\delta,0),\Delta) = \varinjlim \{\Phi((\delta,0),w) : (\delta,0) < w \in W\}.$$

Finally, we define

(9) $T^\#[\kappa] = \cup\{T(\delta) : \delta \in \Delta(T)\}$.

(10) $p\langle\langle T,\kappa\rangle\rangle = \cup\{B(\delta) : \delta \in \Delta(T)\}$.

(11) $X(T,\kappa) = \mathcal{S}t(p\langle\langle T,\kappa\rangle\rangle)$.

When there is no confusion, we simply write $T^{\#}$, p<<T>>, and X(T), above. □

2.3. REMARKS.

(1) If our only interest is the construction of a Parovičenko space, then the cardinal restrictions in 2.1 ((2) and (3)) are unnecessary.
(2) A reasonable modification, for which we have as yet no use, is to allow in Case 1, any tree X satisfying the input conditions for κ and having depth $\Delta(T)$.
(3) If T is a tree satisfying the input conditions for ω_1, then it satisfies the input conditions for ω. However, as witnessed by Case 2, the objects $T^{\#}$ and p<<T>> will be different depending on which κ, ω or ω_1, one decides to use in the machine. For example, if one uses $\kappa = \omega$, Then $T^{\#}$ *will not be* ω_1*-closed*. So to avoid such confusion, the objects (9), (10) and (11) specify the κ for which the machine is being applied.

3. A PAROVIČENKO SPACE.

From [3] a zero-dimensional F-space becomes an F_{ω^+}-space using the generalization: for a cardinal κ, a space is called an F_κ*-space* if each union of less than κ many clopen sets is C^*-embedded in the space. Similarly, we might call a space an *almost* P_κ*-space* if each non-empty intersection of less than κ many open sets has infinite interior. Now suppose that B is a Boolean algebra. Then St(B) is an

(1) F_κ*-space* iff B satisfies the following condition (from [3])

(H_κ) for every $A_0, A_1 \subset B$ such that $|A_0 \cup A_1| < \kappa$, and such that $\forall\, I \in [A_0]^{<\omega}$, $\forall\, J \in [A_1]^{<\omega}$, $\exists\, a_0 \in A_0\ \exists\, a_1 \in A_1$ satisfying $0_B < \Sigma\, I \leq a_0 < a_1 \leq \Pi J < 1_B$;

(2) *almost*-P *space* iff B is atomless and λ-closed for each cardinal $\lambda < \kappa$.

LEMMA 3.1. *Suppose* κ *is an infinite cardinal and* T *is a tree satisfying the input conditions for* κ. *Then*
(1) T *is dense in* p<<T>> *and* $\Delta(T^{\#}) = \Delta(T)$.
(3) p<<T>> *is atomless and* κ*-closed*.
(3) p<<T>> *satisfies condition* (H_{κ^+}).

PROOF. (1) From 2.2 (3), $T(\delta)$ is dense in $B(\delta)$. So $T^{\#}$ is dense in p<<T>>. The second assertion follows since $\Delta(T)$ is regular.

(2) An atom in $p<<T>>$ is, by (1), a minimal element of $T^{\#}$, and hence, a minimal element of $T(\delta)$ for some $\delta \in \Delta(T)$. This contradicts the assertion that $T(\delta)$ is a tree.

Suppose λ is an ordinal, $\omega \leq \lambda \leq \kappa$, and

(i) $\{b(\alpha) : \alpha \in \lambda\} \subseteq p<<T>>$ is such that $\beta < \alpha \Rightarrow b(\alpha) < b(\beta)$.

From (1) above we can find $t(\beta) \in T^{\#}$ for each β with

(ii) $0 < t(\beta) < b(\beta) \cdot b(\alpha)' \;\forall \alpha,\ \beta < \alpha \in \lambda$.

From (9) we can find a first $\gamma \in \Delta(T)$ such that $t(\alpha) \in T(\gamma)$ and $b(\alpha) \in B(\gamma)$ $\forall \alpha \in \lambda$. Now find a first $\delta \in \Delta(T)$, $\gamma < \delta$, such that $t(\alpha) \in T(\gamma) \uparrow \delta$ $\forall \alpha \in \lambda$. From 2.1 (4) and 2.2 (1) we may choose for each $\alpha \in \lambda$,

(iii) $S(\alpha) \in \ell v(T(\gamma),\delta)$ and $s(\alpha) < t(\alpha)$.

Set $r = \Sigma_{B(\delta)} \{s(\alpha) : \alpha \in \lambda\}$ and for each $\alpha \leq \lambda$, set $q(\alpha) = \Sigma_{B(\delta+1)} \{s(\beta) : \beta < \alpha\}$. From 2.2 (8), we have for each $\alpha < \lambda$, $q(\alpha) < q(\lambda) < r$. From (i), (ii) and (iii) above we have in $p<<T>>$

$$0 < r \cdot q(\lambda)' < r \cdot q(\alpha)' < b(\alpha).$$

In particular this means

(iv) $$\prod_{B(\delta)} \{b(\alpha) : \alpha \in \lambda\} < \prod_{B(\delta+1)} \{b(\alpha) : \alpha \in \lambda\} < \prod_{B(\delta+2)} \{b(\alpha) : \alpha \in \lambda\}.$$

(3) Suppose $I, J \subseteq p<<T>>$ such that $I < J$, $|I| + |J| \leq \kappa^{+}$, $I \subseteq p<<T>> \sim \{1\}$ is directed upwards, and $J \subseteq p<<T>> - \{0\}$ is directed downwards. We may find a $\delta \in \Delta(T)$ such that $I \cup J \subseteq B(\delta)$. From 2.2 (2) and (2) (iv) above, we have, in $p<<T>>$,

$$a \leq \sum_{B(\delta)} I \leq \prod_{B(\delta)} J < \prod_{B(\delta+1)} J \leq b \quad \forall a \in I,\ \forall b \in J. \qquad \square$$

<u>THEOREM 3.2</u>. *Suppose κ is an infinite cardinal and T is a tree satisfying the input conditions for κ. Then $X(T)$ is an almost-P_{κ}, F-space of weight ¢.*

<u>PROOF</u>. Except for the weight, the result follows from 3.1. On the other hand, 2.2 (2) shows

$$¢ \leq \text{weight } X(T) = ¢ \cdot \Delta(T) = ¢. \qquad \square$$

4. THE DENSE SUBSPACE

For some (e.g., the Cantor tree) but not all trees T, St(<<T>>) is an orderable space. However, as expected, many results for compact zero-dimensional orderable spaces are true for the spaces St(<<T>>). The next lemma is one of these.

LEMMA 4.1. *Suppose that* T *is a tree and* $a_0, a_1 \in \langle\langle T\rangle\rangle$ *are such that* $a_0 < a_1$. *Then* $\forall\, i \in 2\ \exists\, F(a_i) \in [\langle\langle T\rangle\rangle]^{<\omega}$ *satisfying*:

(1) $a_i = \Sigma\, F(a_i)$ *and* $b, c \in F(a_i) \Rightarrow b \cdot c = 0$;

(2) $b \in F(a_i) \Rightarrow \exists$ *a finite set* $\{t(b)\} \cup G(b) \subseteq T \cup \{0\}$ *such that* $b = t(b) \cdot (\Sigma\, G(b))'$;

(3) $0 \neq b \in F(a_i)$ *and* $s, t \in G(b) \Rightarrow s \cdot t = 0$ *and* $s + t < t(b)$;

(4) *either* (i) $F(a_0) \subseteq F(a_1)$ *or* (ii) $\exists\, b \in F(a_0)\ \exists\, c \in F(a_1)$ *with* $t(b) \leq c$.

PROOF. Suppose $a \in \langle\langle T\rangle\rangle$. Then from [3, 2.14] we can write

$$a = \sum_{i=1}^{n} \prod_{j=1}^{n(i)} s_{ij} \cdot r'_{ij}, \quad \text{where } s_{ij}, r'_{ij} \in T\ \forall\, i,j.$$

Since every pair of elements of T have 0 product or are related by $\leq$, we can choose for each j the smallest of the s_{ij}, say e(j), and a pairwise disjoint collection G(j) of the r'_{ij} such that

$$\prod_{j=1}^{n(i)} s_{ij} \cdot r'_{ij} = e(j)(\Sigma\, G(j))'.$$

Similar re-arranging produces the set F(a) satisfying (1), (2) and (3) with $a = a_i$. For (4) we must again re-arrange $F(a_0)$ by including all non-disjoint products

$$t(b) \cdot t(c) \cdot (\Sigma\, G(b) \cup G(c))'$$

for $b \in F(a_0)$ and $c \in F(a_1)$. □

LEMMA 4.2. *Suppose* κ *is an infinite cardinal and* T *is a tree satisfying the input conditions for* κ. *If* $\delta \in \Delta(T)$ *and if* M *is a branch of* $T(\delta)$ *such that*

$$M \nsubseteq \cup\{T(\gamma) : \gamma < \delta\},$$

then M *is a branch of* $T^{\#}$ *and*

$$u(M) = \{p \in p{<<}T{>>} : \exists\, t \in M,\ t \leq p\}$$

is an ultra-filter in $p{<<}T{>>}$.

<u>PROOF</u>. We use the terminology of (2.2) and make the convention

$$T^{\#} = \cup\{T(w) : w \in W\} \text{ and } p{<<}T{>>} = \cup\{B(w) : w \in W\}.$$

The proof is by induction on W.

<u>CASE 0</u>. *Suppose* M *is a branch of* $T(0,0)$. If $a \in T(0,0)$, $F(a)$ is the decomposition of a given by 4.1, and if $a \cdot t \neq 0\ \forall t \in M$, then $\exists\, b \in F(a)$ such that $b \cdot t \neq 0\ \forall t$ in a dense subset of M. Since M is linearly ordered, $b \cdot t \neq 0\ \forall t \in M$. Since $T(0,0)$ is a tree, $t(b) \in M$ and $\exists\, m \in M$ with $m \cdot g = 0\ \forall g \in G(b)$. So $a \geq m$. Therefore

$$u(M) \cap {<<}T(0,0){>>}$$

is an ultra-filter in ${<<}T(0,0){>>}$.

Suppose $I \in [{<<}T(0,0){>>}]^{\leq\kappa}$ and $a \in B(0,0)$ is such that $a \cdot t \neq 0$ $\forall t \in M$. If $a = \Sigma I$, then for each $t \in M$, $\exists\, i(t) \in I$ with $i(t) \cdot t \neq 0$. Since $T(0,0)$ is κ-closed, $cf(ord[M]) > \kappa$. So the function $i : M \to I$ is constant on a tail of M. From the previous paragraph $a \in u(M)$. If $a = \Pi\, I$, then for each $j \in I$ we can find $m(j) \in M$ with $m(j) \leq j$. But $\exists\, m \in M$ having $m \leq m(j)$ for all $j \in J$. So $m \leq a$. So we have shown that

$$u(M) \cap B(0,0)$$

is an ultra-filter in $B(0,0)$.

Now suppose $w \in W$ and $\forall v < w$ we know that

$(i)_v$ M is a branch of $T(v)$, and

$(ii)_v$ $u(M) \cap B(v)$ is an ultra-filter in $B(v)$.

If w agrees with the restrictions of case 3 of 2.2, then

$$T(w) = \cup\{T(v) : v < w\} \text{ and } B(w) = \kappa cp{<<}\cup\{B(v) : v < w\}{>>}.$$

Clearly M is a branch of $T(w)$. So $(i)_w$ is satisfied. If w agrees with the restrictions of case 2 of 2.2, then M is a branch of $T(w)$ because $cf(ord[M]) > \kappa$. So $(i)_w$ is satisfied. If w agrees with the restrictions of case 1 of 2.2, then M is seen to be a branch of $T(w)$, by observing the order $<$. So $(i)_w$ is satisfied. For each of these three cases, we can show

$(ii)_w$ by mimicing the argument of the previous paragraph.

From our induction hypothesis, it should be obvious that M is a branch of $T^{\#}$ and u(M) is an ultra-filter in p<<Γ>>.

CASE $\delta > 0$. *Suppose* M *is a branch of* T(w) *and*

$$M \nsubseteq \cup\{T(v) : v < w\}.$$

This is impossible if w satisfies the restrictions of case 3 of 2.2. On the other hand, if w agrees with the restrictions of case 1 or case 2, then a tail of M lies in a copy C of T = T(0,0). Clearly

$$C^{\#} = \{t \in T^{\#} : t \in \cap\{m\downarrow : m \in M \cap T(v) \text{ for some } v < w\}\}.$$

So from case 0, M is a branch of $T^{\#}$ and u(M) is an ultra-filter in p<<T>>. □

Recall [3] that for an infinite cardinal Δ, a point x in a space is a P_Δ-*point* if x belongs to the interior of each intersection of less than Δ many neighborhoods of x.

THEOREM 4.3. *Suppose* T *is a tree satisfying the input conditions for an infinite cardinal* κ. *If*

$$\Delta = \sup\{cf(ord[M]) : M \textit{ is a branch of } T\},$$

then X(T) *has a dense orderable subspace of* P_Δ-*points of character* Δ *and* $\Delta \le \Delta(T) \le \Delta^+$.

PROOF. From cases 1 and 2 of 2.2,

$$\Delta \le \sup\{cf(ord[M]) : \exists\, w \in W \text{ such that M is a branch of } T(w)\} \le \le \Delta(T) \le \Delta^+$$

and each $t \in T^{\#}$ is a member of a branch M_t of T(w), for some $w \in W$, such that $\Delta = cf(ord[M_t])$. From 4.2, $D = \{u(M_t) : t \in T^{\#}\}$ consists entirely of P_Δ-points of character Δ. According to 3.1 (1) every non-empty open set of X(T) contains a memner of $T^{\#}$. Thus, D is dense in X(T). To see that D is orderable, use the standard tree-to-line argument (e.g., see [9]). □

5. APPLICATIONS AND REMARKS.

LEMMA 5.1. *Suppose that* T *is a tree satisfying the input conditions for* ω.

If Γ *is a regular uncountable cardinal and if* $\Gamma > \Delta(T)$*, then no element of* $X(T)$ *is a* P_Γ*-point of character* Γ.

PROOF. For convenience set $\Lambda = \Delta(T)$. We need only show that no untra-filter on p<<T>> is generated by a Γ-decreasing sequence. Assume the conatrary, i.e., suppose an ultra-filter u on p<<T>> is generated by a family

$$A = \{a_\alpha : \alpha < \Gamma\}, \text{ where } \beta < \alpha \Rightarrow a_\alpha < a_\beta.$$

From 2.2 (10) there is a first $\delta < \Lambda$ such that $v = u \cap B(\delta)$ is an ultra-filter on $B(\delta)$. We observe that $\Pi\, I \in v$ whenever $I \subseteq v$ and $|I| \leq \omega$.

CASE 1. $\delta = 0$

Since $1_T \in v$, $M = v \cap T$ is a non-empty linearly ordered set. Since Λ is regular, there is a first $\gamma < \Lambda$ with $M \subseteq T \uparrow \gamma$. From 2.1 (3) we choose

(i) $\{t_\mu : \mu < \Lambda\} \subseteq \ell v(T,\gamma)$ with $M = t_\mu \uparrow \forall\, \mu < \Lambda$.

For each $\mu < \Lambda$, we set $s_\mu = \Sigma\{t_\nu : \nu \leq \mu\}$.

Suppose $s_\mu \in v$ for some $\mu < \Lambda$. Then for each $\alpha \in \Gamma$ there is a $\nu(\alpha) < \mu$ with $t_{\nu(\alpha)} \cdot a_\alpha \neq 0$. As μ is countable, some $t_\nu = t_{\nu(\alpha)}$ on a cofinal set in Γ. Since A is decreasing, $t_\nu \in v$. This is a contradiction. Therefore, there is $\alpha < \Gamma$ with

(ii) $a_\alpha < \Pi M \cdot s'_\mu \quad \forall\, \mu < \Lambda$.

Now suppose $a_\alpha \geq \Pi\, I \in v$, for some countable set $I \subset$ <<T>>. We may decompose each $a \in I$ according to 3.3. Since v is an ultra-filter, there is, for each $a \in I$, a finite set $\{t(a)\} \cup G(a) \subseteq T \cup \{0\}$ such that

(iii) $a \geq t(a) \cdot (\Sigma\, G(a))' \in v$

Set $m = \Pi\{t(a) : a \in I\}$ and $n = \Sigma\{\Sigma\, G(a) : a \in I\}$. From (iii) $t(a) \in v$, so $t(a) \in M$ for each $a \in I$. Since v is closed under countable products, $m \cdot n' \in v$. From (i), $s_\mu < m$ for each $\mu < \Lambda$. Hence, from (ii) and (iii)

$$m \cdot s'_\mu \geq \Pi M \cdot s'_\mu > a_\alpha \geq m \cdot n' \geq \Pi M \cdot n' \in v \quad \forall\, \mu < \Lambda.$$

Therefore, $s_\mu < n$. Since T is a tree, $\exists\, a \in I$ and $t \in G(a)$ with $s_\mu < t$. From (i) $t \in M$. From (iii) $t \notin M$.

In the case of $a_\alpha \geq \Sigma\, I \in v$ for some countable set $I \subseteq$ <<T>>, we arrive at a contradiction with an argument similar to but technically simpler than the previous paragraph and the derivation of (ii) above. Therefore, *no* Γ*-decreasing family on* $B(0)$ *generates an ultra-filter.*

CASE 2. δ *is a successor ordinal.*

We enter case 1 of 2.2 and assume without loss of generality that the embeddings of 2.2 (4) and (5) are all inclusions. Now each

$$T(\delta-1,\mu+1) \sim T(\delta-1,\mu)$$

has a greatest element t_μ = (= 1_s in case 1 of 2.2). Further, in $B(\delta,0)$, $\{t_\mu\} \cup (t_\mu{\downarrow})$ is isomorphic to $B(0)$. So case 1 above applies, and we have $t_\mu \in v$. In particular,

(i) $\quad a \in B(\delta-1,\mu+1) \cap v \Rightarrow \exists\, b \in B(\delta-1,\mu) \cap v$ such that $b \le a$.

Identify the pair $(\delta,0)$ with $(\delta-1,\text{¢})$. Suppose we have shown for a given ordinal $\mu \le \text{¢}$ that

$(ii)_\mu$ $\quad \forall\, \nu < \mu$ and $a \in B(\delta-1,\nu) \cap v$, $\exists\, b \in B(\delta-1,0) \cap v$ having $b \le a$.

If μ is a successor ordinal, then (i) and (ii) imply $(ii)_{\mu+1}$. If $cf(\mu) = \omega$, say $\mu = \sup\{\mu(n) : n \in \omega\}$, then $(ii)_\mu$ and the argument for (ii) of case 1 above show

$$\Sigma\, \{t(\mu(n) : n \in \omega\} \notin v.$$

So $(ii)_{\mu+1}$ holds. If $cf(\mu) < \omega$, and if

$$I \subseteq \cup\{B(\delta-1,\nu) : \nu < \mu\} \text{ with } |I| \le \omega,$$

then $\Pi I, \Sigma I \in B(\delta-1,\xi)$ for some $\xi < \mu$. So $(ii)_{\mu+1}$ holds. Therefore, $B(\delta-1) \cap v$ generates u. This is a contradiction.

CASE 3. $cf(\delta) = \omega$.

We enter case 2 of 2.2. List M with $\{M_\mu : \mu < \text{¢}\}$ and for $\mu < \text{¢}$ set $t(\mu) = \Pi M_\mu$ is $B(\delta)$. Then $\{t(\mu)\} \cup (t(\mu){\downarrow})$ in $B(\delta)$ is isomorphic to $B(0)$. So $t(\mu) \notin v \;\forall \mu < \text{¢}$. For each $\mu < \text{¢}$, set

$$C(\mu) = \omega cp{<<}(\cup\{t(\nu){\downarrow} : \nu < \mu\}) \cup (\cup\{B(\gamma) : \gamma < \delta\}){>>}.$$

Then $B(\delta) = \cup\{C(\mu) : \mu < \text{¢}\}$. The argument for this case can now be seen to be identical with the argument for case 2. Thus $cf(\delta) = \omega$ gives a contradiction.

We complete the proof of the Lemma by applying The Pressing Down Lemma to $B(\delta) = \cup\{B(\gamma) : \gamma < \delta\}$ in the obvious manner. □

THEOREM 5.2. *For each regular uncountable cardinal* $\Delta \leq \mathfrak{c}$ *there is a Parovičenko space* X_Δ *of weight* $\mathfrak{c}$ *possessing a dense orderable subspace* L_Δ *such that for each regular uncoutable cardinal* Γ, $\Delta < \Gamma \leq \mathfrak{c}$, L_Γ *is not densely embeddable into* X_Δ.

PROOF. Let T_Δ be the set of all $f \in {}^\alpha 2$ such that $\alpha \in \Delta$, and either $\alpha = 0$ or $cf(\alpha) = \omega$, and f is constant on a tail of β whenever $\omega < cf(\beta) \leq \beta < \alpha$. We order T_Δ by reverse inclusion; therefore, we immediately find that T_Δ is ω-closed, $\Delta(T_\Delta) = \Delta$, and T_Δ satisfies 2.1 (4); 2.1 (3) is satisfied since $f \in T_\Delta$ and $0 < dom(f)$ implies $cf(dom(f)) = \omega$. To see that $|T_\Delta| = \mathfrak{c}$, observe that $T_\Delta \omega_1 = T_{\omega_1}$ and $|T_{\omega_1}| = 2^{\omega_1} = \mathfrak{c}$, and then use induction.

Since T_Δ satisfies the input conditions for ω, we enter it into the machine. Let $X_\Delta = X(T_\Delta,\omega)$ and L_Δ be the dense subspace provided by 4.3. Then the theorem follows from 3.2 and 5.1. □

REMARKS 5.2.

(1) There are "easier" methods of obtaining Parovičenko spaces with dense, orderable subspaces. One such method, described verbally to the author by E. VAN DOUWEN, uses the machinery for lifting maps from $X \times N$ to X up to $\beta(X \times N) - (X \times N)$. Yet another method uses 1.2 (2) for the clopen sets of $\beta N - N$ to form a quotient space by collapsing the intersection of each branch of the tree. However, in the latter method it is not clear how to obtain 5.2.

(2) The reader might observe that 5.2 yields another proof of the main result of [4]. Further, using the limit ordinal levels of the complete binary tree of height ω_2 along with T_{ω_1} proves the main result of [12].

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ERRATA TO "ORDERABILITY AND SUBORDERABILITY RESULTS FOR TOTALLY DISCONNECTED SPACES" IN TOPOLOGY AND ORDER STRUCTURES, I

S. PURISCH

It was stated that the earliest orderability result of which the author is aware if the 1910 article by L.E.J. BROUWER characterizing the Cantor set. This is true for totally disconnected spaces. However in 1905 O. VEBLEN (Theory of plane curves in nonmetrical analysis situs, Trans. Amer. Math. Soc. 6 (1905), 83-98) proved that every metric continuum with exactly two non cut points is homeomorphic to the unit interval. VEBLEN also combined the concepts of ordered set and topological space in defining a simple arc.