Chapter 11

Between Mathematical Programming and Systems Theory: Linear Complementarity Systems

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ABSTRACT *Complementarity systems arise from the interconnection of an input-output system (of the type well known in mathematical systems theory) with a set of complementarity conditions (of the type well known in mathematical programming). It is shown by means of a list of examples that complementarity systems appear quite naturally in a broad range of applications. A solution concept for linear complementarity systems is provided, and conditions for existence and uniqueness of solutions are given.*

11.1 Introduction

Inequalities have played a rather minor role in the powerful development of systems theory that has taken place since about 1960. In contrast to this, they are central to the field of mathematical programming that has likewise seen major advances in the past decades. Of course, systems theory is concerned with differential equations; mixing these with inequalities means giving up the smoothness properties that form the basis of much of the theory of dynamical systems. Technological innovation, however, pushes toward the consideration of systems of a mixed continuous/discrete nature, which are likely to be described by systems of differential equations as well as algebraic equations and inequalities. In fact there are many situations in which there are good reasons to consider dynamics in conjunction with inequalities; think for instance of saturation effects in control systems, unilateral constraints in robotics, piecewise linear dynamics, and so on.

Among the many systems of equations and inequalities that one may imagine, the ones that are in so-called *complementarity* form enjoy particular attention in mathematical programming. More specifically the *linear complementarity problem* (LCP) has been the subject of extensive study because of its wide range of applications; see the book by Cottle et al. [5] for a comprehensive treatment. The LCP may be formulated as follows: given a vector $q \in \mathbb{R}^k$ and a matrix M of size $k \times k$, find vectors y and u

in $\mathbb{R}^k$ that satisfy the affine relation

$$y = q + Mu \tag{11.1}$$

and the complementarity conditions

$$\forall i = 1, \ldots, k: \quad y_i \geq 0, \ \ u_i \geq 0, \quad y_i u_i = 0. \tag{11.2}$$

Under suitable conditions this problem has exactly one solution. This chapter is concerned with the dynamical systems that one obtains when the static relation (11.1) is replaced by a dynamic relation of the form, for instance,

$$\dot{x}(t) = f(x(t), u(t)), \qquad y(t) = h(x(t), u(t)). \tag{11.3}$$

So effectively we have a dynamical input-output system of the type studied in systems and control theory, and we couple it with the complementarity conditions that appear in the LCP. Dynamical systems that are obtained in this way are called *complementarity systems*. Section 11.2 of the paper lists a number of situations in which one finds dynamics of this kind. The fact that the algebraic LCP has unique solutions under suitable circumstances leads to the suggestion that, under certain conditions, complementarity systems may also have unique solutions. Much of this chapter is indeed concerned with finding conditions for existence and uniqueness of solutions of complementarity systems. In particular we consider *linear* complementarity systems in which the dynamic relation (11.3) is specialized to the linear time-homogeneous equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t). \tag{11.4}$$

The idea of applying complementarity conditions to external variables of general dynamical systems originates in [35]. In the specific context of mechanical systems the use of complementarity conditions, which in this case relates to the presence of unilateral constraints, goes back much further and can in fact be traced to work by Fourier and by Farkas for the static (equilibrium) case and papers by Moreau and by Lötstedt for the dynamic case; see [35] for a brief review. The theory of complementarity systems has been further developed and considerably expanded in a number of recent papers, see for instance [36, 21, 20, 29, 7, 16, 18, 19, 17]. It is the purpose of the present chapter to give a survey of results obtained in these papers. Some new material is added in particular with respect to possible applications of complementarity systems.

Parts of this survey have been taken from joint papers with Kanat Çamlıbel, Maurice Heemels, Arjan van der Schaft, and Siep Weiland. It is a pleasure to acknowledge their contributions. Compared to Chapter 4 of [37], we concentrate here on the case of linear complementarity systems.

The chapter is structured as follows. Motivation for the framework of complementarity systems is provided in the next section by means of a number of examples. The issue of existence and uniqueness of solutions is introduced briefly in Section 11.3. We then turn to linear complementarity systems in Section 11.4, where a complete specification of the dynamics (including event rules) is provided. A distributional framework is sketched in Section 11.5, and some results on existence and uniqueness of solutions are given in Section 11.6. Section 11.7 is concerned with an application to relay systems. One of the ways in which linear complementarity systems differ from common nonlinear systems is that there may be discontinuous dependence on initial conditions; a simple example of this is provided in Section 11.8. Conclusions follow in Section 11.9.

For use below we mention here a basic fact about the linear complementarity problem (11.1) and (11.2). The LCP (11.1) and (11.2) has a unique solution (y, u) for all q if and only if all principal minors of the matrix M are positive [34; 5, Theorem 3.3.7]. (Given a matrix M of size $k \times k$ and two nonempty subsets I and J of $\{1, \ldots, k\}$ of equal cardinality, the (I, J)-*minor* of M is the determinant of the square submatrix $M_{IJ} := (m_{ij})_{i \in I, j \in J}$. The *principal minors* are those with $I = J$ [11, p. 2].) A matrix all of whose principal minors are positive is called a P-matrix. For example, all positive definite matrices are P-matrices. This is even true when the term "positive definite" is understood to apply not only to symmetric matrices, but also to nonsymmetric matrices M that are such that $x^T M x > 0$ for all $x \neq 0$.

11.2 Examples

11.2.1 Circuits with ideal diodes

A large amount of electrical network modeling is carried out on the basis of ideal lumped elements: resistors, inductors, capacitors, diodes, and so on. There is not necessarily a one-to-one relation between the elements in a model and the parts of the actual circuit; for instance, a resistor may under some circumstances be better modeled by a parallel connection of an ideal resistor and an ideal capacitor than by an ideal resistor alone. The standard ideal elements should rather be looked at as forming a construction kit from which one can quickly build a variety of models.

Among the standard elements the ideal diode has its own place because of the nonsmoothness of its characteristic. In circuit simulation software that has no ability to cope with mode changes, the ideal diode can not be admitted as a building block and will have to be replaced for instance by a heavily nonlinear resistor; a price will have to be paid in terms of speed of simulation. The alternative is to work with a hybrid system simulator; more specifically, the software will have to be able to deal with complementarity systems.

To write the equations of a network with (say) k ideal diodes in complementarity form, first extract the diodes so that the network appears as a k-port. For each port, we have a choice between denoting voltage by u_i and current by y_i or vice versa (with the appropriate sign conventions). Usually it is possible to make these choices in such a way that the dynamics of the k-port can be written as

$$\dot{x} = f(x,u), \qquad y = h(x,u).$$

For linear networks, one can actually show that it is *always* possible to write the dynamics in this form. To achieve this, it may be necessary to let u_i denote voltage at some ports and current at some other ports; in that case one sometimes speaks of a "hybrid" representation, where of course the term is used in a different sense than in this chapter. Replacing the ports by diodes, we obtain a representation in the semi-explicit complementarity form (11.3).

For electrical networks it would seem reasonable in most cases to assume that there are no jumps in the continuous state variables, so that there is no need to specify event conditions in addition to the flow conditions (11.3). Complementarity systems in general do not always have continuous solutions, so if one wants to prove that electrical networks with ideal diodes do indeed have continuous solutions, one will have to make a connection with certain specific properties of electrical networks. The passivity property is one that immediately comes to mind, and indeed there are certain conclusions that can be drawn from passivity and that are relevant in the study of properties of complementarity systems. To illustrate this, consider the specific case of a linear passive system coupled with a number of ideal diodes. The system is described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ 0 &\leq y \perp u \geq 0, \end{aligned} \tag{11.5}$$

where the last line is a shorthand for the complementarity conditions (11.2). Under the assumption that the system representation is minimal, the passivity property implies (see [39]) that there exists a positive definite matrix Q such that

$$\begin{bmatrix} A^TQ + QA & QB - C^T \\ B^TQ - C & -(D + D^T) \end{bmatrix} \leq 0. \tag{11.6}$$

If now for instance the matrix D is nonsingular, then it follows that D is actually positive definite. Under this condition one can prove that the complementarity system (11.5) has continuous solutions. If on the other hand D is equal to zero, then the passivity condition (11.6) implies that $C = B^TQ$ so that in this case the matrix $CB = B^TQB$ is positive definite (assuming that B has full column rank). One can prove that under this condition the system (11.5) has solutions with continuous state trajectories, if the system is consistently initialized (i. e., the initial condition x_0 satisfies $Cx_0 \geq 0$). The importance of the matrices D and CB derives from the fact that they appear in the power series expansion of the transfer matrix $C(sI - A)^{-1}B + D$ around infinity:

$$C(sI - A)^{-1}B + D \;=\; D + CBs^{-1} + CABs^{-2} + \cdots .$$

We return to this when we discuss linear complementarity systems.

11.2.2 Mechanical systems with unilateral constraints

Mechanical systems with geometric inequality constraints are given by equations of the following form (see [35]), in which $\partial H/\partial p$ and $\partial H/\partial q$ denote column vectors of partial derivatives, and the time arguments of q, p, y, and u have been omitted for brevity.

$$\begin{array}{rcll} \dot{q} & = & \frac{\partial H}{\partial p}(q,p) & q \in \mathbb{R}^n,\ p \in \mathbb{R}^n \\ \dot{p} & = & -\frac{\partial H}{\partial q}(q,p) + \frac{\partial C^T}{\partial q}(q)u & u \in \mathbb{R}^k \\ y & = & C(q) & y \in \mathbb{R}^k \\ 0 & \leq & y \perp u \geq 0. & \end{array} \qquad (11.7)$$

Here, $C(q) \geq 0$ is the column vector of geometric inequality constraints, and $u \geq 0$ is the vector of Lagrange multipliers producing the constraint force vector $(\partial C/\partial q)^T(q)u$. (The expression $(\partial C^T/\partial q)$ denotes an $n \times k$ matrix whose ith column is given by $\partial C_i/\partial q$.) The complementarity conditions in this case express that the i-th component of u_i can be only nonzero if the ith constraint is active; that is, $y_i = C_i(q) = 0$. Furthermore, $u_i \geq 0$ since the constraint forces will be always pushing in the direction of rendering y_i nonnegative. This basic principle of handling geometric inequality constraints can be found, for example, in [32, 26], and dates back to Fourier and Farkas. The Hamiltonian $H(q,p)$ denotes the total energy, generally given as the sum of a kinetic energy $1/2\, p^T M^{-1}(q)p$ (where $M(q)$ denotes the mass matrix, depending on the configuration vector q) and a potential energy $V(q)$. The semiexplicit complementarity system (11.7) is called a Hamiltonian complementarity system, since the dynamics of every mode is Hamiltonian [35]. In particular, every mode is energy-conserving (since the constraint forces are workless); it should be noted though that the model could be easily extended to mechanical systems with dissipation by replacing the second set of equations of (11.7) by

$$\dot{p} = -\frac{\partial H}{\partial q}(q,p) - \frac{\partial R}{\partial \dot{q}}(\dot{q}) + \frac{\partial C^T}{\partial q}(q)u, \qquad (11.8)$$

where $R(\dot{q})$ denotes a Rayleigh dissipation function.

11.2.3 Optimal control with state constraints

The purpose of this subsection is to indicate how one may relate optimal control problems with state constraints to complementarity systems. The study of this subject is far from being complete and so we offer some suggestions rather than present a rigorous treatment. Consider the problem of maximizing a functional of the form

$$\int_0^T F(t, x(t), u(t))dt + F_T(x(T)) \tag{11.9}$$

over a collection of trajectories described by

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = x_0 \tag{11.10}$$

together with the constraints

$$g(t, x(t), u(t)) \geq 0. \tag{11.11}$$

In the above, g may be a vector-valued function, and then the inequalities are taken componentwise. Under suitable conditions (see [13] for much more detailed information), candidates for optimal solutions can be found by solving a system of equations that is obtained as follows. Let λ be a vector variable of the same length as x, and define the *Hamiltonian* $H(t, x, u, \lambda)$ by

$$H(t, x, u, \lambda) = F(t, x, u) + \lambda^T f(t, x, u). \tag{11.12}$$

Also, let η be a vector of the same length as g, and define the *Lagrangian* $L(t, x, u, \lambda, \eta)$ by

$$L(t, x, u, \lambda, \eta) = H(t, x, u, \lambda) + \eta^T g(t, x, u). \tag{11.13}$$

The system referred to before is now the following.

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{11.14a}$$

$$\dot{\lambda}(t) = -\frac{\partial L}{\partial x}(t, x(t), u(t), \lambda(t), \eta(t)) \tag{11.14b}$$

$$u(t) = \arg \max_{\{u \mid g(t,x(t),u) \geq 0\}} L(t, x(t), u, \lambda(t), \eta(t)) \tag{11.14c}$$

$$0 \leq g(t, x(t), u(t)) \perp \eta(t) \geq 0 \tag{11.14d}$$

with initial conditions

$$x(0) = x_0 \tag{11.15}$$

and final conditions

$$\lambda(T) = \frac{\partial F_T}{\partial x}(x(T)). \tag{11.16}$$

Suppose that $u(t)$ can be solved from (11.14c) so that

$$u(t) = u^*(t, x(t), \lambda(t), \eta(t)), \tag{11.17}$$

where $u^*(t, x, \lambda, \eta)$ is a certain function. Then define $g^*(t, x, \lambda, \eta)$ by

$$g^*(t, x, \lambda, \eta) = g(t, x, u^*(t, x, \lambda, \eta), \lambda, \eta) \tag{11.18}$$

and introduce a variable $y(t)$ by

$$y(t) = g^*(t, x(t), \lambda(t), \eta(t)). \tag{11.19}$$

The system (11.14) can now be rewritten as

$$\begin{aligned}
\dot{x}(t) &= f(t, x(t), u^*(t, x(t), \lambda(t), \eta(t))) \\
\dot{\lambda}(t) &= -\tfrac{\partial L}{\partial x}(t, x(t), u^*(t, x(t), \lambda(t), \eta(t)), \lambda(t), \eta(t)) \\
y(t) &= g^*(t, x(t), \lambda(t), \eta(t)) \\
0 \leq y(t) &\perp \eta(t) \geq 0.
\end{aligned} \tag{11.20}$$

Here we have a (time-inhomogeneous) complementarity system with state variables x and λ and complementary variables y and η. The system has mixed boundary conditions (11.15) and (11.16); therefore one will have existence and uniqueness of solutions under conditions that in general will be different from the ones that hold for initial value problems.

A case of special interest is the one in which a quadratic criterion is optimized for a linear time-invariant system, subject to linear inequality constraints on the state. Consider for instance the following problem: minimize

$$\tfrac{1}{2} \int_0^T (x(t)^T Q x(t) + u(t)^T u(t)) dt \tag{11.21}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{11.22}$$

$$Cx(t) \geq 0, \tag{11.23}$$

where A, B, and C are matrices of appropriate sizes, and Q is a nonnegative definite matrix. Following the scheme above leads to the system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0 \tag{11.24a}$$

$$\dot{\lambda} = Qx - A^T\lambda - C^T\eta, \qquad \lambda(T) = 0 \tag{11.24b}$$

$$u = \arg\max\,[-\tfrac{1}{2}u^T u + \lambda^T Bu] \tag{11.24c}$$

$$0 \leq Cx \perp \eta \geq 0, \tag{11.24d}$$

where we have omitted the time arguments for brevity. Solving for u from (11.24c) leads to the equations

$$\frac{d}{dt}\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A & BB^T \\ Q & -A^T \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ -C^T \end{bmatrix}\eta \tag{11.25a}$$

$$y = [C \;\; 0]\begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{11.25b}$$

$$0 \leq y \perp \eta \geq 0. \tag{11.25c}$$

Not surprisingly, this is a linear complementarity system.

The study of optimal control problems subject to state constraints is fraught with difficulties; see Hartl et al. [13] for a discussion. The setting of complementarity systems may be of help at least in categorizing these difficulties.

11.2.4 Variable-structure systems

Consider a nonlinear input-output system of the form

$$\dot{x} = f(x, \bar{u}), \qquad \bar{y} = h(x, \bar{u}) \tag{11.26}$$

in which the input and output variables are adorned with a bar for reasons that become clear in a moment. Suppose that the system is in feedback coupling with a relay element given by the propositional formula

$$\{\{\bar{u} = 1\} \wedge \{\bar{y} \geq 0\}\} \vee \{\{-1 \leq \bar{u} \leq 1\} \wedge \{\bar{y} = 0\}\} \vee \{\{\bar{u} = -1\} \wedge \{\bar{y} \leq 0\}\}. \tag{11.27}$$

Many of the systems considered in the well-known book by Filippov on discontinuous dynamical systems [10] can be rewritten in this form. At first sight, relay systems do not seem to fit in the complementarity framework. However, let us introduce new variables y_1, y_2, u_1, and u_2, together with the following new equations.

$$\begin{aligned} u_1 &= \tfrac{1}{2}(1 - \bar{u}) \\ u_2 &= \tfrac{1}{2}(1 + \bar{u}) \\ \bar{y} &= y_1 - y_2. \end{aligned} \tag{11.28}$$

Instead of considering (11.26) together with (11.27), we can also consider (11.26) together with the standard complementarity conditions for the vectors $y = \mathrm{col}(y_1, y_2)$ and $u = \mathrm{col}(u_1, u_2)$:

$$\{\{\{y_1 = 0\} \wedge \{u_1 \geq 0\}\} \vee \{\{y_1 \geq 0\} \wedge \{u_1 = 0\}\}\} \wedge \{\{\{y_2 = 0\} \wedge \{u_2 \geq 0\}\} \vee \{\{y_2 \geq 0\} \wedge \{u_2 = 0\}\}\}. \tag{11.29}$$

It can be easily verified that the trajectories of (11.26), (11.28), and (11.29) are the same as those of (11.26) and (11.27). Note in particular that, although (11.29) in principle allows four modes, the conditions (11.28) imply that $u_1 + u_2 = 1$ so that the mode in which both u_1 and u_2 vanish is excluded, and the actual number of modes is three.

So it turns out that we can rewrite a relay system as a complementarity system, at least if we are willing to accept that some algebraic equations appear in the system description. It is possible to eliminate the variables $\bar{y}$ and $\bar{u}$ and obtain equations in the form

$$\begin{aligned} \dot{x} &= f(x, u_2 - u_1) \\ y_1 - y_2 &= h(x, u_2 - u_1) \\ u_1 + u_2 &= 1 \end{aligned} \tag{11.30}$$

together with the complementarity conditions (11.29), but (11.30) is not in standard input-state-output form but rather in a DAE type of form

$$F(\dot{x}, x, y, u) = 0. \tag{11.31}$$

If the relay is a part of a model whose equations are built up from submodels, then it is likely anyway that the system description will already be in terms of both differential and algebraic equations, and then it may not be much of a problem to have a few algebraic equations added (depending on how the "index" of the system is affected). Alternatively however one may replace the equations (11.28) by

$$\begin{aligned} u_1 &= \tfrac{1}{2}(1 - \bar{u}) \\ y_2 &= \tfrac{1}{2}(1 + \bar{u}) \\ \bar{y} &= y_1 - u_2 \end{aligned} \tag{11.32}$$

which are the same as (11.28) except that y_2 and u_2 have traded places. The equations (11.30) can now be rewritten as

$$\begin{aligned} \dot{x} &= f(x, 1 - 2u_1) \\ y_1 &= h(x, 1 - 2u_1) + u_2 \\ y_2 &= 1 - u_1 \end{aligned} \tag{11.33}$$

and this system does appear in standard input-output form. The only concession one has to make here is that (11.33) will have a feedthrough term (i. e., the output y depends directly on the input u) even when this is not the case in the original system (11.26).

11.2.5 Piecewise linear systems

Suppose that a linear system is coupled with a control device that switches among several linear low-level controllers depending on the state of the controlled system, as is the case for instance in many gain scheduling controllers; then the closed-loop system may be described as a piecewise linear system. Another way in which piecewise linear systems may arise is as approximations to nonlinear systems. Modeling by means of piecewise linear systems is attractive because it combines the relative tractability of linear dynamics with a flexibility that is often needed for a precise description of dynamics through a range of operating conditions.

There exist definitions of piecewise linear systems at various levels of generality. Here we limit ourselves to systems of the following form (time arguments omitted for brevity).

$$\dot{x} = Ax + Bu \qquad (11.34a)$$

$$y = Cx + Du \qquad (11.34b)$$

$$(y_i, u_i) \in \text{graph}(f_i) \qquad (i = 1, \ldots, k), \qquad (11.34c)$$

where, for each i, f_i is a piecewise linear function from $\mathbb{R}$ to $\mathbb{R}^2$. As is common usage, we use the term "piecewise linear" to refer to functions that would in fact be more accurately described as being piecewise affine. We consider functions f_i that are continuous, although from some points of view it would be natural to include also discontinuous functions; for instance systems in which the dynamics is described by means of piecewise constant functions have attracted attention in hybrid systems theory.

The model (11.34) is natural for instance as a description of electrical networks with a number of piecewise linear resistors. Descriptions of this form are quite common in circuit theory (cf. [28]). Linear relay systems are also covered by (11.34); note that the "sliding mode" corresponding to the vertical part of the relay characteristic is automatically included. Piecewise linear friction models are often used in mechanics (for instance Coulomb friction), which again leads to models like (11.34).

One needs to define a solution concept for (11.34); in particular, one has to say in what function space one will be looking for solutions. With an eye on the intended applications, it seems reasonable to require that the trajectories of the variable x should be continuous and piecewise differentiable. As for the variable u, some applications suggest that it may be too much to require continuity for this variable as well. For an example of this, take a mass point that is connected by a linear spring to a fixed wall, and that can move in one direction subject to Coulomb friction. In a model for this situation the variable u would play the role of the friction force which, according to the Coulomb model, has constant magnitude as long as the mass point is moving, and has sign opposite to the direction of motion. If the mass point is given a sufficiently large initial velocity away from the fixed wall, it will come to a standstill after some time and then immediately be pulled back toward the wall, so that in this case the friction force jumps instantaneously from one end of its interval of possible values to the other. Even allowing jumps in the variable u, we can still define a solution of (11.34) to be a triple (x, u, y) such that (11.34b) and (11.34c) hold for almost all t, and (11.34a) is satisfied in the sense of Carathéodory; that is to say,

$$x(t) = x(0) + \int_0^t [Ax(\tau) + Bu(\tau)]d\tau \qquad (11.35)$$

for all t.

The first question that should be answered in connection with the system (11.34) is whether solutions exist and are unique. For this, one should first of all find conditions under which, for a given initial condition $x(0) = x_0$, there exists a unique continuation in one of the possible "modes" of the systems (corresponding to all possible combinations of the different branches of the piecewise linear characteristics of the system). This can be a highly nontrivial problem; for instance in a mechanical system with many friction points, it may not be so easy to say at which points sliding will take place and at which points stick will occur. It turns out to be possible to address the problem on the basis of the theory of the linear complementarity problem and extensions of it. For the case of Coulomb friction, also in combination with nonlinear dynamics, this is worked out in [33]. The general case can be developed on the basis of a theorem by Kaneko and Pang [25], which states that any piecewise linear characteristic can be described by means of the so-called *extended horizontal linear complementarity problem*. On this basis, the piecewise linear system (11.34) may also be described as an extended horizontal linear complementarity system. Results on the solvability of the EHLCP have been given by Sznajder and Gowda [38]. Using these results, one can obtain sufficient conditions for the existence of unique solution starting at a given initial state; see [4] for details.

11.2.6 Projected dynamical systems

The concept of equilibrium is central to mathematical economics. For instance, one may consider an oligopolistic market in which several competitors determine their production levels so as to maximize their profits; it is of interest to study the equilibria that may exist in such a situation. On a wider scale, one may discuss general economic equilibrium involving production, consumption, and prices of commodities. In fact in all kinds of competitive systems the notion of equilibrium is important.

The term "equilibrium" can actually be understood in several ways. For instance, the celebrated Nash equilibrium concept of game theory is defined as a situation in which no player can gain by unilaterally changing his position. Similar notions in mathematical economics lead to concepts of equilibria that can be characterized in terms of systems of *algebraic* equations and inequalities. On the other hand, we have the classical notion of equilibrium in the theory of dynamical systems, where the concept is defined in terms of a given set of *differential* equations. It is natural to expect, though, that certain relations can be found between the static and dynamic equilibrium concepts.

In 1993, Dupuis and Nagurney [9] have proposed a general strategy to embed a given static equilibrium problem into a dynamic system. Dupuis and Nagurney assume that the static equilibrium problem can be formulated in terms of a *variational equality*; that is to say, the problem is specified by giving a closed convex subset K of $\mathbb{R}^k$ and a function F from K to $\mathbb{R}^k$, and $\bar{x} \in K$ is an equilibrium if

$$\langle F(\bar{x}), x - \bar{x} \rangle \geq 0 \tag{11.36}$$

for all $x \in K$. The formulation in such terms is standard within mathematical programming. With the variational problem they associate a discontinuous dynamical system that is defined by $\dot{x} = -F(x)$ on the interior of K but which is defined differently on the boundary of K in such a way as to make sure that solutions will not leave the convex set K. They then prove that the stationary points of the so-defined dynamical system coincide with the solutions of the variational equality.

In some more detail, the construction proposed by Dupuis and Nagurney can be described as follows. The space $\mathbb{R}^k$ in which state vectors take their values is taken as a Euclidean space with the usual inner product. Let P denote the mapping that assigns to a given point x in $\mathbb{R}^k$ the (uniquely defined) point in K that is closest to x; that is to say,

$$P(x) = \arg \min_{z \in K} ||x - z||. \tag{11.37}$$

For $x \in K$ and a velocity vector $v \in \mathbb{R}^k$, let

$$\pi(x, v) = \lim_{\delta \to 0} \frac{P(x + \delta v) - x}{\delta}. \tag{11.38}$$

If x is in the interior of K, then clearly $\pi(x, v) = v$; however if x is on the boundary of K and v points outwards then $\pi(x, v)$ is a modification of v. The dynamical system considered by Dupuis and Nagurney is now defined by

$$\dot{x} = \pi(x, -F(x)) \tag{11.39}$$

with initial condition x_0 in K. The right-hand side of this equation is in general subject to a discontinuous change when the state vector reaches the boundary of K. The state may then follow the boundary along a $(k-1)$-dimensional surface or a part of the boundary characterized by more than one constraint, and after some time it may reenter the interior of K after which it may again reach the boundary, and so on.

In addition to the expression (11.38) Dupuis and Nagurney also employ a different formulation which has been used in [8]. For this, first introduce the set of inward normals that is defined, for a boundary point x of K, by

$$n(x) = \{\gamma \mid \|\gamma\| = 1, \text{ and } \langle \gamma, x - y \rangle \leq 0, \ \forall y \in K\}. \tag{11.40}$$

If K is a convex polyhedron then we may write

$$\pi(x, v) = v + \langle v, -\gamma^* \rangle \gamma^*, \tag{11.41}$$

where γ^* is defined by

$$\gamma^* := \arg \max_{\gamma \in n(x)} \langle v, -\gamma \rangle. \tag{11.42}$$

A further reformulation is possible by introducing the "cone of admissible velocities." To formulate this concept, first recall that a *curve* in $\mathbb{R}^k$ is a smooth mapping from an interval, say $(-1, 1)$, to $\mathbb{R}^k$. An *admissible velocity* at a point x with respect to the closed convex set $K \subset \mathbb{R}^k$ is any vector that appears as a directional derivative at 0 of a C^∞ function $f(t)$ that satisfies $f(0) = x$ and $f(t) \in K$ for $t \geq 0$. One can show that the set of admissible velocities is a closed convex cone for any x in the boundary of K; of course, the set of admissible velocities is empty when $x \notin K$ and coincides with $\mathbb{R}^k$ if x belongs to the interior of K. One can furthermore show (see [22]) that the mapping defined in (11.41) for given x is in fact just the projection to the cone of admissible velocities. In this way we get an alternative definition of projected dynamical systems. The new formulation is more "intrinsic" in a differential-geometric sense than the original one which is based on the standard coordinatization of k-dimensional Euclidean space. Indeed it would be possible in this way to formulate projected dynamics for systems defined on Riemannian manifolds; the inner product on tangent spaces that is provided by the Riemannian structure makes it possible to define the required projection. One possible application would be the use of projected gradient flows to find minima subject to constraints (cf. for instance [23] for the unconstrained case).

Assume now that the set K is given as an intersection of convex sets of the form $\{x \mid h_i(x) \geq 0\}$ where the h_is are smooth functions. This is actually the situation that one typically finds in applications. It is then possible to reformulate the projected dynamical system as a complementarity system. The construction is described in [22] and we summarize it briefly here. Let $H(x)$ denote the gradient matrix defined by the functions $h_i(x)$; that is to say, the (i, j)th element of $H(x)$ is

$$(H(x))_{ij} = \frac{\partial h_i}{\partial x_j}(x).$$

For $x \in K$, let $I(x)$ be the set of "active" indices; that is,

$$I(x) = \{i \mid h_i(x) = 0\}.$$

We denote by $H_{I(x)\bullet}$ the matrix formed by the rows of $H(x)$ whose indices are active; it will be assumed that this matrix has full row rank for all x in the boundary of K ("independent constraints"). One can then show that for each $x \in K$ the cone of admissible velocities is given by $\{v \mid H_{I(x)\bullet}v \geq 0\}$. Moreover, the set of inward normals as defined in (11.40) is given by $\{\gamma \mid ||\gamma|| = 1 \text{ and } \gamma = H^T_{I(x)\bullet}u \text{ for some } u\}$. Consequently, the projection of an arbitrary vector v_0 on the cone of admissible velocities is obtained by solving the minimization problem

$$\min_v \{||v_0 - v|| \mid H_{I(x)\bullet}v \geq 0\}.$$

By standard methods, one finds that the minimizer is given by $H^T_{I(x)\bullet}u$ where u is the (unique) solution of the complementarity problem

$$0 \leq H_{I(x)\bullet}v_0 + H_{I(x)\bullet}H^T_{I(x)\bullet}u \perp u \geq 0. \tag{11.43}$$

Now, compare the projected dynamical system (11.39) to the complementarity system defined by

$$\dot{x} = -F(x) + H^T(x)u \tag{11.44a}$$

$$y = h(x) \tag{11.44b}$$

$$0 \leq y \perp u \geq 0, \tag{11.44c}$$

where $h(x)$ is a vector defined in the obvious manner by $(h(x))_i = h_i(x)$, and where the trajectories of all variables are required to be continuous. Suppose that the system is initialized at $t = 0$ at a point x_0 in K. For indices i such that $h_i(x_0) > 0$, the complementarity conditions imply that we must have $u_i(0) = 0$. For indices that are active at x_0 we have $y_i(0) = 0$; to satisfy the inequality constraints also for positive t we need $\dot{y}_i(0) \geq 0$. Moreover, it follows from the complementarity conditions and the continuity conditions that we must have $u_i(0) = 0$ for indices i such that $\dot{y}_i(0) = 0$, and, vice versa, $\dot{y}_i(0) = 0$ for indices i such that $u_i(0) > 0$. Since

$$\begin{aligned} \dot{y}_{I(x_0)}(0) &= H_{I(x_0)\bullet}(-F(x_0) + H^T u(0)) \\ &= H_{I(x_0)\bullet}(-F(x_0) + H^T_{I(x_0)\bullet}u_{I(x_0)}(0)) \end{aligned}$$

the vector $u_{I(x_0)}(0)$ must be a solution of the complementarity problem (11.43). It follows that $H^T u(0)$ is of the form appearing in (11.41). The reverse conclusion follows as well, and moreover one can show that "local" equality of solutions as just shown implies "global" equality [22].

11.2.7 Diffusion with a free boundary

In this subsection we consider a situation in which a complementarity system arises as an approximation to a partial differential equation with a free boundary. We take a specific example that arises in the theory of option pricing. For this we first need to introduce some terminology. A *European put option* is a contract that gives the holder the right, but not the obligation, to sell a certain asset to the counterparty in the contract for a specified price (the "exercise price") at a specified time in the future ("time of maturity"). The underlying asset can for instance be a certain amount of stocks, or a certain amount of foreign currency. For a concrete example, consider an investor who has stocks that are worth 110 now and who would like to turn these stocks into cash in one year's time. Of course it is hard to predict what the value of the stocks will be at that time; to make sure that the proceeds will be at least 100, the investor may buy a put option with exercise price 100 that matures in one year. In this way the investor is sure that she can sell the stocks for at least 100.

Of course one has to pay a price to buy such protection, and it is the purpose of option theory to determine "reasonable" option prices. The modern theory of option pricing started in the early 1970s with the seminal work by Black, Scholes, and Merton. This theory is not based on the law of large numbers, but rather on the observation that the risk that goes with conferring an option contract is not as big as it would seem to be at first sight. By following an active trading strategy in the underlying asset, the seller ("writer") of the option will be able to reduce the risk. Under suitable model assumptions the risk can even be completely eliminated; that is to say, the cost of providing protection becomes independent of the evolution of the value of the underlying asset and hence can be predicted in advance. The "no-arbitrage" argument then states that this fixed cost must, by the force of competition, be the market price of the option. The model assumptions under which one can show that the risk of writing an option can be eliminated are too strong to be completely realistic; nevertheless, they provide a good guideline for devising strategies that at least are able to reduce risk substantially.

One of the assumptions made in the original work of Black and Scholes [3] is that the price paths of the underlying asset may be described by a stochastic differential equation of the form

$$dS(t) \;=\; \mu S(t)dt + \sigma S(t)dw(t), \tag{11.45}$$

where $w(t)$ denotes a standard Wiener process. (See any textbook on SDEs, for instance [31], for the meaning of the above.) Under a number of additional assumptions (for instance: the underlying asset can be traded continuously and without transaction costs, and there is one fixed interest rate r that holds both for borrowing and for lending), Black and Scholes derived a partial differential equation that describes the price of the option at any time before maturity as a function of two variables, namely, time t and the price S of the underlying asset. The Black–Scholes equation for the option price $C(S,t)$ is (with omission of the arguments)

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{11.46}$$

with end condition for time of maturity T and exercise price K

$$C(S,T) = \max(K - S, 0) \tag{11.47}$$

and boundary conditions

$$C(0,t) = e^{-r(T-t)}K, \qquad \lim_{S\to\infty} C(S,t) = 0. \tag{11.48}$$

It turns out that the "drift" parameter μ in equation (11.45) is immaterial, whereas the "volatility" parameter σ is very important since it determines the diffusion coefficient in the PDE (11.46).

So far we have been discussing a *European* put option. An *American* put option is the same except that the option may be exercised at any time until the maturity date, rather than only at the time of maturity. (The terms "European" and "American" just serve as a way of distinction; both types of options are traded on both continents.) The possibility of early exercise brings a discrete element into the discussion since at any time the option may be in two states: "alive" or "exercised." For American options, the Black–Scholes equation (11.46) is replaced by an inequality

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC \le 0 \tag{11.49}$$

in which equality holds if the option is not exercised, that is, if its value exceeds the revenues of exercise. For the put option, this is expressed by the inequality

$$C(S,t) > \max(K - S, 0). \tag{11.50}$$

The Black–Scholes equation (11.46) is a nonlinear partial differential equation. A simple substitution, however, will transform it to a linear PDE. To this end, express the price of the underlying asset S in terms of a new (dimensionless) independent variable x by

$$S = Ke^x.$$

To make the option price dimensionless as well, introduce $v := C/K$. We also change the final value problem (11.46) and (11.47) to an initial value problem by setting set $\tau = T - t$. After some computation, we find that the equation (11.46) is replaced by

$$-\frac{\partial v}{\partial \tau} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + (r - \tfrac{1}{2}\sigma^2)\frac{\partial v}{\partial x} - rv = 0 \tag{11.51}$$

with the initial and boundary conditions (for the European put option)

$$v(x,0) = \max(1-e^x,0), \quad \lim_{x\to-\infty} v(x,\tau) = e^{-r\tau}, \quad \lim_{x\to\infty} v(x,\tau) = 0. \tag{11.52}$$

For the American put option, we get the set of inequalities

$$\frac{\partial v}{\partial \tau} - \tfrac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} - (r - \tfrac{1}{2}\sigma^2)\frac{\partial v}{\partial x} + rv \geq 0 \tag{11.53}$$

$$v \geq \max(1-e^x,0) \tag{11.54}$$

with the boundary conditions

$$\lim_{x\to\infty} v(x,\tau) = 0 \tag{11.55}$$

and

$$v(x,\tau) = 1 - e^x \quad \text{for } x \leq x_f(\tau), \tag{11.56}$$

where $x_f(\tau)$ is the location at time τ of the free boundary that should be determined as part of the problem on the basis of the so-called "smooth pasting" or "high contact" conditions which require that v and $\partial v/\partial x$ should be continuous as functions of x across the free boundary.

Define the function $g(x)$ by

$$g(x) = \max(1-e^x,0). \tag{11.57}$$

The partial differential inequality (11.53) and its associated boundary conditions may then be written in the following form which is implicit with respect to the free boundary.

$$(\tfrac{\partial v}{\partial \tau} - \tfrac{1}{2}\sigma^2 \tfrac{\partial^2 v}{\partial x^2} - (r - \tfrac{1}{2}\sigma^2)\tfrac{\partial v}{\partial x} + rv)(v-g) = 0, \tag{11.58a}$$

$$\tfrac{\partial v}{\partial \tau} - \tfrac{1}{2}\sigma^2 \tfrac{\partial^2 v}{\partial x^2} - (r - \tfrac{1}{2}\sigma^2)\tfrac{\partial v}{\partial x} + rv \geq 0, \quad v - g \geq 0. \tag{11.58b}$$

This already suggests a complementarity formulation. Indeed, the above might be considered as an example of an infinite-dimensional complementarity system, both in the sense that the state space dimension is infinite and in the sense that there are infinitely many complementary variables. A complementary system of the type studied here is obtained by approximating the infinite-dimensional system by a finite-dimensional system. For the current application it seems reasonable to carry out the approximation by what is known in numerical analysis as the "method of lines." In this approach the space variable is discretized but the time variable is not. Specifically, take a grid of, say, N points in the space variable (in our case this is the dimensionless price variable x), and associate with each grid point x_i a variable $x_i(t)$ which is intended to be an approximation to $v(x_i, t)$. The action of the linear differential operator

$$v \mapsto \tfrac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + (r - \tfrac{1}{2}\sigma^2)\frac{\partial v}{\partial x} - rv \tag{11.59}$$

can be approximated by a linear mapping acting on the space spanned by the variables $x_1, \ldots, x_N$. For instance, on an evenly spaced grid an approximation to the first-order differential operator $\partial/\partial x$ is given by

$$A_1 = \frac{1}{2h}\begin{bmatrix} -2 & 2 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & 0 & & \vdots \\ 0 & -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & -1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & -2 & 2 \end{bmatrix}, \tag{11.60}$$

where h is the mesh size, and the second-order differential operator $\partial^2/\partial x^2$ is approximated by

$$A_2 = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & & \vdots \\ 0 & 1 & -2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{bmatrix}. \tag{11.61}$$

The mapping (11.59) is then approximated by the matrix

$$A = \tfrac{1}{2}\sigma^2 A_2 + (r - \tfrac{1}{2}\sigma^2)A_1 - rI. \tag{11.62}$$

The function $g(x)$ is represented by the vector g with entries $g_i = g(x_i)$. Consider now the linear complementary system with $N+1$ state variables and N pairs of complementary variables given by

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} I \\ 0 \end{bmatrix} u \tag{11.63a}$$

$$y = [\, I \;\; -g \,]\, x \tag{11.63b}$$

$$0 \leq y \perp u \geq 0. \tag{11.63c}$$

The system is initialized at

$$x_i(0) = g_i \qquad (i = 1, \ldots, N), \qquad x_{N+1}(0) = 1. \tag{11.64}$$

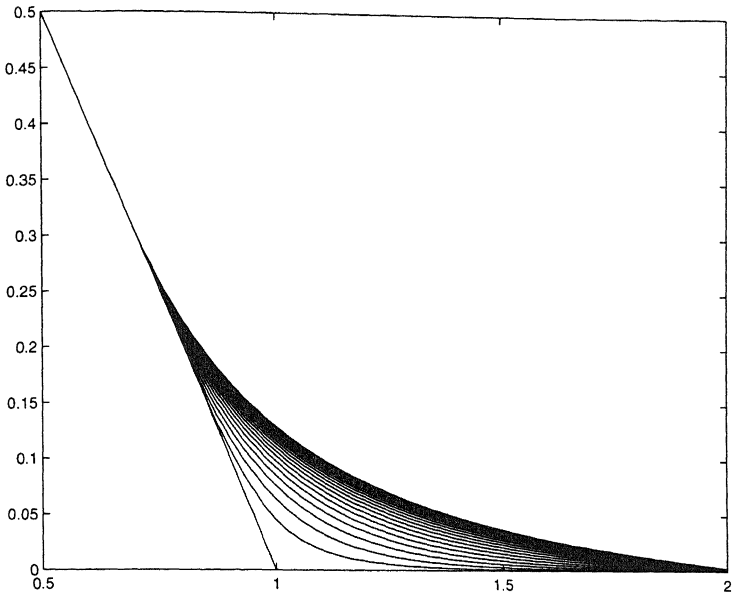

FIGURE 11.1. Value of an American put option at different times before expiry.

The complementarity system defined in this way is a natural candidate for providing an approximate solution to the diffusion equation with a free boundary that corresponds to the Black–Scholes equation for an American put option.

A natural idea for generating approximate solutions of complementarity systems is to use an implicit Euler method. For linear complementarity systems of the form (11.4) and (11.2), the method may be written as follows.

$$\frac{x((k+1)\Delta t) - x(k\Delta t)}{\Delta t} = Ax((k+1)\Delta t) + Bu((k+1)\Delta t) \quad (11.65a)$$

$$y((k+1)\Delta t) = Cx((k+1)\Delta t) + Du((k+1)\Delta t) \quad (11.65b)$$

$$0 \leq y((k+1)\Delta t) \perp u((k+1)\Delta t) \geq 0. \quad (11.65c)$$

At each step this gives rise to a complementarity problem which under suitable assumptions has a unique solution. The results of applying the above method to the equations for an American put option are shown in Figure 11.1, which presents solutions for various times before expiry as a function of the value of the underlying asset (the variable S).

A more standard numerical method for dealing with the American-style Black-Scholes equation is the finite difference method in which both the time and space variables are discretized; see, for instance, [40]. In general, an advantage of a "semidiscretization" approach over an approach in which all variables are discretized simultaneously is that one may make use of the highly developed theory of step-size control for numerical solutions of ODEs, rather than using a uniform grid. We are of course working here with a complementarity system rather than an ODE, and it must be said that the theory of step-size control for complementarity systems is at an early stage of development. Moreover, the theory of approximation of free boundary problems by complementarity systems has been presented here only for a special case and on the basis of plausibility rather than formal proof, and much further work on this topic is needed.

11.2.8 Max-plus systems

From the fact that the relation

$$z = \max(x, y) \tag{11.66}$$

may also be written in the form

$$z = x + a = y + b, \qquad 0 \leq a \perp b \geq 0 \tag{11.67}$$

it follows that any system that can be written in terms of linear operations and the "max" operation can also be written as a complementarity system. In particular it follows that the so-called max-plus systems (see [1]), which are closely related to timed Petri nets, can be written as complementarity systems. The resulting equations appear in discrete time, as opposed to the other examples in this section which are all in continuous time; note however that the "time" parameter in a max-plus system is in the standard applications a cycle counter rather than actual time. For further discussion of the relation between the max algebra and the complementarity problem see [6].

11.3 Existence and uniqueness of solutions

Complementarity systems can be looked at as a class of *hybrid* systems; this term is understood here as referring to systems that are described in terms of both continuous (real-valued) and discrete (finite-valued) variables. Hybrid systems provide a rather wide modeling context, so that there are no easily verifiable necessary and sufficient conditions for well-posedness of general hybrid dynamical systems. It is already of interest to give sufficient conditions for well-posedness of particular classes of hybrid systems, such as complementarity systems. The advantage of considering special classes is that one can hope for conditions that are relatively easy to verify. In a number of special cases, such as mechanical systems or electrical network models, there are moreover natural candidates for such sufficient conditions.

Uniqueness of solutions is always understood below in the sense of what is sometimes called *right uniqueness*, that is, uniqueness of solutions defined on an interval $[t_0, t_1)$ given an initial state at t_0. It can easily happen in general hybrid systems and even in complementarity systems that uniqueness holds in one direction of time but not in the other; this is one of the points in which discontinuous dynamical systems differ from smooth systems. We also allow for the possibility of an initial jump.

We have to distinguish between *local* and *global* existence and uniqueness. Local existence and uniqueness, for solutions starting at t_0, holds if there exists an $\varepsilon > 0$ such that on $[t_0, t_0 + \varepsilon)$ there is a unique solution starting at the given initial condition. For global existence and uniqueness, we require that for any given initial condition there is a unique solution on $[t_0, \infty)$. If local uniqueness holds for all initial conditions and existence holds globally, then uniqueness must also hold globally since there is no point at which solutions can split. However local existence does not imply global existence. This phenomenon is already well known in the theory of smooth dynamical systems; for instance the differential equation $\dot{x}(t) = x^2(t)$ with $x(0) = x_0$ has the unique solution $x(t) = x_0(1 - x_0 t)^{-1}$ which for positive x_0 is defined only on the interval $[0, x_0^{-1})$. Some growth conditions have to be imposed to prevent this "escape to infinity." In hybrid systems, there may be additional complications; in particular we may have an accumulation of mode switches such as in the following example that has been adapted from Filippov [10, p. 116].

Example 11.1. Consider the relay system given by

$$\begin{aligned} \dot{x}_1 &= -\operatorname{sgn} x_1 + 2 \operatorname{sgn} x_2 \\ \dot{x}_2 &= -2 \operatorname{sgn} x_1 - \operatorname{sgn} x_2. \end{aligned} \tag{11.68}$$

This is a *piecewise constant* system; in each quadrant of the (x_1, x_2)-plane the right-hand side is a constant vector. From each initial point there exists locally a unique solution. The solutions are spiraling towards the origin, which is an equilibrium point. It can be verified that $d/dt\,(|x_1(t)| + |x_2(t)|) = -2$ which means that solutions starting at $\operatorname{col}(x_{10}, x_{20})$ can not stay away from the origin for longer than $(|x_{10}| + |x_{20}|)/2$ units of time. However, solutions can not arrive at the origin without going through an infinite number of mode switches; since these mode switches have to occur in a finite time interval, we do not have a global solution in the space of piecewise differentiable functions (if only finitely many pieces are allowed). Some solutions of the system (11.68) are plotted in Figure 11.2.

Although the occurrence of an accumulation of mode switches would seem to be exceptional, no general conditions are known at present that exclude this phenomenon. Below we use the term *well-posedness* to refer to local existence and uniqueness of solutions for all feasible initial conditions (i. e., initial conditions for which none of the inequality constraints are

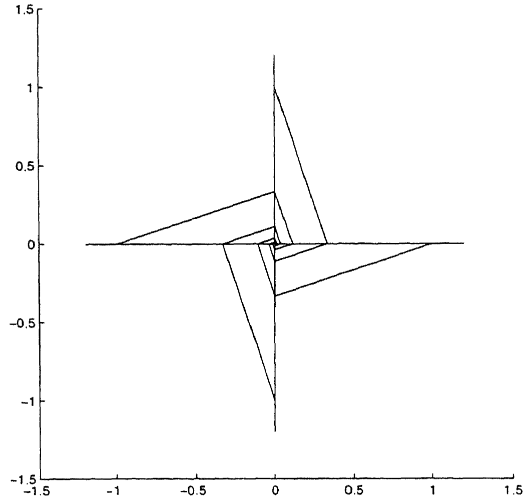

FIGURE 11.2. Solutions of Filippov's example (11.68).

violated).

11.4 Linear complementarity systems

Consider the following system of linear differential and algebraic equations and inequalities

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{11.69a}$$

$$y(t) = Cx(t) + Du(t) \tag{11.69b}$$

$$0 \leq y(t) \perp u(t) \geq 0. \tag{11.69c}$$

The equations (11.69a) and (11.69b) constitute a linear system in state space form; the number of inputs is taken equal to the number of outputs. The relations (11.69c) are the usual *complementarity conditions*. The set of indices for which $y_i(t) = 0$ (we call this the *active index set*) need not be constant in time, so that the system may switch from one "operating mode" to another. To define the dynamics of (11.69) completely, we have to specify when these mode switches occur, what their effect will be on the state variables, and how a new mode will be selected. A proposal for answering these questions (cf. [21]) is explained below. The specification of the complete dynamics of (11.69) defines a class of dynamical systems called *linear complementarity systems*.

Let n denote the length of the vector $x(t)$ in the equations (11.69a) and (11.69b) and let k denote the number of inputs and outputs. There are then 2^k possible choices for the active index set. The equations of motion when the active index set is I are given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ y_i(t) &= 0, \quad i \in I \\ u_i(t) &= 0, \quad i \in I^c, \end{aligned} \tag{11.70}$$

where I^c denotes the index set that is complementary to I; that is, $I^c = \{i \in \{1, \ldots, k\} \mid i \notin I\}$. We say that the above equations represent the system in *mode* I. An equivalent and somewhat more explicit form is given by the (generalized) state equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_{\bullet I} u_I(t) \\ 0 &= C_{I\bullet} x(t) + D_{II} u_I(t) \end{aligned} \tag{11.71}$$

together with the output equations

$$\begin{aligned} y_{I^c}(t) &= C_{I^c\bullet} x(t) + D_{I^c I} u_I(t) \\ u_{I^c}(t) &= 0. \end{aligned} \tag{11.72}$$

Here and below, the notation $M_{\bullet I}$, where M is a matrix of size $m \times k$ and I is a subset of $\{1, \ldots ,k\}$, denotes the submatrix of M formed by taking the columns of M whose indices are in I. The notation $M_{I\bullet}$ denotes the submatrix obtained by taking the rows with indices in the index set I.

The system (11.71) in general does not have solutions in a classical sense for all possible initial conditions. The initial values of the variable x for which there does exist a continuously differentiable solution are called *consistent states*. Under conditions that are specified below, each consistent initial state gives rise to a unique solution of (11.71). The system (11.69) follows the path of such a solution (it "stays in mode I") as long as the variables $u_I(t)$ defined implicitly by (11.71) and the variables $y_{I^c}(t)$ defined by (11.72) are all nonnegative. As soon as continuation in mode I would lead to a violation of these inequality constraints, a switch to a different mode has to occur. If moreover the value of the variable $x(t)$ at which violation of the constraints has become imminent is not a consistent state for the new mode, then a state jump is necessary. So both concerning the dynamics in a given mode and concerning the transitions between different modes there are a number of questions to be answered. For this we rely on the geometric theory of linear systems (see [41, 2, 4] for the general background).

Denote by V_I the *consistent subspace* of mode I; that is to say, V_I is the set of initial conditions x_0 for which there exist smooth functions $x(\cdot)$ and $u_I(\cdot)$, with $x(0) = x_0$, such that (11.71) is satisfied. The space V_I can be computed as the limit of the sequence defined by

$$V_I^0 = \mathbb{R}^n$$

$$V_I^{i+1} = \{x \in V_I^i \mid \exists u \in \mathbb{R}^{|I|} \text{ s.t. } Ax + B_{\bullet I} u \in V_I^i,\ C_{I\bullet} x + D_{II} u = 0\}. \tag{11.73}$$

There exists a linear mapping F_I such that (11.71) will be satisfied for $x_0 \in V_I$ by taking $u_I(t) = F_I x(t)$. The mapping F_I is uniquely determined, and more generally the function $u_I(\cdot)$ that satisfies (11.71) for given $x_0 \in V_I$ is uniquely determined, if the full column rank condition

$$\ker \begin{bmatrix} B_{\bullet I} \\ D_{II} \end{bmatrix} = \{0\} \tag{11.74}$$

holds and moreover we have

$$V_I \cap T_I = \{0\}, \tag{11.75}$$

where T_I is the subspace that can be computed as the limit of the following sequence.

$$\begin{aligned} T_I^0 &= \{0\} \\ T_I^{i+1} &= \{x \in \mathbb{R}^n \mid \exists \tilde{x} \in T_I^i,\ \tilde{u} \in \mathbb{R}^{|I|} \text{ s.t.} \\ &\qquad x = A\tilde{x} + B_{\bullet I}\tilde{u},\ C_{I\bullet}\tilde{x} + D_{II}\tilde{u} = 0\}. \end{aligned} \tag{11.76}$$

As indicated below, the subspace T_I is best thought of as the *jump space* associated with mode I, that is, as the space along which fast motions will occur that take an inconsistent initial state instantaneously to a point in the consistent space V_I; note that under the condition (11.75) this projection is uniquely determined. The projection can be used to define a *jump rule*. However, there are 2^k possible projections, corresponding to all possible subsets of $\{1, \ldots, k\}$; which one of these to choose should be determined by a *mode selection rule*.

For the formulation of a mode selection rule we have to relate index sets in some way to continuous states. Such a relation can be established on the basis of the so-called *rational complementarity problem* (RCP). The RCP is defined as follows. Let a rational vector $q(s)$ of length k and a rational matrix $M(s)$ of size $k \times k$ be given. The rational complementarity problem is to find a pair of rational vectors $y(s)$ and $u(s)$ (both of length k) such that

$$y(s) = q(s) + M(s)u(s) \tag{11.77}$$

and moreover for all indices $1 \leq i \leq k$ we have either $y_i(s) = 0$ and $u_i(s) > 0$ for all sufficiently large s, or $u_i(s) = 0$ and $y_i(s) > 0$ for all sufficiently large s. The vector $q(s)$ and the matrix $M(s)$ are called the *data* of the RCP, and we write RCP($q(s)$,$M(s)$). We also consider an RCP whose data are a quadruple of constant matrices (A, B, C, D) (such as could be used to define (11.69a) and (11.69b)) and a constant vector x_0, namely, by setting

$$q(s) = C(sI - A)^{-1}x_0 \quad \text{and} \quad M(s) = C(sI - A)^{-1}B + D.$$

We say that an index set $I \subset \{1,\ldots,k\}$ *solves* the RCP (11.77) if there exists a solution $(y(s), u(s))$ with $y_i(s) = 0$ for $i \in I$ and $u_i(s) = 0$ for $i \notin I$. The collection of index sets I that solve $\mathrm{RCP}(A, B, C, D; x_0)$ is denoted $\mathcal{S}(A, B, C, D; x_0)$ or simply $\mathcal{S}(x_0)$ if the quadruple (A, B, C, D) is given by the context.

After these preparations, we can now proceed to a specification of the complete dynamics of linear complementarity systems. We assume that a quadruple (A, B, C, D) is given whose transfer matrix $G(s) = C(sI - A)^{-1}B + D$ is *totally invertible*; that is, for each index set I the $k \times k$ matrix $G_{II}(s)$ is nonsingular. Under this condition (see Theorem 11.1), the two subspaces V_I and T_I as defined above form for all I a direct sum decomposition of the state space $\mathbb{R}^n$, so that the projection along T_I onto V_I is well defined. We denote this projection P_I. The interpretation that we give to the equations (11.69) is the following.

$$\begin{array}{llll} \dot{x} = Ax + Bu, & y = Cx + Du & & \\ u_I \geq 0, & y_I = 0, & u_{I^c} = 0, & y_{I^c} \geq 0 \\ I^\sharp \in \mathcal{S}(x), & x^\sharp = P_{I^\sharp}x, & & \end{array} \tag{11.78}$$

where the symbol $\sharp$ denotes "next value." In the expression above, we use a continuous state x and the continuous complementary variables y and u, as well as a discrete state I. The discrete state must switch when this is necessary to prevent violation of the inequality constraints; in general we allow the continuous state to jump, possibly even several times at the same instant (in this case we have an "event with multiplicity"). Below we always consider the system (11.69) in the interpretation (11.78).

11.5 A distributional interpretation

The interpretation of T_I as a jump space can be made precise by introducing the class of *impulsive-smooth distributions* that was studied by Hautus [14] (see also [15, 12]). The general form of an impulsive-smooth distribution ϕ is

$$\phi \;=\; p(d/dt)\delta \;+\; f \tag{11.79}$$

where $p(\cdot)$ is a polynomial, d/dt denotes the distributional derivative, δ is the delta distribution with support at zero, and f is a distribution that can be identified with the restriction to $(0, \infty)$ of some function in $C^\infty(\mathbb{R})$. The class of such distributions is denoted C_{imp}. For an element of C_{imp} of the form (11.79), we write $\phi(0^+)$ for the limit value $\lim_{t \downarrow 0} f(t)$. Having introduced the class C_{imp}, we can replace the system of equations (11.71) by its distributional version

$$\begin{aligned} \tfrac{d}{dt}x &= Ax + B_{\bullet I}u_I + x_0\delta \\ 0 &= C_{I\bullet}x + D_{II}u_I \end{aligned} \tag{11.80}$$

in which the initial condition x_0 appears explicitly, and we can look for a solution of (11.80) in the class of vector-valued impulsive-smooth distributions. It was shown in [15] that if the conditions (11.74) and (11.75) are satisfied, then there exists a unique solution $(x, u_I) \in C_{\mathrm{imp}}^{n+|I|}$ to (11.80) for each $x_0 \in V_I + T_I$; moreover, the solution is such that $x(0^+)$ is equal to $P_I x_0$, the projection of x_0 onto V_I along T_I. The solution is most easily written down in terms of its Laplace transform:

$$\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B_{\bullet I}\hat{u}_I(s) \tag{11.81}$$

$$\hat{u}_I(s) = -G_{II}^{-1}(s)C_{I\bullet}(sI - A)^{-1}x_0, \tag{11.82}$$

where

$$G_{II}(s) := C_{I\bullet}(sI - A)^{-1}B_{\bullet I} + D_{II}. \tag{11.83}$$

Note that the notation is consistent in the sense that $G_{II}(s)$ can also be viewed as the (I, I)-submatrix of the transfer matrix $G(s) := C(sI - A)^{-1}B + D$. It is shown in [15] (see also [30]) that the transfer matrix $G_{II}(s)$ associated with the system parameters in (11.71) is left invertible when (11.74) and (11.75) are satisfied. Since the transfer matrices $G_{II}(s)$ that we consider are square, left invertibility is enough to imply invertibility, and so (by duality) we also have $V_I + T_I = \mathbb{R}^n$. Summarizing, we can list the following equivalent conditions.

Theorem 11.1. *Consider a time-invariant linear system with k inputs and k outputs, given by standard state space parameters (A, B, C, D). The following conditions are equivalent.*

(i) For each index set $I \subset \bar{k}$, the associated system (11.71) *admits for each $x_0 \in V_I$ a unique smooth solution (x, u) such that $x(0) = x_0$.*

(ii) For each index set $I \subset \bar{k}$, the associated distributional system (11.80) *admits for each initial condition x_0 a unique impulsive-smooth solution (x, u).*

(iii) The conditions (11.74) *and* (11.75) *are satisfied for all $I \subset \bar{k}$.*

(iv) The transfer matrix $G(s) = C(sI - A)^{-1}B + D$ is totally invertible (as a matrix over the field of rational functions).

In connection with the system (11.69) it makes sense to introduce the following definitions.

Definition 11.1. *An impulsive-smooth distribution $\phi = p(d/dt)\delta + f$ as in* (11.79) *is called* initially nonnegative *if the leading coefficient of the polynomial $p(\cdot)$ is positive, or, in the case $p = 0$, the smooth function f is nonnegative on an interval of the form $(0, \varepsilon)$ with $\varepsilon > 0$. A vector-valued impulsive-smooth* *distribution is called* initially nonnegative *if each of its components is initially nonnegative in the above sense.*

Definition 11.2. *A triple of vector-valued impulsive-smooth distributions* (u, x, y) *is called an* initial solution *to* (11.69) *with* initial state x_0 *and* solution mode I *if*

(i) the triple (u, x, y) *satisfies the distributional equations*

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu + x_0\delta \\ y &= Cx + Du; \end{aligned}$$

(ii) both u *and* y *are initially nonnegative;*

(iii) $y_i = 0$ *for all* $i \in I$ *and* $u_i = 0$ *for all* $i \notin I$.

For an impulsive-smooth distribution w that has a rational Laplace transform $\hat{w}(s)$ (such as in (11.81) and (11.82)), we have that w is initially nonnegative if and only if $\hat{w}(s)$ is nonnegative for all sufficiently large real values of s. From this it follows that the collection of index sets I for which there exists an initial solution to (11.69) with initial state x_0 and solution mode I is exactly $\mathcal{S}(x_0)$ as we defined this set before in terms of the rational complementarity problem.

11.6 Well-posedness

Given the interpretation (11.78) of the dynamics (11.69), of course the first question to ask is under which conditions we have solutions for all initial conditions, and under which conditions these solutions are unique. Well-posedness is understood here in the sense of existence of a unique solution on some interval of nonzero length for all initial conditions. Uniqueness is understood in the sense of piecewise differentiable solutions whose points of nondifferentiability do not accumulate (sometimes called "non-Zeno" solutions). Actually it is in some sense natural to allow accumulations "forward in time" and to exclude accumulations "backward in time," but we do not enter this discussion here; see [37] for more information.

We start with a result from [35], which gives necessary and sufficient conditions for well-posedness be it only for a rather limited class of linear complementarity systems. The result is concerned with *bimodal* linear complementarity systems, that is, systems with only two modes ($k = 1$). Such a system of the form (11.69) has a transfer function $g(s) = C(sI - A)^{-1}B + D$ which is a rational function. In this case the conditions of Theorem 11.1 apply if $g(s)$ is nonzero. The system is said to have *no feedthrough term* if the matrix D vanishes. The system is called *degenerate* if the transfer matrix $g(s)$ is of the form $g(s) = 1/q(s)$ where $q(s)$ is a polynomial; in this case the consistent subspace in the constrained mode is just the origin. The *Markov parameters* of the system are the coefficients of the expansion of $g(s)$ around infinity,

$$g(s) = g_0 + g_1 s^{-1} + g_2 s^{-2} + \cdots .$$

The *leading Markov parameter* is the first parameter in the sequence g_0, $g_1, \ldots$ that is nonzero. Having introduced this terminology, we can now formulate the following result [35, Theorem 4.8].

Theorem 11.2. *A nondegenerate bimodal linear complementarity system without feedthrough term and with nonzero transfer function is well-posed if and only if its leading Markov parameter is positive.*

It is typical to find that well-posedness of complementarity systems is linked to a positivity condition. If the number of pairs of complementary variables is larger than one, an appropriate matrix version of the positivity condition has to be used. As might be expected, the type of positivity that we need is the "P-matrix" property from the theory of the LCP. Recall (see the end of Section 11.1) that a square real matrix is said to be a P-matrix if all its principal minors are positive.

To state a result on well-posedness for multivariable linear complementarity systems, we again need some concepts from linear systems theory. Recall (see, for instance, [24, p. 384] or [27, p. 24]) that a square rational matrix $G(s)$ is said to be *row proper* if it can be written in the form

$$G(s) \;=\; \Delta(s)B(s), \tag{11.84}$$

where $\Delta(s)$ is a diagonal matrix whose diagonal entries are of the form s^k for some integer k that may be different for different entries, and $B(s)$ is a proper rational matrix that has a proper rational inverse (i.e., $B(s)$ is *bicausal*). A proper rational matrix $B(s) = B_0 + B_1 s^{-1} + \cdots$ has a proper rational inverse if and only if the constant matrix B_0 is invertible. This constant matrix is uniquely determined in a factorization of the above form; it is called the *leading row coefficient matrix* of $G(s)$. In a completely similar way one defines the notions of column properness and of the leading column coefficient matrix. We can now state the following result [21, Theorem 6.3].

Theorem 11.3. *The linear complementarity system* (11.69) *is well-posed if the associated transfer matrix* $G(s) = C(sI - A)^{-1}B + D$ *is both row and column proper, and if both the leading row coefficient matrix and the leading column coefficient matrix are P-matrices. Moreover, in this case the multiplicity of events is at most one; that is, at most one reinitialization takes place at times when a mode change occurs.*

An alternative sufficient condition for well-posedness can be based on the rational complementarity problem (RCP) that was already used above (Section 11.4). For a given set of linear system parameters (A, B, C, D), we denote by RCP(x_0) the rational complementarity problem RCP$(q(s), M(s))$ with data $q(s) = C(sI - A)^{-1}x_0$ and $M(s) = C(sI - A)^{-1}B + D$. For the purposes of simplicity, the following result is stated under somewhat stronger hypotheses than were used in the original paper [20, Theorems 5.10, 5.16].

Theorem 11.4. *Consider the linear complementarity system* (11.69), *and assume that the associated transfer matrix is totally invertible. The system* (11.69) *is well-posed if the problem RCP(x_0) has a unique solution for all x_0.*

A connection between the rational complementarity problem and the standard linear complementarity problem can be established in the following way [20, Theorem 4.1, Corollary 4.10].

Theorem 11.5. *For given $q(s) \in \mathbb{R}^k(s)$ and $M(s) \in \mathbb{R}^{k\times k}(s)$, the problem RCP($q(s), M(s)$) is uniquely solvable if and only if there exists $\mu \in \mathbb{R}$ such that for all $\lambda > \mu$ the problem LCP($q(\lambda), M(\lambda)$) is uniquely solvable.*

The above theorem provides a convenient way of proving well-posedness for several classes of linear complementarity systems. The following example is taken from [20].

Example 11.2. A linear mechanical system may be described by equations of the form

$$M\ddot{q} + D\dot{q} + Kq = 0, \tag{11.85}$$

where q is the vector of generalized coordinates, M is the generalized mass matrix, D is the damping matrix, and K is the elasticity matrix. The mass matrix M is positive definite. Suppose now that we subject the above system to unilateral constraints of the form

$$Fq \geq 0, \tag{11.86}$$

where F is a given matrix. Under the assumption of inelastic collisions, the dynamics of the resulting system may be described by

$$M\ddot{q} + D\dot{q} + Kq = F^T u, \qquad y = Fq \tag{11.87}$$

together with complementarity conditions between y and u. The associated RCP is the following.

$$y(s) = F(s^2M + sD + K)^{-1}[(sM + D)q_0 + M\dot{q}_0] + F(s^2M + sD + K)^{-1}F^T u(s). \tag{11.88}$$

If F has full row rank, then the matrix $F(s^2M+sD+K)^{-1}F^T$ is positive definite (although not necessarily symmetric) for all sufficiently large s because the term with s^2 becomes dominant. By combining the standard result on solvability of LCPs with Theorem 11.5, it follows that RCP is solvable and we can use this to prove the well-posedness of the constrained mechanical system; this provides some confirmation for the validity of the model that has been used, since physical intuition certainly suggests that a unique solution should exist.

In the above example, one can easily imagine cases in which the matrix F does not have full row rank so that the fulfillment of some constraints already implies that some other constraints will also be satisfied; think, for instance, of a chair having four legs on the ground. In such cases the basic result on solvability of LCPs does not provide enough information, but there are alternatives available that make use of the special structure that is present in equations like (11.88). On the basis of this, one can still prove well-posedness; in particular the trajectories of the coordinate vector $q(t)$ are uniquely determined, even though the trajectories of the constraint force $u(t)$ are not.

11.7 Relay systems

For piecewise linear relay systems of the form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du, \qquad u_i = -\operatorname{sgn} y_i \quad (i = 1, \ldots, k) \tag{11.89}$$

Theorem 11.5 can be applied, but the application is not straightforward for the following reason. As noted above, it is possible to rewrite a relay system as a complementarity system (in several ways actually). Using the method (11.28), one arrives at a relation between the new inputs $\operatorname{col}(u_1, u_2)$ and the new outputs $\operatorname{col}(y_1, y_2)$ that may be written in the frequency domain as follows (ι denotes the vector all of whose entries are 1, and $G(s)$ denotes the transfer matrix $C(sI - A)^{-1}B + D$).

$$\begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} = \begin{bmatrix} -G^{-1}(s)C(sI - A)^{-1}x_0 + s^{-1}\iota \\ G^{-1}(s)C(sI - A)^{-1}x_0 + s^{-1}\iota \end{bmatrix} + \\ + \begin{bmatrix} G^{-1}(s) & -G^{-1}(s) \\ -G^{-1}(s) & G^{-1}(s) \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix}. \tag{11.90}$$

The matrix that appears on the right-hand side is singular for all s and so the corresponding LCP does not always have a unique solution. However the vector that we find at the right-hand side is of a special form and we only need to ensure existence of a unique solution for vectors of this particular form. On the basis of this observation, the following result is obtained.

Theorem 11.6. [29, 20] *The piecewise linear relay system* (11.89) *is well-posed if the transfer matrix* $G(s)$ *is a P-matrix for all sufficiently large* s.

This result gives a criterion that is straightforward to verify (compute the determinants of all principal minors of $G(s)$, and check the signs of the leading Markov parameters), but that is restricted to piecewise linear systems. Filippov [10, §2.10] gives a criterion for well-posedness that works for general nonlinear systems, but needs to be verified on a point-by-point basis.

11.8 Discontinuous dependence on initial conditions

Finally, let us briefly consider a property that for smooth dynamical systems is often taken into the definition of well-posedness, namely, the continuous dependence of solutions on initial conditions. In a hybrid system context this property would obviously have to be restricted to the trajectories of the continuous variables. Even then, the continuous dependence on initial conditions can easily be violated, even within such a limited class as linear Hamiltonian complementarity systems. This is shown in the following example from [21].

Example 11.3. Consider the equations

$$\ddot{x}_1 = -2x_1 + x_2 + u_1 + u_2 \tag{11.91}$$

$$\ddot{x}_2 = x_1 - x_2 - u_2 \tag{11.92}$$

$$y_1 = x_1 \tag{11.93}$$

$$y_2 = x_1 - x_2 \tag{11.94}$$

together with the standard complementarity conditions $0 \leq y \perp u \geq 0$. These equations arise from a system with two carts connected by springs to each other and to a fixed wall. The variables x_1 and x_2 indicate the positions of the first and the second cart, respectively. There are inelastic stops in the system that prevent x_1 and $x_1 - x_2$ from becoming negative. There are arbitrarily close initial conditions for which the two constraints become active in a different order, which results in quite different trajectories.

The phenomenon of discontinuous dependence on initial conditions should be viewed as a result of an idealization, reflecting a very strong dependence on initial conditions in a corresponding smooth model. Certainly the fact that such discontinuities appear is a problem in numerical simulation, but numerical problems would also occur when, for instance, the strict unilateral constraints in the example above would be replaced by very stiff springs. So the hybrid model in itself cannot be held responsible for the (near-)discontinuity problem; one should rather say that it clearly exposes this problem.

11.9 Conclusions

In this chapter we have discussed the class of nonsmooth dynamical systems that arises from connecting a set of complementarity conditions to an input-output dynamical system. It has been shown by a number of examples that systems of this type arise naturally in a number of applications, and that in some other cases a connection can be made through suitable rewriting operations. While a solution concept for complementarity systems with general nonlinear dynamics is not available at present due to the difficulties associated with state jumps in a nonlinear context, it is possible to formulate a complete specification of the dynamics when we restrict to linear complementarity systems under suitable hypotheses. Concepts of linear systems theory play an important role in the development of the theory of linear complementarity systems; for instance, the field of rational functions is used extensively just as in the linear systems case, with an extra element that comes from the imposition of an order structure on this field.

For a proper understanding of linear complementarity systems it should be emphasized that they are in fact nonlinear systems and may well exhibit, for instance, chaotic behavior. Moreover, they are discontinuous systems whose trajectories are nonsmooth; state jumps may occur as well as accumulations of event times. On the other hand, linear complementarity systems are close to linear systems and can be analyzed using methods that have been developed in linear systems theory over the past decades. It is a pleasure to make this observation in a volume dedicated to one of the foremost proponents of this field of study. So, here's to Didi.

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