Coordination Control of Linear Systems

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Coordination Control of Linear Systems

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door

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geboren te Hannover, Duitsland

promotoren: prof. dr. ir. J. H. van Schuppen
prof. dr. A. C. M. Ran

Acknowledgments

In September 2008, a few days after handing in my M.Sc. thesis, I started my Ph.D. at the VU and at CWI. Now it is four years (and one month) later, and the thesis is (hopefully) complete and going to the printer tomorrow.

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My sister Paula drew the picture on the cover and helped with the cover design, and the idea behind the cover (explained on p. 1) is due to Henk Nijmeijer.

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Pia Kempker
Amsterdam, 2012

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1 Introduction

When dealing with large dynamical systems consisting of many components, we are often unable to analyze or control the whole system at once – both on a conceptual and computational level. This problem, and the often modular nature of the system itself, suggest a decentralized approach to large-scale systems: We split the system into subsystems –or modules–, and then try to derive some insight into the overall system by analyzing each module separately, and by taking into account their interconnections. Similarly, we often strive to leave the control of the overall system to local controllers instead of one centralized controller[1]. Whether this approach – analyzing or controlling each module in isolation, and then glueing the results back together according to the network topology – actually leads to good results, largely depends on the structure (or topology) of the system, and on the objective we are trying to achieve, or the type of system property we would like to analyze. The advantages and limitations of decentralized analysis and control of large systems are illustrated in a few real-world examples:

- The example depicted on the cover of this thesis is an orchestra, coordinated by a conductor: While all musicians have their own local information in terms of the notes they should play, the conductor has a better overview over the orchestra as a whole, and controls when and at which speed the different musicians should be playing their parts.

- The basic principle of democracy, that every citizen's opinion should weigh equally in the government's decision process, is infeasible in practice since collecting and analyzing feedback information from all citizens is impossible. Instead, representative democracies are implemented as more feasible alternatives: Groups of citizens choose one or more representatives, who should then report a collated version of the citizens' feedbacks to the government. This form of representation often consists of several layers (e.g. in federal republics). The principle of representative democracy is thus a compromise between direct democracy, with direct feedback from all citizens, and dictatorship, with no feedback from the citizens. The details of the corresponding electoral system – or in other words, which type of bottom-to-top feedback should be sent at which times and in which form – is an interesting question both from a mathematical and political perspective.[2]

[1] There are many possible reasons for this choice: The system will be more robust to the failure (or corruption) of controllers, controlling one module is conceptually and computationally easier than controlling the whole system at once, if the system topology changes (i.e. a module is added or removed) the system does not have to be reconfigured from scratch, obtaining local information necessary for control is easier than requiring information from another location, etc.

[2] The engineering version of democracy is called *consensus control*: several subsystems (e.g. temperature sensors at different locations) communicate with each other until they agree on a common value (e.g. the average temperature).

- In contrast to the democratic structure based on feedback through voting, hierarchical personnel structures with little or no bottom-to-top feedback are very common in organizations – traditionally in the army, but also in many companies. In this type of structures, control of the overall system is based on the chain-of-command principle, allowing the top of the hierarchy to better control and predict the state of the overall system, and facilitating quick responses to changing conditions since no consensus needs to be found. While typically undesirable for humans, its efficiency makes this structure a useful topology for decentralized engineering systems.

- The human body – as well as many other biological entities – is an inherently modular system: In a nutshell, it is an interconnection of several organs, each with its own limited task and functionality. These modules are interconnected by the nerve system, and controlled partly via a central controller (the brain) and partly via local controllers (e.g. local reflexes). Only through the coordination of these different modules is it possible to achieve complex tasks, such as playing tennis, which none of the modules could achieve independently.

- Another example for a decentralized system in which some form of coordination is inevitable is traffic: Each car is an independent entity, with its own objective (to reach a destination) and its own local controller (the driver using the gas pedal and steering wheel). These entities are interconnected by the fact that two cars should never be at the same place at the same time (i.e. vehicles should not collide). This consideration gave rise to control measures such as traffic lights, which coordinate the different cars passing through the same intersection. Moreover, different traffic lights along the same major road often cooperate to allow for green waves, while traffic lights in different parts of the country are independent of each other.

From these examples, we can already derive some basic principles about decentralized control:

- decentralization – i.e. splitting the system into parts and controlling each part locally – is usually desirable but not always possible,

- hierarchies naturally arise from practicalities, and are often preferable to other types of system structures,

- in decentralized control, it is essential *where* information is available, and *how* (i.e. at which location) we can exert control on the system.

The aim of this thesis is to contribute to the mathematical formalization and exploration of some of these principles.

1.1 Introduction to coordination control

The class of decentralized dynamical systems considered in this thesis is that of coordinated linear systems, a special class of hierarchical systems. Coordinated linear systems are structured linear systems consisting of one coordinator system and two or more subsystems, each with their own input and output. The coordinator state and input may influence the subsystem states and outputs. The state and input of each subsystem have no influence on the coordinator state, input or output, and neither can they influence the state, input or output of the other subsystem(s). In other words,

- the coordinator subsystem influences the other subsystems but is not influenced by them,
- and when disregarding the influence of the coordinator, the subsystems are independent.

This corresponds to a hierarchical system with two layers and a top-to-bottom information structure, as illustrated in Figure 1.1.

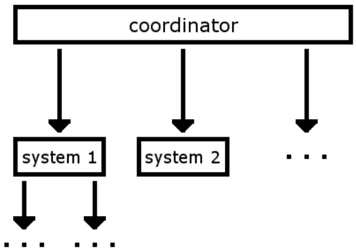

Figure 1.1: Scheme of a coordinated system

Possible applications of coordinated linear systems arise when several subsystems require interaction (i.e. coordination) to meet a joint control objective. This may apply to linear systems with an inherent hierarchical structure, but also other types of interconnected systems, which permit a hierarchical modeling approach.

Inherently hierarchical systems include traffic networks and power networks, where the major roads or power lines are at a higher level than the side streets or local distribution lines. Other examples are groups of autonomous vehicles with a leader-follower structure, such as vehicle platoons and formations: Platoons are typically modeled by chain structures, with the first vehicle at the highest level, and in formations the first vehicle may have several direct followers.

Other interconnected systems can be transformed into coordinated systems, where the coordinator consists of those parts of each system that are relevant to the other systems, and the subsystems consist of the remaining parts of each system. This corresponds to imposing a hierarchy on the different parts of a decentralized system, in order to facilitate decentralized control synthesis. Moreover, large-scale monolithic systems can be decomposed into subsystems with a hierarchical information structure in order to reduce the computational effort needed for control synthesis.[3]

This thesis develops an in-depth mathematical analysis of coordinated linear systems, focusing on the following questions:

(1) How can we construct coordinated linear systems, from large monolithic systems or decentralized systems with non-hierarchical information structures (e.g. interconnected systems with two-way communication)?

(2) Given a coordinated linear system, is this system 'as decentralized as possible', i.e. are all interactions allowed by the system structure actually required? Is all communication actually necessary? And can centralized measurement or control actions be replaced by local ones?

(3) Which part of each subsystem is controllable by which input – can control be done locally, or is coordination required to meet the control objective?

(4) A similar question arises for observability: Is all measurement data which is necessary for implementing a given control law available locally, or is communication of measurement data required?

(5) Given a coordinated linear system and an achievable control objective, how can we synthesize a control law which achieves this control objective, but also respects the given information structure? How does the performance of such a control law compare to its centralized counterpart? Will performance improve if bottom-to-top communication is permitted on an event-driven basis?

(6) Can we extend concepts and results derived for coordinated linear systems to related classes of coordinated and hierarchical systems?

1.2 Literature review

Concerning questions (1)-(6) above, this section summarizes some of the previous work, and relates it to the contributions of this thesis.

[3]Other criteria for the decomposition of dynamical systems into several parts include geographical proximity and different time scales in the system evolution.

System decompositions

Most previous work on decompositions of linear systems is based on structured matrices and graph-theoretic approaches ([9, 57]): The system matrices are reduced to binary (structured) form, for each entry specifying whether it is zero or non-zero. The dependencies among the different state, input and output variables can then be represented by a directed graph, with the different variables as nodes, and directed edges among them whenever the corresponding entries in the structured system matrices are non-zero. The graph-theoretic concepts of reachability and co-reachability can then be used to decompose the system, and to analyze the interconnections among the subsystems. A major drawback of this approach is that it ignores the different possible structures corresponding to a given linear system under transformed state, input and output spaces.

A complementary approach for decomposing large linear systems is based on the strength of the interactions among the different subsystems: Weak interactions are identified e.g. via dissipation inequalities ([1]), and then removed, leading to a more decentralized approximation of the original system. Other decomposition approaches are based on different time scales, different geographic regions, etc. ([5]).

Previous work on the special case of decompositions into hierarchical linear systems includes decompositions based on aggregation ([34, 43]): Lower-order approximations of the original system (or subsystems) are used on the higher level, in order to reduce the complexity of the control synthesis procedure. A geometric approach to coordination control, in which a system is decomposed using feedback compensation, can be found in [64]. The goal of this decomposition is to identify a coordinator and several subsystems, with the coordinator controlling the system-wide performance, while the subsystems do local control. The compensating feedback is chosen such that system becomes a hierarchical system.

The approach used in Chapter 4 of this thesis differs from existing approaches in the sense that it uses the geometric (i.e. basis-independent) concepts of controllability and observability subspaces, and that the original system and interconnections are neither changed nor aggregated, and the option of having compensating feedback is not taken into account. Another original contribution of Chapter 4 is the development of concepts and results concerning the minimality of a given decomposition.

In the related field of team theory, the decomposition of a linear system according to the observations and influence of two independent decision makers was studied in [51] – this is a special case of the decomposition derived in Section 4.3.4.

Controllability and observability

While the classical concepts of controllability and observability ([21]) for unstructured systems are characterized in terms of invariant subspaces of the state space, most literature on controllability and observability in decentralized settings is based on graph-theoretic concepts: A system is called structurally controllable if every state variable is reachable from at least one input variable in the corresponding graph representation ([8, 9, 57]). Structural controllability is a basis-dependent concept, and it is necessary, but in general not sufficient for controllability. The dual concept is structural observability, defined via the co-reachability of the state variables from at least one output variable. In [35], driver nodes are identified, which can control the whole network (given as linear system).

Early work in the field of team theory discusses the controllability and observability of a linear system via multiple decision makers with partial observations ([2, 16]), using invariant subspaces of the state space. In Chapter 5, this approach is generalized to coordinated linear systems. Together with the novel distinction between independently and jointly reachable subspaces, and between completely and independently indistinguishable subspaces, this allowed for a systematic approach to the problem of defining concepts of controllability and observability for coordinated linear systems.

LQ optimal control

For monolithic systems, the LQ control problem was introduced and solved in [20]. Early decentralized versions of the LQ problem appeared in the field of team theory (a subfield of game theory), where several decision makers, each with partial observations of the state of a linear system, aim at minimizing a joint quadratic control objective ([2, 16, 45]). Team theory problems with delayed communication among the decision makers are discussed in [49]. A different setup is that of Stackelberg games (also stemming from game theory), where the decision makers are one leader and one follower: First the leader makes a control decision, and then the follower bases its control decision on information about the leader's decision ([17, 68]).

In the field of control theory, early work on decentralized control methods for large scale and hierarchical systems is surveyed in [52], and an early survey of leader-follower strategies is given in [6]. A linear-quadratic coordination control problem was described in [3]. In this setup, the aspect of coordination was not related to the information structure, but to the control objective: The coordinator minimizes a global control objective, taking into account the subsystem control laws, and the subsystems minimize local control objectives. Local or structured control feedback synthesis for decentralized LQ control problems was also discussed in [58], [57] and [53].

In general, decentralized LQ control problems are much more involved than their monolithic counterparts: In [71] it is shown that in a decentralized setting in which different subsystems have access to different observation sets, the optimal control law may not be a linear controller. In light of this problem, the identification of special system structures, for which the LQ problem simplifies, has been considered: First characterizations of structured systems, for which local controllers can achieve global stability, are discussed in [57]. In the input-output framework, the concept of quadratic invariance was introduced in [50], characterizing convex problems in decentralized LQ control.

The class of poset-causal systems, introduced in [55, 56], consists of all structured linear systems whose information structure is consistent with a partial order relation on the subsystems. For this class, the problem of finding structure-preserving optimal controllers is convex in the input-output framework. In the state space representation, the optimal control law is a dynamic state feedback: The controller for each subsystem includes observers for all its direct or indirect followers.

Coordinated linear systems are a subclass of poset-causal systems; however, in contrast to the approach of [55, 56], we restrict attention to static state feedback in Chapter 6. This choice was made in the interest of scalability of the results with respect to the number of subsystems. In accordance with the results of [55, 56], we found that the optimal static state feedback for each subsystem only depends on its own dynamics and on its direct or indirect followers, but not on the rest of the hierarchy. This result allowed us to approach the problem in a bottom-to-top manner. The novelty of our approach is the derivation of this control synthesis procedure restricted to static state feedback, making use of linear and quadratic matrix equations and numerical optimization.

Control with event-based feedback

Control with event-based feedback –or event-triggered control– is a relatively new topic, aimed at minimizing the amount of communication necessary for control, while still achieving the desired performance levels. Rather than having regular or continuous feedback from the plant to the controller, feedback is sent only when the difference between the actual state of the plant and the observer estimate of the state at the controller exceeds a fixed threshold ([13–15, 36]). This leads to an ultimately-bounded closed-loop system. First attempts to incorporate the concept of adaptive listening ([19]) in order to further reduce the total cost of communication can be found in [38], and an event-based feedback scheme for decentralized control was derived in [60].

In Chapter 7 we incorporate event-based bottom-to-top feedback in the set of admissible control laws for our LQ control problem: The coordinator system implements a piecewise-constant approximation of the optimal state feedback for the centralized (i.e. unstructured) problem. Each lower-level subsystem sends its current state to the coordinator whenever the approximation error, caused by using the last communicated value instead of the current value of the subsystem state at the coordinator level, exceeds a certain threshold. Novelties of this approach are the use of an exponentially decaying threshold, which leads to an exponentially stable closed-loop system, and bounding the approximation error instead of the difference between the current and last received state.

1.3 Contents of the thesis

This thesis is structured as follows:

As prerequisite material, some elements of the classical theory of linear systems are summarized in Chapter 2. In Chapter 3, the concept of a coordinated linear system is defined and characterized, several basic properties of coordinated linear systems are derived, and an overview of related decentralized systems is given.

Chapter 4 deals with questions (1) and (2): In Section 4.1 we give some construction procedures for the transformation of monolithic and interconnected linear systems into coordinated linear systems. Based on the considerations summarized in question (2), several concepts of minimality of a given coordinated linear system decomposition are introduced and characterized in Section 4.2, and some results concerning the construction of a minimal decomposition are given.

Questions (3) and (4) are discussed in Chapter 5: In Section 5.2, the concept of reachability is refined to distinguish among the different inputs and the different parts of the overall system state, and based on this, a controllability decomposition for coordinated linear systems is derived, and several different concepts of controllability are defined and characterized. A similar approach is used for the concepts of indistinguishability and observability in Section 5.3. We then illustrate how to combine these concepts, and derive equivalent conditions for stabilizability via dynamic measurement feedback.

While question (5), in the generality in which it is formulated above, is easier posed than answered, its restriction to LQ control problems is the topic of Chapters 6 and 7: The LQ problem over all structure-preserving static state feedbacks is discussed in Chapter 6. The overall control problem is separated into conditionally independent subproblems, a numerical approach to their solution is derived, and the behavior and performance of the resulting control law are illustrated in examples. Chapter 7 focuses on the last part of question (5): We approximate the centralized (i.e. not structure-preserving) optimum by introducing event-based bottom-to-top feedback, and derive bounds on the stability of the resulting closed-loop system and on the corresponding costs.

As an illustration of the theory developed in this thesis and its potential practical purposes, Chapter 8 discusses two case studies of coordination control: In Section 8.1, a formation flying problem for autonomous underwater vehicles (AUVs) is introduced and solved using the coordination control framework developed in the previous chapters. Section 8.2 deals with coordinated ramp metering, i.e. the coordinated control of on-ramp metering devices at two neighboring on-ramps of a highway.

Chapter 9 summarizes the main results of this thesis, and points to some possible directions of extending the results to related classes of systems, as suggested in question (6).

2 Prerequisites

In this chapter, some elements of the classical theory of monolithic (i.e. unstructured) linear systems are summarized as background material necessary for the later chapters.

2.1 Notation

The notation for system-theoretic concepts used in this thesis complies in large parts with e.g. [63]. The direct sum of two independent linear spaces will be denoted by $\dot{+}$, i.e.

$$V \dot{+} W = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \middle| \, v \in V, \ w \in W \right\} = \begin{bmatrix} I \\ 0 \end{bmatrix} V + \begin{bmatrix} 0 \\ I \end{bmatrix} W.$$

For notational simplicity, we restrict attention to coordinated linear systems with one coordinator and two subsystems. The two subsystems are indexed by 1 and 2, and the coordinator is indexed by c. The index i is used for the subsystems only (i.e. $i = 1, 2$), and the index j denotes all three parts of the system (i.e. $j = 1, 2, c$). State spaces are denoted by X, input spaces by U, and output spaces by Y. Their dimensions are denoted by $n = \dim X, \ m = \dim U$ and $p = \dim Y$.

The state, input and output space of a coordinated linear system are composed of the state, input and output spaces of the subsystems and coordinator, i.e.

$$X = X_1 \dot{+} X_2 \dot{+} X_c, \quad U = U_1 \dot{+} U_2 \dot{+} U_c, \quad Y = Y_1 \dot{+} Y_2 \dot{+} Y_c,$$

with dimensions $n_1 + n_2 + n_c = n$, $m_1 + m_2 + m_c = m$ and $p_1 + p_2 + p_c = p$.

Note that throughout this thesis, we will use the notation X_1 both for the linear space X_1 of dimension n_1, and for the n_1-dimensional linear subspace of the n-dimensional space X. In other words, we use the notation X_1 both for the space itself and for its natural embedding $\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X_1$ into $X = X_1 \dot{+} X_2 \dot{+} X_c$. In particular, for $M \in \mathbb{R}^{n \times n}$ and S a linear subspace of X, we use the notation MX_1 for the image space $M \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X_1 \subseteq X$, and the notation $X_1 \cap S$ for the intersection space $\left(\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X_1 \right) \cap S \subseteq \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X_1$. The same holds for the spaces X_2 and X_c and their embeddings $\begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} X_2$ and $\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} X_c$ into X, and for the corresponding input and output spaces.

The state, input and output of a coordinated linear systems are then denoted by

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix}, \; u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_c(t) \end{bmatrix} \text{ and } y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_c(t) \end{bmatrix}.$$

In some parts of this thesis, we restrict attention to one subsystem and one coordinator, in which case the subsystem is indexed by s, and the state, input and output vectors are denoted by

$$x(t) = \begin{bmatrix} x_s(t) \\ x_c(t) \end{bmatrix}, \; u(t) = \begin{bmatrix} u_s(t) \\ u_c(t) \end{bmatrix} \text{ and } y(t) = \begin{bmatrix} y_s(t) \\ y_c(t) \end{bmatrix}.$$

2.2 Monolithic linear systems

The following sections summarize some of the theory for linear time-invariant systems that will be needed in the following chapters. We primarily work with continuous-time systems, but some examples and simulations use the discrete-time equivalent. Here we consider the class of all linear systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2.1}$$

with state space X, input space U and output space Y, and with initial state $x(0) = x_0$. If the input trajectory $u : [0, t] \to U$ is a piecewise-continuous function then the integral on the right hand side of (2.2) is well-defined as a Riemann integral, and the state $x(t)$ at time t is then given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{2.2}$$

The output $y(t)$ is given by

$$y(t) = Cx(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau.$$

Taking the Laplace transform, we get the input-output relation

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s), \; s \in \mathbb{C}$$

in the frequency domain, with **transfer function**

$$\hat{G}(s) = C(sI - A)^{-1}B.$$

The transfer function is a rational matrix function, which characterizes the input-output behavior of a continuous-time linear system without the need of a state variable, and hence independently of the choice of the state space and its basis.

2.3 Controllability and observability

For linear systems, the concepts of reachability and controllability are defined as follows (see e.g. [63]):

> A state $\bar{x} \in X$ is called **reachable** (from the initial state $x_0 = 0$) if there exists a finite terminal time $\bar{t} < \infty$ and a piecewise-continuous input trajectory $u : [0, \bar{t}] \to U$ such that the state trajectory of the linear system with $x_0 = 0$ satisfies $x(\bar{t}) = \bar{x}$. The set of all reachable states will be denoted by $\mathfrak{R}$. A linear system (or, equivalently, the matrix pair (A, B)) is called **controllable** if $X = \mathfrak{R}$.

The reachable set $\mathfrak{R}$ is the smallest A-invariant subspace of X containing im B, see [63, 72]. This subspace is unique, and is given by

$$\mathfrak{R} = \text{im} \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}, \tag{2.3}$$

where n is the state space dimension $\dim X$. The matrix $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ is called the controllability matrix. Observe that $\mathfrak{R}$ is an A-invariant subspace by the Cayley-Hamilton theorem.

From these properties, we can derive the **Kalman controllability decomposition** (see e.g. [21]): Let $X_1 = \mathfrak{R}$, and choose for X_2 any complement of X_1 in X, then with respect to the decomposition $X = X_1 \dot{+} X_2$ the system has the form

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} x(t). \end{aligned} \tag{2.4}$$

The matrix pair (A_{11}, B_1) is a controllable pair.

We call a matrix $M \in \mathbb{R}^{n\times n}$ exponentially **stable** if its spectrum lies in the open left half plane, i.e. if $\sigma(M) \subset \mathbb{C}_- = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$. The related concept of stabilizability is then defined as follows:

> A linear system (or, equivalently, the matrix pair (A, B)) is called **stabilizable** if there exists a linear state feedback $F \in \mathbb{R}^{m\times n}$ such that the closed-loop system $\dot{x}(t) = (A + BF)x(t)$, obtained by applying the input $u(t) = Fx(t)$, is stable.

Stabilizability is equivalent to the matrix A_{22} in (2.4) being a stable matrix. The exponential of a stable matrix is bounded in norm by a negative scalar exponential, i.e.

$$M \text{ stable} \implies \exists\, \alpha > 0, c \in \mathbb{R} \text{ s.t. } \left\|e^{Mt}\right\| \le ce^{-\alpha t}\ \forall t \in \mathbb{R}.$$

Applying the stabilizing state feedback $u(\cdot) = Fx(\cdot)$ leads to the closed-loop state trajectory $x(t) = e^{(A+BF)t}x_0$, satisfying

$$\|x(t)\| = \left\|e^{(A+BF)t}x_0\right\| \le \left\|e^{(A+BF)t}\right\| \|x_0\| \le ce^{-\alpha t} \|x_0\|.$$

Hence the closed-loop state trajectory goes to zero exponentially for $t \to \infty$.

The concepts of indistinguishability and observability are typically defined as follows (see e.g. [63]):

> A pair $(\bar{x}, \bar{\bar{x}})$ of states in X is called **indistinguishable** if the outputs $\bar{y}(t)$ and $\bar{\bar{y}}(t)$, generated by the linear system with input trajectory $u \equiv 0$ and initial conditions $x_0 = \bar{x}$ and $x_0 = \bar{\bar{x}}$, respectively, have $\bar{y}(t) = \bar{\bar{y}}(t)$ for all $t \in [0, \infty)$. The set of all states $\bar{x} \in X$ such that $(\bar{x}, 0)$ is indistinguishable will be called $\mathfrak{I}$. A linear system (or, equivalently, the matrix pair (C, A)) is called **observable** if $\mathfrak{I} = \{0\}$.

Note that for linear systems, the pair $(\bar{x}, \bar{\bar{x}})$ is indistinguishable if and only if the pair $(\bar{x} - \bar{\bar{x}}, 0)$ is indistinguishable. Hence, when studying the observability properties of linear systems, we can restrict attention to pairs of the form $(\bar{x}, 0)$. In the following, and with some abuse of notation, a state $\bar{x} \in X$ will be called **indistinguishable** if the pair $(\bar{x}, 0)$ is indistinguishable in the sense defined above.

The set of indistinguishable states $\mathfrak{I}$ is the largest A-invariant subspace of X contained in $\ker C$, see [63, 72]. This subspace is unique, and is given by

$$\mathfrak{I} = \ker \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}, \tag{2.5}$$

with $n = \dim X$. The matrix $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is called the observability matrix. A-invariance of $\mathfrak{I}$ again follows from the Cayley-Hamilton theorem.

The A-invariance property leads to the Kalman observability decomposition (see e.g. [21]): Let $X_2 = \mathfrak{I}$ and choose for X_1 any complement of X_2 in X, then with respect to the decomposition $X = X_1 \dot{+} X_2$, the system has the form

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} C_1 & 0 \end{bmatrix} x(t). \end{aligned} \tag{2.6}$$

The pair (C_1, A_{11}) is an observable pair.

In analogy to stabilizability, the concept of detectability is defined as follows:

> A linear system (or, equivalently, the matrix pair (C, A)) is called **detectable** if there exists a linear state observer matrix $K \in \mathbb{R}^{n \times p}$ such that the system describing the observer error $\dot{e}(t) = (A - KC)e(t)$ is stable.

Detectability is equivalent to the matrix A_{22} in (2.6) being a stable matrix.

2.4 LQ optimal control

We consider unstructured linear time-invariant deterministic systems. The infinite-horizon, undiscounted linear-quadratic (LQ) control problem is given by

$$\min_{u(\cdot) \text{ piecewise continuous}} J(x_0, u(\cdot)), \tag{2.7}$$

with cost function

$$J(x_0, u(\cdot)) = \int_{t_0}^{\infty} x^T(t) Q x(t) + u^T(t) R u(t) \, dt, \tag{2.8}$$

subject to the system dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0. \tag{2.9}$$

If $Q \geq 0$ and $R > 0$ then the problem is a well-defined minimization problem, i.e. there exists a piecewise continuous $u(\cdot)$ such that the minimum is attained.

In other words, our control objective in LQ optimal control is to minimize a quadratic cost function, representing a trade-off: The cost function penalizes the weighted norm of the state trajectory on the one hand, and the weighted norm of the control effort on the other hand.

The solution of this problem is well-known (see e.g. [63]): If (A, B) is a stabilizable pair and (Q, A) is a detectable pair then the algebraic Riccati equation

$$XBR^{-1}B^TX - A^TX - XA - Q = 0 \tag{2.10}$$

has a unique solution X such that $A - BR^{-1}B^TX$ is stable. This solution X is also the largest positive semidefinite solution. The optimal control law is then the state feedback $u(\cdot) = Gx(\cdot)$, where $G = -R^{-1}B^TX$. The closed-loop system is given by

$$\dot{x}(t) = (A + BG)x(t) = (A - BR^{-1}B^TX)x(t),$$

with $A - BR^{-1}B^TX$ stable by the choice of X. The corresponding cost is given by

$$J(x_0, Gx(\cdot)) = x_0^TXx_0. \tag{2.11}$$

The control law $u(\cdot) = Gx(\cdot)$ derived above has the following properties:

- the optimal input trajectory is a linear state feedback, i.e. it is of the form $u(t) = Gx(t)$ where G is a matrix and $x(t)$ is the current state,
- the feedback matrix G is independent of the initial state x_0,
- the entries of G, and also the corresponding cost $J(x_0, Gx(\cdot))$, can be computed offline.

In Chapter 6, we will derive the corresponding results for the case of coordinated linear systems, and compare the properties of the coordination control law with the properties given here.

2.4.1 Relation between costs and control laws

The following result quantifies the relative cost increase caused by using other state feedbacks than the optimal one, and will be useful in Chapter 6, when we restrict the set of admissible feedback matrices to those respecting the underlying information structure.

This theorem is a slight variation of Lemma 16.3.2 in [33], and a proof is given for convenience:

2.4.1. Theorem. *We consider a system of the form (2.9) and the optimal control problem (2.7), with cost function (2.8). We assume that $Q \geq 0$, $R > 0$, (A, B) is a stabilizable pair, and (Q, A) is a detectable pair. Let X be the stabilizing solution of (2.10), and let*

$G = -R^{-1}B^TX$. For any other stabilizing state feedback matrix F the difference in cost is given by

$$J(x_0, Fx(\cdot)) - J(x_0, Gx(\cdot)) = \int_0^\infty \|R^{1/2}(F - G)e^{(A+BF)t}x_0\|^2 dt.$$

Proof. We have $J(x_0, Gx(\cdot)) = x_0^T X x_0$. The cost corresponding to any other stabilizing feedback F is given by the solution Y of the Lyapunov equation

$$(A + BF)^T Y + Y(A + BF) + F^T RF + Q = 0 :$$

For this choice of Y, and noting that $\lim_{t\to\infty} e^{(A+BF)t} = 0$, we have

$$\begin{aligned}
J(x_0, Fx(\cdot)) &= \int_0^\infty x(t)^T(Q + F^T RF)x(t)\, dt \\
&= \int_0^\infty -x(t)^T((A + BF)^T Y + Y(A + BF))x(t)\, dt \\
&= x_0^T \int_0^\infty -\frac{d}{dt}\left(e^{(A+BF)^T t} Y e^{(A+BF)t}\right) dt\, x_0 \\
&= x_0^T \left(- e^{(A+BF)^T t} Y e^{(A+BF)t}\Big|_{t\to\infty} + e^{(A+BF)^T t} Y e^{(A+BF)t}\Big|_{t=0}\right) x_0 = x_0^T Y x_0.
\end{aligned}$$

In the following, we derive a Lyapunov equation for the difference $Y - X$ of the costs, using the Riccati equation for X and the Lyapunov equation for Y:

$$\begin{aligned}
&(A + BF)^T(Y - X) + (Y - X)(A + BF) \\
&= -F^T RF - Q - A^T X - F^T B^T X - XA - XBF \\
&= -F^T RF - XBR^{-1}B^T X - F^T B^T X - XBF \\
&= -(F + R^{-1}B^T X)^T R(F + R^{-1}B^T X) \\
&= -(F - G)^T R(F - G).
\end{aligned}$$

Using this, we can now derive an expression for the cost difference:

$$\begin{aligned}
&J(x_0, Fx(\cdot)) - J(x_0, Gx(\cdot)) = x_0^T(Y - X)x_0 \\
&= x_0^T \int_0^\infty -\frac{d}{dt}\left(e^{(A+BF)^T t}(Y - X)e^{(A+BF)t}\right) dt\, x_0 \\
&= x_0^T \int_0^\infty -\left(e^{(A+BF)^T t}((A + BF)(Y - X) + (Y - X)(A + BF))e^{(A+BF)t}\right) dt\, x_0
\end{aligned}$$

$$
\begin{aligned}
&= x_0^T \int_0^\infty \left(e^{(A+BF)^T t} (F-G)^T R (F-G) e^{(A+BF)t} \right) dt \, x_0 \\
&= \int_0^\infty \| R^{1/2}(F-G) x_F(t) \|^2 dt,
\end{aligned}
$$

where $x_F(\cdot)$ is the state trajectory of the closed-loop system obtained from applying the feedback $Fx(\cdot)$, i.e. $x_F(t) = e^{(A+BF)t}x_0$. □

From the theorem above we see that the difference in cost between the optimal solution and another stabilizing solution can be described in terms of the corresponding feedback matrices. If no special restrictions are imposed on the feedback F considered here, then minimizing $x_0^T(Y - X)x_0$ trivially leads to $F = G$, with $Y = X$.

However, in decentralized control it is often necessary, or preferable, that F complies with the underlying information structure of the system. Our result above states that for any non-empty subset $\mathfrak{F} \subseteq \{F \in \mathbb{R}^{m\times n} | \sigma(A+BF) \subset \mathbb{C}^-\}$, the problem

$$\inf_{F\in\mathfrak{F}} x_0^T Y x_0$$

has a solution (if the unrestricted problem has a solution, i.e. if X exists), and this solution can be found by solving

$$\inf_{F\in\mathfrak{F}} \int_0^\infty \| R^{1/2}(F-G) x_F(t) \|^2 dt,$$

or equivalently

$$\inf_{F\in\mathfrak{F}} \int_0^\infty \| R^{1/2}(F-G) e^{(A+BF)t} x_0 \|^2 dt.$$

3 Coordinated Linear Systems

An intuitive description of coordinated linear systems was given in Section 1.1. In the following, coordinated linear systems will be defined, and several of their basic properties will be discussed.

3.1 Definition

For the purposes of this thesis, and in contrast to [48], we define coordinated linear systems with inputs and outputs in terms of independence and invariance properties of the state, input and output spaces. This geometric approach to linear systems was developed in [72].

3.1.1. Definition. Let a continuous-time, time-invariant linear system with inputs and outputs, of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t)\end{aligned}$$

be given. Moreover, let the state space, input space and output space of the system be decomposed as

$$X = X_1 \dot{+} X_2 \dot{+} X_c,\ U = U_1 \dot{+} U_2 \dot{+} U_c \text{ and } Y = Y_1 \dot{+} Y_2 \dot{+} Y_c.$$

Then we call the system a **coordinated linear system** if we have that

(1) X_1 and X_2 are A-invariant,[1]

(2) $BU_1 \subseteq X_1$ and $BU_2 \subseteq X_2$,

(3) and $CX_1 \subseteq Y_1$ and $CX_2 \subseteq Y_2$.

In this definition, the subspaces X_c, U_c and Y_c are the state, input and output spaces of the coordinator system, the subspaces X_1, U_1 and Y_1 correspond to subsystem 1, and the subspaces X_2, U_2 and Y_2 correspond to subsystem 2. Conditions (1), (2) and (3) in Definition 3.1.1 imply that the state and input of each subsystem have no influence on the states or the outputs of the coordinator or the other subsystem.

With respect to the decompositions $X = X_1 \dot{+} X_2 \dot{+} X_c$, $U = U_1 \dot{+} U_2 \dot{+} U_c$ and $Y = Y_1 \dot{+} Y_2 \dot{+} Y_c$, the system is then of the form

[1] Note that we use X_1 to denote both the space X_1 and the subspace $\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X_1 \subseteq X$ (see Section 2.1).

$$\dot{x}(t) = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} x(t) + \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} C_{11} & 0 & C_{1c} \\ 0 & C_{22} & C_{2c} \\ 0 & 0 & C_{cc} \end{bmatrix} x(t). \tag{3.1}$$

The structure of the system matrices in (3.1) follows directly from Conditions (1), (2) and (3) in Definition 3.1.1. Note that, with the trivial choices

$$\begin{aligned} X_1 &= \{0\}, & X_2 &= \{0\}, & X_c &= X, \\ U_1 &= \{0\}, & U_2 &= \{0\}, & U_c &= U, \\ Y_1 &= \{0\}, & Y_2 &= \{0\}, & Y_c &= Y, \end{aligned}$$

any linear system qualifies as a coordinated linear system.

The interconnections between the different variables of a coordinated linear system are illustrated in Figure 3.1.

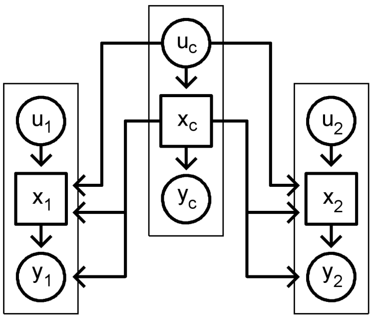

Figure 3.1: A coordinated linear system with inputs and outputs

In this figure, we recognize the strict top-to-bottom information structure described in Section 1.1.

For the special case of linear systems without inputs and outputs, we give an alternative formulation of Condition (1) of Definition 3.1.1, in terms of projections. This leads to a more constructive description of all possible coordinated linear system representations of a given linear system without inputs and outputs, in terms of systems of quadratic matrix equations. A linear map $P : X \to X$ is called a **projection** if $P^2 = P$ (see e.g. [12]). Using the relation between projections and invariant subspaces (see [12, 72]), we can state the following result:

3.1.2. Proposition. *For the projections $P_1 : X \to X$ and $P_2 : X \to X$, the linear subspaces $X_1 = P_1 X$ and $X_2 = P_2 X$ of X are independent and satisfy Condition (1) of Definition 3.1.1, if and only if P_1 and P_2 satisfy*

$$P_1 A P_1 = A P_1,\ P_2 A P_2 = A P_2, \tag{3.2}$$

$$P_1 P_2 = 0,\ P_2 P_1 = 0. \tag{3.3}$$

Proof. This follows directly from the fact that a subspace S of X is A-invariant if and only if $PAP = AP$ for some (and equivalently, any) projector $P : X \to X$ with $\operatorname{im} P = S$. Condition (3.3) is equivalent to

$$X_1 \cap X_2 = P_1 X \cap P_2 X = \{0\}.$$

□

Extending Proposition 3.1.2 to linear systems with inputs and outputs is conceptually straightforward but notationally more involved.

3.2 Basic properties

The set of matrices

$$\mathbb{R}_{\text{CLS}} = \left\{ \begin{bmatrix} M_{11} & 0 & M_{1c} \\ 0 & M_{22} & M_{2c} \\ 0 & 0 & M_{cc} \end{bmatrix},\ M_{ij} \in \mathbb{R}^{n_i \times n_j},\ i, j = 1, 2, c \right\}$$

forms an invertible algebraic ring (i.e. it is closed with respect to taking linear combinations, matrix multiplication, and matrix inversion):

linear combinations:

$$\alpha \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} + \beta \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} = \begin{bmatrix} \alpha A_{11} + \beta B_{11} & 0 & \alpha A_{1c} + \beta B_{1c} \\ 0 & \alpha A_{22} + \beta B_{22} & \alpha A_{2c} + \beta B_{2c} \\ 0 & 0 & \alpha A_{cc} + \beta B_{cc} \end{bmatrix}.$$

matrix multiplication:

$$\begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} = \begin{bmatrix} A_{11} B_{11} & 0 & A_{11} B_{1c} + A_{1c} B_{cc} \\ 0 & A_{22} B_{22} & A_{22} B_{2c} + A_{2c} B_{cc} \\ 0 & 0 & A_{cc} B_{cc} \end{bmatrix}.$$

matrix inversion: Suppose $M \in \mathbb{R}_{\text{CLS}}$ is invertible, then M_{11}, M_{22}, M_{cc} are invertible because M is block upper-triangular. M^{-1} is given by

$$M^{-1} = \begin{bmatrix} M_{11}^{-1} & 0 & -M_{11}^{-1} M_{1c} M_{cc}^{-1} \\ 0 & M_{22}^{-1} & -M_{22}^{-1} M_{2c} M_{cc}^{-1} \\ 0 & 0 & M_{cc}^{-1} \end{bmatrix} \in \mathbb{R}_{\text{CLS}}$$

In particular, e^M is of the form

$$\exp\left(\begin{bmatrix} M_{11} & 0 & M_{1c} \\ 0 & M_{22} & M_{2c} \\ 0 & 0 & M_{cc} \end{bmatrix}\right) = \begin{bmatrix} e^{M_{11}} & 0 & \star_{1c} \\ 0 & e^{M_{22}} & \star_{2c} \\ 0 & 0 & e^{M_{cc}} \end{bmatrix}, \tag{3.4}$$

where the entries denoted by $\star$ are not specified further. Hence the information structure imposed by the invariance properties of Definition 3.1.1 is left unchanged over time by the system dynamics.

A natural consequence of this invariance property is that the transfer function is of the form

$$\begin{aligned} \hat{G}(z) &= C(zI - A)^{-1}B \\ &= \begin{bmatrix} C_{11}(zI - A_{11})^{-1}B_{11} & 0 & \star_{1c} \\ 0 & C_{22}(zI - A_{22})^{-1}B_{22} & \star_{2c} \\ 0 & 0 & C_{cc}(zI - A_{cc})^{-1}B_{cc} \end{bmatrix}, \end{aligned}$$

where

$$\star_{ic} = C_{ii}(zI - A_{ii})^{-1}B_{ic} + (C_{ic} - C_{ii}(zI - A_{ii})^{-1}A_{ic})(zI - A_{cc})^{-1}B_{cc}.$$

Note that the diagonal entries of the linear combination, product and inverse are just the linear combination, product and inverse of the corresponding diagonal entries of the original matrices, respectively. This means that these operations also preserve the structure of matrices corresponding to more nested hierarchies: If $A \in \mathbb{R}_{\text{CLS}}$ with a diagonal entry $A_{ii} \in \mathbb{R}_{\text{CLS}}$, then operations as above will yield matrices in $\mathbb{R}_{\text{CLS}}$ with the ii-th entry again in $\mathbb{R}_{\text{CLS}}$.

Hence coordinated linear systems can act as building blocks for constructing linear systems with a more complex hierarchical structure: An extension to an arbitrary number of subsystems is straightforward, and nested hierarchies can be modeled by using another coordinated linear system as one of the subsystems of a coordinated linear system. Hierarchical systems that are modeled by such a combination of coordinated linear systems can again be shown to have an information structure that is invariant with respect to the system dynamics. Two of

these extensions are illustrated below:
Add a third subsystem:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{\boldsymbol{x}}_\mathbf{2} \\ \dot{\boldsymbol{x}}_\mathbf{3} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 & A_{1c} \\ 0 & \boldsymbol{A}_\mathbf{22} & \mathbf{0} & A_{2c} \\ 0 & \mathbf{0} & \boldsymbol{A}_\mathbf{33} & A_{3c} \\ 0 & 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ \boldsymbol{x}_\mathbf{2} \\ \boldsymbol{x}_\mathbf{3} \\ x_c \end{bmatrix} + \begin{bmatrix} B_{11} & 0 & 0 & B_{1c} \\ 0 & \boldsymbol{B}_\mathbf{22} & \mathbf{0} & B_{2c} \\ 0 & \mathbf{0} & \boldsymbol{B}_\mathbf{33} & B_{3c} \\ 0 & 0 & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1 \\ \boldsymbol{u}_\mathbf{2} \\ \boldsymbol{u}_\mathbf{3} \\ u_c \end{bmatrix}$$

Add another level:

$$\begin{bmatrix} \dot{\boldsymbol{x}}_{\tilde{\mathbf{1}}} \\ \dot{\boldsymbol{x}}_{\tilde{\mathbf{2}}} \\ \dot{\boldsymbol{x}}_{\tilde{\mathbf{c}}} \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{\tilde{\mathbf{1}}\tilde{\mathbf{1}}} & \mathbf{0} & \boldsymbol{A}_{\tilde{\mathbf{1}}\tilde{\mathbf{c}}} & 0 & A_{\tilde{1}c} \\ \mathbf{0} & \boldsymbol{A}_{\tilde{\mathbf{2}}\tilde{\mathbf{2}}} & \boldsymbol{A}_{\tilde{\mathbf{2}}\tilde{\mathbf{c}}} & 0 & A_{\tilde{2}c} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{A}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}} & 0 & A_{\tilde{c}c} \\ 0 & 0 & 0 & A_{22} & A_{2c} \\ 0 & 0 & 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{\tilde{\mathbf{1}}} \\ \boldsymbol{x}_{\tilde{\mathbf{2}}} \\ \boldsymbol{x}_{\tilde{\mathbf{c}}} \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_{\tilde{\mathbf{1}}\tilde{\mathbf{1}}} & \mathbf{0} & \boldsymbol{B}_{\tilde{\mathbf{1}}\tilde{\mathbf{c}}} & 0 & B_{\tilde{1}c} \\ \mathbf{0} & \boldsymbol{B}_{\tilde{\mathbf{2}}\tilde{\mathbf{2}}} & \boldsymbol{B}_{\tilde{\mathbf{2}}\tilde{\mathbf{c}}} & 0 & B_{\tilde{2}c} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{B}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}} & 0 & B_{\tilde{c}c} \\ 0 & 0 & 0 & B_{22} & B_{2c} \\ 0 & 0 & 0 & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\tilde{\mathbf{1}}} \\ \boldsymbol{u}_{\tilde{\mathbf{2}}} \\ \boldsymbol{u}_{\tilde{\mathbf{c}}} \\ u_2 \\ u_c \end{bmatrix}$$

It is also possible to decompose the state space X_1 of subsystem 1 into $X_{\tilde{1}} \dot{+} X_{\tilde{2}} \dot{+} X_{\tilde{c}}$ but leave the input space U_1 unchanged – in the second example above, this would correspond to $U_1 = U_{\tilde{c}}$ and $B_{11} = \begin{bmatrix} B_{\tilde{1}\tilde{c}} \\ B_{\tilde{2}\tilde{c}} \\ B_{\tilde{c}\tilde{c}} \end{bmatrix}$.

3.3 Related distributed systems

In the following, several related classes of systems are described. For a more complete overview of different classes of hierarchical systems, see [10, 57, 67].

Leader-follower systems

This type of systems (strongly related to the concept of Stackelberg games in economics, see [17, 68]) is the most basic example of a hierarchical system, with one leader system on the higher level and one follower system on the lower level. Decentralized control synthesis for this class of systems was discussed in e.g. [61]. For the purposes of this thesis, we define leader-follower systems to be linear time-invariant systems with a representation of the form

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sc} \\ 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix} + \begin{bmatrix} B_{ss} & B_{sc} \\ 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_s \\ u_c \end{bmatrix}, \quad \begin{bmatrix} x_s(0) \\ x_c(0) \end{bmatrix} = \begin{bmatrix} x_{0,s} \\ x_{0,c} \end{bmatrix}. \tag{3.5}$$

In compliance with our notation for coordinated systems, the subscript s stands for ‘subsystem’, and c stands for ‘coordinator’.

Note that coordinated linear systems are a special type of leader-follower systems, with $A_{ss} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ and $B_{ss} = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$ (or, equivalently, leader-follower systems are a special type of coordinated linear systems, with only one subsystem). For notational simplicity, some of the theory about LQ optimal control in Chapter 6 will first be developed for leader-follower systems, and then extended to coordinated linear systems.

Poset-causal systems

The class of poset-causal systems, introduced and analyzed in [54–56], consists of all distributed linear systems for which the underlying information structure is invariant under the system dynamics, i.e. for which the set of corresponding system matrices forms an algebraic ring. This class is characterized by partial orderings on the set of subsystems (i.e. $\text{subsystem}_1 \leq \text{subsystem}_2$ if subsystem_2 influences subsystem_1), and includes all hierarchical systems that can be formed by composing coordinated linear systems, as described in Section 3.2.

When viewing the underlying information structure of a decentralized system as a graph, the poset-condition imposed on this class of systems can be restated as follows:

- The information structure has no loops (this corresponds to the antisymmetry property of partial orderings),
- and wherever there is a path, there is also a link (this corresponds to the transitivity property).

The condition that there should be no loops is crucial for decentralized control synthesis: Any system of this class can be written in such a way that the system matrices are block upper-triangular (by arranging the subsystems according to the partial ordering), and hence eigenvalue assignment problems for the global system can easily be reduced to their local counterparts (see Section 5.2.3.2).

Research on decentralized control for this class of systems has focused on using the partial ordering among the subsystems to determine which observers to include in which location for control purposes, an approach complementary to the one used in this thesis.

Coordinated Gaussian systems

Coordinated linear systems are straightforwardly extended to include Gaussian noise terms (see [41, 67]): In the discrete-time formulation, coordinated Gaussian systems are defined to have a state space representation of the form

$$\begin{aligned}
\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_c(t+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_c(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} M_{11} & 0 & M_{1c} \\ 0 & M_{22} & M_{2c} \\ 0 & 0 & M_{cc} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_c(t) \end{bmatrix}, \\
\begin{bmatrix} y_1(t+1) \\ y_2(t+1) \\ y_c(t+1) \end{bmatrix} &= \begin{bmatrix} C_{11} & 0 & C_{1c} \\ 0 & C_{22} & C_{2c} \\ 0 & 0 & C_{cc} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} N_{11} & 0 & N_{1c} \\ 0 & N_{22} & N_{2c} \\ 0 & 0 & N_{cc} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_c(t) \end{bmatrix},
\end{aligned}$$

with $v_1,\ v_2,\ v_c$ Gaussian white noises. An LQG (linear-quadratic Gaussian) control problem, minimizing the infinite-horizon average cost for coordinated Gaussian systems, is discussed in [41], and shows many similarities with the theory in Chapter 6.

Coordinated discrete-event systems

Coordinated discrete-event systems, i.e. distributed discrete-event systems with several subsystems and a coordinator, have been studied in [27–29]. The theory of coordinated supervisory control for this class of systems shows some similarities with coordinated linear systems. A case study involving a paint factory is described in [4].

Systems with several dynamic controllers

If we take the transposes of all system matrices in a coordinated linear system, we get one lower-level subsystem (corresponding to the original coordinator) with two higher-level coordinators (corresponding to the original subsystems)[2]. The higher-level systems can be thought of as two dynamic controllers. The two controllers cannot interact with each other, just like the subsystems in the original system were independent of each other.

This class of systems is related to team theory in economics (see [45]): In team decision problems, several agents have different partial observations of the same system, and make control decisions based on a common control objective (the common objective distinguishes team theory from the broader field of game theory). The overall performance of the agents in team decision problems can often be improved by allowing for communication among the agents (see e.g. [49] and the references therein). A concrete example of how the global performance can

[2] Transposing all system matrices is equivalent to reversing the causality relation among the different subsystems, i.e. all arrows in the underlying information structure switch directions.

be improved by including communication among the controllers on a nearest-neighbor basis is discussed in [39], where different local voltage controllers jointly try to keep the voltage in a large-scale power network within the safety limits.

Mammillary systems

In mammillary systems, different subsystems have bidirectional interconnections with a joint coordinator, but are otherwise independent of each other. This type of interconnection is common in compartmental systems observed in biological applications (see [18]), and in particular is it used to model the role of the blood in mammals as a coordinating agent among the organs. Gaussian mammillary systems are of the form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_s \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & 0 & A_{1c} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{ss} & A_{sc} \\ A_{c1} & \cdots & A_{cs} & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_c \end{bmatrix} + \begin{bmatrix} B_{11} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & B_{ss} & 0 \\ 0 & \cdots & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_s \\ u_c \end{bmatrix}$$
$$+ \begin{bmatrix} M_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & M_{ss} & 0 \\ 0 & \cdots & 0 & M_{cc} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_s \\ v_c \end{bmatrix},$$

with $v_1, \dots, v_s,\ v_c$ Gaussian white noises. The main difference to coordinated systems is that feedback from each subsystem to the coordinator is allowed (i.e. $A_{ci} \neq 0$) – this destroys the ring structure of the system matrices, and hence the invariance of the information structure under the system dynamics.

Nearest-neighbor systems

The class of nearest-neighbor systems arises naturally from applications with spatially distributed networks of systems, by imposing that each system communicates only with its nearest neighbors. In the special case of linear systems in a string formation, i.e. each system has one neighbor to its left and one to its right, this leads to linear system representations with tri-diagonal system matrices, of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & \cdots & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{23} & \cdots & 0 & 0 & 0 \\ 0 & A_{32} & A_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{n-2,n-2} & A_{n-2,n-1} & 0 \\ 0 & 0 & 0 & \cdots & A_{n-2,n-1} & A_{n-1,n-1} & A_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & A_{n,n-1} & A_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$$
$$+ \begin{bmatrix} B_{11} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & B_{22} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & B_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_{n-2,n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & B_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & B_{n,n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \\ u_n \end{bmatrix}$$

The controllability properties of the class of systems with nearest-neighbor interconnections have been studied in the case of one leader/controller (see [62] and [37]), using graph theory to describe properties of the interconnection structure. The work was extended to allow for several leaders/controllers (see [46] and [44]). A common result is that connectivity in general, and cyclic interconnections in particular, seem to destroy the controllability of the network – a result in favor of hierarchical interconnection structures.

4 Construction and Minimality

This chapter deals with the construction and minimality of coordinated linear systems. Construction procedures are given to transform unstructured or interconnected systems into coordinated linear systems. Several concepts of minimality for coordinated linear systems are suggested and characterized. A few results of this chapter were published in [23].

4.1 Construction from interconnected systems

Suppose we are given an interconnected system, consisting of two linear systems and linear interconnection relations, of the form

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + B_{11}u_1 + B_{12}z_2, & \dot{x}_2 &= A_{22}x_2 + B_{22}u_2 + B_{21}z_1, \\ y_1 &= C_{11}x_1 + D_{11}u_1 + D_{12}z_2, & y_2 &= C_{22}x_2 + D_{22}u_2 + D_{21}z_1, \\ z_1 &= P_{11}x_1 + Q_{11}u_1, & z_2 &= P_{22}x_2 + Q_{22}u_2. \end{aligned} \tag{4.1}$$

The variables z_1 and z_2 connect systems 1 and 2, as illustrated in Figure 4.1.

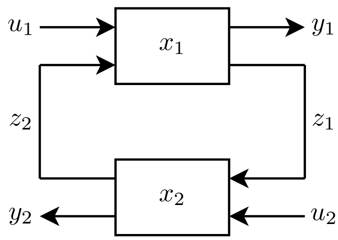

Figure 4.1: An interconnected system with inputs and outputs

Equations (4.1) can be combined to describe the dynamics of the overall state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of the interconnected system,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \end{aligned} \tag{4.2}$$

with the state, input and output spaces given by $X = X_1 \dot{+} X_2$, $U = U_1 \dot{+} U_2$ and $Y = Y_1 \dot{+} Y_2$, and with A, B, C and D given by

$$A = \begin{bmatrix} A_{11} & B_{12}P_{22} \\ B_{21}P_{11} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12}Q_{22} \\ B_{21}Q_{11} & B_{22} \end{bmatrix},$$
$$C = \begin{bmatrix} C_{11} & D_{12}P_{22} \\ D_{21}P_{11} & C_{22} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12}Q_{22} \\ D_{21}Q_{11} & D_{22} \end{bmatrix}. \tag{4.3}$$

The problem we consider in this section is how to transform an interconnected system of the form (4.2) into a coordinated linear system. In order to achieve this, the part of system 1 which influences system 2 via z_1 will have to be in the coordinator, and the same holds for system 2.

In other words, we want to decompose the state, input and output space of the interconnected system into three parts each, forming a coordinated linear system. The new subsystem spaces will be denoted by $X_{i\backslash c}$, $U_{i\backslash c}$, $Y_{i\backslash c}$, $i = 1, 2$, the subscript $i\backslash c$ indicating that part of the original system has been moved to the coordinator. The new decomposition should respect the original one, in the sense that the original system 1 will be part of the new subsystem $1\backslash c$ and the coordinator, but not of subsystem $2\backslash c$, and vice versa.

4.1.1 Simple case without inputs and outputs

We first give a procedure for the construction of a coordinated linear system for the special case of an interconnected system without inputs and outputs. The principle of the procedure is that each of the subsystems undergoes a state space transformation, such that only part of the subsystem is observable to the other subsystem.

4.1.1. Procedure. *Construction of a coordinated linear system from an interconnected system, without inputs and outputs.*

Consider a linear system consisting of the interconnection of two subsystems, with representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}.$$

(1) Apply a state space transformation such that the matrix pairs (A_{21}, A_{11}) and (A_{12}, A_{22}) are transformed to the Kalman observable form,

$$\left[\begin{array}{c} \dot{x}_1^{\text{obs}}(t) \\ \dot{x}_1^{\text{unobs}}(t) \\ \hline \dot{x}_2^{\text{obs}}(t) \\ \dot{x}_2^{\text{unobs}}(t) \end{array}\right] = \left[\begin{array}{cc|cc} A_{11,11} & 0 & A_{12,11} & 0 \\ A_{11,21} & A_{11,22} & A_{12,21} & 0 \\ \hline A_{21,11} & 0 & A_{22,11} & 0 \\ A_{21,21} & 0 & A_{22,21} & A_{22,22} \end{array}\right] \left[\begin{array}{c} x_1^{\text{obs}}(t) \\ x_1^{\text{unobs}}(t) \\ \hline x_2^{\text{obs}}(t) \\ x_2^{\text{unobs}}(t) \end{array}\right],$$

corresponding to the decompositions $X_1 = X_1^{\text{obs}} \dot{+} X_1^{\text{unobs}}$ and $X_2 = X_2^{\text{obs}} \dot{+} X_2^{\text{unobs}}$.

(2) Define the subsystem state spaces $X_{1\backslash c} = X_1^{\text{unobs}}$ and $X_{2\backslash c} = X_2^{\text{unobs}}$ and the coordinator state space $X_c = X_1^{\text{obs}} \dot{+} X_2^{\text{obs}}$. This results in the coordinated linear system

$$\left[\begin{array}{c} \dot{x}_1^{\text{unobs}}(t) \\ \hline \dot{x}_2^{\text{unobs}}(t) \\ \hline \dot{x}_1^{\text{obs}}(t) \\ \dot{x}_2^{\text{obs}}(t) \end{array}\right] = \left[\begin{array}{c|c|cc} A_{11,22} & 0 & A_{11,21} & A_{12,21} \\ \hline 0 & A_{22,22} & A_{21,21} & A_{22,21} \\ \hline 0 & 0 & A_{11,11} & A_{12,11} \\ 0 & 0 & A_{21,11} & A_{22,11} \end{array}\right] \left[\begin{array}{c} x_1^{\text{unobs}}(t) \\ \hline x_2^{\text{unobs}}(t) \\ \hline x_1^{\text{obs}}(t) \\ x_2^{\text{obs}}(t) \end{array}\right],$$

with state space $X = X_{1\backslash c} \dot{+} X_{2\backslash c} \dot{+} X_c$.

Note that the unobservable parts of the interconnection of the two original subsystems now form the local subsystems while the observable parts of the two original subsystems form the coordinator.

This construction is possible for any interconnected system – the dimensions of the new state spaces however depend on the 'interconnectedness' of the system, and several may be zero.

4.1.2 General case with inputs and outputs

The problem of finding a coordinated linear system representation for an interconnected system with inputs and outputs can be stated as follows:

4.1.2. Problem. Given an interconnected system of the form (4.2), with state space $X = X_1 \dot{+} X_2$, input space $U = U_1 \dot{+} U_2$ and output space $Y = Y_1 \dot{+} Y_2$, find all decompositions $X = X_{1\backslash c} \dot{+} X_{2\backslash c} \dot{+} X_c$, $U = U_{1\backslash c} \dot{+} U_{2\backslash c} \dot{+} U_c$ and $Y = Y_{1\backslash c} \dot{+} Y_{2\backslash c} \dot{+} Y_c$ such that the following properties hold:[1]

$$AX_{1\backslash c} \subseteq X_{1\backslash c}, \quad AX_{2\backslash c} \subseteq X_{2\backslash c}, \tag{4.4}$$
$$BU_{1\backslash c} \subseteq X_{1\backslash c}, \quad BU_{2\backslash c} \subseteq X_{2\backslash c}, \tag{4.5}$$
$$CX_{1\backslash c} \subseteq Y_{1\backslash c}, \quad CX_{2\backslash c} \subseteq Y_{2\backslash c}, \tag{4.6}$$
$$DU_{1\backslash c} \subseteq Y_{1\backslash c}, \quad DU_{2\backslash c} \subseteq Y_{2\backslash c}, \tag{4.7}$$
$$X_{1\backslash c} \subseteq X_1, \quad X_{2\backslash c} \subseteq X_2, \tag{4.8}$$
$$U_{1\backslash c} \subseteq U_1, \quad U_{2\backslash c} \subseteq U_2, \tag{4.9}$$
$$Y_{1\backslash c} \subseteq Y_1, \quad Y_{2\backslash c} \subseteq Y_2. \tag{4.10}$$

Properties (4.4)-(4.7) are equivalent to saying that, with respect to the decompositions $X = X_{1\backslash c} \dot{+} X_{2\backslash c} \dot{+} X_c$, $U = U_{1\backslash c} \dot{+} U_{2\backslash c} \dot{+} U_c$ and $Y = Y_{1\backslash c} \dot{+} Y_{2\backslash c} \dot{+} Y_c$, the

[1] For possible ambiguities in our notation for linear subspaces, see Section 2.1.

system is a coordinated linear system. Properties (4.8)-(4.10) are additional constraints on the possible decompositions of X, U and Y, particular to Problem 4.1.2: Since the interconnected system we start out with already has an inherent decomposition into two subsystems, we require that this original structure is preserved, in the sense that no part of the original system 1 will be moved to subsystem 2 of the coordinated linear system, and no part of the original system 2 will be moved to subsystem 1.

With respect to the given decompositions $X = X_1 \dotplus X_2$, $U = U_1 \dotplus U_2$ and $Y = Y_1 \dotplus Y_2$ of the state, input and output space of the interconnected system, we can write the system as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \end{aligned} \tag{4.11}$$

For simplicity, this representation will be used in the rest of this section.

The following characterization of properties (4.4)-(4.10) will be useful in constructing a coordinated linear system from an interconnected system:

4.1.3. Proposition. *For a system of the form (4.11), properties (4.4)-(4.10) are equivalent to the set of properties*

$$Y_{1\backslash c} \subseteq Y_1, \tag{4.12}$$

$$X_{1\backslash c} \textit{ is an } A_{11}\textit{-invariant subspace of } \ker \begin{bmatrix} A_{21} \\ C_{21} \end{bmatrix} \cap C_{11}^{-P} Y_{1\backslash c}, \tag{4.13}$$

$$U_{1\backslash c} \subseteq \ker \begin{bmatrix} B_{21} \\ D_{21} \end{bmatrix} \cap B_{11}^{-P} X_{1\backslash c} \cap D_{11}^{-P} Y_{1\backslash c}, \tag{4.14}$$

$$Y_{2\backslash c} \subseteq Y_2, \tag{4.15}$$

$$X_{2\backslash c} \textit{ is an } A_{22}\textit{-invariant subspace of } \ker \begin{bmatrix} A_{12} \\ C_{12} \end{bmatrix} \cap C_{22}^{-P} Y_{2\backslash c}, \tag{4.16}$$

$$U_{2\backslash c} \subseteq \ker \begin{bmatrix} B_{12} \\ D_{12} \end{bmatrix} \cap B_{22}^{-P} X_{2\backslash c} \cap D_{22}^{-P} Y_{2\backslash c}, \tag{4.17}$$

where $\cdot^{-P}$ *denotes the preimage, i.e.* $C_{11}^{-P} Y_{1\backslash c} = \{x_1 \in X_1 \mid C_{11} x_1 \in Y_{1\backslash c}\}$.

Proof. First we show that (4.12)-(4.17) imply (4.4)-(4.10). Properties (4.4)-(4.10) are only derived for subsystem 1, noting that the same holds for subsystem 2. By (4.13), it follows that $X_{1\backslash c} \subseteq X_1$ (since $C_{11} : X_1 \to Y_1$), so (4.8) holds. To see (4.4), note that

$$AX_{1\backslash c} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} X_{1\backslash c} = \begin{bmatrix} A_{11} X_{1\backslash c} \\ 0 \end{bmatrix} \subseteq X_{1\backslash c}.$$

Similarly, (4.6) follows from

$$CX_{1\backslash c} = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} X_{1\backslash c} = \begin{bmatrix} C_{11}X_{1\backslash c} \\ 0 \end{bmatrix} \subseteq Y_{1\backslash c}.$$

From (4.14) we derive that $U_{1\backslash c} \subseteq U_1$ (since $B_{11} : U_1 \to X_1$), and hence (4.9) holds. Properties (4.5) and (4.7) follow from

$$BU_{1\backslash c} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} U_{1\backslash c} = \begin{bmatrix} B_{11}U_{1\backslash c} \\ 0 \end{bmatrix} \subseteq X_{1\backslash c}$$

and

$$DU_{1\backslash c} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} U_{1\backslash c} = \begin{bmatrix} D_{11}U_{1\backslash c} \\ 0 \end{bmatrix} \subseteq Y_{1\backslash c},$$

respectively. (4.10) is the same as (4.12).

Conversely, we need to show that (4.4)-(4.10) imply (4.12)-(4.17). To show (4.13) we note that, since $AX_{1\backslash c} \subseteq X_{1\backslash c} \subseteq X_1$ by (4.4) and (4.8), and $CX_{1\backslash c} \subseteq Y_{1\backslash c} \subseteq Y_1$ by (4.6) and (4.10), we also have that

$$\begin{bmatrix} A \\ C \end{bmatrix} X_{1\backslash c} = \begin{bmatrix} A_{11} \\ A_{21} \\ C_{11} \\ C_{21} \end{bmatrix} X_{1\backslash c} \subseteq X_1 \dot{+} Y_1.$$

From (4.4) and (4.6) it then also follows that $\begin{bmatrix} A_{21} \\ C_{21} \end{bmatrix} X_{1\backslash c} = 0$, and hence also $AX_{1\backslash c} = \begin{bmatrix} A_{11}X_{1\backslash c} \\ 0 \end{bmatrix} \subseteq X_{1\backslash c}$ and $CX_{1\backslash c} = \begin{bmatrix} C_{11}X_{1\backslash c} \\ 0 \end{bmatrix} \subseteq Y_{1\backslash c}$, i.e. $X_{1\backslash c}$ is A_{11}-invariant and $C_{11}X_{1\backslash c} \subseteq Y_{1\backslash c}$.

(4.14) follows from a similar argument: We have $BU_{1\backslash c} \subseteq X_{1\backslash c} \subseteq X_1$ by (4.5) and (4.8), and $DU_{1\backslash c} \subseteq Y_{1\backslash c} \subseteq Y_1$ by (4.7) and (4.10), but also $U_{1\backslash c} \subseteq U_1$ by (4.9), so

$$\begin{bmatrix} B \\ D \end{bmatrix} U_{1\backslash c} = \begin{bmatrix} B_{11} \\ B_{21} \\ D_{11} \\ D_{21} \end{bmatrix} U_{1\backslash c} \subseteq X_1 \dot{+} Y_1,$$

which gives $\begin{bmatrix} B_{21} \\ D_{21} \end{bmatrix} U_{1\backslash c} = 0$. From this it follows that $BU_{1\backslash c} = \begin{bmatrix} B_{11}U_{1\backslash c} \\ 0 \end{bmatrix}$ and $DU_{1\backslash c} = \begin{bmatrix} D_{11}U_{1\backslash c} \\ 0 \end{bmatrix}$, and together with (4.5) and (4.7) this gives $B_{11}U_{1\backslash c} \subseteq X_{1\backslash c}$ and $D_{11}U_{1\backslash c} \subseteq Y_{1\backslash c}$.

(4.15)-(4.17) follow from the same arguments for subsystem 2. □

Using Proposition 4.1.3, the construction of a coordinated linear system from the interconnected system (4.11) can now be done as follows:

For the construction of subsystem 1 of the coordinated linear system we first choose an output space $Y_{1\backslash c} \subseteq Y_1$. This space can be chosen freely. Given $Y_{1\backslash c}$, we can choose a state space $X_{1\backslash c}$ satisfying (4.13). Since the choice $X_{1\backslash c} = \{0\}$ satisfies (4.13), a valid state space $X_{1\backslash c}$ always exists. Now, given $Y_{1\backslash c}$ and $X_{1\backslash c}$, we can choose an input space $U_{1\backslash c}$ satisfying (4.14). Again, $U_{1\backslash c} = \{0\}$ is always a valid choice.

Note that properties (4.15)-(4.17) for subsystem 2 are independent of the choice of $X_{1\backslash c}$, $U_{1\backslash c}$ and $Y_{1\backslash c}$. This is due to the separation of the two subsystems by properties (4.8)-(4.10), and does in general not hold for the construction of a coordinated linear system from other classes of systems than the one considered here. Hence, in the setting of this section, we can choose the output space $Y_{2\backslash c}$, the state space $X_{2\backslash c}$ and the input space $U_{2\backslash c}$ of subsystem 2 as we did for subsystem 1.

Properties (4.12)-(4.17) contain no restrictions on the choice of the coordinator spaces X_c, U_c and Y_c. Hence, given the state spaces, input spaces and output spaces of the subsystems, we are free to choose any complements X_c, U_c and Y_c such that $X = X_{1\backslash c} \dot{+} X_{2\backslash c} \dot{+} X_c$, $U = U_{1\backslash c} \dot{+} U_{2\backslash c} \dot{+} U_c$ and $Y = Y_{1\backslash c} \dot{+} Y_{2\backslash c} \dot{+} Y_c$.

The following proposition identifies one possible decomposition of the system given in (4.11), according to Properties (4.12)-(4.17).

4.1.4. Proposition. *In the notation of (4.11), the choice*

$$Y_{1\backslash c} = Y_1, \tag{4.18}$$

$$X_{1\backslash c} = \ker \text{obsmat}\left(\begin{bmatrix} A_{21} \\ C_{21} \end{bmatrix}, A_{11} \right), \tag{4.19}$$

$$U_{1\backslash c} = \ker \begin{bmatrix} B_{21} \\ D_{21} \end{bmatrix} \cap \ker \left(\text{obsmat}\left(\begin{bmatrix} A_{21} \\ C_{21} \end{bmatrix}, A_{11} \right) B_{11} \right), \tag{4.20}$$

$$Y_{2\backslash c} = Y_2, \tag{4.21}$$

$$X_{2\backslash c} = \ker \text{obsmat}\left(\begin{bmatrix} A_{12} \\ C_{12} \end{bmatrix}, A_{22} \right), \tag{4.22}$$

$$U_{2\backslash c} = \ker \left(\begin{bmatrix} B_{12} \\ D_{12} \end{bmatrix} \right) \cap \ker \left(\text{obsmat}\left(\begin{bmatrix} A_{12} \\ C_{12} \end{bmatrix}, A_{22} \right) B_{22} \right) \tag{4.23}$$

is a solution to Problem 4.1.2, with $\dim X_c$*,* $\dim Y_c$ *and* $\dim U_c$ *minimal.*

Here, obsmat (C, A) refers to the observability matrix, i.e.

$$\text{obsmat}\,(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix},$$

with $n = \dim X$ (see Section 2.3).

Proof. The choices $Y_{1\backslash c} = Y_1$ and $Y_{2\backslash c} = Y_2$ satisfy (4.12) and (4.15) and maximize $\dim Y_{1\backslash c} + \dim Y_{2\backslash c}$, thus minimizing $\dim Y_c$. With these output spaces we have $C_{11}^{-P} Y_{1\backslash c} = X_1$, $C_{22}^{-P} Y_{2\backslash c} = X_2$, $D_{11}^{-P} Y_{1\backslash c} = U_1$ and $D_{22}^{-P} Y_{2\backslash c} = U_2$, and hence these choices also lead to the least restrictive conditions in (4.13), (4.14), (4.16) and (4.17).

Recall from Section 2.3 that the indistinguishable subspace

$$\ker \text{obsmat}\left(\begin{bmatrix} B_{21} \\ D_{21} \end{bmatrix}, A_{11}\right)$$

is the largest A_{11}-invariant subspace in $\ker \begin{bmatrix} B_{21} \\ D_{21} \end{bmatrix}$, and thus the largest subspace satisfying (4.13). The same holds for $X_{2\backslash c}$ as given in (4.22), and hence $\dim X_c = \dim X - \dim \left(X_{1\backslash c} \dot{+} X_{2\backslash c}\right)$ is minimal. Again, choosing the largest possible $X_{1\backslash c}$ and $X_{2\backslash c}$ automatically leads to the least restrictive conditions (4.14) and (4.17) on $U_{1\backslash c}$ and $U_{2\backslash c}$.

For the choice of $U_{1\backslash c}$ given in (4.20), we note that

$$\begin{aligned} &\ker \left(\text{obsmat}\left(\begin{bmatrix} B_{21} \\ D_{21} \end{bmatrix}, A_{11}\right) B_{11}\right) = B_{11}^{-P} \ker \text{obsmat}\left(\begin{bmatrix} B_{21} \\ D_{21} \end{bmatrix}, A_{11}\right) \\ &= B_{11}^{-P} X_{1\backslash c}, \end{aligned}$$

and hence $U_{1\backslash c}$ is the maximal subspace of U_1 satisfying (4.14). The same holds for $U_{2\backslash c}$ as given in (4.23), and hence $\dim U_c = \dim U - \dim \left(U_{1\backslash c} \dot{+} U_{2\backslash c}\right)$ is minimal. □

If we use the subsystem spaces given in Proposition 4.1.4, and if we choose for X_{1c} a complement of $X_{1\backslash c}$ in X_1, for U_{1c} a complement of $U_{1\backslash c}$ in U_1, and do the same for X_{2c} and U_{2c}, then $X_c = X_{1c} \dot{+} X_{2c}$ is a coordinator state space, and

$U_c = U_{1c} \dot{+} U_{2c}$ is a coordinator input space. With this choice, (4.11) yields the state-space representation

$$\left[\begin{array}{c} \dot{x}_{1\backslash c} \\ \dot{x}_{1c} \\ \hline \dot{x}_{2\backslash c} \\ \dot{x}_{2c} \end{array}\right] = \left[\begin{array}{cc|cc} A^{11}_{11} & A^{12}_{11} & 0 & A^{12}_{12} \\ 0 & A^{22}_{11} & 0 & A^{22}_{12} \\ \hline 0 & A^{12}_{21} & A^{11}_{22} & A^{12}_{22} \\ 0 & A^{22}_{21} & 0 & A^{22}_{22} \end{array}\right] \left[\begin{array}{c} x_{1\backslash c} \\ x_{1c} \\ \hline x_{2\backslash c} \\ x_{2c} \end{array}\right] + \left[\begin{array}{cc|cc} B^{11}_{11} & B^{12}_{11} & 0 & B^{12}_{12} \\ 0 & B^{22}_{11} & 0 & B^{22}_{12} \\ \hline 0 & B^{12}_{21} & B^{11}_{22} & B^{12}_{22} \\ 0 & B^{22}_{21} & 0 & B^{22}_{22} \end{array}\right] \left[\begin{array}{c} u_{1\backslash c} \\ u_{1c} \\ \hline u_{2\backslash c} \\ u_{2c} \end{array}\right],$$

$$\left[\begin{array}{c} y_1 \\ y_2 \end{array}\right] = \left[\begin{array}{cc|cc} C^1_{11} & C^2_{11} & 0 & C^2_{12} \\ 0 & C^2_{21} & C^1_{22} & C^2_{22} \end{array}\right] \left[\begin{array}{c} x_{1\backslash c} \\ x_{1c} \\ \hline x_{2\backslash c} \\ x_{2c} \end{array}\right] + \left[\begin{array}{cc|cc} D^1_{11} & D^2_{11} & 0 & D^2_{12} \\ 0 & D^2_{21} & D^1_{22} & D^2_{22} \end{array}\right] \left[\begin{array}{c} u_{1\backslash c} \\ u_{1c} \\ \hline u_{2\backslash c} \\ u_{2c} \end{array}\right],$$

which, after rearranging the blocks according to $X = X_{1\backslash c} \dot{+} X_{2\backslash c} \dot{+} X_c$, $U = U_{1\backslash c} \dot{+} U_{2\backslash c} \dot{+} U_c$ and $Y = Y_{1\backslash c} \dot{+} Y_{2\backslash c} \dot{+} Y_c$, becomes

$$\left[\begin{array}{c} \dot{x}_{1\backslash c} \\ \hline \dot{x}_{2\backslash c} \\ \hline \dot{x}_{1c} \\ \dot{x}_{2c} \end{array}\right] = \left[\begin{array}{c|c|cc} A^{11}_{11} & 0 & A^{12}_{11} & A^{12}_{12} \\ \hline 0 & A^{11}_{22} & A^{12}_{21} & A^{12}_{22} \\ \hline 0 & 0 & A^{22}_{11} & A^{22}_{12} \\ 0 & 0 & A^{22}_{21} & A^{22}_{22} \end{array}\right] \left[\begin{array}{c} x_{1\backslash c} \\ \hline x_{2\backslash c} \\ \hline x_{1c} \\ x_{2c} \end{array}\right] + \left[\begin{array}{c|c|cc} B^{11}_{11} & 0 & B^{12}_{11} & B^{12}_{12} \\ \hline 0 & B^{11}_{22} & B^{12}_{21} & B^{12}_{22} \\ \hline 0 & 0 & B^{22}_{11} & B^{22}_{12} \\ 0 & 0 & B^{22}_{21} & B^{22}_{22} \end{array}\right] \left[\begin{array}{c} u_{1\backslash c} \\ \hline u_{2\backslash c} \\ \hline u_{1c} \\ u_{2c} \end{array}\right],$$

$$\left[\begin{array}{c} y_1 \\ \hline y_2 \end{array}\right] = \left[\begin{array}{c|c|cc} C^1_{11} & 0 & C^2_{11} & C^2_{12} \\ \hline 0 & C^1_{22} & C^2_{21} & C^2_{22} \end{array}\right] \left[\begin{array}{c} x_{1\backslash c} \\ \hline x_{2\backslash c} \\ \hline x_{1c} \\ x_{2c} \end{array}\right] + \left[\begin{array}{c|c|cc} D^1_{11} & 0 & D^2_{11} & D^2_{12} \\ \hline 0 & D^1_{22} & D^2_{21} & D^2_{22} \end{array}\right] \left[\begin{array}{c} u_{1\backslash c} \\ \hline u_{2\backslash c} \\ \hline u_{1c} \\ u_{2c} \end{array}\right].$$

The coordinator spaces X_c and U_c are still free to choose (except that they have to be complements of the subsystem spaces). A good choice for X_c and U_c would be one that minimizes the amount of communication necessary from the coordinator to the subsystems – this will be formalized in Section 4.3.

4.2 Decompositions of monolithic linear systems

This section discusses how to find coordinated linear system decompositions for a given monolithic linear system.

4.2.1 Problem formulation

We first give a formal description of the problem considered in this section:

4.2.1. Problem. Consider the monolithic linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{4.24}$$

with state space $X = \mathbb{R}^n$, input space $U = \mathbb{R}^m$, output space $Y = \mathbb{R}^p$ and initial state $x_0 \in X$. Find the number $s \in \mathbb{N}$ of subsystems, and decompositions

$$\begin{aligned} X &= X_{1\backslash c} \dot{+} \ldots \dot{+} X_{s\backslash c} \dot{+} X_c, \\ U &= U_{1\backslash c} \dot{+} \ldots \dot{+} U_{s\backslash c} \dot{+} U_c, \\ Y &= Y_{1\backslash c} \dot{+} \ldots \dot{+} Y_{s\backslash c} \dot{+} Y_c, \end{aligned}$$

such that for all $j = 1, \ldots, s$:

$$AX_{j\backslash c} \subseteq X_{j\backslash c},\ BU_{j\backslash c} \subseteq X_{j\backslash c} \text{ and } CX_{j\backslash c} \subseteq Y_{j\backslash c}.^{2} \tag{4.25}$$

When dealing with linear systems, one usually assumes that B has full column rank and C has full row rank. This assumption is natural for monolithic linear systems:

- If B is not of full column rank then the input space U can be reduced without changing the controllability properties of the system.
- If C is not of full row rank then the output space Y can be reduced without changing the observability properties of the system.

For decentralized systems, these assumptions are not useful: For example, the state of a subsystem may be controllable both via its local input or via the coordinator input, so if we were to restrict attention to the usual concept of controllability then one of these inputs is irrelevant to the system. However, for local controllability it is important that the subsystem is controllable via the local input, and for coordinator controllability it is important that the subsystem is controllable via the coordinator input (these concepts will be defined and discussed in Chapter 5). Even though both inputs can control the same part of the state, their different locations in the decentralized system distinguish them, and (in general) neither of them should be removed from the system.

Similarly, different outputs in a decentralized system may have different roles in the system even though they both observe the same part of the state: If a subsystem output and a coordinator output both observe the same part of the coordinator state, then this state information is available both locally and remotely, and hence both the coordinator itself and the subsystem observing this part can use

[2]Note that such a decomposition always exists: The (trivial) decomposition $X = X_c$, $Y = Y_c$, $U = U_c$ satisfies these conditions.

this information for control purposes, without the coordinator having to communicate its observations.

The difference between Problem 4.2.1 and Problem 4.1.2 from the previous section is that no a priori decomposition of the original system into two parts is given, i.e. conditions (4.8)-(4.10) are dropped from the problem. These conditions separated the overall decomposition problem into two independent subproblems (i.e. each original subsystem has to be separated into a local part and a coordinator part). Dropping these conditions means that we have more freedom in choosing our decompositions, but also that we lose the independence property of the different subproblems. In fact, part of Problem 4.2.1 is to first identify the different subproblems – this also means that choosing the number of subsystems s is part of the problem.

Our approach in the following is to first decompose the state space X into several subsystems according to the invariance properties of A, and then applying a result similar to Proposition 4.1.3.

4.2.2 Systems without inputs and outputs

The first problem we need to consider is how to split the monolithic system into different subsystems, and how many subsystems to expect. An important property of subsystems is that their state spaces should be A-invariant – this suggests that we consider the Jordan normal form of A (see [40]):

There exists a decomposition[3]

$$X = X_1 \dot{+} X_2 \dot{+} \dots \dot{+} X_s,$$

and let the transformed system be given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_s \end{bmatrix} = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix},$$

where the J_j are the different Jordan blocks. We notice that the Jordan normal form of A naturally splits the system into s independent subsystems, one for each Jordan block. This decomposition is 'as decentralized as possible', i.e. splitting any Jordan block into two subsystems will destroy the A-invariance of the subsystem. Hence, to summarize, by taking the Jordan decomposition, we have

[3] If A is assumed to be non-derogatory then the number of different A-invariant spaces in X is finite. In that case, the Jordan decomposition of X is unique up to reordering.

found the number of subsystems s and the state spaces of the different subsystems, and any further decomposition would lose the A-invariance property.

For systems without inputs and outputs, this is the most straightforward way of splitting up a monolithic system into subsystems. Since we consider systems with inputs and outputs, the condition $AX_j \subseteq X_j$ is not the only one the state space has to satisfy, and hence a further decomposition of the state space will be necessary.

4.2.3 Systems with inputs and outputs

We first rewrite conditions (4.25) into a more constructive form, similar to conditions (4.12)-(4.17) for interconnected systems:

4.2.2. Proposition. *With $i \in \{1,\ldots,s\}$, there exist linearly independent subspaces $X_{i\backslash c} \subseteq X$, $U_{i\backslash c} \subseteq U$ and $Y_{i\backslash c} \subseteq Y$, such that (4.25) holds, if and only if there exist K_i, K_c, L_i, L_c, $X_{i\backslash c}$, X_c, $U_{i\backslash c}$, U_c, $Y_{i\backslash c}$ and Y_c such that*

$$Y = Y_{1\backslash c} \dot{+} \ldots \dot{+} Y_{s\backslash c} \dot{+} Y_c, \tag{4.26}$$

$$\ker C = K_1 \dot{+} \ldots \dot{+} K_s \dot{+} K_c, \tag{4.27}$$

$$\ker B = L_1 \dot{+} \ldots \dot{+} L_s \dot{+} L_c, \tag{4.28}$$

$$\text{For } j = 1,\ldots,s,\ X_{j\backslash c} \text{ is an } A\text{-invariant subspace of } C^{-P} Y_{j\backslash c} \text{ with } X_{j\backslash c} \cap \ker C \subseteq K_j, \tag{4.29}$$

$$X_c \text{ is a complement of } X_{1\backslash c} \dot{+} \ldots \dot{+} X_{s\backslash c} \text{ in } X, \tag{4.30}$$

$$\text{For } j = 1,\ldots,s,\ U_{j\backslash c} \text{ is a subspace of } B^{-P} X_{j\backslash c} \text{ with } U_{j\backslash c} \cap \ker B \subseteq L_j, \tag{4.31}$$

$$U_c \text{ is a complement of } U_{1\backslash c} \dot{+} \ldots \dot{+} U_{s\backslash c} \text{ in } U. \tag{4.32}$$

Proof. Conditions (4.26)-(4.32) indeed lead to decompositions of X, U and Y: For $j \neq k \in \{1,\ldots,s\}$ we have

$$X_{j\backslash c} \cap X_{k\backslash c} \subseteq C^{-P}\left(Y_{j\backslash c} \cap Y_{k\backslash c}\right) = C^{-P}\{0\} = \ker C,$$

but $X_{j\backslash c} \cap \ker C \subseteq K_j$ and hence $X_{j\backslash c} \cap X_{k\backslash c} \subseteq K_j \cap K_k = \{0\}$. Similarly,

$$U_{j\backslash c} \cap U_{k\backslash c} \subseteq B^{-P}\left(X_{j\backslash c} \cap X_{k\backslash c}\right) = B^{-P}\{0\} = \ker B,$$

but $U_{j\backslash c} \cap \ker B \subseteq L_j$ and hence $U_{j\backslash c} \cap U_{k\backslash c} \subseteq L_j \cap L_k = \{0\}$. For Y we have a decomposition given in (4.26). In addition, (4.25) is clearly satisfied, by (4.29) and (4.31).

Conversely, let decompositions of X, Y and U be given such that (4.25) holds. Then (4.26), (4.30) and (4.32) are automatically satisfied. Observe that for $i \neq j$ we have

$$(X_{i\backslash c} \cap \ker C) \cap (X_{j\backslash c} \cap \ker C) = \{0\}.$$

Hence there exists a decomposition (4.27) such that (4.29) holds. Likewise, since for $i \neq j$ we have

$$(U_{i\backslash c} \cap \ker B) \cap (U_{j\backslash c} \cap \ker B) = \{0\},$$

there exists a decomposition (4.28) such that (4.31) holds. □

From Proposition 4.2.2, we can now read off a general procedure for the construction of a coordinated linear system decomposition from a monolithic linear system:

4.2.3. Procedure.

(1) Choose the number of subsystems $s \in \mathbb{N}$, and pick any decompositions (4.26), (4.27) and (4.28) of Y, $\ker C$ and $\ker B$.

(2) For each $j = 1, \ldots, s$, pick a complement M_j of $\ker C$ in $C^{-P}Y_{j\backslash c}$, and choose for $X_{j\backslash c}$ an A-invariant subspace of $M_j \dot{+} K_j$.

(3) For each $j = 1, \ldots, s$, pick a complement N_j of $\ker B$ in $B^{-P}X_{j\backslash c}$, and choose for $U_{j\backslash c}$ any subspace of $N_j \dot{+} L_j$.

(4) Pick for X_c and U_c any complements of $X_{1\backslash c} \dot{+} \ldots \dot{+} X_{s\backslash c}$ and $U_{1\backslash c} \dot{+} \ldots \dot{+} U_{s\backslash c}$ in X and U.

At each step of this procedure there are many choices involved, and it is not clear from the procedure which choices would lead to "good" decompositions (e.g. decompositions with $X_{j\backslash c} \neq \{0\}$ for all $j = 1, \ldots, s$). The most restrictive condition in Proposition 4.2.2 is the A-invariance of the state spaces $X_{j\backslash c}$: If A is non-derogatory then this gives us only finitely many choices, and the chance that we picked spaces $Y_{j\backslash c}$ and K_j in step (1) in such a way that $M_j \dot{+} K_j$ contains a non-trivial A-invariant space in step (2) is very small. Hence we approach the problem differently: We first decompose X according to the Jordan normal form of A, and then find decompositions of Y and U for this fixed state space decomposition.

Given the decomposition $X = X_1 \dot{+} \ldots \dot{+} X_s$ constructed in the previous section, our system is of the form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_s \end{bmatrix} &= \begin{bmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_s \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \end{bmatrix} + \begin{bmatrix} B_1 \\ \vdots \\ B_s \end{bmatrix} u, \\ y &= \begin{bmatrix} C_1 & \cdots & C_s \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \end{bmatrix}. \end{aligned} \tag{4.33}$$

In the following, we will first consider the problem of finding appropriate output space decompositions, and then we will give a procedure for finding the corresponding input space decompositions, noting that by Proposition 4.2.2, the conditions on possible input space decompositions depend on the choice of the output space decomposition, but not vice versa: The characterization of $U_{j\backslash c}$ in (4.31) depends on $X_{j\backslash c}$, which in turn depends on $Y_{j\backslash c}$ in (4.29). The characterization of $Y_{j\backslash c}$ given in (4.26) is independent of the choices for $X_{j\backslash c}$ and $U_{j\backslash c}$.

4.2.3.1 Output space decompositions

We discuss the problem of finding output space decompositions for a given state space decomposition:

4.2.4. Problem. Given a system of the form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_s \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A_{11} & \cdots & 0 & A_{1c} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{ss} & A_{sc} \\ 0 & \cdots & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_c \end{bmatrix}, \\ y &= \begin{bmatrix} C_1 & \cdots & C_s & C_c \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_c \end{bmatrix}, \end{aligned} \tag{4.34}$$

with state space $X = X_1 \dot{+} \ldots \dot{+} X_s \dot{+} X_c$ and output space Y, find an output space decomposition

$$Y = Y_1 \dot{+} \ldots \dot{+} Y_s \dot{+} Y_c$$

such that $CX_j \subseteq Y_j$ for $j = 1, \ldots, s$.

For a fixed state space decomposition, appropriate output space decompositions do not always exist:

4.2.5. Proposition. *For the existence of an output space decomposition as described in Problem 4.2.4, it is necessary and sufficient that*

$$\operatorname{rank} \begin{bmatrix} C_1 & \ldots & C_s \end{bmatrix} = \sum_{j=1,\ldots,s} \operatorname{rank} C_j. \tag{4.35}$$

Proof. The condition $CX_j \subseteq Y_j$ can only be satisfied by a linearly independent set $Y_1, \ldots, Y_s$ of local output spaces if the set of images $\{CX_j\}_{j=1,\ldots,s}$ is linearly independent. This is the case if and only if

$$\begin{aligned} \text{rank } \begin{bmatrix} C_1 & \ldots & C_s \end{bmatrix} &= \dim\, C\,(X_1 \dot{+} \ldots \dot{+} X_s) = \dim\,(CX_1 + \cdots + CX_s) \\ &= \dim\, CX_1 + \cdots + \dim\, CX_s = \sum_{j=1,\ldots,s} \text{rank } C_j. \end{aligned} \qquad \square$$

If (4.35) holds then $\text{im } C_j \cap \text{im } C_k = \{0\}$ for $j \neq k$, and a decomposition of the output space is straightforward:

4.2.6. Procedure.

(1) If (4.35) holds then we set $Y_j = \text{im } C_j$ for $j = 1, \ldots, s$ and pick for Y_c any complement of $Y_1 \dot{+} \ldots \dot{+} Y_s$ in Y. The resulting decomposition

$$Y = Y_1 \dot{+} \ldots \dot{+} Y_s \dot{+} Y_c$$

satisfies $CX_j \subseteq Y_j$ for all $j = 1, \ldots, s$.

(2) With respect to this decomposition, the system representation is

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_s \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & 0 & A_{1c} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{ss} & A_{sc} \\ 0 & \cdots & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_c \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ \vdots \\ y_s \\ y_c \end{bmatrix} = \begin{bmatrix} C_{11} & \cdots & 0 & C_{1c} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & C_{ss} & C_{sc} \\ 0 & \ldots & 0 & C_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_c \end{bmatrix}.$$

We now return to the general case (4.34) without the assumption (4.35), and give a procedure to construct a decomposition of the output space and the state space such that $CX_{j\backslash c} \subset Y_{j\backslash c}$ holds.

4.2.7. Procedure.

(1) Output space decomposition:

(a) For each subsystem $j = 1, \ldots, s$ we set

$$Y_j^{\cap} = \text{im } C_j \cap \left(\sum_{k=1,\ldots,s,\ k\neq j} \text{im } C_k \right),$$

and let Y_j be any complement of $Y_j^{\cap}$ in $\text{im } C_j$.

(b) With respect to the output space decomposition

$$Y = Y_1 \dot{+} \ldots \dot{+} Y_s \dot{+} \left(\sum_{j=1,\ldots,s} Y_j^{\cap} \right) \dot{+} Y^{\text{cpl}},$$

where Y^{cpl} is any complement of im $\begin{bmatrix} C_1 & \cdots & C_s \end{bmatrix}$ in Y, the system is then of the form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_s \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & 0 & A_{1c} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{ss} & A_{sc} \\ 0 & \cdots & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_c \end{bmatrix},$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_s \\ y^{\cap} \\ y^{\text{cpl}} \end{bmatrix} = \begin{bmatrix} C_{11} & \cdots & 0 & C_{1c} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & C_{ss} & C_{sc} \\ C_{\cap 1} & \cdots & C_{\cap s} & C_{cc}^{\cap} \\ 0 & \cdots & 0 & C_{cc}^{\text{cpl}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_c \end{bmatrix}.$$

(2) State space decomposition:

(a) For each subsystem $j = 1, \ldots, s$ we find the observability decomposition of the pair $(C_{\cap j}, A_{jj})$.

(b) With respect to the decomposition $X_j = X_j^{\text{unobs}} \dot{+} X_j^{\text{obs}}$, we then have

$$(C_{\cap j}, A_{jj}) = \left(\begin{bmatrix} 0 & C_{\cap j}^{\text{obs}} \end{bmatrix}, \begin{bmatrix} A_{jj}^{\text{unobs}} & A_{jj}^{\nwarrow} \\ 0 & A_{jj}^{\text{obs}} \end{bmatrix} \right).$$

The overall system is given by

$$\left[\begin{array}{c} \dot{x}_1^{\text{unobs}} \\ \dot{x}_1^{\text{obs}} \\ \hline \vdots \\ \vdots \\ \hline \dot{x}_s^{\text{unobs}} \\ \dot{x}_s^{\text{obs}} \\ \hline \dot{x}_c \end{array}\right] = \left[\begin{array}{cc|cc|cc|c} A_{11}^{\text{unobs}} & A_{11}^{\nwarrow} & \cdots & \cdots & 0 & 0 & A_{1c}^{\text{unobs}} \\ 0 & A_{11}^{\text{obs}} & \cdots & \cdots & 0 & 0 & A_{1c}^{\text{obs}} \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & \cdots & A_{ss}^{\text{unobs}} & A_{ss}^{\nwarrow} & A_{sc}^{\text{unobs}} \\ 0 & 0 & \cdots & \cdots & 0 & A_{ss}^{\text{obs}} & A_{sc}^{\text{obs}} \\ \hline 0 & 0 & \cdots & \cdots & 0 & 0 & A_{cc} \end{array}\right] \left[\begin{array}{c} x_1^{\text{unobs}} \\ x_1^{\text{obs}} \\ \hline \vdots \\ \vdots \\ \hline x_s^{\text{unobs}} \\ x_s^{\text{obs}} \\ \hline x_c \end{array}\right],$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_s \\ y^{\cap} \\ y^{\mathrm{cpl}} \end{bmatrix} = \left[\begin{array}{cc|cc|cc|c} C_{11}^{\mathrm{unobs}} & C_{11}^{\mathrm{obs}} & \cdots & \cdots & 0 & 0 & C_{1c} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & C_{ss}^{\mathrm{unobs}} & C_{ss}^{\mathrm{obs}} & C_{sc} \\ 0 & C_{\cap 1}^{\mathrm{obs}} & \cdots & \cdots & 0 & C_{\cap s}^{\mathrm{obs}} & C_{cc}^{\cap} \\ 0 & 0 & \cdots & \cdots & 0 & 0 & C_{cc}^{\mathrm{cpl}} \end{array}\right] \left[\begin{array}{c} x_1^{\mathrm{unobs}} \\ x_1^{\mathrm{obs}} \\ \hline \vdots \\ \vdots \\ \hline x_s^{\mathrm{unobs}} \\ x_s^{\mathrm{obs}} \\ \hline x_c \end{array}\right].$$

(3) We now define

$$\widetilde{X}_1 = X_1^{\mathrm{unobs}},\ \ldots,\ \widetilde{X}_s = X_s^{\mathrm{unobs}},\ \widetilde{X}_c = X_1^{\mathrm{obs}} \dot{+} \ldots \dot{+} X_s^{\mathrm{obs}} \dot{+} X_c$$

and

$$\widetilde{Y}_1 = Y_1,\ \ldots,\ \widetilde{Y}_s = Y_s,\ \widetilde{Y}_c = \left(\sum_{j=1,\ldots,s} Y_j^{\cap} \right) \dot{+} Y^{\mathrm{cpl}}.$$

With respect to this decomposition, the system is then a coordinated linear system, of the form

$$\left[\begin{array}{c} \dot{x}_1^{\mathrm{unobs}} \\ \hline \vdots \\ \hline \dot{x}_s^{\mathrm{unobs}} \\ \hline \dot{x}_1^{\mathrm{obs}} \\ \vdots \\ \dot{x}_s^{\mathrm{obs}} \\ \dot{x}_c \end{array}\right] = \left[\begin{array}{c|c|c|cccc} A_{11}^{\mathrm{unobs}} & \cdots & 0 & A_{11}^{\nwarrow} & \cdots & 0 & A_{1c}^{\mathrm{unobs}} \\ \hline \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & \cdots & A_{ss}^{\mathrm{unobs}} & 0 & \cdots & A_{ss}^{\nwarrow} & A_{sc}^{\mathrm{unobs}} \\ \hline 0 & \cdots & 0 & A_{11}^{\mathrm{obs}} & \cdots & 0 & A_{1c}^{\mathrm{obs}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & A_{ss}^{\mathrm{obs}} & A_{sc}^{\mathrm{obs}} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & A_{cc} \end{array}\right] \left[\begin{array}{c} x_1^{\mathrm{unobs}} \\ \hline \vdots \\ \hline x_s^{\mathrm{unobs}} \\ \hline x_1^{\mathrm{obs}} \\ \vdots \\ x_s^{\mathrm{obs}} \\ x_c \end{array}\right],$$

$$\left[\begin{array}{c} y_1 \\ \hline \vdots \\ \hline y_s \\ \hline y^{\cap} \\ y^{cpl} \end{array}\right] = \left[\begin{array}{c|c|c|cccc} C_{11}^{\mathrm{unobs}} & \cdots & 0 & C_{11}^{\mathrm{obs}} & \cdots & 0 & C_{1c} \\ \hline \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & \cdots & C_{ss}^{\mathrm{unobs}} & 0 & \cdots & C_{ss}^{\mathrm{obs}} & C_{sc} \\ \hline 0 & \cdots & 0 & C_{\cap 1}^{\mathrm{obs}} & \cdots & C_{\cap s}^{\mathrm{obs}} & C_{cc}^{\cap} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & C_{cc}^{cpl} \end{array}\right] \left[\begin{array}{c} x_1^{\mathrm{unobs}} \\ \hline \vdots \\ \hline x_s^{\mathrm{unobs}} \\ \hline x_1^{\mathrm{obs}} \\ \vdots \\ x_s^{\mathrm{obs}} \\ x_c \end{array}\right].$$

This procedure does not necessarily lead to a decomposition with a minimal coordinator state space: Moving $X_1^{\text{obs}} \dot{+} \ldots \dot{+} X_s^{\text{obs}}$ to the coordinator state space is sufficient for satisfying (4.35), but may not be necessary.

4.2.3.2 Input space decompositions

The problem of finding appropriate input space decompositions for a given state space decomposition will be discussed next:

4.2.8. Problem. Given a system of the form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_s \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & 0 & A_{1c} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{ss} & A_{sc} \\ 0 & \cdots & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_c \end{bmatrix} + \begin{bmatrix} B_1 \\ \vdots \\ B_s \\ B_c \end{bmatrix} u,$$

with state space $X = X_1 \dot{+} \ldots \dot{+} X_s \dot{+} X_c$ and input space U, find an input space decomposition

$$U = U_1 \dot{+} \ldots \dot{+} U_s \dot{+} U_c$$

such that $BU_j \subseteq X_j$ for $j = 1, \ldots, s$.[4]

This problem is straightforwardly solved by applying the following procedure:

4.2.9. Procedure.

(1) Pick an arbitrary decomposition

$$\ker B = U_1^{\ker} \dot{+} \ldots \dot{+} U_s^{\ker} \dot{+} U_c^{\ker}$$

of $\ker B$.

(2) For each subsystem $j = 1, \ldots, s$, let U_j^{cpl} be any complement of $\ker B$ in the space

$$\bigcap_{k=1,\ldots,s,c,\ k \neq j} \ker B_k.$$

(3) Now we set the subsystem input spaces to $U_j = U_j^{\ker} \dot{+} U_j^{\text{cpl}}$. With U_c any complement of $U_1 \dot{+} \ldots \dot{+} U_s$ in U, we now have

$$U = U_1 \dot{+} \ldots \dot{+} U_s \dot{+} U_c,$$

and $BU_j = BU_j^{\ker} + BU_j^{\text{cpl}} = BU_j^{\text{cpl}} = B_j U_j^{\text{cpl}} \subseteq X_j$.

[4]Such an input space decomposition always exists: Take $U = U_c$.

(4) With respect to this decomposition, the system is now of the form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_s \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & 0 & A_{1c} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{ss} & A_{sc} \\ 0 & \cdots & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_c \end{bmatrix} + \begin{bmatrix} B_{11} & \cdots & 0 & B_{1c} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & B_{ss} & B_{sc} \\ 0 & \dots & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_s \\ u_c \end{bmatrix},$$

with each subsystem input space U_j satisfying $BU_j \subseteq X_j$.

Note that the spaces $U_1, \dots, U_s$ are indeed linearly independent: Suppose w.l.o.g. $u \in U_1 \cap (U_2 + \cdots + U_s)$. Since $U_2 + \cdots + U_s \subseteq \ker B_1$, we have

$$u \in \left(\bigcap_{k=2,\dots,s,c} \ker B_k \right) \cap \ker B_1 = \bigcap_{k=1,\dots,s,c} \ker B_k = \ker B,$$

but $U_j \cap \ker B = U_j^{\ker}$, and hence

$$u \in U_1^{\ker} \cap \left(\bigcap_{k=2,\dots,s,c} U_k^{\ker} \right) = \{0\}.$$

4.2.3.3 Combined procedure for systems with inputs and outputs

The following procedure shows how to combine the different procedures given above to find a coordinated linear system representation of a monolithic linear system with inputs and outputs:

4.2.10. Procedure.

(1) Decompose the state space X such that A is in Jordan normal form. The system is now of the form (4.33).

(2) Check condition (4.35):

- If (4.35) holds then apply Procedure 4.2.6 to decompose the output space,
- otherwise apply Procedure 4.2.7 to decompose the output space and change the state space decomposition.

(3) For decomposing the input space, apply Procedure 4.2.9.

4.3 Concepts of minimality

This section describes how to transform a given coordinated linear system decomposition into a minimal decomposition – three different concepts of minimal decompositions are introduced and characterized, and several transformation procedures are given.

4.3.1 Problem formulation

In this section we consider coordinated linear system decompositions of the form

$$X = X_1 \dot{+} X_2 \dot{+} X_c,\ U = U_1 \dot{+} U_2 \dot{+} U_c \text{ and } Y = Y_1 \dot{+} Y_2 \dot{+} Y_c, \tag{4.36}$$

of a given linear system defined by system matrices A, B, C, D, i.e. with respect to the decomposition the system is of the form

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_c \end{bmatrix}, \\
\begin{bmatrix} y_1 \\ y_2 \\ y_c \end{bmatrix} &= \begin{bmatrix} C_{11} & 0 & C_{1c} \\ 0 & C_{22} & C_{2c} \\ 0 & 0 & C_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} D_{11} & 0 & D_{1c} \\ 0 & D_{22} & D_{2c} \\ 0 & 0 & D_{cc} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_c \end{bmatrix}.
\end{aligned} \tag{4.37}$$

Before we can formally define the problem discussed in this section, we need to introduce and define possible concepts of minimality.

4.3.1.1 Minimal coordinator

We say that a coordinated linear system decomposition has a minimal coordinator if the state space, input space and output space dimensions of the coordinator system are as small as possible:

4.3.1. Definition. Decomposition (4.36) is said to have a **minimal coordinator** if for all possible coordinated linear system decompositions $X_1' \dot{+} X_2' \dot{+} X_c'$, $U_1' \dot{+} U_2' \dot{+} U_c'$ and $Y_1' \dot{+} Y_2' \dot{+} Y_c'$ of X, U and Y we have

$$\begin{aligned}
\dim X_c &\leq \dim X_c', \\
\dim U_c &\leq \dim U_c', \\
\dim Y_c &\leq \dim Y_c'.
\end{aligned}$$

This concept corresponds to the system being 'as decentralized as possible': The coordinator system, and hence the centralized part of the system, is reduced to a minimum, in the sense of dimension. The existence of a minimal coordinator is in no way obvious – we require the dimensions of X_c, U_c and Y_c to all be minimal at the same time. Alternative formulations may require only $\dim X_c$ to be minimal, or the sum of the three dimensions.

4.3.1.2 Minimal communication

A coordinated linear system requires minimal communication if the dimension of the vector which has to be sent from the coordinator to the subsystems is as small as possible:

4.3.2. Definition. Decomposition (4.36) is said to require **minimal communication** if for all possible coordinated linear system decompositions $X_1' \dot{+} X_2' \dot{+} X_c'$, $U_1' \dot{+} U_2' \dot{+} U_c'$ and $Y_1' \dot{+} Y_2' \dot{+} Y_c'$ of X, U and Y, with system matrices A', B', C' and D', we have

$$\text{rank} \begin{bmatrix} A_{1c} & B_{1c} \\ C_{1c} & D_{1c} \end{bmatrix} \leq \text{rank} \begin{bmatrix} A'_{1c} & B'_{1c} \\ C'_{1c} & D'_{1c} \end{bmatrix},$$
$$\text{rank} \begin{bmatrix} A_{2c} & B_{2c} \\ C_{2c} & D_{2c} \end{bmatrix} \leq \text{rank} \begin{bmatrix} A'_{2c} & B'_{2c} \\ C'_{2c} & D'_{2c} \end{bmatrix}.$$

This definition is based on the fact that the subsystem state $\dot{x}_i$ depends on $A_{ic}x_c + B_{ic}u_c$, and the subsystem output y_i depends on $C_{ic}x_c + D_{ic}u_c$. Hence the coordinator needs to send a vector

$$\begin{bmatrix} A_{ic}x_c + B_{ic}u_c \\ C_{ic}x_c + D_{ic}u_c \end{bmatrix} = \begin{bmatrix} A_{ic} & B_{ic} \\ C_{ic} & D_{ic} \end{bmatrix} \begin{bmatrix} x_c \\ u_c \end{bmatrix}$$

to subsystem i.

4.3.1.3 Local controllability and observability

Another concept of minimality which will be important in later chapters is local controllability and/or observability: If a coordinated linear system is locally controllable and observable then the coordination required to control or measure the system is minimal.

4.3.3. Definition. Decomposition (4.36) is **locally controllable** if the pairs

$$(A_{11}, B_{11}),\ (A_{22}, B_{22}) \text{ and } (A_{cc}, B_{cc})$$

are controllable pairs; it is called **locally observable** if the pairs

$$(C_{11}, A_{11}),\ (C_{22}, A_{22}) \text{ and } (C_{cc}, A_{cc})$$

are observable pairs.

This definition is a short version of the definitions and characterizations in Sections 5.2.3.2 and 5.3.3.2, where these concepts are described in more detail.

4.3.1.4 Problem formulation

Given the concepts of minimality defined above, we can now formulate the problems considered in this section:

4.3.4. Problem. Given a coordinated linear system of the form (4.37), with respect to a decomposition of the form (4.36), and given the concepts of minimality defined in Definitions 4.3.1, 4.3.2 and 4.3.3,

- under which conditions do minimal decompositions exist,
- and how can the given decomposition be transformed into a minimal decomposition?

Some considerations and partial results concerning this problem are given in the following subsections.

4.3.2 Minimality of the coordinator

For interconnected systems, we have already found a decomposition with a minimal coordinator in Proposition 4.1.4. However, this was based on the extra condition that the original decompositions $X = X_1 \dot{+} X_2$, $U = U_1 \dot{+} U_2$ and $Y = Y_1 \dot{+} Y_2$ are respected.

For the more general case of decomposing monolithic linear systems, we found that a construction procedure similar to the one given for interconnected systems is not useful in practice, and Procedure 4.2.10 generally does not lead to a minimal coordinator. In particular, if condition (4.35) is not satisfied then Procedure 4.2.7 has to be applied, which leads to an unnecessarily large coordinator state space. Moreover, reducing the number of subsystems s may lead to smaller coordinator spaces – in fact, if $s = 1$, i.e. there is only one subsystem, then no coordination is required at all.

In the following we assume that the decompositions (4.36) are already fixed, and we describe how to reduce the coordinator spaces, noting again that the state space decomposition depends on the output space decomposition but not vice versa, and the input space decomposition depends on the state space and output space decompositions but not vice versa.

The minimal coordinator output space is always $Y_c = \{0\}$: We can always move the coordinator output space to a (new or existing) subsystem without violating the underlying information structure. This would however lead to a locally unobservable system, as discussed in Section 4.3.4.

In the following we give a procedure for reducing the coordinator state space for an arbitrary coordinated linear system with outputs:

4.3.5. Procedure.

(1) We find the observability decomposition of the pair $\left(\begin{bmatrix} A_{1c} \\ A_{2c} \\ C_{1c} \\ C_{2c} \end{bmatrix}, A_{cc}\right)$: The unobservable subspace $X_c^{\not o}$ has no influence on the rest of the system, and hence we can move it from the coordinator into a new subsystem $\widetilde{X}_3 = X_c^{\not o}$ (or into one of the existing subsystems). The coordinator state space decreases to the observable part: $\widetilde{X}_c = X_c^o$.

(2) We find the observability decomposition of the pair $\left(\begin{bmatrix} A_{1c} \\ C_{1c} \end{bmatrix}, A_{cc}\right)$: The corresponding unobservable space $X_c^{\not o,1}$ only influences subsystem 2, and hence can be moved to that subsystem: $\widetilde{X}_2 = X_2 \dot{+} X_c^{\not o,1}$. The coordinator state space reduces to the observable part: $\widetilde{X}_c = X_c^{o,1}$.

(3) We repeat this process for the pair $\left(\begin{bmatrix} A_{2c} \\ C_{2c} \end{bmatrix}, A_{cc}\right)$, leading to $\widetilde{X}_1 = X_1 \dot{+} X_c^{\not o,2}$ and $\widetilde{X}_c = X_c^{o,2}$.

Setting C_{1c} and C_{2c} to zero in this procedure, we get a procedure for reducing the coordinator state space of a coordinated linear system without inputs and outputs.

The state space decomposition resulting from this procedure, and even the resulting $\dim X_c$, may depend on the order in which steps (2) and (3) are applied.

Given the output space and state space decompositions, the coordinator input space can be reduced as follows:

- The subspace $U_c^{\ker,1} = \ker \begin{bmatrix} B_{1c} \\ D_{1c} \\ B_{cc} \\ D_{cc} \end{bmatrix}$ of U_c only concerns subsystem 2, and can hence be moved to U_2: With $U_c = U_c^{\ker,1} \dot{+} U_c^{\text{rest},1}$, this gives $\widetilde{U}_c = U_c^{\text{rest},1}$ and $\widetilde{U}_2 = U_2 \dot{+} U_c^{\ker,1}$.
- Similarly, the subspace $U_c^{\ker,2} = \ker \begin{bmatrix} B_{2c} \\ D_{2c} \\ B_{cc} \\ D_{cc} \end{bmatrix}$ can be moved to subsystem 1.

The new input spaces and their dimensions may depend on the order in which these steps are applied, since $U_c^{\ker,1} \cap U_c^{\ker,2} = \ker \begin{bmatrix} B_{\cdot c} \\ D_{\cdot c} \end{bmatrix}$ may be non-zero.

So far we have only considered reductions of a fixed coordinator subspace X_c. However, given subspaces $X_{1\backslash c}$ and $X_{2\backslash c}$, there are many choices for their complement X_c. Note that the state space transformation

$$S = \begin{bmatrix} I & 0 & S_1 \\ 0 & I & S_2 \\ 0 & 0 & I \end{bmatrix} \tag{4.38}$$

yields a new coordinator subspace $\bar{X}_c$ with $X = X_{1\backslash c} \dot{+} X_{2\backslash c} \dot{+} \bar{X}_c$, while leaving $X_{1\backslash c}$ and $X_{2\backslash c}$ unchanged, for all S_1 and S_2 of the appropriate dimensions.

In the following we will characterize minimal coordinator state spaces of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix}, \tag{4.39}$$

using transformations of the form (4.38).

For the proposition below, we note that for a system of the form (4.39), and for S of the form (4.38),

$$S^{-1}AS = \begin{bmatrix} A_{11} & 0 & A_{11}S_1 - S_1A_{cc} + A_{1c} \\ 0 & A_{22} & A_{22}S_2 - S_2A_{cc} + A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix}. \tag{4.40}$$

We also note that, for a system without outputs, the part of the coordinator state space which can be moved to a new subsystem in step (1) of Procedure 4.3.5 is exactly the largest A-invariant subspace of X_c. Similarly, with the notation

$$A^{(1)} = \begin{bmatrix} A_{11} & A_{1c} \\ 0 & A_{cc} \end{bmatrix} \text{ and } A^{(2)} = \begin{bmatrix} A_{22} & A_{2c} \\ 0 & A_{cc} \end{bmatrix},$$

the largest $A^{(1)}$-invariant subspace of X_c was moved to subsystem 2, and the largest $A^{(2)}$-invariant subspace was moved to subsystem 1.

4.3.6. Proposition. *Given a system of the form (4.39),*

(1) *no complement X_c of $X_{1\backslash c} \dot{+} X_{2\backslash c}$ contains an A-invariant subspace if and only if all eigenvectors $\begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} \in X_{1\backslash c} \dot{+} X_{2\backslash c} \dot{+} X_c$ of A have $x_c = 0$, and*

(2) *no complement* X_c *of* $X_{1\backslash c}\dot{+}X_{2\backslash c}$ *contains an* $A^{(1)}$*-invariant or* $A^{(2)}$*-invariant subspace if and only if all eigenvectors* $\begin{bmatrix} x_1 \\ x_c \end{bmatrix} \in X_{1\backslash c}\dot{+}X_c$ *of* $A^{(1)}$ *have* $x_c = 0$ *and all eigenvectors* $\begin{bmatrix} x_2 \\ x_c \end{bmatrix} \in X_{2\backslash c}\dot{+}X_c$ *of* $A^{(2)}$ *have* $x_c = 0$.

Proof. By (4.40) and step (1) of Procedure 4.3.5, no complement contains an A-invariant subspace if and only if the pair

$$\left(\begin{bmatrix} A_{11}S_1 - S_1 A_{cc} + A_{1c} \\ A_{22}S_2 - S_2 A_{cc} + A_{2c} \end{bmatrix}, A_{cc} \right)$$

is observable for all $S_1,\ S_2$. Applying the Hautus test, this is equivalent to

$$\begin{bmatrix} (A_{11}S_1 - S_1 A_{cc} + A_{1c})x \\ (A_{22}S_2 - S_2 A_{cc} + A_{2c})x \end{bmatrix} = 0 \ \Rightarrow\ x = 0$$

for all $S_1,\ S_2$ and all $x \in X_c$ such that $A_{cc}x = \lambda x$. Substituting λx for $A_{cc}x$, this is equivalent to

$$\begin{bmatrix} (A_{11} - \lambda I)S_1 x + A_{1c}x \\ (A_{22} - \lambda I)S_2 x + A_{2c}x \\ (A_{cc} - \lambda I)x \end{bmatrix} = 0 \ \Rightarrow\ x = 0$$

for all $S_1,\ S_2$ and all $x \in X_c$. Setting $x_1 = S_1 x$ and $x_2 = S_2 x$ gives

$$\begin{bmatrix} A_{11} - \lambda I & 0 & A_{1c} \\ 0 & A_{22} - \lambda I & A_{2c} \\ 0 & 0 & A_{cc} - \lambda I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x \end{bmatrix} = 0 \ \Rightarrow\ x = 0,$$

which is equivalent to saying that all eigenvectors $\begin{bmatrix} x_1 \\ x_2 \\ x \end{bmatrix}$ of A have $x = 0$.

The proof of part (2) follows from the same argumentation. □

4.3.3 Minimal communication

For the problem of finding coordinated linear systems which require minimal communication, we have not yet found any results. If all subspaces of the decomposition are free to choose (e.g. because the coordinated linear system is constructed from a monolithic linear system), minimal communication is trivially achieved by moving the whole system to the coordinator – in that case, no communication is required at all.

If we assume that the state, input and output spaces of the subsystems are fixed, and in line with the previous subsection, we can reduce the problem to finding complements X_c of $X_{1\backslash c}\dot{+}X_{2\backslash c}$ in X, U_c of $U_{1\backslash c}\dot{+}U_{2\backslash c}$ in U, and Y_c of $Y_{1\backslash c}\dot{+}Y_{2\backslash c}$ in Y, such that the resulting decomposition requires minimal communication:

Define the transformations

$$R = \begin{bmatrix} I & 0 & R_{1c} \\ 0 & I & R_{2c} \\ 0 & 0 & I \end{bmatrix} : Y \to Y, \;\; S = \begin{bmatrix} I & 0 & S_{1c} \\ 0 & I & S_{2c} \\ 0 & 0 & I \end{bmatrix} : X \to X,$$

$$\text{and } T = \begin{bmatrix} I & 0 & T_{1c} \\ 0 & I & T_{2c} \\ 0 & 0 & I \end{bmatrix} : U \to U,$$

where the submatrices R_{ic}, S_{ic} and T_{ic} are free to choose. The problem we want to solve can then be written as

$$\min_{R,S,T} \text{ rank } \begin{bmatrix} (SAS^{-1})_{1c} & (SBT^{-1})_{1c} \\ (RCS^{-1})_{1c} & (RDT^{-1})_{1c} \end{bmatrix}, \tag{4.41}$$

$$\min_{R,S,T} \text{ rank } \begin{bmatrix} (SAS^{-1})_{2c} & (SBT^{-1})_{2c} \\ (RCS^{-1})_{2c} & (RDT^{-1})_{2c} \end{bmatrix}. \tag{4.42}$$

For A, B, C, D given in (4.37), we have

$$\begin{bmatrix} (SAS^{-1})_{1c} & (SBT^{-1})_{1c} \\ (RCS^{-1})_{1c} & (RDT^{-1})_{1c} \end{bmatrix} = \begin{bmatrix} A_{1c} - A_{11}S_{1c} + S_{1c}A_{cc} & B_{1c} - B_{11}T_{1c} + S_{1c}B_{cc} \\ C_{1c} - C_{11}S_{1c} + R_{1c}C_{cc} & D_{1c} - D_{11}T_{1c} + R_{1c}D_{cc} \end{bmatrix},$$

$$\begin{bmatrix} (SAS^{-1})_{2c} & (SBT^{-1})_{2c} \\ (RCS^{-1})_{2c} & (RDT^{-1})_{2c} \end{bmatrix} = \begin{bmatrix} A_{2c} - A_{22}S_{2c} + S_{2c}A_{cc} & B_{2c} - B_{22}T_{2c} + S_{2c}B_{cc} \\ C_{2c} - C_{22}S_{2c} + R_{2c}C_{cc} & D_{2c} - D_{22}T_{2c} + R_{2c}D_{cc} \end{bmatrix}.$$

From this we can see that the two subproblems decouple – in (4.41) we need to minimize over R_{1c}, S_{1c}, T_{1c} and (4.42) is a minimization problem over R_{2c}, S_{2c}, T_{2c}.

For the special case of a system without inputs and outputs, the minimization problem given in (4.41) reduces to

$$\min_{S_{1c}:X_c \to X_1} \text{ rank } \left(A_{1c} - A_{11}S_{1c} + S_{1c}A_{cc}\right). \tag{4.43}$$

The solution to this problem follows directly from [47, Theorem 2.1], a simplified version of which is quoted for convenience:

Define $s(A_1, A_2) = \max_{\lambda \in \mathbb{C}} \min \{\dim \ker (A_1 - \lambda I), \dim \ker (A_2 - \lambda I)\}$.
Consider the linear Sylvester map $T : \mathbb{C}^{p \times q} \to \mathbb{C}^{p \times q}$ defined by

$$T(S) = SA_2 - A_1 S, \quad S \in \mathbb{C}^{p \times q},$$

(a) Every matrix $X \in \mathbb{C}^{p \times q}$ can be written in the form

$$X = T(S) + Y,$$

for some $S \in \mathbb{C}^{p \times q}$ and $Y \in \mathbb{C}^{p \times q}$ with rank $Y \leq s(A_1, A_2)$.

(b) Assume that $s(A_1, A_2) \neq 0$. Then for fixed A_1 and A_2 there is a Zariski open nonempty set Ω of $\mathbb{C}^{p \times q}$ such that for every $X \in \Omega$ there is no representation of X in the form

$$X = T(S) + Y,$$

where $S, Y \in \mathbb{C}^{p \times q}$ are such that rank $Y < s(A_1, A_2)$.

Translated to the minimization problem in (4.43), this means that

$$\min_{S_{1c}: X_c \to X_1} \text{rank } (A_{1c} - A_{11} S_{1c} + S_{1c} A_{cc}) \leq s(A_{11}, A_{cc}),$$

and that this upper bound is attained for all A_{1c} in a Zariski open nonempty subset of $\mathbb{C}^{n_1 \times n_c}$ (i.e. for "almost all" A_{1c}).

4.3.4 Local observability and controllability

The following result concerns the existence of locally controllable or observable decompositions:

4.3.7. Proposition. *For the existence of a locally controllable coordinated linear system decomposition it is necessary and sufficient that the pair (A, B) is a controllable pair. The existence of a local observable decomposition is equivalent to the pair (C, A) being an observable pair.*

Proof. By the block-triangular structure of A and B, local controllability is a stronger concept than controllability of (A, B). Conversely, if (A, B) is a controllable pair then setting $X_c = X$ and $U_c = U$ trivially leads to a locally controllable system. The same argument holds for local observability, using the block-triangular structure of C and A. □

For coordinated linear systems with inputs, a state space transformation which renders the system locally controllable is straightforward:

4.3.8. Procedure. We assume that the pair (A, B) is a controllable pair.

(1) For each subsystem $i = 1, 2$ we find the controllability decomposition of (A_{ii}, B_{ii}), i.e. we decompose the subsystem state space into $X_i = X_i^c \dotplus X_c^{\not c}$ such that with respect to this decomposition we have

$$(A_{ii}, B_{ii}) = \left(\begin{bmatrix} A_{ii}^c & A_{ii}^{\nwarrow} \\ 0 & A_{ii}^{\not c} \end{bmatrix}, \begin{bmatrix} B_{ii}^c \\ 0 \end{bmatrix} \right),$$

with (A_{ii}^c, B_{ii}^c) a controllable pair.

(2) We reduce the subsystem state spaces to their controllable parts and move the uncontrollable parts to the coordinator, i.e. we reset

$$X_1 := X_1^c,\ X_2 := X_2^c, \text{ and } X_c := X_1^{\not c} \dotplus X_2^{\not c} \dotplus X_c.$$

(3) With respect to the new state space decomposition, the system representation becomes

$$\begin{bmatrix} \dot{x}_1^c \\ \dot{x}_2^c \\ \hline \dot{x}_1^{\not c} \\ \dot{x}_2^{\not c} \\ \dot{x}_c \end{bmatrix} = \left[\begin{array}{cc|ccc} A_{11}^c & 0 & A_{11}^{\nwarrow} & 0 & A_{1c}^c \\ 0 & A_{22}^c & 0 & A_{22}^{\nwarrow} & A_{2c}^c \\ \hline 0 & 0 & A_{11}^{\not c} & 0 & A_{1c}^{\not c} \\ 0 & 0 & 0 & A_{22}^{\not c} & A_{2c}^{\not c} \\ 0 & 0 & 0 & 0 & A_{cc} \end{array}\right] \begin{bmatrix} x_1^c \\ x_2^c \\ \hline x_1^{\not c} \\ x_2^{\not c} \\ x_c \end{bmatrix} + \left[\begin{array}{cc|c} B_{11}^c & 0 & B_{1c}^c \\ 0 & B_{22}^c & B_{2c}^c \\ \hline 0 & 0 & B_{1c}^{\not c} \\ 0 & 0 & B_{2c}^{\not c} \\ 0 & 0 & B_{cc} \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ \hline u_c \end{bmatrix}.$$

Now the subsystems are locally controllable by construction. Suppose there is a locally uncontrollable mode in the coordinator, then this mode is uncontrollable in the usual sense (since the only option for controlling the coordinator state is via the coordinator input), but the system is controllable in the usual sense, which contradicts the assumption. Hence the system resulting from this procedure is locally controllable.

Note that in Procedure 4.3.8 we only applied a state space transformation, but left the input spaces unchanged. In some situations it might be useful to move parts of the coordinator input to a subsystem in order to achieve local controllability, instead of moving parts of the subsystem state to the coordinator. However, if we start out with a decomposition in which the coordinator input space has minimal dimension then the procedure above leads to a locally controllable system with minimal coordinator state and input spaces.

For the transformation of a coordinated linear system into a locally observable form, and in line with the previous section, we find that there are different choices to consider:

4.3.9. Procedure. We assume that pair (C, A) is an observable pair.

(1) We find the observability decomposition of the coordinator pair (C_{cc}, A_{cc}), i.e. we decompose the coordinator state space into $X_c = X_c^{\not o} \dot{+} X_c^{o}$, such that

$$(C_{cc}, A_{cc}) = \left(\begin{bmatrix} 0 & C_{cc}^{o} \end{bmatrix}, \begin{bmatrix} A_{cc}^{\not o} & A_{cc}^{\nwarrow} \\ 0 & A_{cc}^{o} \end{bmatrix} \right),$$

with (C_{cc}^{o}, A_{cc}^{o}) an observable pair.

(2) If $X_c^{\not o}$ is $\begin{bmatrix} A_{1c} \\ C_{1c} \end{bmatrix}$-unobservable then $X_c^{\not o}$ can move to subsystem 2. Similarly, it may move to subsystem 1 if it is $\begin{bmatrix} A_{2c} \\ C_{2c} \end{bmatrix}$-unobservable.

(3) Otherwise, pick decompositions $Y_1 = Y_1^{\not m} \dot{+} Y_1^{m}$ and $Y_2 = Y_2^{\not m} \dot{+} Y_2^{m}$ of the subsystem output spaces, such that $X_c^{\not o}$ is observable from $Y_1^{m} \dot{+} Y_2^{m}$.

(4) We reset the output spaces,

$$Y_1 := Y_1^{\not m},\ Y_2 := Y_2^{\not m},\ Y_c := Y_c \dot{+} Y_1^{m} \dot{+} Y_2^{m},$$

and the state spaces,

$$\begin{aligned} &X_1 \text{ is any } A\text{-invariant subspace of } C^{-P} Y_1, \\ &X_2 \text{ is any } A\text{-invariant subspace of } C^{-P} Y_2, \\ &X_c \text{ is any complement of } X_1 \dot{+} X_2 \text{ in } X. \end{aligned}$$

Now the coordinator system is locally observable by construction. The subsystems are locally observable since (C, A) was assumed to be observable. Picking the decompositions in step (3) is a difficult problem – in practice, this can be done by choosing one output vector at a time and testing whether the unobservable subspace $X_c^{\not o}$ would decrease if that vector were in the coordinator. Moreover, the dimensions of the coordinator spaces resulting from this procedure depend on which choices are made.

A combined procedure for transforming a coordinated linear system with inputs and outputs into a locally controllable and observable system would be to iterate Procedures 4.3.8 and 4.3.9: Several iterations may be needed since applying Procedure 4.3.9 to a locally controllable system may render the system locally uncontrollable, and Procedure 4.3.8 may destroy local observability.

5 Controllability and Observability

This chapter deals with the controllability and observability properties of coordinated linear systems, and was published as [25].

5.1 Introduction

Control theory is typically concerned with controlling global behavior of a system. For large systems consisting of several interconnected parts, it is of particular interest whether these global properties can be achieved via *local* control: The idea is to solve a control problem of lower complexity for each part of the system separately, and ideally these solutions would combine to a control law achieving the desired global property. The structure of coordinated systems allows for local control synthesis in a top-to-bottom manner, by first finding a control law for the coordinator, and then controlling each subsystem separately. For this, it is relevant which part of the system is controllable by using *which input*: Subsystem i, with state space X_i, may be controllable via its local input u_i, or via the coordinator input u_c, or both. The coordinator can only be controllable via the coordinator input u_c.

Moreover, decentralized systems are typically set up in such a way that each part of the system only has access to partial information about the system state: Each part of the system can observe (part of) its *local* state, and possibly the state of other, usually neighboring, parts of the system. In order to obtain information about the *global* state of the system, it is then necessary for the different parts to communicate their observations to each other. This communication, and hence the availability of global state information, is restricted by the information structure imposed on the system. In particular, the coordinator of a coordinated linear system can only observe (part of) its own state. The subsystems, on the other hand, can observe (part of) their local state and the coordinator state, where the coordinator state can be observed by subsystem i either indirectly via its influence on the local state x_i, or directly via communication from the coordinator.

In this chapter, we study the controllability and observability properties of coordinated linear systems, taking into account the different locations and roles of the available inputs and outputs, in order to provide a conceptual framework for future research on control synthesis for coordinated linear systems.

The results in this chapter make extensive use of the properties of reachable and indistinguishable subspaces summarized in Section 2.3, and the controllability and observability decompositions derived for coordinated linear systems combine the corresponding Kalman decompositions with the special structure of the system matrices in (3.1).

Note that for linear systems, the concepts of controllability and observability are dual to each other: The matrix pair (A, B) is a controllable pair if and only if the transposed pair (B^T, A^T) is an observable pair. However, for coordinated linear systems, this duality cannot be used to reduce one of the two concepts to the other one: If the matrix pair (A, B) corresponds to a coordinated linear system then the transposed pair (B^T, A^T) does not represent a coordinated linear system, unless A and B are block-diagonal. Hence, the concepts of controllability and observability for coordinated linear systems are treated separately in this chapter.

The outline of the chapter is as follows: In Section 5.2, we refine the usual concept of reachability, taking into account which input is used to reach a state, and which part of the system the reachable state corresponds to. Using these concepts, we then derive a controllability decomposition for coordinated linear systems, and discuss several possibilities for defining the concept of controllability in this setting. In Section 5.3 the usual concept of indistinguishability is refined in a similar manner, an observability decomposition for coordinated linear systems is derived, and several possible concepts of observability for this class of systems are defined and discussed. The combination of the different concepts of controllability and observability, for the purpose of stabilization via measurement feedback or the characterization of other forms of output controllability, is discussed in Section 5.4, and some conclusions are given in Section 5.5.

5.2 Controllability

In this section we consider coordinated linear systems with inputs and without outputs, i.e. systems of the form

$$\dot{x}(t) = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} x(t) + \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} u(t), \tag{5.1}$$

with $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix}$ and $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_c(t) \end{bmatrix}$.

In the following we will refine the usual concept of reachability, taking into account which input is used to reach a state, and which part of the system the reachable state corresponds to. Using these concepts, we will then derive a controllability decomposition for coordinated linear systems, and discuss several possibilities for defining the concept of controllability in this setting.

5.2.1 Reachability

For monolithic linear systems, the concept of reachability describes whether there exists a piecewise-continuous input trajectory, such that a given state $x \in X$ can be reached from the zero initial state in finite time (see e.g. [21]).

For coordinated linear systems it is not only interesting whether a state can be reached, but also which input is used to reach it: While a subsystem state x_i may be reachable via the local input u_i, or the coordinator input u_c, or a combination of the two, a coordinator state x_c can only be reached via the coordinator input u_c. This restriction is due to the condition $BU_i \subseteq X_i$ in Definition 3.1.1. For this reason, the usual definition of reachability is not satisfactory for coordinated linear systems. We introduce several concepts which are related to, but different from the definition of reachability as quoted in Section 2.3. In these new concepts, special attention is paid to specifying whether a state is reachable using the local input or the coordinator input.

5.2.1. Definition. We define the following concepts of reachability:[1]

- For $i = 1, 2$, a state $\bar{x}_i \in X_i$ is called $\boldsymbol{u_i}$**-reachable** (i.e. reachable using the local input u_i) if there exist a finite terminal time $\bar{t} \in [0, \infty)$ and a piecewise-continuous input trajectory $u_i : [0, \bar{t}] \to U_i$ such that the system

$$\dot{x}_i(t) = A_{ii}x_i(t) + B_{ii}u_i(t),\ x_i(0) = 0$$

  has a state trajectory $x_i : [0, \bar{t}] \to X_i$ satisfying $x_i(\bar{t}) = \bar{x}_i$. The set of all u_i-reachable states $x_i \in X_i$ will be denoted by $\mathfrak{R}_i$.

- A state $\bar{x} \in X$ is called $\boldsymbol{u_c}$**-reachable** (i.e. reachable using the coordinator input u_c) if there exist a finite terminal time $\bar{t} \in [0, \infty)$ and a piecewise-continuous input trajectory $u_c : [0, \bar{t}] \to U_c$ such that the system

$$\dot{x}(t) = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} x(t) + \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix} u_c(t),\ x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

  has a state trajectory $x : [0, \bar{t}] \to X$ satisfying $x(\bar{t}) = \bar{x}$. The set of all u_c-reachable states $x \in X$ will be denoted by $\mathfrak{R}_c$.

Note that $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_c$ are reachable subspaces of three different linear systems, and hence these subspaces have the following properties (see [72]):

- $\mathfrak{R}_1$ is the smallest A_{11}-invariant subspace of X_1 containing im B_{11},
- $\mathfrak{R}_2$ is the smallest A_{22}-invariant subspace of X_2 containing im B_{22},

[1] Note that we use X_1 to denote both the space X_1 and the subspace $\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X_1 \subseteq X$ (see Section 2.1).

- and $\mathfrak{R}_c$ is the smallest A-invariant subspace of X containing im $\begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix}$.

Recall from Section 2.3 that the linear subspace $\mathfrak{R} \subseteq X$ was defined as the set of all reachable states in the usual sense. The concepts of Definition 5.2.1 relate to the usual concept of reachability as follows:

5.2.2. Lemma. *For the set $\mathfrak{R}$ of all reachable states, the following relation holds:*

$$\mathfrak{R} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathfrak{R}_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathfrak{R}_2 + \mathfrak{R}_c. \tag{5.2}$$

Lemma 5.2.2 implies that our choice of definitions of $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_c$ complies with our intuitive conception of reachability: A state is reachable if and only if it can be reached via a combination of the available control inputs, and this is the case if and only if it is a combination of several states, each of which is reachable via one of the control inputs.

Proof. By (2.3) and with the notation $\mathfrak{C}(A,B) = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$, we have

$$\begin{aligned}
\mathfrak{R} &= \operatorname{im} \mathfrak{C}(A,B) = \operatorname{im} \mathfrak{C}\left(A, \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix}\right) \\
&= \operatorname{im} \mathfrak{C}\left(A, \begin{bmatrix} B_{11} \\ 0 \\ 0 \end{bmatrix}\right) + \operatorname{im} \mathfrak{C}\left(A, \begin{bmatrix} 0 \\ B_{22} \\ 0 \end{bmatrix}\right) + \operatorname{im} \mathfrak{C}\left(A, \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix}\right) \\
&= \operatorname{im} \begin{bmatrix} B_{11} & A_{11}B_{11} & \dots & A_{11}^{n-1}B_{11} \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 & 0 & \dots & 0 \\ B_{22} & A_{22}B_{22} & \dots & A_{22}^{n-1}B_{22} \\ 0 & 0 & \dots & 0 \end{bmatrix} \\
&\quad + \operatorname{im} \mathfrak{C}\left(A, \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix}\right) \\
&= \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \operatorname{im} \mathfrak{C}(A_{11}, B_{11}) + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \operatorname{im} \mathfrak{C}(A_{22}, B_{22}) + \operatorname{im} \mathfrak{C}\left(A, \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix}\right) \\
&= \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathfrak{R}_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathfrak{R}_2 + \mathfrak{R}_c.
\end{aligned}$$

□

So far we have split up the reachable subspace $\mathfrak{R}$ according to the different input spaces of a coordinated linear system. We still need to decompose the resulting subspaces $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_c$ according to the three different state spaces: Since $\mathfrak{R}_1 \subseteq X_1$ and $\mathfrak{R}_2 \subseteq X_2$, no further decomposition of these two subspaces is needed. A further decomposition of $\mathfrak{R}_c$ according to $X_1 \dot{+} X_2 \dot{+} X_c$ is more involved, since the same input trajectory u_c is used for all parts of the system. A simple example illustrates this:

5.2.3. Example. Let

$$\dot{x}(t) = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] x(t) + \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] u(t),$$

with $X_1 = \text{span}\{e_1, e_2\}$, $X_2 = \text{span}\{e_3, e_4\}$, $X_c = \text{span}\{e_5\}$, $U_1 = U_2 = U_c = \mathbb{R}$, as indicated by the lines in the matrices above. This system has $\mathfrak{R}_1 = \text{span}\{e_1\} \subset X_1$ and $\mathfrak{R}_2 = \text{span}\{e_3\} \subset X_2$. The u_c-reachable space $\mathfrak{R}_c$ can be found by looking at the controllability matrix of $\left(\begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix}, \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix} \right)$:

$$\begin{aligned} \mathfrak{R}_c &= \text{im} \begin{bmatrix} B_{1c} & A_{11}B_{1c} + A_{1c}B_{cc} & A_{11}^2 B_{1c} + A_{11}A_{1c}B_{cc} + A_{1c}A_{cc}B_{cc} & \dots \\ B_{2c} & A_{22}B_{2c} + A_{2c}B_{cc} & A_{22}^2 B_{2c} + A_{22}A_{2c}B_{cc} + A_{2c}A_{cc}B_{cc} & \dots \\ B_{cc} & A_{cc}B_{cc} & A_{cc}^2 B_{cc} & \dots \end{bmatrix} \\ &= \text{im} \begin{bmatrix} 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \end{bmatrix} = \text{span}\{e_1 + e_3, e_2 + e_4, e_5\}. \end{aligned}$$

We see that $X_c \subset \mathfrak{R}_c$, and hence the coordinator is u_c-reachable[2]. Since we have $X_1 \cap \mathfrak{R}_c = \{0\}$ and $X_2 \cap \mathfrak{R}_c = \{0\}$, no part of either of the subsystems is u_c-reachable. However,

$$(X_1 \dot{+} X_2) \cap \mathfrak{R}_c = \text{span}\{e_1 + e_3, e_2 + e_4\} \neq \{0\},$$

so $X_1 \dot{+} X_2$ has a non-trivial u_c-reachable subspace. In fact, any state in X_1 can be reached via u_c, but then subsystem 2 will arrive at the same state, and vice versa.

[2] In Definition 5.2.4 we will distinguish independently and jointly u_c-reachable subspaces – in this example, the coordinator is independently u_c-reachable.

In light of the properties of u_c-reachability illustrated in Example 5.2.3, we define the following spaces related to $\mathfrak{R}_c$:

5.2.4. Definition. The subspaces

$$X_1 \cap \mathfrak{R}_c,\ X_2 \cap \mathfrak{R}_c,\ X_c \cap \mathfrak{R}_c$$

will be called the **independently u_c-reachable** subspaces of X_1, X_2 and X_c, respectively.

The subspaces

$$\begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c,\ \begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathfrak{R}_c,\ \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c$$

will be called the **jointly u_c-reachable** subspaces of X_1, X_2 and X_c, respectively.

The term 'independently u_c-reachable' means that a state is reachable via an input trajectory $u_c : [0,t] \to U_c$ which leaves the rest of the system unaffected at time t; in other words, a state $x_1 \in X_1$ is independently u_c-reachable if the state $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \in X$ is u_c-reachable. The same holds for states in X_2 and X_c.

The term 'jointly u_c-reachable' is used for states that are u_c-reachable, but not necessarily independently u_c-reachable; for example, a state $x_1 \in X_1$ is jointly u_c-reachable if there exist $x_2 \in X_2$ and $x_c \in X_c$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix}$ is u_c-reachable, but these x_2 and x_c may be non-zero.

5.2.5. Remark. Note that independent u_c-reachability of $x_1 \in X_1$, in the sense of Definition 5.2.4, only means that the state $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$ is u_c-reachable in finite time; it does not mean that there exists an admissible input trajectory $u_c : [0,t] \to U_c$ such that the system state remains in the subspace $\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X_1$ at all times $\tau \in [0,t]$: The second concept is a stronger alternative of Definition 5.2.4, and will not be discussed here. In the following, we will always use the term 'independently u_c-reachable' to indicate that other parts of the system reach state 0 (from the initial state 0) at a finite time t, not that they remain in state 0 over the interval $[0,t]$.

The subspaces of Definition 5.2.4 can be combined to form over- and under-approximations of $\mathfrak{R}_c$: It immediately follows from Definition 5.2.4 that

$$(X_1 \cap \mathfrak{R}_c)\dot{+}(X_2 \cap \mathfrak{R}_c)\dot{+}(X_c \cap \mathfrak{R}_c) \subseteq \mathfrak{R}_c, \tag{5.3}$$

$$\mathfrak{R}_c \subseteq \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c \dot{+} \begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathfrak{R}_c \dot{+} \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c. \tag{5.4}$$

Note that the inclusions in (5.3) and (5.4) are in general not equalities.

5.2.2 Controllability decomposition

Using the definitions and results of Section 5.2.1, we will now derive a decomposition of the state space X according to the reachable spaces $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_c$, and the underlying decomposition $X = X_1 \dot{+} X_2 \dot{+} X_c$.

We start by looking at the state space X_1 of subsystem 1: We have that

$$\mathfrak{R}_1 \subseteq X_1 \text{ and } X_1 \cap \mathfrak{R}_c \subseteq \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c \subseteq X_1, \tag{5.5}$$

where $\mathfrak{R}_1$ is the u_1-reachable space, $X_1 \cap \mathfrak{R}_c$ is the independently u_c-reachable subspace of X_1, and $\begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$ is the jointly u_c-reachable subspace of X_1.

We decompose X_1 according to the following procedure:

5.2.6. Procedure.

(1) Let $X_1^1 = \mathfrak{R}_1 \cap (X_1 \cap \mathfrak{R}_c)$, and observe that then $X_1^1 \subseteq \mathfrak{R}_1 \cap \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$.

(2) Take for X_1^2 a complement of X_1^1 in $\mathfrak{R}_1 \cap \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$.

(3) Let X_1^3 be a complement of $X_1^1 \dot{+} X_1^2$ in $\mathfrak{R}_1$. Note that now $X_1^1 \dot{+} X_1^2 \dot{+} X_1^3 = \mathfrak{R}_1$.

(4) Likewise, as $X_1^1 \subseteq X_1 \cap \mathfrak{R}_c$, we can let X_1^4 be a complement of X_1^1 in $X_1 \cap \mathfrak{R}_c$. Observe that $X_1^4 \cap \mathfrak{R}_1 = \{0\}$, and $\mathfrak{R}_1 + (X_1 \cap \mathfrak{R}_c) = X_1^1 \dot{+} X_1^2 \dot{+} X_1^3 \dot{+} X_1^4$.

(5) Next, we let X_1^5 be a complement of $X_1^1 \dot{+} X_1^2 \dot{+} X_1^4$ in $\begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$, observing that $X_1^1 \dot{+} X_1^2 \dot{+} X_1^4$ is indeed a subspace of $\begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$, since $X_1^4 \subseteq X_1 \cap \mathfrak{R}_c \subseteq \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$ by (5.5).

(6) X_1^6 is a complement of $X_1^1 \dot{+} X_1^2 \dot{+} X_1^3 \dot{+} X_1^4 \dot{+} X_1^5$ in X_1.

This construction can be done numerically, by first picking a basis B_1 for $X_1^1 = \mathfrak{R}_1 \cap (X_1 \cap \mathfrak{R}_c)$, and then extending B_1 to obtain bases for the other subspaces in the decomposition.

Now we can write

$$X_1 = X_1^1 \dot{+} X_1^2 \dot{+} X_1^3 \dot{+} X_1^4 \dot{+} X_1^5 \dot{+} X_1^6.$$

Note that one or more of these subspaces can be $\{0\}$.

The reachability properties of the elements of this decomposition are given in the following table:

subspace	u_1-reachable	u_c-reachable
X_1^1	yes	independently
X_1^2	yes	only jointly
X_1^3	yes	no
X_1^4	no	independently
X_1^5	no	only jointly
X_1^6	no	no

The decomposition of X_2 is analogous, with the same reachability properties.

For the decomposition of the coordinator state space, we have to take into account the subspaces $X_c \cap \mathfrak{R}_c \subseteq \begin{bmatrix}0 & 0 & I\end{bmatrix} \mathfrak{R}_c \subseteq X_c$.

5.2.7. Procedure.

(1) We set $X_c^1 = X_c \cap \mathfrak{R}_c$.

(2) We let X_c^2 be a complement of X_c^1 in $\begin{bmatrix}0 & 0 & I\end{bmatrix} \mathfrak{R}_c$.

(3) For X_c^3 we choose any complement of $X_c^1 \dot{+} X_c^2$ in X_c.

Now we can write

$$X_c = X_c^1 \dot{+} X_c^2 \dot{+} X_c^3.$$

Now X_c^1 is independently u_c-reachable, X_c^2 is only jointly u_c-reachable, and X_c^3 is not reachable at all.

The following theorem is the main result of this section. It describes the invariance and controllability properties of the system given in (5.1), using the decomposition derived above.

5.2.8. Theorem. *With respect to the state space decomposition*

$$X = (X_1^1 \dot{+} \ldots \dot{+} X_1^6) \dot{+} (X_2^1 \dot{+} \ldots \dot{+} X_2^6) \dot{+} (X_c^1 \dot{+} X_c^2 \dot{+} X_c^3) \tag{5.6}$$

defined by Procedures 5.2.6 and 5.2.7, the system given in (5.1) has the form given in Table 5.1.

Table 5.1: The controllability decomposition

$$A = \left[\begin{array}{cccccc|cccccc|ccc}
A_{11}^{11} & A_{11}^{12} & A_{11}^{13} & A_{11}^{14} & A_{11}^{15} & A_{11}^{16} & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{11} & A_{1c}^{12} & A_{1c}^{13} \\
0 & A_{11}^{22} & A_{11}^{23} & 0 & A_{11}^{25} & A_{11}^{26} & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{21} & A_{1c}^{22} & A_{1c}^{23} \\
0 & A_{11}^{32} & A_{11}^{33} & 0 & A_{11}^{35} & A_{11}^{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{32} & A_{1c}^{33} \\
0 & 0 & 0 & A_{11}^{44} & A_{11}^{45} & A_{11}^{46} & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{41} & A_{1c}^{42} & A_{1c}^{43} \\
0 & 0 & 0 & 0 & A_{11}^{55} & A_{11}^{56} & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{51} & A_{1c}^{52} & A_{1c}^{53} \\
0 & 0 & 0 & 0 & A_{11}^{65} & A_{11}^{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{62} & A_{1c}^{63} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{22}^{13} & A_{22}^{14} & A_{22}^{15} & A_{22}^{16} & A_{2c}^{11} & A_{2c}^{12} & A_{2c}^{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{22} & A_{22}^{23} & 0 & A_{22}^{25} & A_{22}^{26} & A_{2c}^{21} & A_{2c}^{22} & A_{2c}^{23} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{32} & A_{22}^{33} & 0 & A_{22}^{35} & A_{22}^{36} & 0 & A_{2c}^{32} & A_{2c}^{33} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{44} & A_{22}^{45} & A_{22}^{46} & A_{2c}^{41} & A_{2c}^{42} & A_{2c}^{43} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{55} & A_{22}^{56} & A_{2c}^{51} & A_{2c}^{52} & A_{2c}^{53} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{65} & A_{22}^{66} & 0 & A_{2c}^{62} & A_{2c}^{63} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{11} & A_{cc}^{12} & A_{cc}^{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{21} & A_{cc}^{22} & A_{cc}^{23} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{33}
\end{array}\right] \tag{5.7}$$

$$B = \left[\begin{array}{c|c|c}
B_{11}^{1} & 0 & B_{1c}^{1} \\
B_{11}^{2} & 0 & B_{1c}^{2} \\
B_{11}^{3} & 0 & 0 \\
0 & 0 & B_{1c}^{4} \\
0 & 0 & B_{1c}^{5} \\
0 & 0 & 0 \\
\hline
0 & B_{22}^{1} & B_{2c}^{1} \\
0 & B_{22}^{2} & B_{2c}^{2} \\
0 & B_{22}^{3} & 0 \\
0 & 0 & B_{2c}^{4} \\
0 & 0 & B_{2c}^{5} \\
0 & 0 & 0 \\
\hline
0 & 0 & B_{cc}^{1} \\
0 & 0 & B_{cc}^{2} \\
0 & 0 & 0
\end{array}\right] \tag{5.8}$$

In the notation of Table 5.1, and with $i = 1, 2$, the following pairs are controllable pairs:

$$\left(\begin{bmatrix} A_{ii}^{11} & A_{ii}^{12} & A_{ii}^{13} \\ 0 & A_{ii}^{22} & A_{ii}^{23} \\ 0 & A_{ii}^{32} & A_{ii}^{33} \end{bmatrix}, \begin{bmatrix} B_{ii}^{1} \\ B_{ii}^{2} \\ B_{ii}^{3} \end{bmatrix}\right), \left(\left[\begin{array}{cc|cc|c} A_{11}^{11} & A_{11}^{14} & 0 & 0 & A_{1c}^{1} \\ 0 & A_{11}^{44} & 0 & 0 & A_{1c}^{4} \\ \hline 0 & 0 & A_{22}^{11} & A_{22}^{14} & A_{2c}^{1} \\ 0 & 0 & 0 & A_{22}^{44} & A_{2c}^{4} \\ \hline 0 & 0 & 0 & 0 & A_{cc}^{11} \end{array}\right], \left[\begin{array}{c} B_{1c}^{1} \\ B_{1c}^{4} \\ \hline B_{2c}^{1} \\ B_{2c}^{4} \\ \hline B_{cc}^{1} \end{array}\right]\right) \tag{5.9}$$

$$\left(\begin{bmatrix} A_{cc}^{11} & A_{cc}^{12} \\ A_{cc}^{21} & A_{cc}^{22} \end{bmatrix}, \begin{bmatrix} B_{cc}^{1} \\ B_{cc}^{2} \end{bmatrix}\right).$$

Proof. Recall from Definitions 3.1.1 and 5.2.1 that the subspaces X_1 and $\mathfrak{R}_c$ are A-invariant. Since $\mathfrak{R}_1 \subseteq X_1$ is A_{11}-invariant, its embedding into X is A-invariant. Now $X_1^1 = \mathfrak{R}_1 \cap (X_1 \cap \mathfrak{R}_c)$ is an intersection of A-invariant spaces, and hence A-invariant, which gives the first column of A in Table 5.1. The second and third column follow from $X_1^1 \dot{+} X_1^2 \dot{+} X_1^3 = \mathfrak{R}_1$ being A-invariant (as a subspace of X). The A-invariance of $X_1 \cap \mathfrak{R}_c = X_1^1 \dot{+} X_1^4$ explains the fourth column. The fifth and sixth column follow from the A-invariance of X_1. Similar arguments establish columns 7 - 12 for subsystem 2.

For establishing columns 13 and 14 of A, we note that

$$\begin{aligned} X_c^1 \dot{+} X_c^2 &= \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \operatorname{im} \mathfrak{C}\left(A, \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix}\right) \\ &= \operatorname{im} \begin{bmatrix} B_{cc} & A_{cc}B_{cc} & A_{cc}^2 B_{cc} & \dots \end{bmatrix} = \operatorname{im} \mathfrak{C}\left(A_{cc}, B_{cc}\right), \end{aligned}$$

and hence $\begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c = X_c^1 \dot{+} X_c^2$ is A_{cc}-invariant. Moreover,

$$\begin{aligned} A(X_c \cap \mathfrak{R}_c) &\subseteq \mathfrak{R}_c \subseteq \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c \dot{+} \begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathfrak{R}_c \dot{+} \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c \\ &= (X_1^1 \dot{+} X_1^2 \dot{+} X_1^4 \dot{+} X_1^5) \dot{+} (X_2^1 \dot{+} X_2^2 \dot{+} X_2^4 \dot{+} X_2^5) \dot{+} (X_c^1 \dot{+} X_c^2), \end{aligned}$$

which gives that no part of $(X_1^3 \dot{+} X_1^6) \dot{+} (X_2^3 \dot{+} X_2^6) \dot{+} X_c^3$ is u_c-reachable.

The structure of the B-matrix follows from

$$\begin{aligned} &\operatorname{im} B_{11} \subseteq \mathfrak{R}_1 = X_1^1 \dot{+} X_1^2 \dot{+} X_1^3, \ \operatorname{im} B_{22} \subseteq \mathfrak{R}_2 = X_2^1 \dot{+} X_2^2 \dot{+} X_2^3, \\ &\operatorname{im} \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix} \subseteq \mathfrak{R}_c \subseteq (X_1^1 \dot{+} X_1^2 \dot{+} X_1^4 \dot{+} X_1^5) \dot{+} (X_2^1 \dot{+} X_2^2 \dot{+} X_2^4 \dot{+} X_2^5) \dot{+} (X_c^1 \dot{+} X_c^2). \end{aligned}$$

The first controllable pair follows directly from the first part of Definition 5.2.1 and $\mathfrak{R}_i = X_i^1 \dot{+} X_i^2 \dot{+} X_i^3$. The second controllable pair follows from $(X_1^1 \dot{+} X_1^4) \dot{+} (X_2^1 \dot{+} X_2^4) \dot{+} X_c^1 \subseteq \mathfrak{R}_c$ and the second part of Definition 5.2.1. The last controllable pair follows from $X_c^1 \dot{+} X_c^2 = \operatorname{im} \mathfrak{C}(A_{cc}, B_{cc})$, which was derived earlier in this proof. □

An additional decomposition of the input space U_c of the coordinator, specifying which subsystem is influenced by which part of the input, would lead to a more refined controllability decomposition with respect to the B-matrix.

5.2.3 Concepts of controllability

In this section, we define several concepts of controllability for systems of the form (5.1), and express these concepts in terms of the controllability decomposition of Section 5.2.2. The concepts we introduce in this section were chosen because of their relevance for different applications; many other concepts are possible.

The most important concepts introduced in this section are weak local controllability (Definition 5.2.11) and independent controllability (Definition 5.2.17): Weak local controllability is necessary and sufficient for pole placement, and independent controllability is a locally verifiable concept replacing the usual concept of controllability.

5.2.3.1 Coordinator controllability

Coordinator controllability will be defined as follows:

5.2.9. Definition. We call a system of the form (5.1) **coordinator controllable** if all states $x \in X$ are u_c-reachable, i.e. if $\mathfrak{R}_c = X$.

Coordinator controllability is a very strong condition (see Proposition 5.2.19). In the case of coordinator controllability, control synthesis can be done by only specifying a control signal for the coordinator. This may be useful if the subsystems correspond to physical entities with limited computing capabilities, since no local control synthesis is necessary.

5.2.10. Proposition. *A system of the form (5.1) is coordinator controllable if and only if, in the notation of (5.1), the pair* $\left(A, \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix}\right)$ *is a controllable pair.*

Proof. If the system is coordinator controllable, i.e. if $\mathfrak{R}_c = X$, then in particular $X_1, X_2, X_c \subseteq \mathfrak{R}_c$, and conversely if $X_1, X_2, X_c \subseteq \mathfrak{R}_c$ then $X = X_1 \dot{+} X_2 \dot{+} X_c = \mathfrak{R}_c$.

Hence, in terms of decomposition (5.6), coordinator controllability is equivalent to

$$X_1 = X_1 \cap \mathfrak{R}_c = X_1^1 \dot{+} X_1^4, \quad X_2 = X_2 \cap \mathfrak{R}_c = X_2^1 \dot{+} X_2^4, \quad X_c = X_c \cap \mathfrak{R}_c = X_c^1.$$

The representation in Table 5.1 then reduces to

$$A = \left[\begin{array}{cc|cc|c} A_{11}^{11} & A_{11}^{14} & 0 & 0 & A_{1c}^{11} \\ 0 & A_{11}^{44} & 0 & 0 & A_{1c}^{41} \\ \hline 0 & 0 & A_{22}^{11} & A_{22}^{14} & A_{2c}^{11} \\ 0 & 0 & 0 & A_{22}^{44} & A_{2c}^{41} \\ \hline 0 & 0 & 0 & 0 & A_{cc}^{11} \end{array}\right], \; B = \left[\begin{array}{c|c|c} B_{11}^{1} & 0 & B_{1c}^{1} \\ 0 & 0 & B_{1c}^{4} \\ \hline 0 & B_{22}^{1} & B_{2c}^{1} \\ 0 & 0 & B_{2c}^{4} \\ \hline 0 & 0 & B_{cc}^{1} \end{array}\right], \tag{5.10}$$

and the second controllable pair in (5.9) coincides with the controllable pair given in Proposition 5.2.10. □

5.2.3.2 Weak local controllability

We define weak local controllability as follows:

5.2.11. Definition. We call a system of the form (5.1) **weakly locally controllable** if for $j = 1, 2, c$, all $x_j \in X_j$ are u_j-reachable, i.e. if

$$\mathfrak{R}_1 = X_1, \; \mathfrak{R}_2 = X_2 \textit{ and } X_c \subseteq \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c.$$

The term 'weak local controllability' is used to distinguish this concept from the slightly stronger concept of strong local controllability, defined in subsection 5.2.3.3. Weak local controllability means that control synthesis can be done locally: The control law for the coordinator can be found independently of the subsystems, and once the coordinator input is fixed, control synthesis for the subsystems can be done locally. In large systems, this is useful because the complexities of the local control problems may be much lower than that of the combined control problem.

5.2.12. Proposition. *A system of the form (5.1) is weakly locally controllable if and only if, in the notation of (5.1), the following pairs are controllable pairs:*

$$(A_{11}, B_{11}), \; (A_{22}, B_{22}), \; (A_{cc}, B_{cc}).$$

Proposition 5.2.12 can be proven directly without using the controllability decomposition. We choose the longer proof here since it gives more insight in the controllability structure.

Proof. In the notation of decomposition (5.6), weak local controllability amounts to

$$X_1 = \mathfrak{R}_1 = X_1^1 \dot{+} X_1^2 \dot{+} X_1^3,\ X_2 = \mathfrak{R}_2 = X_2^1 \dot{+} X_2^2 \dot{+} X_2^3,$$
$$X_c = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c = X_c^1 \dot{+} X_c^2.$$

The representation in Table 5.1 then reduces to

$$A = \left[\begin{array}{ccc|ccc|cc} A_{11}^{11} & A_{11}^{12} & A_{11}^{13} & 0 & 0 & 0 & A_{1c}^{11} & A_{1c}^{12} \\ 0 & A_{11}^{22} & A_{11}^{23} & 0 & 0 & 0 & A_{1c}^{21} & A_{1c}^{22} \\ 0 & A_{11}^{32} & A_{11}^{33} & 0 & 0 & 0 & 0 & A_{1c}^{32} \\ \hline 0 & 0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{22}^{13} & A_{2c}^{11} & A_{2c}^{12} \\ 0 & 0 & 0 & 0 & A_{22}^{22} & A_{22}^{23} & A_{2c}^{21} & A_{2c}^{22} \\ 0 & 0 & 0 & 0 & A_{22}^{32} & A_{22}^{33} & 0 & A_{2c}^{32} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{11} & A_{cc}^{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{21} & A_{cc}^{22} \end{array}\right], \quad B = \left[\begin{array}{c|c|c} B_{11}^1 & 0 & B_{1c}^1 \\ B_{11}^2 & 0 & B_{1c}^2 \\ B_{11}^3 & 0 & 0 \\ \hline 0 & B_{22}^1 & B_{2c}^1 \\ 0 & B_{22}^2 & B_{2c}^2 \\ 0 & B_{22}^3 & 0 \\ \hline 0 & 0 & B_{cc}^1 \\ 0 & 0 & B_{cc}^2 \end{array}\right], \tag{5.11}$$

and the corresponding controllable pairs in (5.9) are

$$\left(\begin{bmatrix} A_{ii}^{11} & A_{ii}^{12} & A_{ii}^{13} \\ 0 & A_{ii}^{22} & A_{ii}^{23} \\ 0 & A_{ii}^{32} & A_{ii}^{33} \end{bmatrix}, \begin{bmatrix} B_{ii}^1 \\ B_{ii}^2 \\ B_{ii}^3 \end{bmatrix} \right),\ i = 1,2 \ \text{ and } \ \left(\begin{bmatrix} A_{cc}^{11} & A_{cc}^{12} \\ A_{cc}^{21} & A_{cc}^{22} \end{bmatrix}, \begin{bmatrix} B_{cc}^1 \\ B_{cc}^2 \end{bmatrix} \right). \tag{5.12}$$

These are exactly the pairs $(A_{11}, B_{11}),\ (A_{22}, B_{22}),\ (A_{cc}, B_{cc})$. □

Note that weak local controllability is necessary and sufficient for pole placement (see [48]): For coordinated linear systems, admissible state feedback matrices must be of the form $F = \begin{bmatrix} F_{11} & 0 & F_{1c} \\ 0 & F_{22} & F_{2c} \\ 0 & 0 & F_{cc} \end{bmatrix}$, since feedback matrices of any other form would destroy the information structure imposed on the systems. Applying a state feedback of this type leads to the closed-loop system

$$\dot{x}(t) = \begin{bmatrix} A_{11} + B_{11}F_{11} & 0 & A_{1c} + B_{11}F_{1c} + B_{1c}F_{cc} \\ 0 & A_{22} + B_{22}F_{22} & A_{2c} + B_{22}F_{2c} + B_{2c}F_{cc} \\ 0 & 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix} x(t),$$

with spectrum

$$\sigma(A + BF) = \sigma(A_{11} + B_{11}F_{11}) \cup \sigma(A_{22} + B_{22}F_{22}) \cup \sigma(A_{cc} + B_{cc}F_{cc}).$$

Now the spectrum of the closed-loop system matrix can be assigned freely if and only if the pairs (A_{11}, B_{11}), (A_{22}, B_{22}) and (A_{cc}, B_{cc}) are controllable pairs, and this is equivalent to weak local controllability of the coordinated linear system.

5.2.3.3 Strong local controllability

The concept of strong local controllability will be defined as follows:

5.2.13. Definition. We call a system of the form (5.1) **strongly locally controllable** if for $i = 1, 2$, all states $x_i \in X_i$ are u_i-reachable, and for all $x_c \in X_c$, the state $\begin{bmatrix} 0 \\ 0 \\ x_c \end{bmatrix}$ is u_c-reachable, i.e. if

$$\mathfrak{R}_1 = X_1,\ \mathfrak{R}_2 = X_2 \ \text{ and } \ \mathfrak{R}_c \cap X_c = X_c.$$

An important detail in this definition is that *for all* $x_c \in X_c$, $\begin{bmatrix} 0 \\ 0 \\ x_c \end{bmatrix}$ is u_c-reachable: For weak local controllability we only require that for all $x_c \in X_c$ there exist $x_1 \in X_1$ and $x_2 \in X_2$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix}$ is reachable, but these x_1 and x_2 may depend on the value of x_c.

The difference between the concepts of weak and strong local controllability is that in the case of strong local controllability, it is possible to control each part of the system locally and *independently*, i.e. without influencing the rest of the system (see Remark 5.2.5). This means that control synthesis can be done in a fully decentralized manner: As in the case of weak local controllability, each part of the system can be controlled locally. In contrast to weak local controllability, control synthesis for the subsystems can be done in such a way that the subsystem state at a fixed time $\bar{t}$ is not influenced by the coordinator.

To illustrate this independence, we outline the control synthesis procedure for a strongly locally controllable system: Suppose we want the system to reach the state $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_c \end{bmatrix}$ at time $\bar{t}$, from the initial state $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ at time 0. This can be done as follows:

- Fix input trajectories $u_1 : [0, \bar{t}] \to U_1$ and $u_2 : [0, \bar{t}] \to U_2$ such that the local systems described by

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + B_{11}u_1, \quad x_1(0) = 0, \\ \dot{x}_2 &= A_{22}x_2 + B_{22}u_2, \quad x_2(0) = 0, \end{aligned}$$

satisfy $x_1(\bar{t}) = \bar{x}_1$ and $x_2(\bar{t}) = \bar{x}_2$.

- Fix an input trajectory $u_c : [0, \bar{t}] \to U_c$ such that the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix} u_c, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_c(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

satisfies $\begin{bmatrix} x_1(\bar{t}) \\ x_2(\bar{t}) \\ x_c(\bar{t}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{x}_c \end{bmatrix}$.

The existence of these input trajectories follows directly from strong local controllability.[3] Note again that (as in Remark 5.2.5) by applying u_c to the system, we get $x_1(\bar{t}) = 0$ and $x_2(\bar{t}) = 0$, but not necessarily $x_1(t) = 0$ and $x_2(t) = 0$ for all $t \in [0, \bar{t}]$. Now applying the combined input trajectory $u = \begin{bmatrix} u_1 \\ u_2 \\ u_c \end{bmatrix} : [0, \bar{t}] \to U$ yields

$$\begin{aligned}
\begin{bmatrix} x_1(\bar{t}) \\ x_2(\bar{t}) \\ x_c(\bar{t}) \end{bmatrix} &= \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \\ u_c(\tau) \end{bmatrix} d\tau \\
&= \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} \begin{bmatrix} B_{11} \\ 0 \\ 0 \end{bmatrix} u_1(\tau) d\tau + \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} \begin{bmatrix} 0 \\ B_{22} \\ 0 \end{bmatrix} u_2(\tau) d\tau + \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix} u_c(\tau) d\tau \\
&= \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \int_0^{\bar{t}} e^{A_{11}(\bar{t}-\tau)} B_{11} u_1(\tau) d\tau + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \int_0^{\bar{t}} e^{A_{22}(\bar{t}-\tau)} B_{22} u_2(\tau) d\tau \\
&+ \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix} u_c(\tau) d\tau \\
&= \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \bar{x}_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \bar{x}_2 + \begin{bmatrix} 0 \\ 0 \\ \bar{x}_c \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_c \end{bmatrix}.
\end{aligned}$$

[3] By definition, reachability only implies the existence of a finite time t at which a state $\bar{x}$ can be reached via an input trajectory $u : [0, t] \to U$. However, for linear systems this input trajectory can be transformed into a trajectory $\bar{u} : [0, \bar{t}] \to U$ such that $x(\bar{t}) = \bar{x}$, for any given finite $\bar{t} > 0$ (see e.g. [63]).

Hence we have found a suitable open-loop controller for reaching $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_c \end{bmatrix}$. The problems of finding the local input trajectories $u_1(\cdot)$ and $u_2(\cdot)$ are independent of the rest of the system, and $u_c(\cdot)$ can be found without knowledge of $u_1(\cdot)$ and $u_2(\cdot)$.

5.2.14. Proposition. *A system of the form (5.1) is strongly locally controllable if and only if there exist decompositions of X_1 and X_2, such that the corresponding system representation is of the form*

$$A = \left[\begin{array}{ccc|ccc|c} A_{11}^{11} & A_{11}^{12} & A_{11}^{33} & 0 & 0 & 0 & A_{1c}^{1} \\ 0 & A_{11}^{22} & A_{11}^{23} & 0 & 0 & 0 & A_{1c}^{2} \\ 0 & A_{11}^{32} & A_{11}^{33} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{22}^{13} & A_{2c}^{1} \\ 0 & 0 & 0 & 0 & A_{22}^{22} & A_{22}^{23} & A_{2c}^{2} \\ 0 & 0 & 0 & 0 & A_{22}^{32} & A_{22}^{33} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & A_{cc} \end{array}\right], \; B = \left[\begin{array}{c|c|c} B_{11}^{1} & 0 & B_{1c}^{1} \\ B_{11}^{2} & 0 & B_{1c}^{2} \\ B_{11}^{3} & 0 & 0 \\ \hline 0 & B_{22}^{1} & B_{2c}^{1} \\ 0 & B_{22}^{2} & B_{2c}^{2} \\ 0 & B_{22}^{3} & 0 \\ \hline 0 & 0 & B_{cc} \end{array}\right]. \tag{5.13}$$

In the notation of (5.13), the following tuples are controllable pairs:

$$\left(\begin{bmatrix} A_{ii}^{11} & A_{ii}^{12} & A_{ii}^{13} \\ 0 & A_{ii}^{22} & A_{ii}^{23} \\ 0 & A_{ii}^{32} & A_{ii}^{33} \end{bmatrix}, \begin{bmatrix} B_{ii}^{1} \\ B_{ii}^{2} \\ B_{ii}^{3} \end{bmatrix} \right), \quad \left(\left[\begin{array}{c|c|c} A_{11}^{11} & 0 & A_{1c}^{1} \\ \hline 0 & A_{22}^{11} & A_{2c}^{1} \\ \hline 0 & 0 & A_{cc} \end{array}\right], \left[\begin{array}{c} B_{1c}^{1} \\ \hline B_{2c}^{1} \\ \hline B_{cc} \end{array}\right] \right), \tag{5.14}$$

for $i = 1, 2$.

Proof. In the notation of decomposition (5.6), strong local controllability is equivalent to

$$X_1 = X_1^1 \dot{+} X_1^2 \dot{+} X_1^3, \quad X_2 = X_2^1 \dot{+} X_2^2 \dot{+} X_2^3, \quad X_c = X_c^1.$$

The representation in Table 5.1 then reduces to a representation of the form (5.13), and the corresponding controllable pairs in (5.9) reduce to the ones in (5.14). □

5.2.3.4 Joint controllability

Joint controllability will be defined as follows:

5.2.15. Definition. We call a system of the form (5.1) **jointly controllable** if all states $x \in X$ are $\begin{bmatrix} u_1 \\ u_2 \\ u_c \end{bmatrix}$-reachable, i.e. if

$$\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathfrak{R}_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathfrak{R}_2 + \mathfrak{R}_c = X.$$

By (5.2), this definition is equivalent to $X = \mathfrak{R}$, and hence joint controllability is equivalent to controllability in the usual sense (see Section 2.3).

From the characterizations given in Sections 5.2.3.1, 5.2.3.2 and 5.2.3.3, it follows that coordinator controllability, weak local controllability and strong local controllability can be checked in a decentralized way, by looking at each subsystem separately. As the example in Section 5.2.4 will illustrate, this is not necessarily the case for joint controllability. We can however formulate a necessary condition and a sufficient condition for joint controllability, both of which can be checked for each subsystem separately:

5.2.16. Lemma. *For joint controllability, it is necessary that*

$$X_1 = \mathfrak{R}_1 + \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c, \ X_2 = \mathfrak{R}_2 + \begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathfrak{R}_c \text{ and } X_c = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c;$$

it is sufficient that

$$X_1 = \mathfrak{R}_1 + (X_1 \cap \mathfrak{R}_c), \ X_2 = \mathfrak{R}_2 + (X_2 \cap \mathfrak{R}_c) \text{ and } X_c = X_c \cap \mathfrak{R}_c.$$

Proof. From inclusions (5.3) and (5.4) it follows directly that

$$\begin{aligned}
&(\mathfrak{R}_1 + (X_1 \cap \mathfrak{R}_c)) \dotplus (\mathfrak{R}_2 + (X_2 \cap \mathfrak{R}_c)) \dotplus (X_c \cap \mathfrak{R}_c) \\
&= \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathfrak{R}_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathfrak{R}_2 + ((X_1 \cap \mathfrak{R}_c) \dotplus (X_2 \cap \mathfrak{R}_c) \dotplus (X_c \cap \mathfrak{R}_c)) \\
&\subseteq \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathfrak{R}_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathfrak{R}_2 + \mathfrak{R}_c \\
&= \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathfrak{R}_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathfrak{R}_2 + \left(\begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c \dotplus \begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathfrak{R}_c \dotplus \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c\right) \\
&\subseteq \left(\mathfrak{R}_1 + \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c\right) \dotplus \left(\mathfrak{R}_2 + \begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathfrak{R}_c\right) \dotplus \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c.
\end{aligned}$$

□

For coordinated linear systems, joint controllability is not a very natural concept: If the sufficient condition is not satisfied then control requires some parts of the system to reverse the effects caused by controlling other parts of the system; if, for example, $\mathfrak{R}_c = \text{span}\left\{\begin{bmatrix} I \\ I \\ I \end{bmatrix}\right\}$ then the state $\begin{bmatrix} 0 \\ 0 \\ \bar{x}_c \end{bmatrix}$ can only be reached by an input trajectory $\begin{bmatrix} u_1(\cdot) \\ u_2(\cdot) \\ u_c(\cdot) \end{bmatrix}$, where neither $u_1(\cdot)$ nor $u_2(\cdot)$ can be identically zero. Since for the subsystems, both the initial state and the target state are zero, the only purpose of the non-zero input trajectories is to counteract the influence of the coordinator on the subsystems.

5.2.3.5 Independent controllability

A more natural concept for coordinated linear systems is independent controllability, which will be defined as follows:

5.2.17. Definition. We call a system of the form (5.1) **independently controllable** if for all $x_1 \in X_1$ the state $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} u_1 \\ u_c \end{bmatrix}$-reachable, for all $x_2 \in X_2$ the state $\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} u_2 \\ u_c \end{bmatrix}$-reachable, and for all $x_c \in X_c$ the state $\begin{bmatrix} 0 \\ 0 \\ x_c \end{bmatrix}$ is u_c-reachable, i.e. if

$$(\mathfrak{R}_1 + (X_1 \cap \mathfrak{R}_c)) \dot{+} (\mathfrak{R}_2 + (X_2 \cap \mathfrak{R}_c)) \dot{+} (X_c \cap \mathfrak{R}_c) = X.$$

In the case of independent controllability, each part of the system can be controlled in such a way that at a finite time t, no other part of the system is influenced by that control signal (see Remark 5.2.5). Control is possible both via the local inputs and the coordinator input. As is immediate from Definition 5.2.17, independent controllability coincides with the sufficient condition for joint controllability, and hence can be checked by looking at each part of the system separately.

5.2.18. Proposition. *A system of the form (5.1) is independently controllable if and only if there exist decompositions of X_1 and X_2, such that the corresponding system representation is of the form*

$$A=\left[\begin{array}{cccc|cccc|c} A_{11}^{11} & A_{11}^{12} & A_{11}^{13} & A_{11}^{14} & 0 & 0 & 0 & 0 & A_{1c}^{1} \\ 0 & A_{11}^{22} & A_{11}^{23} & 0 & 0 & 0 & 0 & 0 & A_{1c}^{2} \\ 0 & A_{11}^{32} & A_{11}^{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{11}^{44} & 0 & 0 & 0 & 0 & A_{1c}^{4} \\ \hline 0 & 0 & 0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{22}^{13} & A_{22}^{14} & A_{2c}^{1} \\ 0 & 0 & 0 & 0 & 0 & A_{22}^{22} & A_{22}^{23} & 0 & A_{2c}^{2} \\ 0 & 0 & 0 & 0 & 0 & A_{22}^{32} & A_{22}^{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{44} & A_{2c}^{4} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc} \end{array}\right],\ B=\left[\begin{array}{c|c|c} B_{11}^{1} & 0 & B_{1c}^{1} \\ B_{11}^{2} & 0 & B_{1c}^{2} \\ B_{11}^{3} & 0 & 0 \\ 0 & 0 & B_{1c}^{4} \\ \hline 0 & B_{22}^{1} & B_{2c}^{1} \\ 0 & B_{22}^{2} & B_{2c}^{2} \\ 0 & B_{22}^{3} & 0 \\ 0 & 0 & B_{2c}^{4} \\ \hline 0 & 0 & B_{cc} \end{array}\right] \qquad (5.15)$$

with controllable pairs

$$\left(\begin{bmatrix} A_{ii}^{11} & A_{ii}^{12} & A_{ii}^{13} \\ 0 & A_{ii}^{22} & A_{ii}^{23} \\ 0 & A_{ii}^{32} & A_{ii}^{33} \end{bmatrix}, \begin{bmatrix} B_{ii}^{1} \\ B_{ii}^{2} \\ B_{ii}^{3} \end{bmatrix}\right), \left(\left[\begin{array}{cc|cc|c} A_{11}^{11} & A_{11}^{14} & 0 & 0 & A_{1c}^{1} \\ 0 & A_{11}^{44} & 0 & 0 & A_{1c}^{4} \\ \hline 0 & 0 & A_{22}^{11} & A_{22}^{14} & A_{2c}^{1} \\ 0 & 0 & 0 & A_{22}^{44} & A_{2c}^{4} \\ \hline 0 & 0 & 0 & 0 & A_{cc} \end{array}\right], \left[\begin{array}{c} B_{1c}^{1} \\ B_{1c}^{4} \\ \hline B_{2c}^{1} \\ B_{2c}^{4} \\ \hline B_{cc} \end{array}\right]\right). \qquad (5.16)$$

Proof. In the notation of (5.6), independent controllability is equivalent to

$$\begin{aligned} X_1 &= \mathfrak{R}_1 + (X_1 \cap \mathfrak{R}_c) = X_1^1 \dot{+} X_1^2 \dot{+} X_1^3 \dot{+} X_1^4, \\ X_2 &= \mathfrak{R}_2 + (X_2 \cap \mathfrak{R}_c) = X_2^1 \dot{+} X_2^2 \dot{+} X_2^3 \dot{+} X_2^4, \\ X_c &= X_c \cap \mathfrak{R}_c = X_c^1. \end{aligned}$$

The representation in Table 5.1 then reduces to a representation of the form (5.15), and the corresponding controllable pairs in (5.9) reduce to the ones in (5.16). □

5.2.3.6 Relations between the concepts of controllability

In the following, some relations between the different concepts of controllability, as defined in Sections 5.2.3.1-5.2.3.5, are established.

5.2.19. Proposition. *The concepts of controllability defined in Sections 5.2.3.1-5.2.3.5 are related as follows:*

(1) *Coordinator controllability implies independent controllability,*

(2) *strong local controllability implies independent controllability,*

(3) *strong local controllability implies weak local controllability,*

(4) *independent controllability implies joint controllability,*

(5) *and weak local controllability implies joint controllability.*

Proof. The first three items follow directly from the reduced state space decompositions given in the proofs of Propositions 5.2.10, 5.2.12, 5.2.14 and 5.2.18.

The fourth item follows from the fact that the definition of independent controllability coincides with the sufficient condition for joint controllability given in Proposition 5.2.16.

For the last item, we note that by Definition 5.2.11, weak local controllability corresponds to $\mathfrak{R}_1 = X_1$, $\mathfrak{R}_2 = X_2$ and $\begin{bmatrix}0 & 0 & I\end{bmatrix}\mathfrak{R}_c = X_c$. This implies

$$\begin{aligned} X =& X_1 \dot{+} X_2 \dot{+} X_c = X_1 \dot{+} X_2 \dot{+} \begin{bmatrix}0 & 0 & I\end{bmatrix}\mathfrak{R}_c \\ =& \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} X_2 + \mathfrak{R}_c = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathfrak{R}_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathfrak{R}_2 + \mathfrak{R}_c = \mathfrak{R}, \end{aligned}$$

and this implies joint controllability by Definition 5.2.15. □

Weak local controllability is not a special case of independent controllability because in the case of weak local controllability, the coordinator input u_c may have a joint influence on the coordinator state and the subsystem states. In terms of decomposition (5.6), for independent controllability it is necessary that $X_c^2 = \{0\}$, while for weak local controllability this is not necessary.

5.2.4 Example

To illustrate the theory developed in this chapter, consider the system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

with

$$A = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{c|c|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

This example, with subsystems $X_1 = \text{span}\{e_1, e_2, e_3\}$ and $X_2 = \text{span}\{e_4, e_5, e_6\}$ and coordinator $X_c = \text{span}\{e_7, e_8, e_9\}$, has the following reachable subspaces:

- $\mathfrak{R}_1 = \text{span}\{e_1, e_2, e_3\} = X_1$
- $\mathfrak{R}_2 = \text{span}\{e_4, e_5\} \subsetneq X_2$
- $\mathfrak{R}_c = \text{span}\{e_1, e_2 + e_4, e_3 + e_5 + e_6, e_7, e_8, e_9\} \subsetneq X$
- $\mathfrak{R} = \text{span}\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\} = X$
- $X_1 \cap \mathfrak{R}_c = \text{span}\{e_1\} \subsetneq X_1$
- $X_2 \cap \mathfrak{R}_c = \{0\} \subsetneq X_2$
- $X_c \cap \mathfrak{R}_c = \text{span}\{e_7, e_8, e_9\} = X_c$
- $\begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c = \text{span}\{e_1, e_2, e_3\} = X_1$
- $\begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathfrak{R}_c = \text{span}\{e_4, e_5 + e_6\} \subsetneq X_2$
- $\begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c = \text{span}\{e_7, e_8, e_9\} = X_c$

Since $X_c \cap \mathfrak{R}_c = X_c$, the coordinator is strongly locally controllable. Since $\mathfrak{R}_1 = X_1$ and $\mathfrak{R}_2 \neq X_2$, subsystem 1 is locally controllable, while subsystem 2 is not. The system is not coordinator controllable since $\mathfrak{R}_c \neq X$. However, since $\mathfrak{R} = X$, the system is jointly controllable. Independent controllability fails because $\mathfrak{R}_2 + X_2 \cap \mathfrak{R}_c = \{e_4, e_5\} \neq X_2$.

Now suppose that B is given by

$$B = \left[\begin{array}{c|c|ccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

i.e. the 1 in the first column, third row moves to the first column, second row. In this case, e_3 is no longer locally reachable. The reachable spaces change as follows:

- $\mathfrak{R}_1 = \text{span}\{e_1, e_2\} \subsetneq X_1$
- $\mathfrak{R} = \text{span}\{e_1, e_2, e_3 + e_6, e_4, e_5, e_7, e_8, e_9\} \subsetneq X$,

The other subspaces stay the same. Now $\mathfrak{R} \neq X$, and hence the system is no longer jointly controllable.

Note that for this example, controllability cannot be checked by looking at the subsystems separately: For both choices of B, the sufficient condition of Lemma 5.2.16 is not satisfied, but the necessary condition holds.

5.3 Observability

In this section we consider coordinated linear systems without inputs and with outputs, i.e. systems of the form

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} x(t), \\ y(t) &= \begin{bmatrix} C_{11} & 0 & C_{1c} \\ 0 & C_{22} & C_{2c} \\ 0 & 0 & C_{cc} \end{bmatrix} x(t), \end{aligned} \tag{5.17}$$

with $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix}$ and $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_c(t) \end{bmatrix}$.

In the following, we will refine the usual concept of indistinguishability, as discussed in Section 2.3, taking into account which output can distinguish a state from the zero state, and which part of the system the distinguishable state belongs to. Based on these new concepts, we will derive an observability decomposition for coordinated linear systems, and introduce several possible concepts of observability for systems of the form (5.17).

5.3.1 Indistinguishability

For the case of unstructured linear systems, the concept of indistinguishability describes whether a given initial state can be distinguished from the zero initial state via the output trajectory in finite time (see e.g. [21]).

For coordinated linear systems, it is not only relevant whether a state is indistinguishable from the zero state, but also which output is able or unable to distinguish this state from 0. Since for coordinated linear systems we have $CX_i \subseteq Y_i$, the state x_i of subsystem i can possibly be observed by looking at the corresponding local output y_i, while x_i is indistinguishable from 0 at the other subsystem or the coordinator. However, the state x_c of the coordinator system may be distinguishable from 0 at y_1, y_2 or y_c. In order to separate these different cases, we introduce the following refined concepts of indistinguishability:

5.3.1. Definition. We define the following concepts of indistinguishability:

- For $i = 1, 2$, a state $\begin{bmatrix} \bar{x}_i \\ \bar{x}_c \end{bmatrix} \in X_i \dot{+} X_c$ is called ***y_i*-indistinguishable** (i.e. indistinguishable from the zero state by the local output y_i) if the system

$$\begin{bmatrix} \dot{x}_i \\ x_c \end{bmatrix} = \begin{bmatrix} A_{ii} & A_{ic} \\ 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_i \\ x_c \end{bmatrix}, \; y_i = \begin{bmatrix} C_{ii} & C_{ic} \end{bmatrix} \begin{bmatrix} x_i \\ x_c \end{bmatrix},$$

  with initial state $\begin{bmatrix} x_i \\ x_c \end{bmatrix}(0) = \begin{bmatrix} \bar{x}_i \\ \bar{x}_c \end{bmatrix}$, has $y_i(t) = 0$ for all $t \in T$. The set of all y_i-indistinguishable states $\begin{bmatrix} \bar{x}_i \\ \bar{x}_c \end{bmatrix} \in X_i \dot{+} X_c$ will be denoted by $\mathfrak{I}_i$.

- A state $\bar{x} \in X_c$ is called ***y_c*-indistinguishable** (i.e. indistinguishable from the zero state by the coordinator output y_c) if the system

$$\dot{x}_c = A_{cc} x_c, \; y_c = C_{cc} x_c,$$

  with initial state $x_c(0) = \bar{x}_c$, has $y_c(t) = 0$ for all $t \in T$. The set of all y_c-indistinguishable states $\bar{x}_c \in X_c$ will be denoted by $\mathfrak{I}_c$.

The spaces $\mathfrak{I}_1$, $\mathfrak{I}_2$ and $\mathfrak{I}_c$ are indistinguishable subspaces of three different linear systems. Hence they have the following properties (see [72]):

- $\mathfrak{I}_1$ is the largest $\begin{bmatrix} A_{11} & A_{1c} \\ 0 & A_{cc} \end{bmatrix}$-invariant subspace of $X_1 \dot{+} X_c$ contained in $\ker \begin{bmatrix} C_{11} & C_{1c} \end{bmatrix}$,
- $\mathfrak{I}_2$ is the largest $\begin{bmatrix} A_{22} & A_{2c} \\ 0 & A_{cc} \end{bmatrix}$-invariant subspace of $X_2 \dot{+} X_c$ contained in $\ker \begin{bmatrix} C_{22} & C_{2c} \end{bmatrix}$,
- and $\mathfrak{I}_c$ is the largest A_{cc}-invariant subspace of X_c contained in $\ker C_{cc}$.

Recall from Section 2.3 that the linear subspace $\mathfrak{I} \subseteq X$ was defined as the set of all indistinguishable states in the usual sense. The different indistinguishable spaces of Definition 5.3.1 relate to the usual concept of indistinguishability as follows:

5.3.2. Lemma. *For the indistinguishable subspace $\mathfrak{I}$ defined in Section 2.3, the following relation holds:*

$$\mathfrak{I} = (\mathfrak{I}_1 \dot{+} X_2) \cap (X_1 \dot{+} \mathfrak{I}_2) \cap (X_1 \dot{+} X_2 \dot{+} \mathfrak{I}_c) \tag{5.18}$$

The purpose of Lemma 5.3.2 is to verify that the different indistinguishable subspaces of Definition 5.3.1 combine to the indistinguishable subspace in the usual sense: This should be expected since a state is indistinguishable if none of the available outputs can distinguish it from the zero state.

Proof. By (2.5) and with the notation $\mathfrak{O}(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$, we have

$$
\begin{aligned}
\mathfrak{I} &= \ker \mathfrak{O}(C,A) \\
&= \ker \mathfrak{O}\left(\begin{bmatrix} C_{11} & 0 & C_{1c} \\ 0 & C_{22} & C_{2c} \\ 0 & 0 & C_{cc} \end{bmatrix}, A \right) \\
&= \ker \mathfrak{O}\left(\begin{bmatrix} C_{11} & 0 & C_{1c} \end{bmatrix}, A \right) \cap \ker \mathfrak{O}\left(\begin{bmatrix} 0 & C_{22} & C_{2c} \end{bmatrix}, A \right) \cap \ker \mathfrak{O}\left(\begin{bmatrix} 0 & 0 & C_{cc} \end{bmatrix}, A \right) \\
&= \left(X_2 \dotplus \ker \mathfrak{O}\left(\begin{bmatrix} C_{11} & C_{1c} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{1c} \\ 0 & A_{cc} \end{bmatrix} \right) \right) \\
&\quad \cap \left(X_1 \dotplus \ker \mathfrak{O}\left(\begin{bmatrix} C_{22} & C_{2c} \end{bmatrix}, \begin{bmatrix} A_{22} & A_{2c} \\ 0 & A_{cc} \end{bmatrix} \right) \right) \cap \ker \begin{bmatrix} 0 & 0 & C_{cc} \\ 0 & 0 & C_{cc}A_{cc} \\ \vdots & \vdots & \vdots \\ 0 & 0 & C_{cc}A_{cc}^{n-1} \end{bmatrix} \\
&= (X_2 \dotplus \mathfrak{I}_1) \cap (X_1 \dotplus \mathfrak{I}_2) \cap (X_1 \dotplus X_2 \dotplus \ker \mathfrak{O}(C_{cc}, A_{cc})) \\
&= (X_2 \dotplus \mathfrak{I}_1) \cap (X_1 \dotplus \mathfrak{I}_2) \cap (X_1 \dotplus X_2 \dotplus \mathfrak{I}_c).
\end{aligned}
$$

□

We have refined the concept of indistinguishability according to the different output spaces of a coordinated linear system. In order to preserve our original decomposition of the state space of a coordinated linear system according to $X_1 \dotplus X_2 \dotplus X_c$, we need to split up $\mathfrak{I}_1 \subseteq X_1 \dotplus X_c$ and $\mathfrak{I}_2 \subseteq X_2 \dotplus X_c$ further. To show that it is possible that $\mathfrak{I}_i \subseteq X_i \dotplus X_c$ but $\mathfrak{I}_i \cap X_i = \{0\}$ and $\mathfrak{I}_i \cap X_c = \{0\}$, consider the following example:

5.3.3. Example. Consider the system

$$
\dot{x}(t) = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] x(t), \quad y(t) = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array} \right] x(t),
$$

with decomposition $X_1 = \text{span}\{e_1, e_2\}$, $X_2 = \{0\}$ and $X_c = \text{span}\{e_3, e_4\}$. We can find the y_1-indistinguishable space by looking at the observability matrix of the pair $\left(\begin{bmatrix} C_{11} & C_{1c} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{1c} \\ 0 & A_{cc} \end{bmatrix}\right)$:

$$\begin{aligned} \mathfrak{I}_1 &= \ker \begin{bmatrix} C_{11} & C_{1c} \\ C_{11}A_{11} & C_{11}A_{1c} + C_{1c}A_{cc} \\ C_{11}A_{11}^2 & C_{11}A_{11}A_{1c} + C_{11}A_{1c}A_{cc} + C_{1c}A_{cc}^2 \\ \vdots & \vdots \end{bmatrix} \\ &= \ker \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \text{span}\{e_2 - e_3\} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right\}. \end{aligned}$$

This gives the following subspaces:

$$\mathfrak{I}_1 \cap X_1 = \{0\}, \quad \begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_1 = \text{span}\{e_2\}, \quad \mathfrak{I}_1 \cap X_c = \{0\}, \quad \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_1 = \text{span}\{e_3\}.$$

While neither X_1 nor X_c contain y_1-indistinguishable subspaces, with notation $x_1 = \begin{bmatrix} (x_1)_1 \\ (x_1)_2 \end{bmatrix}$ and $x_c = \begin{bmatrix} (x_c)_1 \\ (x_c)_2 \end{bmatrix}$ we have that x_1 is y_1-indistinguishable whenever $(x_1)_1 = 0$ and $(x_c)_1 = -(x_1)_2$, and x_c is y_1-indistinguishable whenever $(x_c)_2 = 0$ and $(x_1)_2 = -(x_c)_1$.

Since $\mathfrak{I}_i$ can in general not be decomposed according to X_i and X_c, we need to work with under- and overapproximations of $\mathfrak{I}_i$, $i = 1, 2$. In analogy with the case of reachable subspaces in Section 5.2.1, these will be defined as follows:

5.3.4. Definition. We call the following spaces **completely indistinguishable subspaces**:

$$\mathfrak{I}_1 \cap X_1, \quad \mathfrak{I}_1 \cap X_c, \quad \mathfrak{I}_2 \cap X_2, \quad \mathfrak{I}_2 \cap X_c.$$

The following spaces will be called **independently indistinguishable subspaces**:

$$\begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_1, \quad \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_1, \quad \begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_2, \quad \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_2.$$

The completely indistinguishable subspaces are subspaces of the indistinguishable spaces $\mathfrak{I}_1$ and $\mathfrak{I}_2$. This means for example that a state $x_1 \in X_1$ is completely y_1-indistinguishable if the state $\begin{bmatrix} x_1 \\ 0 \end{bmatrix} \in X_1 \dot{+} X_c$ is y_1-indistinguishable. The term 'independently y_j-indistinguishable' means that a state is y_j-indistinguishable from 0 if no further information from the other outputs is available: A state $x_1 \in X_1$ is independently y_1-indistinguishable if there exists a state $x_c \in X_c$ such that $\begin{bmatrix} x_1 \\ x_c \end{bmatrix}$ is y_i-indistinguishable. However, x_c might not be 0, so it is possible that x_c is observable from y_2 or y_c, and given the value of x_c we may be able to derive the value of x_1. In other words, a state is independently y_j-indistinguishable if it is not distinguishable from 0 by the output y_j alone.

It immediately follows that

$$(\mathfrak{I}_i \cap X_i) \dotplus (\mathfrak{I}_i \cap X_c) \subseteq \mathfrak{I}_i \subseteq \begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_i \dotplus \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_i,$$

and hence by (5.18), we have

$$(\mathfrak{I}_1 \cap X_1) \dotplus (\mathfrak{I}_2 \cap X_2) \dotplus ((\mathfrak{I}_1 \cap X_c) \cap (\mathfrak{I}_2 \cap X_c) \cap \mathfrak{I}_c) \subseteq \mathfrak{I}, \tag{5.19}$$

$$\mathfrak{I} \subseteq \begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_1 \dotplus \begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_2 \dotplus \left(\begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_1 \cap \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_2 \cap \mathfrak{I}_c \right). \tag{5.20}$$

5.3.2 Observability decomposition

We will decompose the state space X according to the observability properties of the system, by first decomposing $X_1 \dotplus X_c$, and then including X_2 in the decomposition. In the following, we call a subspace of the state space y_j-distinguishable if it contains no non-zero y_j-indistinguishable states.

In X_i, $i = 1, 2$, we have the y_i-indistinguishable subspaces $\mathfrak{I}_i \cap X_i$ and $\begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_i$, with $\mathfrak{I}_i \cap X_i \subseteq \begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_i$. Hence, we can decompose X_i as follows:

5.3.5. Procedure.

(1) Let $X_i^3 = \mathfrak{I}_i \cap X_i$.

(2) Let X_i^2 be any complement of X_i^3 in $\begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_i$.

(3) Finally, let X_i^1 be any complement of $X_i^2 \dotplus X_i^3$ in X_i.

Now $X_i = X_i^1 \dotplus X_i^2 \dotplus X_i^3$, with the following distinguishability properties:

subspace	y_i-distinguishable
X_i^1	yes
X_i^2	depends on x_c
X_i^3	no

For the decomposition of X_c according to the observability properties of the system corresponding to $X_i \dotplus X_c$, we have to take into account the indistinguishable subspaces $\mathfrak{I}_c$, $\mathfrak{I}_i \cap X_c$ and $\begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_i$, with $\mathfrak{I}_i \cap X_c \subseteq \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_i$. We decompose X_c as follows:

5.3.6. Procedure.

(1) Let $X^6_{c,i} = \mathfrak{I}_c \cap (\mathfrak{I}_i \cap X_c)$.

(2) Let $X^5_{c,i}$ be any complement of $X^6_{c,i}$ in $\mathfrak{I}_c \cap \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_i$.

(3) For $X^4_{c,i}$, we choose a complement of $X^5_{c,i} \dot{+} X^6_{c,i}$ in $\mathfrak{I}_c$. Note that now $X^4_{c,i} \dot{+} X^5_{c,i} \dot{+} X^6_{c,i} = \mathfrak{I}_c$.

(4) Let $X^3_{c,i}$ be a complement of $X^6_{c,i}$ in $\mathfrak{I}_i \cap X_c$.

(5) $X^2_{c,i}$ is any complement of $X^3_{c,i} \dot{+} X^5_{c,i} \dot{+} X^6_{c,i}$ in $\begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_i$.

(6) Finally, let $X^1_{c,i}$ be any complement of $X^2_{c,i} \dot{+} X^3_{c,i} \dot{+} X^4_{c,i} \dot{+} X^5_{c,i} \dot{+} X^6_{c,i}$ in X_c.

Then $X_c = X^1_{c,i} \dot{+} X^2_{c,i} \dot{+} X^3_{c,i} \dot{+} X^4_{c,i} \dot{+} X^5_{c,i} \dot{+} X^6_{c,i}$, with distinguishability properties as described in the following table:

subspace	y_c-distinguishable	y_i-distinguishable
$X^1_{c,i}$	yes	yes
$X^2_{c,i}$	yes	depends on x_i
$X^3_{c,i}$	yes	no
$X^4_{c,i}$	no	yes
$X^5_{c,i}$	no	depends on x_i
$X^6_{c,i}$	no	no

Suppose a decomposition of $X_1 \dot{+} X_c$ as described above is given, then we extend it to a decomposition of X by including the y_2-distinguishability properties of X_2: The subspace X_2 will be decomposed into $X_2 = X^1_2 \dot{+} X^2_2 \dot{+} X^3_2$ as above, with $X^3_2 = \mathfrak{I}_2 \cap X_2$ and $X^2_2 \dot{+} X^3_2 = \begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_2$. Now for $k = 1, \ldots, 6$, decompose $X^k_{c,1}$ as follows:

5.3.7. Procedure.

(1) $X^{3k}_c = X^k_{c,1} \cap (X_c \cap \mathfrak{I}_2)$,

(2) X^{3k-1}_c is a complement of X^{3k}_c in $X^k_{c,1} \cap \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_2$,

(3) X^{3k-2}_c is a complement of $X^{3k-1}_c \dot{+} X^{3k}_c$ in $X^k_{c,1}$.

The subspaces X^{3k-2}_c are y_2-distinguishable, the subspaces X^{3k-1}_c are only y_2-distinguishable for some values of x_2, and the subspaces X^{3k}_c are y_2-indistinguishable. Now $X_c = X^1_c \dot{+} \ldots \dot{+} X^{18}_c$, and the overall decomposition of X is

$$X = \left(X^1_1 \dot{+} X^2_1 \dot{+} X^3_1\right) \dot{+} \left(X^1_2 \dot{+} X^2_2 \dot{+} X^3_2\right) \dot{+} \left(X^1_c \dot{+} X^2_c \dot{+} \ldots \dot{+} X^{18}_c\right). \tag{5.21}$$

In the above decomposition we first considered the subspace $X_1 \dot{+} X_c$, and then extended the decomposition of $X_1 \dot{+} X_c$ by considering the distinguishability properties of X_2. However, the properties of the resulting decomposition (5.21) are unaffected by the order in which we consider X_1 and X_2: By setting

$$X_{c,2}^k = X_c^k \dot{+} X_c^{k+3} \dot{+} X_c^{k+6}, \quad k = 1, 2, 3,$$
$$X_{c,2}^k = X_c^{k+6} \dot{+} X_c^{k+9} \dot{+} X_c^{k+12}, \quad k = 4, 5, 6,$$

and using the decomposition of X_2 as above, we get a decomposition of $X_2 \dot{+} X_c$, with the same indistinguishability properties as the given decomposition of $X_1 \dot{+} X_c$. The indistinguishability properties of X_c in decomposition (5.21) are given in the following table:

subspace	y_c-distinguishable	y_1-distinguishable	y_2-distinguishable
X_c^1	yes	yes	yes
X_c^2	yes	yes	depends on x_2
X_c^3	yes	yes	no
X_c^4	yes	depends on x_1	yes
X_c^5	yes	depends on x_1	depends on x_2
X_c^6	yes	depends on x_1	no
X_c^7	yes	no	yes
X_c^8	yes	no	depends on x_2
X_c^9	yes	no	no
X_c^{10}	no	yes	yes
X_c^{11}	no	yes	depends on x_2
X_c^{12}	no	yes	no
X_c^{13}	no	depends on x_1	yes
X_c^{14}	no	depends on x_1	depends on x_2
X_c^{15}	no	depends on x_1	no
X_c^{16}	no	no	yes
X_c^{17}	no	no	depends on x_2
X_c^{18}	no	no	no

The main result of this section concerns the invariance and observability properties of the different subspaces of the decompositions derived above:

5.3.8. Theorem. *With respect to the decomposition (5.21) of X, system (5.17) is of the form given in Table 2, with observable pairs*

$$\left(\begin{bmatrix} C_{11}^1 & \vline & C_{1c}^1 & C_{1c}^2 & C_{1c}^3 & C_{1c}^{10} & C_{1c}^{11} & C_{1c}^{12} \end{bmatrix}, \left[\begin{array}{c|cccccc} A_{11}^{11} & A_{1c}^{11} & A_{1c}^{12} & A_{1c}^{13} & A_{1c}^{1,10} & A_{1c}^{1,11} & A_{1c}^{1,12} \\ \hline 0 & A_{cc}^{11} & A_{cc}^{12} & 0 & 0 & 0 & 0 \\ 0 & A_{cc}^{21} & A_{cc}^{22} & 0 & 0 & 0 & 0 \\ 0 & A_{cc}^{31} & A_{cc}^{32} & A_{cc}^{33} & 0 & 0 & 0 \\ 0 & A_{cc}^{10,1} & A_{cc}^{10,2} & 0 & A_{cc}^{10,10} & A_{cc}^{10,11} & 0 \\ 0 & A_{cc}^{11,1} & A_{cc}^{11,2} & 0 & A_{cc}^{11,10} & A_{cc}^{11,11} & 0 \\ 0 & A_{cc}^{12,1} & A_{cc}^{12,2} & A_{cc}^{12,3} & A_{cc}^{12,10} & A_{cc}^{12,11} & A_{cc}^{12,12} \end{array}\right] \right), \tag{5.22}$$

$$\left(\begin{bmatrix} C_{22}^1 & \vline & C_{2c}^1 & C_{2c}^4 & C_{2c}^7 & C_{2c}^{10} & C_{2c}^{13} & C_{2c}^{16} \end{bmatrix}, \left[\begin{array}{c|cccccc} A_{22}^{11} & A_{2c}^{11} & A_{2c}^{14} & A_{2c}^{17} & A_{2c}^{1,10} & A_{2c}^{1,13} & A_{2c}^{1,16} \\ \hline 0 & A_{cc}^{11} & A_{cc}^{14} & 0 & 0 & 0 & 0 \\ 0 & A_{cc}^{41} & A_{cc}^{44} & 0 & 0 & 0 & 0 \\ 0 & A_{cc}^{71} & A_{cc}^{74} & A_{cc}^{77} & 0 & 0 & 0 \\ 0 & A_{cc}^{10,1} & A_{cc}^{10,4} & 0 & A_{cc}^{10,10} & A_{cc}^{10,13} & 0 \\ 0 & A_{cc}^{13,1} & A_{cc}^{13,4} & 0 & A_{cc}^{13,10} & A_{cc}^{13,13} & 0 \\ 0 & A_{cc}^{16,1} & A_{cc}^{16,4} & A_{cc}^{16,7} & A_{cc}^{16,10} & A_{cc}^{16,13} & A_{cc}^{16,16} \end{array}\right] \right), \tag{5.23}$$

$$\left(\begin{bmatrix} C_{11}^1 & C_{11}^2 \end{bmatrix}, \begin{bmatrix} A_{11}^{11} & A_{11}^{12} \\ A_{11}^{21} & A_{11}^{22} \end{bmatrix} \right), \left(\begin{bmatrix} C_{22}^1 & C_{22}^2 \end{bmatrix}, \begin{bmatrix} A_{22}^{11} & A_{22}^{12} \\ A_{22}^{21} & A_{22}^{22} \end{bmatrix} \right), \tag{5.24}$$

$$\left(\begin{bmatrix} C_{cc}^1 & C_{cc}^2 & C_{cc}^3 & C_{cc}^4 & C_{cc}^5 & C_{cc}^6 & C_{cc}^7 & C_{cc}^8 & C_{cc}^9 \end{bmatrix}, \begin{bmatrix} A_{cc}^{11} & A_{cc}^{12} & 0 & A_{cc}^{14} & A_{cc}^{15} & 0 & 0 & 0 & 0 \\ A_{cc}^{21} & A_{cc}^{22} & 0 & A_{cc}^{24} & A_{cc}^{25} & 0 & 0 & 0 & 0 \\ A_{cc}^{31} & A_{cc}^{32} & A_{cc}^{33} & A_{cc}^{34} & A_{cc}^{35} & A_{cc}^{36} & 0 & 0 & 0 \\ A_{cc}^{41} & A_{cc}^{42} & 0 & A_{cc}^{44} & A_{cc}^{45} & 0 & 0 & 0 & 0 \\ A_{cc}^{51} & A_{cc}^{52} & 0 & A_{cc}^{54} & A_{cc}^{55} & 0 & 0 & 0 & 0 \\ A_{cc}^{61} & A_{cc}^{62} & A_{cc}^{63} & A_{cc}^{64} & A_{cc}^{65} & A_{cc}^{66} & 0 & 0 & 0 \\ A_{cc}^{71} & A_{cc}^{72} & 0 & A_{cc}^{74} & A_{cc}^{75} & 0 & A_{cc}^{77} & A_{cc}^{78} & 0 \\ A_{cc}^{81} & A_{cc}^{82} & 0 & A_{cc}^{84} & A_{cc}^{85} & 0 & A_{cc}^{87} & A_{cc}^{88} & 0 \\ A_{cc}^{91} & A_{cc}^{92} & A_{cc}^{93} & A_{cc}^{94} & A_{cc}^{95} & A_{cc}^{96} & A_{cc}^{97} & A_{cc}^{98} & A_{cc}^{99} \end{bmatrix} \right). \tag{5.25}$$

Proof. The first two columns of A follow from the A-invariance of X_1 (see Definition 3.1.1). Since $\mathfrak{I}_1 \subseteq X_1 \dot{+} X_c$ is $\begin{bmatrix} A_{11} & A_{1c} \\ 0 & A_{cc} \end{bmatrix}$-invariant and X_2 is A-invariant, the subspace $\mathfrak{I}_1 \dot{+} X_2 \subseteq X$ is A-invariant. Now $X_1^3 = \mathfrak{I}_1 \cap X_1 = (\mathfrak{I}_1 \dot{+} X_2) \cap X_1$ is A-invariant, which establishes the third column of A. Columns 4-6 follow from a similar argument for X_2.

$$
A = \left[\begin{array}{ccc|ccc|cccccccccccccccccc}
A_{11}^{11} & A_{11}^{12} & 0 & 0 & 0 & 0 & A_{1c}^{11} & A_{1c}^{12} & A_{1c}^{13} & A_{1c}^{14} & A_{1c}^{15} & A_{1c}^{16} & 0 & 0 & 0 & A_{1c}^{1,10} & A_{1c}^{1,11} & A_{1c}^{1,12} & A_{1c}^{1,13} & A_{1c}^{1,14} & A_{1c}^{1,15} & 0 & 0 & 0 \\
A_{11}^{21} & A_{11}^{22} & 0 & 0 & 0 & 0 & A_{1c}^{21} & A_{1c}^{22} & A_{1c}^{23} & A_{1c}^{24} & A_{1c}^{25} & A_{1c}^{26} & A_{1c}^{27} & A_{1c}^{28} & A_{1c}^{29} & A_{1c}^{2,10} & A_{1c}^{2,11} & A_{1c}^{2,12} & A_{1c}^{2,13} & A_{1c}^{2,14} & A_{1c}^{2,15} & A_{1c}^{2,16} & A_{1c}^{2,17} & A_{1c}^{2,18} \\
A_{11}^{31} & A_{11}^{32} & A_{11}^{33} & 0 & 0 & 0 & A_{1c}^{31} & A_{1c}^{32} & A_{1c}^{33} & A_{1c}^{34} & A_{1c}^{35} & A_{1c}^{36} & A_{1c}^{37} & A_{1c}^{38} & A_{1c}^{39} & A_{1c}^{3,10} & A_{1c}^{3,11} & A_{1c}^{3,12} & A_{1c}^{3,13} & A_{1c}^{3,14} & A_{1c}^{3,15} & A_{1c}^{3,16} & A_{1c}^{3,17} & A_{1c}^{3,18} \\
\hline
0 & 0 & 0 & A_{22}^{11} & A_{22}^{12} & 0 & A_{1c}^{11} & A_{1c}^{12} & 0 & A_{1c}^{14} & A_{1c}^{15} & 0 & A_{1c}^{17} & A_{1c}^{18} & 0 & A_{1c}^{1,10} & A_{1c}^{1,11} & 0 & A_{1c}^{1,13} & A_{1c}^{1,14} & 0 & A_{1c}^{1,16} & A_{1c}^{1,17} & 0 \\
0 & 0 & 0 & A_{22}^{21} & A_{22}^{22} & 0 & A_{2c}^{21} & A_{2c}^{22} & A_{2c}^{23} & A_{2c}^{24} & A_{2c}^{25} & A_{2c}^{26} & A_{2c}^{27} & A_{2c}^{28} & A_{2c}^{29} & A_{2c}^{2,10} & A_{2c}^{2,11} & A_{2c}^{2,12} & A_{2c}^{2,13} & A_{2c}^{2,14} & A_{2c}^{2,15} & A_{2c}^{2,16} & A_{2c}^{2,17} & A_{2c}^{2,18} \\
0 & 0 & 0 & A_{22}^{31} & A_{22}^{32} & A_{22}^{33} & A_{2c}^{31} & A_{2c}^{32} & A_{2c}^{33} & A_{2c}^{34} & A_{2c}^{35} & A_{2c}^{36} & A_{2c}^{37} & A_{2c}^{38} & A_{2c}^{39} & A_{2c}^{3,10} & A_{2c}^{3,11} & A_{2c}^{3,12} & A_{2c}^{3,13} & A_{2c}^{3,14} & A_{2c}^{3,15} & A_{2c}^{3,16} & A_{2c}^{3,17} & A_{2c}^{3,18} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{11} & A_{cc}^{12} & 0 & A_{cc}^{14} & A_{cc}^{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{21} & A_{cc}^{22} & 0 & A_{cc}^{24} & A_{cc}^{25} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{31} & A_{cc}^{32} & A_{cc}^{33} & A_{cc}^{34} & A_{cc}^{35} & A_{cc}^{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{41} & A_{cc}^{42} & 0 & A_{cc}^{44} & A_{cc}^{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{51} & A_{cc}^{52} & 0 & A_{cc}^{54} & A_{cc}^{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{61} & A_{cc}^{62} & A_{cc}^{63} & A_{cc}^{64} & A_{cc}^{65} & A_{cc}^{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{71} & A_{cc}^{72} & 0 & A_{cc}^{74} & A_{cc}^{75} & 0 & A_{cc}^{77} & A_{cc}^{78} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{81} & A_{cc}^{82} & 0 & A_{cc}^{84} & A_{cc}^{85} & 0 & A_{cc}^{87} & A_{cc}^{88} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{91} & A_{cc}^{92} & A_{cc}^{93} & A_{cc}^{94} & A_{cc}^{95} & A_{cc}^{96} & A_{cc}^{97} & A_{cc}^{98} & A_{cc}^{99} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{10,1} & A_{cc}^{10,2} & 0 & A_{cc}^{10,4} & A_{cc}^{10,5} & 0 & 0 & 0 & 0 & A_{cc}^{10,10} & A_{cc}^{10,11} & 0 & A_{cc}^{10,13} & A_{cc}^{10,14} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{11,1} & A_{cc}^{11,2} & 0 & A_{cc}^{11,4} & A_{cc}^{11,5} & 0 & 0 & 0 & 0 & A_{cc}^{11,10} & A_{cc}^{11,11} & 0 & A_{cc}^{11,13} & A_{cc}^{11,14} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{12,1} & A_{cc}^{12,2} & A_{cc}^{12,3} & A_{cc}^{12,4} & A_{cc}^{12,5} & A_{cc}^{12,6} & 0 & 0 & 0 & A_{cc}^{12,10} & A_{cc}^{12,11} & A_{cc}^{12,12} & A_{cc}^{12,13} & A_{cc}^{12,14} & A_{cc}^{12,15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{13,1} & A_{cc}^{13,2} & 0 & A_{cc}^{13,4} & A_{cc}^{13,5} & 0 & 0 & 0 & 0 & A_{cc}^{13,10} & A_{cc}^{13,11} & 0 & A_{cc}^{13,13} & A_{cc}^{13,14} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{14,1} & A_{cc}^{14,2} & 0 & A_{cc}^{14,4} & A_{cc}^{14,5} & 0 & 0 & 0 & 0 & A_{cc}^{14,10} & A_{cc}^{14,11} & 0 & A_{cc}^{14,13} & A_{cc}^{14,14} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{15,1} & A_{cc}^{15,2} & A_{cc}^{15,3} & A_{cc}^{15,4} & A_{cc}^{15,5} & A_{cc}^{15,6} & 0 & 0 & 0 & A_{cc}^{15,10} & A_{cc}^{15,11} & A_{cc}^{15,12} & A_{cc}^{15,13} & A_{cc}^{15,14} & A_{cc}^{15,15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{16,1} & A_{cc}^{16,2} & 0 & A_{cc}^{16,4} & A_{cc}^{16,5} & 0 & A_{cc}^{16,7} & A_{cc}^{16,8} & 0 & A_{cc}^{16,10} & A_{cc}^{16,11} & 0 & A_{cc}^{16,13} & A_{cc}^{16,14} & 0 & A_{cc}^{16,16} & A_{cc}^{16,17} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{17,1} & A_{cc}^{17,2} & 0 & A_{cc}^{17,4} & A_{cc}^{17,5} & 0 & A_{cc}^{17,7} & A_{cc}^{17,8} & 0 & A_{cc}^{17,10} & A_{cc}^{17,11} & 0 & A_{cc}^{17,13} & A_{cc}^{17,14} & 0 & A_{cc}^{17,16} & A_{cc}^{17,17} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{18,1} & A_{cc}^{18,2} & A_{cc}^{18,3} & A_{cc}^{18,4} & A_{cc}^{18,5} & A_{cc}^{18,6} & A_{cc}^{18,7} & A_{cc}^{18,8} & A_{cc}^{18,9} & A_{cc}^{18,10} & A_{cc}^{18,11} & A_{cc}^{18,12} & A_{cc}^{18,13} & A_{cc}^{18,14} & A_{cc}^{18,15} & A_{cc}^{18,16} & A_{cc}^{18,17} & A_{cc}^{18,18}
\end{array}\right],
$$

$$
C = \left[\begin{array}{ccc|ccc|cccccccccccccccccc}
C_{11}^{1} & C_{11}^{2} & 0 & 0 & 0 & 0 & C_{1c}^{1} & C_{1c}^{2} & C_{1c}^{3} & C_{1c}^{4} & C_{1c}^{5} & C_{1c}^{6} & 0 & 0 & 0 & C_{1c}^{10} & C_{1c}^{11} & C_{1c}^{12} & C_{1c}^{13} & C_{1c}^{14} & C_{1c}^{15} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & C_{22}^{1} & C_{22}^{2} & 0 & C_{2c}^{1} & C_{2c}^{2} & 0 & C_{2c}^{4} & C_{2c}^{5} & 0 & C_{2c}^{7} & C_{2c}^{8} & 0 & C_{2c}^{10} & C_{2c}^{11} & 0 & C_{2c}^{13} & C_{2c}^{14} & 0 & C_{2c}^{16} & C_{2c}^{17} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & C_{cc}^{1} & C_{cc}^{2} & C_{cc}^{3} & C_{cc}^{4} & C_{cc}^{5} & C_{cc}^{6} & C_{cc}^{7} & C_{cc}^{8} & C_{cc}^{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

For the remaining columns of A, we first note that

$$\mathfrak{I}_c = X_c^{10} \dot{+} X_c^{11} \dot{+} X_c^{12} \dot{+} X_c^{13} \dot{+} X_c^{14} \dot{+} X_c^{15} \dot{+} X_c^{16} \dot{+} X_c^{17} \dot{+} X_c^{18}$$

is A_{cc}-invariant, which explains the lower-triangular block structure of A_{cc}. Apart from this, columns 7, 8, 10, 11, 16, 17, 19 and 20 have no special structure.

Columns 13, 14, 22 and 23 follow from

$$X_c^7 \dot{+} X_c^8 \dot{+} X_c^9 \dot{+} X_c^{16} \dot{+} X_c^{17} \dot{+} X_c^{18} = \mathfrak{I}_1 \cap X_c$$

being A_{cc}-invariant (since X_c is trivially A_{cc}-invariant), and from

$$\begin{bmatrix} A_{11} & A_{1c} \\ 0 & A_{cc} \end{bmatrix} (\mathfrak{I}_1 \cap X_c) \subseteq \begin{bmatrix} A_{11} & A_{1c} \\ 0 & A_{cc} \end{bmatrix} \mathfrak{I}_1 \subseteq \mathfrak{I}_1 \subseteq (X_1^2 \dot{+} X_1^3) \dot{+} X_c.$$

Similarly, columns 9, 12, 18 and 21 follow from

$$X_c^3 \dot{+} X_c^6 \dot{+} X_c^9 \dot{+} X_c^{12} \dot{+} X_c^{15} \dot{+} X_c^{18} = \mathfrak{I}_2 \cap X_c$$

being A_{cc}-invariant, and from

$$\begin{bmatrix} A_{22} & A_{2c} \\ 0 & A_{cc} \end{bmatrix} (\mathfrak{I}_2 \cap X_c) \subseteq \begin{bmatrix} A_{22} & A_{2c} \\ 0 & A_{cc} \end{bmatrix} \mathfrak{I}_2 \subseteq \mathfrak{I}_2 \subseteq (X_2^2 \dot{+} X_2^3) \dot{+} X_c.$$

Finally, columns 15 and 24 of A follow from $X_c^9 \dot{+} X_c^{18} \subseteq (\mathfrak{I}_1 \cap X_c) \cap (\mathfrak{I}_2 \cap X_c)$ and the arguments for $\mathfrak{I}_1 \cap X_c$ and $\mathfrak{I}_2 \cap X_c$ above.

The structure of the C-matrix follows from

$$\begin{aligned} &X_1^3 \dot{+} \left(X_c^7 \dot{+} X_c^8 \dot{+} X_c^9 \dot{+} X_c^{16} \dot{+} X_c^{17} \dot{+} X_c^{18}\right) \subseteq \mathfrak{I}_1 \subseteq \ker \begin{bmatrix} C_{11} & C_{1c} \end{bmatrix}, \\ &X_2^3 \dot{+} \left(X_c^3 \dot{+} X_c^6 \dot{+} X_c^9 \dot{+} X_c^{12} \dot{+} X_c^{15} \dot{+} X_c^{18}\right) \subseteq \mathfrak{I}_2 \subseteq \ker \begin{bmatrix} C_{22} & C_{2c} \end{bmatrix}, \\ &X_c^{10} \dot{+} X_c^{11} \dot{+} X_c^{12} \dot{+} X_c^{13} \dot{+} X_c^{14} \dot{+} X_c^{15} \dot{+} X_c^{16} \dot{+} X_c^{17} \dot{+} X_c^{18} = \mathfrak{I}_c \subseteq \ker C_{cc}. \end{aligned}$$

The first two observable pairs follow from Definition 5.3.1, and from

$$\begin{aligned} \mathfrak{I}_1 \cap \left(X_1^1 \dot{+} X_c^1 \dot{+} X_c^2 \dot{+} X_c^3 \dot{+} X_c^{10} \dot{+} X_c^{11} \dot{+} X_c^{12}\right) &= \{0\}, \\ \mathfrak{I}_2 \cap \left(X_2^1 \dot{+} X_c^1 \dot{+} X_c^4 \dot{+} X_c^7 \dot{+} X_c^{10} \dot{+} X_c^{13} \dot{+} X_c^{16}\right) &= \{0\}. \end{aligned}$$

The third and fourth observable pairs are due to

$$X_i^3 = \mathfrak{I}_i \cap X_i = \ker \mathfrak{O}\left(\begin{bmatrix} C_{ii} & C_{ic} \end{bmatrix}, \begin{bmatrix} A_{ii} & A_{ic} \\ 0 & A_{cc} \end{bmatrix}\right) \cap X_i = \ker \begin{bmatrix} C_{ii} & C_{1c} \\ C_{ii}A_{ii} & \star \\ C_{ii}A_{ii}^2 & \star \\ \vdots & \vdots \end{bmatrix} \cap X_1$$

$$= \ker \mathfrak{O}(C_{ii}, A_{ii}), \ i = 1, 2,$$

and the last observable pair is due to

$$\mathfrak{I}_c = X_c^{10} \dot{+} X_c^{11} \dot{+} X_c^{12} \dot{+} X_c^{13} \dot{+} X_c^{14} \dot{+} X_c^{15} \dot{+} X_c^{16} \dot{+} X_c^{17} \dot{+} X_c^{18}. \qquad \square$$

An additional decomposition of the output spaces Y_1 and Y_2 of the subsystems, specifying which part of the output observes (part of) the local subsystem state or the coordinator state, would induce some additional structure on the C-matrix. This is not considered here.

5.3.3 Concepts of observability

In this section, the observability decomposition derived in the previous section will be used for the characterization of several possible concepts of observability.

In analogy to Section 3.3, the two concepts of observability most relevant for practical purposes are weak local observability (Definition 5.3.11) and independent observability (Definition 5.3.17).

5.3.3.1 Subsystem observability

Subsystem observability will be defined as follows:

5.3.9. Definition. We call a system of the form (5.17) **subsystem observable** if for $i = 1, 2$, all non-zero states $\begin{bmatrix} x_i \\ x_c \end{bmatrix} \in X_1 \dot{+} X_c$ are y_i-distinguishable, i.e. if $\mathfrak{I}_i = \{0\}$.

Subsystem observability is a rather strong condition (see Proposition 5.3.19). It is a useful concept if one aims at constructing a coordinated system that is as decentralized as possible: If the system is subsystem observable then the coordinator is both y_1-observable and y_2-observable. These observations can then be used for local control synthesis, as described in Section 5.4. On the other hand, if a coordinated system is not subsystem observable and if one aims at having a coordinator of minimal dimension, then all parts of the coordinator that are not both y_1-observable and y_2-observable can be moved to a subsystem (see [23]).

5.3.10. Proposition. *A system of the form (5.17) is subsystem observable if and only if the following pairs are observable pairs:*

$$\left(\begin{bmatrix}C_{11} & C_{1c}\end{bmatrix}, \begin{bmatrix}A_{11} & A_{1c} \\ 0 & A_{cc}\end{bmatrix}\right) \text{ and } \left(\begin{bmatrix}C_{22} & C_{2c}\end{bmatrix}, \begin{bmatrix}A_{22} & A_{2c} \\ 0 & A_{cc}\end{bmatrix}\right).$$

Proof. Subsystem observability is equivalent to $\mathfrak{I}_i = \{0\}$, which means $\begin{bmatrix}I & 0\end{bmatrix}\mathfrak{I}_i = \{0\}$ and $\begin{bmatrix}0 & I\end{bmatrix}\mathfrak{I}_i = \{0\}$. Hence, in terms of decomposition (5.21), subsystem observability is equivalent to

$$X_1 = X_1^1, \quad X_2 = X_2^1, \quad X_c = X_c^1 \dot{+} X_c^{10}.$$

The representation in Table 2 then reduces to a representation of the form

$$\begin{aligned}
\dot{x}(t) &= \left[\begin{array}{c|c|cc} A_{11}^{11} & 0 & A_{1c}^{11} & A_{1c}^{1,10} \\ \hline 0 & A_{22}^{11} & A_{2c}^{11} & A_{2c}^{1,10} \\ \hline 0 & 0 & A_{cc}^{11} & 0 \\ 0 & 0 & A_{cc}^{10,1} & A_{cc}^{10,10} \end{array}\right] x(t), \\
y(t) &= \left[\begin{array}{c|c|cc} C_{11}^{1} & 0 & C_{1c}^{1} & C_{1c}^{10} \\ \hline 0 & C_{22}^{1} & C_{2c}^{1} & C_{2c}^{10} \\ \hline 0 & 0 & C_{cc}^{1} & 0 \end{array}\right] x(t),
\end{aligned} \tag{5.26}$$

and the corresponding observable pairs in (5.22)-(5.25) reduce to

$$\left(\left[\begin{array}{c|cc} C_{ii}^{1} & C_{ic}^{1} & C_{ic}^{10} \end{array}\right], \left[\begin{array}{c|cc} A_{ii}^{11} & A_{ic}^{11} & A_{ic}^{1,10} \\ \hline 0 & A_{cc}^{11} & 0 \\ 0 & A_{cc}^{10,1} & A_{cc}^{10,10} \end{array}\right]\right), \; i = 1, 2. \tag{5.27}$$

These are exactly the pairs given in Proposition 5.3.10. □

5.3.3.2 Weak local observability

Weak local observability will be defined as follows:

5.3.11. Definition. We call a system of the form (5.17) **weakly locally observable** if all non-zero states $x_c \in X_c$ are y_c-distinguishable, and for $i = 1, 2$, all non-zero states $x_i \in X_i$ have that $\begin{bmatrix}x_i \\ 0\end{bmatrix} \in X_i \dot{+} X_c$ is y_i-distinguishable. This means that $\mathfrak{I}_i \cap X_i = \{0\}$ and $\mathfrak{I}_c = \{0\}$.

The term 'weak local observability' distinguishes this concept from the slightly stronger concept of strong local observability, defined in Subsection 5.3.3.3. While weak local observability implies that the coordinator state is observable from the coordinator output, the subsystem states are only required to be observable from their local outputs if $x_c = 0$; for non-zero coordinator states the influence of $A_{ic}x_c$ on x_i may render the subsystem state unobservable.

The following proposition states that weak local observability enables us to reconstruct the system state via local observers:

5.3.12. Proposition. *A system of the form (5.17) is weakly locally observable if and only if, in the notation of (5.17), the following pairs are observable pairs:*

$$(C_{11}, A_{11}),\ (C_{22}, A_{22}) \text{ and } (C_{cc}, A_{cc}).$$

Proof. If the system (5.17) is weakly locally observable, i.e. if $\mathfrak{I}_i \cap X_i = \{0\}$ and $\mathfrak{I}_c = 0$, then in the notation of decomposition (5.21), we have

$$X_1 = X_1^1 \dot{+} X_1^2,\ X_2 = X_2^1 \dot{+} X_2^2,\ X_c = X_c^1 \dot{+} X_c^2 \dot{+} \ldots \dot{+} X_c^9.$$

The representation in Table 2 then reduces to

$$A = \left[\begin{array}{cc|cc|ccccccccc}
A_{11}^{11} & A_{11}^{12} & 0 & 0 & A_{1c}^{11} & A_{1c}^{12} & A_{1c}^{13} & A_{1c}^{14} & A_{1c}^{15} & A_{1c}^{16} & 0 & 0 & 0 \\
A_{11}^{21} & A_{11}^{22} & 0 & 0 & A_{1c}^{21} & A_{1c}^{22} & A_{1c}^{23} & A_{1c}^{24} & A_{1c}^{25} & A_{1c}^{26} & A_{1c}^{27} & A_{1c}^{28} & A_{1c}^{29} \\
\hline
0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{2c}^{11} & A_{2c}^{12} & 0 & A_{2c}^{14} & A_{2c}^{15} & 0 & A_{2c}^{17} & A_{2c}^{18} & 0 \\
0 & 0 & A_{22}^{21} & A_{22}^{22} & A_{2c}^{21} & A_{2c}^{22} & A_{2c}^{23} & A_{2c}^{24} & A_{2c}^{25} & A_{2c}^{26} & A_{2c}^{27} & A_{2c}^{28} & A_{2c}^{29} \\
\hline
0 & 0 & 0 & 0 & A_{cc}^{11} & A_{cc}^{12} & 0 & A_{cc}^{14} & A_{cc}^{15} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{cc}^{21} & A_{cc}^{22} & 0 & A_{cc}^{24} & A_{cc}^{25} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{cc}^{31} & A_{cc}^{32} & A_{cc}^{33} & A_{cc}^{34} & A_{cc}^{35} & A_{cc}^{36} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{cc}^{41} & A_{cc}^{42} & 0 & A_{cc}^{44} & A_{cc}^{45} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{cc}^{51} & A_{cc}^{52} & 0 & A_{cc}^{54} & A_{cc}^{55} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{cc}^{61} & A_{cc}^{62} & A_{cc}^{63} & A_{cc}^{64} & A_{cc}^{65} & A_{cc}^{66} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{cc}^{71} & A_{cc}^{72} & 0 & A_{cc}^{74} & A_{cc}^{75} & 0 & A_{cc}^{77} & A_{cc}^{78} & 0 \\
0 & 0 & 0 & 0 & A_{cc}^{81} & A_{cc}^{82} & 0 & A_{cc}^{84} & A_{cc}^{85} & 0 & A_{cc}^{87} & A_{cc}^{88} & 0 \\
0 & 0 & 0 & 0 & A_{cc}^{91} & A_{cc}^{92} & A_{cc}^{93} & A_{cc}^{94} & A_{cc}^{95} & A_{cc}^{96} & A_{cc}^{97} & A_{cc}^{98} & A_{cc}^{99}
\end{array}\right], \tag{5.28}$$

$$C = \left[\begin{array}{cc|cc|ccccccccc}
C_{11}^{1} & C_{11}^{2} & 0 & 0 & C_{1c}^{1} & C_{1c}^{2} & C_{1c}^{3} & C_{1c}^{4} & C_{1c}^{5} & C_{1c}^{6} & 0 & 0 & 0 \\
\hline
0 & 0 & C_{22}^{1} & C_{22}^{2} & C_{2c}^{1} & C_{2c}^{2} & 0 & C_{2c}^{4} & C_{2c}^{5} & 0 & C_{2c}^{7} & C_{2c}^{8} & 0 \\
\hline
0 & 0 & 0 & 0 & C_{cc}^{1} & C_{cc}^{2} & C_{cc}^{3} & C_{cc}^{4} & C_{cc}^{5} & C_{cc}^{6} & C_{cc}^{7} & C_{cc}^{8} & C_{cc}^{9}
\end{array}\right],$$

and the observable pairs (5.22)-(5.25) reduce to

$$\left(\begin{bmatrix} C_{11}^1 & C_{11}^2 \end{bmatrix}, \begin{bmatrix} A_{11}^{11} & A_{11}^{12} \\ A_{11}^{21} & A_{11}^{22} \end{bmatrix}\right), \left(\begin{bmatrix} C_{22}^1 & C_{22}^2 \end{bmatrix}, \begin{bmatrix} A_{22}^{11} & A_{22}^{12} \\ A_{22}^{21} & A_{22}^{22} \end{bmatrix}\right),$$

$$\left(\begin{array}{c} \begin{bmatrix} C_{cc}^1 & C_{cc}^2 & C_{cc}^3 & C_{cc}^4 & C_{cc}^5 & C_{cc}^6 & C_{cc}^7 & C_{cc}^8 & C_{cc}^9 \end{bmatrix}, \\ \begin{bmatrix} A_{cc}^{11} & A_{cc}^{12} & 0 & A_{cc}^{14} & A_{cc}^{15} & 0 & 0 & 0 & 0 \\ A_{cc}^{21} & A_{cc}^{22} & 0 & A_{cc}^{24} & A_{cc}^{25} & 0 & 0 & 0 & 0 \\ A_{cc}^{31} & A_{cc}^{32} & A_{cc}^{33} & A_{cc}^{34} & A_{cc}^{35} & A_{cc}^{36} & 0 & 0 & 0 \\ A_{cc}^{41} & A_{cc}^{42} & 0 & A_{cc}^{44} & A_{cc}^{45} & 0 & 0 & 0 & 0 \\ A_{cc}^{51} & A_{cc}^{52} & 0 & A_{cc}^{54} & A_{cc}^{55} & 0 & 0 & 0 & 0 \\ A_{cc}^{61} & A_{cc}^{62} & A_{cc}^{63} & A_{cc}^{64} & A_{cc}^{65} & A_{cc}^{66} & 0 & 0 & 0 \\ A_{cc}^{71} & A_{cc}^{72} & 0 & A_{cc}^{74} & A_{cc}^{75} & 0 & A_{cc}^{77} & A_{cc}^{78} & 0 \\ A_{cc}^{81} & A_{cc}^{82} & 0 & A_{cc}^{84} & A_{cc}^{85} & 0 & A_{cc}^{87} & A_{cc}^{88} & 0 \\ A_{cc}^{91} & A_{cc}^{92} & A_{cc}^{93} & A_{cc}^{94} & A_{cc}^{95} & A_{cc}^{96} & A_{cc}^{97} & A_{cc}^{98} & A_{cc}^{99} \end{bmatrix} \end{array}\right). \tag{5.29}$$

But these are exactly the pairs (C_{11}, A_{11}), (C_{22}, A_{22}) and (C_{cc}, A_{cc}). □

If A is antistable (i.e. if $\sigma(A) \subset \{\lambda \in \mathbb{C} | \mathrm{Re}(\lambda) > 0\}$) then weak local observability is necessary and sufficient for state reconstruction via linear state observers: For coordinated linear systems, we have to restrict the admissible observer matrices to matrices of the form $G = \begin{bmatrix} G_{11} & 0 & G_{1c} \\ 0 & G_{22} & G_{2c} \\ 0 & 0 & G_{cc} \end{bmatrix}$ in order to preserve the information structure we have imposed. This gives

$$A - GC = \begin{bmatrix} A_{11} - G_{11}C_{11} & 0 & A_{1c} - G_{11}C_{1c} - G_{1c}C_{cc} \\ 0 & A_{22} - G_{22}C_{22} & A_{2c} - G_{22}C_{2c} - G_{2c}C_{cc} \\ 0 & 0 & A_{cc} - G_{cc}C_{cc} \end{bmatrix}.$$

Now the eigenvalues of this matrix are

$$\sigma(A - GC) = \sigma(A_{11} - G_{11}C_{11}) \cup \sigma(A_{22} - G_{22}C_{22}) \cup \sigma(A_{cc} - G_{cc}C_{cc}),$$

so the matrix $A-GC$, describing the dynamics of the observer error, is stable if and only if the blocks on the diagonal are stable. In the case that A is antistable, this is equivalent to weak local observability of (C, A). Hence, just like in the case of pole placement, state reconstruction concerns each part of the system separately.

5.3.3.3 Strong local observability

The concept of strong local observability will be defined as follows:

5.3.13. Definition. We call a system of the form (5.17) **strongly locally observable** if all non-zero states $x_c \in X_c$ are y_c-distinguishable, and for $i = 1, 2$, all non-zero states $x_i \in X_i$ have that $\begin{bmatrix} x_i \\ x_c \end{bmatrix} \in X_i \dotplus X_c$ is y_i-distinguishable for all $x_c \in X_c$, i.e. if $\mathfrak{I}_i \subseteq X_c$ and $\mathfrak{I}_c = \{0\}$.

In the case of strong local observability, each part of the system observes its own state. Compared to subsystem observability, observations are more decentralized: Each part of the system has full information about their local state, and the subsystems may or may not have information about the coordinator state. Unlike in the case of weak local observability, the local observations of x_i are *independent* of the value of x_c: the coordinator state cannot interfere with the subsystem dynamics in a way that would render the subsystem state unobservable.

5.3.14. Proposition. *A system of the form (5.17) is strongly locally observable if and only if there exists a decomposition of X_c resulting in a system representation of the form*

$$A = \left[\begin{array}{c|c|cccc} \Lambda_{11} & 0 & \Lambda_{1c}^1 & \Lambda_{1c}^2 & 0 & 0 \\ \hline 0 & \Lambda_{22} & \Lambda_{2c}^1 & 0 & \Lambda_{2c}^3 & 0 \\ \hline 0 & 0 & \Lambda_{cc}^{11} & 0 & 0 & 0 \\ 0 & 0 & \Lambda_{cc}^{21} & \Lambda_{cc}^{22} & 0 & 0 \\ 0 & 0 & \Lambda_{cc}^{31} & 0 & \Lambda_{cc}^{33} & 0 \\ 0 & 0 & \Lambda_{cc}^{41} & \Lambda_{cc}^{42} & \Lambda_{cc}^{43} & \Lambda_{cc}^{44} \end{array}\right], \; C = \left[\begin{array}{c|c|cccc} \Gamma_{11} & 0 & \Gamma_{1c}^1 & \Gamma_{1c}^2 & 0 & 0 \\ \hline 0 & \Gamma_{22} & \Gamma_{2c}^1 & 0 & \Gamma_{2c}^3 & 0 \\ \hline 0 & 0 & \Gamma_{cc}^1 & \Gamma_{cc}^2 & \Gamma_{cc}^3 & \Gamma_{cc}^4 \end{array}\right]. \tag{5.30}$$

The following tuples are observable pairs:

$$\begin{aligned} &\left(\left[\begin{array}{c|cc} \Gamma_{11} & \Gamma_{1c}^1 & \Gamma_{1c}^2 \end{array}\right], \left[\begin{array}{c|cc} \Lambda_{11} & \Lambda_{1c}^1 & \Lambda_{1c}^2 \\ \hline 0 & \Lambda_{cc}^{11} & 0 \\ 0 & \Lambda_{cc}^{21} & \Lambda_{cc}^{22} \end{array}\right] \right), \\ &\left(\left[\begin{array}{c|cc} \Gamma_{22} & \Gamma_{2c}^1 & \Gamma_{2c}^3 \end{array}\right], \left[\begin{array}{c|cc} \Lambda_{22} & \Lambda_{2c}^1 & \Lambda_{2c}^3 \\ \hline 0 & \Lambda_{cc}^{11} & 0 \\ 0 & \Lambda_{cc}^{31} & \Lambda_{cc}^{33} \end{array}\right] \right), \\ &\left(\begin{bmatrix} \Gamma_{cc}^1 & \Gamma_{cc}^2 & \Gamma_{cc}^3 & \Gamma_{cc}^4 \end{bmatrix}, \begin{bmatrix} \Lambda_{cc}^{11} & 0 & 0 & 0 \\ \Lambda_{cc}^{21} & \Lambda_{cc}^{22} & 0 & 0 \\ \Lambda_{cc}^{31} & 0 & \Lambda_{cc}^{33} & 0 \\ \Lambda_{cc}^{41} & \Lambda_{cc}^{42} & \Lambda_{cc}^{43} & \Lambda_{cc}^{44} \end{bmatrix} \right). \end{aligned} \tag{5.31}$$

Proof. The conditions $\mathfrak{I}_i \subseteq X_c$ and $\mathfrak{I}_c = \{0\}$ characterizing strong local observability are equivalent to the conditions $\begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_i = \{0\}$, $\begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_i = \mathfrak{I}_i \cap X_c$ and $\mathfrak{I}_c = \{0\}$. In terms of decomposition (5.21), this means

$$X_1 = X_1^1, \; X_2 = X_2^1, \; X_c = X_c^1 \dotplus X_c^3 \dotplus X_c^7 \dotplus X_c^9.$$

The representation in Table 2 then reduces to a representation of the form (5.30), and the corresponding observable pairs in (5.22)-(5.25) reduce to the ones in (5.31). □

An interesting generalization of both subsystem observability and strong local observability is to require that for some matrices D_{1c}, D_{2c} and D_{cc} of appropriate sizes, $\begin{bmatrix} I & 0 \\ 0 & D_{1c} \end{bmatrix} \begin{bmatrix} x_1 \\ x_c \end{bmatrix}$ is observable at subsystem 1, $\begin{bmatrix} I & 0 \\ 0 & D_{2c} \end{bmatrix} \begin{bmatrix} x_2 \\ x_c \end{bmatrix}$ is observable at subsystem 2, and $D_{cc}x_c$ is observable at the coordinator. The interpretation of this concept is that each subsystem observes, in addition to its own state, a particular part of the coordinator state. The observable part of the coordinator can be different for each subsystem.

This concept is equivalent to $\mathfrak{I}_i \subseteq \ker D_{ic}$ for $i = 1, 2$, and $\mathfrak{I}_c \subseteq \ker D_{cc}$. For subsystem observability, we have $D_{ic} = I$, $i = 1, 2$ and $D_{cc} = 0$. Strong local observability corresponds to the other extreme, with $D_{ic} = 0$ for $i = 1, 2$, and $D_{cc} = I$.

5.3.3.4 Joint observability

We define joint observability as follows:

5.3.15. Definition. We call a system of the form (5.17) **jointly observable** if all non-zero states $x \in X$ are $\begin{bmatrix} y_1 \\ y_2 \\ y_c \end{bmatrix}$-distinguishable, i.e. if

$$(\mathfrak{I}_1 \dot{+} X_2) \cap (\mathfrak{I}_2 \dot{+} X_1) \cap (\mathfrak{I}_c \dot{+} X_1 \dot{+} X_2) = \{0\}.$$

Since the characterization of joint observability is equivalent to $\mathfrak{I} = \{0\}$ by Lemma 5.3.2, the concept of joint observability coincides with the usual concept of observability. For coordinated linear systems, this concept is not very useful: If the system is jointly observable but not independently observable (see Definition 5.3.17), then the overall state of the system can only be observed by combining the observations of the different parts of the system; the combination of these observations requires communication among different parts of the system, which does not comply with the information structure we imposed on coordinated systems. This difficulty is illustrated by an example in Section 5.3.4.

Since the concept of joint observability cannot be characterized by separate conditions for each part of the system (which reflects the need for communication between the different parts of the system for the observation of the system state), we cannot give a reduced decomposition for this concept. We can however give a necessary condition and a sufficient condition for joint observability, both of which separate into conditions on the different parts of the system:

5.3.16. Lemma. *For joint observability of a coordinated linear system of the form (5.17), and hence for observability in the usual sense, it is necessary that*

$$\mathfrak{I}_1 \cap X_1 = \{0\},\ \mathfrak{I}_2 \cap X_2 = \{0\},\ (\mathfrak{I}_1 \cap X_c) \cap (\mathfrak{I}_2 \cap X_c) \cap \mathfrak{I}_c = \{0\};$$

it is sufficient that

$$\mathfrak{I}_1 \subseteq X_c,\ \mathfrak{I}_2 \subseteq X_c,\ \begin{bmatrix}0 & I\end{bmatrix}\mathfrak{I}_1 \cap \begin{bmatrix}0 & I\end{bmatrix}\mathfrak{I}_2 \cap \mathfrak{I}_c = \{0\}.$$

Proof. The necessary condition follows directly from (5.19). For the sufficient condition, note that

$$\begin{bmatrix}I & 0\end{bmatrix}\mathfrak{I}_1 = \{0\},\ \begin{bmatrix}I & 0\end{bmatrix}\mathfrak{I}_2 = \{0\},\ \begin{bmatrix}0 & I\end{bmatrix}\mathfrak{I}_1 \cap \begin{bmatrix}0 & I\end{bmatrix}\mathfrak{I}_2 \cap \mathfrak{I}_c = \{0\}$$

is sufficient for $\mathfrak{I} = \{0\}$ by (5.20), but $(\begin{bmatrix}I & 0\end{bmatrix}\mathfrak{I}_i = \{0\}) \Leftrightarrow (\mathfrak{I}_i \subseteq X_c)$, which gives the sufficient condition stated above. □

5.3.3.5 Independent observability

Independent observability will be defined as follows:

5.3.17. Definition. We call a system of the form (5.17) **independently observable** if all non-zero states $x_c \in X_c$ have that $\begin{bmatrix}x_1\\x_2\\x_c\end{bmatrix} \in X_1 \dot{+} X_2 \dot{+} X_c$ is $\begin{bmatrix}y_1\\y_2\\y_c\end{bmatrix}$-distinguishable for any $x_1 \in X_1$ and $x_2 \in X_2$, and for $i = 1, 2$ all non-zero states $x_i \in X_i$ have that the state $\begin{bmatrix}x_i\\x_c\end{bmatrix} \in X_1 \dot{+} X_c$ is y_i-distinguishable for all $x_c \in X_c$, i.e. if

$$\begin{bmatrix}I & 0\end{bmatrix}\mathfrak{I}_i = \{0\},\ i = 1, 2,\quad \begin{bmatrix}0 & I\end{bmatrix}\mathfrak{I}_1 \cap \begin{bmatrix}0 & I\end{bmatrix}\mathfrak{I}_2 \cap \mathfrak{I}_c = \{0\}.$$

By Lemma 5.3.16, independent observability coincides with the sufficient condition for joint observability. Hence it is a stronger condition than joint observability, and more useful in the setting of coordinated linear systems because all parts of the system state are observable *independently* of the value of x_c: no communication among the different parts of the system is required to observe the subsystem states. The coordinator state can only be observed jointly by the different outputs.

5.3.18. Proposition. *A system of the form (5.17) is independently observable if and only if there exists a decomposition of X_c resulting in a system representation of the form*

$$
\dot{x}(t) = \left[\begin{array}{c|c|ccccccc}
\Lambda_{11} & 0 & \Lambda_{1c}^{1} & \Lambda_{1c}^{2} & 0 & 0 & \Lambda_{1c}^{5} & \Lambda_{1c}^{6} & 0 \\
\hline
0 & \Lambda_{22} & \Lambda_{2c}^{1} & 0 & \Lambda_{2c}^{3} & 0 & \Lambda_{2c}^{5} & 0 & \Lambda_{2c}^{7} \\
\hline
0 & 0 & \Lambda_{cc}^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Lambda_{cc}^{21} & \Lambda_{cc}^{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Lambda_{cc}^{31} & 0 & \Lambda_{cc}^{33} & 0 & 0 & 0 & 0 \\
0 & 0 & \Lambda_{cc}^{41} & \Lambda_{cc}^{42} & \Lambda_{cc}^{43} & \Lambda_{cc}^{44} & 0 & 0 & 0 \\
0 & 0 & \Lambda_{cc}^{51} & 0 & 0 & 0 & \Lambda_{cc}^{55} & 0 & 0 \\
0 & 0 & \Lambda_{cc}^{61} & \Lambda_{cc}^{62} & 0 & 0 & \Lambda_{cc}^{65} & \Lambda_{cc}^{66} & 0 \\
0 & 0 & \Lambda_{cc}^{71} & 0 & \Lambda_{cc}^{73} & 0 & \Lambda_{cc}^{75} & 0 & \Lambda_{cc}^{77}
\end{array}\right] x(t),
$$

$$
y(t) = \left[\begin{array}{c|c|ccccccc}
\Gamma_{11} & 0 & \Gamma_{1c}^{1} & \Gamma_{1c}^{2} & 0 & 0 & \Gamma_{1c}^{5} & \Gamma_{1c}^{6} & 0 \\
\hline
0 & \Gamma_{22} & \Gamma_{2c}^{1} & 0 & \Gamma_{2c}^{3} & 0 & \Gamma_{2c}^{5} & 0 & \Gamma_{2c}^{7} \\
\hline
0 & 0 & \Gamma_{cc}^{1} & \Gamma_{cc}^{2} & \Gamma_{cc}^{3} & \Gamma_{cc}^{4} & 0 & 0 & 0
\end{array}\right] x(t), \tag{5.32}
$$

and such that the following tuples are observable pairs:

$$
\left(\left[\begin{array}{c|cccc} \Gamma_{11} & \Gamma_{1c}^{1} & \Gamma_{1c}^{2} & \Gamma_{1c}^{5} & \Gamma_{1c}^{6} \end{array}\right], \left[\begin{array}{c|cccc}
\Lambda_{11} & \Lambda_{1c}^{1} & \Lambda_{1c}^{2} & \Lambda_{1c}^{5} & \Lambda_{1c}^{6} \\
\hline
0 & \Lambda_{cc}^{11} & 0 & 0 & 0 \\
0 & \Lambda_{cc}^{21} & \Lambda_{cc}^{22} & 0 & 0 \\
0 & \Lambda_{cc}^{51} & 0 & \Lambda_{cc}^{55} & 0 \\
0 & \Lambda_{cc}^{61} & \Lambda_{cc}^{62} & \Lambda_{cc}^{65} & \Lambda_{cc}^{66}
\end{array}\right] \right), \tag{5.33}
$$

$$
\left(\left[\begin{array}{c|cccc} \Gamma_{22} & \Gamma_{2c}^{1} & \Gamma_{2c}^{3} & \Gamma_{2c}^{5} & \Gamma_{2c}^{7} \end{array}\right], \left[\begin{array}{c|cccc}
\Lambda_{22} & \Lambda_{2c}^{1} & \Lambda_{2c}^{3} & \Lambda_{2c}^{5} & \Lambda_{2c}^{7} \\
\hline
0 & \Lambda_{cc}^{11} & 0 & 0 & 0 \\
0 & \Lambda_{cc}^{31} & \Lambda_{cc}^{33} & 0 & 0 \\
0 & \Lambda_{cc}^{51} & 0 & \Lambda_{cc}^{55} & 0 \\
0 & \Lambda_{cc}^{71} & \Lambda_{cc}^{73} & \Lambda_{cc}^{75} & \Lambda_{cc}^{77}
\end{array}\right] \right), \tag{5.34}
$$

$$
\left(\left[\begin{array}{cccc} \Gamma_{cc}^{1} & \Gamma_{cc}^{2} & \Gamma_{cc}^{3} & \Gamma_{cc}^{4} \end{array}\right], \left[\begin{array}{cccc}
\Lambda_{cc}^{11} & 0 & 0 & 0 \\
\Lambda_{cc}^{21} & \Lambda_{cc}^{22} & 0 & 0 \\
\Lambda_{cc}^{31} & 0 & \Lambda_{cc}^{33} & 0 \\
\Lambda_{cc}^{41} & \Lambda_{cc}^{42} & \Lambda_{cc}^{43} & \Lambda_{cc}^{44}
\end{array}\right] \right). \tag{5.35}
$$

Proof. In terms of decomposition (5.21), we have $\begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_i = X_i^2 \dot{+} X_i^3 = \{0\}$, and since $\begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_i = \{0\}$ implies that $\begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_i = \mathfrak{I}_i \cap X_c$, we also have

$$
X_c^2,\ X_c^4,\ X_c^5,\ X_c^6,\ X_c^8,\ X_c^{11},\ X_c^{13},\ X_c^{14},\ X_c^{15},\ X_c^{17} = \{0\}.
$$

Now the second condition for independent observability reduces to $\begin{bmatrix}0 & I\end{bmatrix} \mathfrak{I}_1 \cap \begin{bmatrix}0 & I\end{bmatrix} \mathfrak{I}_2 \cap \mathfrak{I}_c = X_c^{18} = \{0\}$. Hence, independent observability is equivalent to

$$X_1 = X_1^1,\ X_2 = X_2^1,\ X_c = X_c^1 \dot{+} X_c^3 \dot{+} X_c^7 \dot{+} X_c^9 \dot{+} X_c^{10} \dot{+} X_c^{12} \dot{+} X_c^{16}.$$

The representation in Table then reduces to a representation of the form (5.32), and the corresponding observable pairs in (5.22)-(5.25) reduce to the ones in (5.33)-(5.35). □

5.3.3.6 Relations between the concepts of observability

For the different concepts of observability defined in Sections 5.3.3.1-5.3.3.5, we have the following relations:

5.3.19. Proposition.

- *Subsystem observability implies independent observability,*
- *strong local observability implies independent observability,*
- *strong local observability implies weak local observability,*
- *independent observability implies joint observability,*
- *and weak local observability implies joint observability.*

Proof. The first three items follow directly from the reduced state space decompositions given in the proofs of Propositions 5.3.10, 5.3.12, 5.3.14 and 5.3.18.

The definition of independent observability in 5.3.17 coincides with the sufficient condition for joint observability in 5.3.16, which gives the fourth item.

Weak local observability corresponds to $\mathfrak{I}_i \cap X_i = \{0\}$ for $i = 1, 2$ and $\mathfrak{I}_c = \{0\}$ by Definition 5.3.11, and since

$$\begin{aligned} \mathfrak{I} &= (\mathfrak{I}_1 \dot{+} X_2) \cap (\mathfrak{I}_2 \dot{+} X_1) \cap (\mathfrak{I}_c \dot{+} X_1 \dot{+} X_2) \\ &= (\mathfrak{I}_1 \dot{+} X_2) \cap (\mathfrak{I}_2 \dot{+} X_1) \cap (X_1 \dot{+} X_2) \\ &= ((\mathfrak{I}_1 \cap X_1) \dot{+} X_2) \cap ((\mathfrak{I}_2 \cap X_2) \dot{+} X_1) = X_2 \cap X_1 = \{0\}, \end{aligned}$$

this implies joint observability by Definition 5.3.15. □

In the case of weak local observability, the observability of the subsystem states x_i might depend on the value of x_c, and hence the observability properties of the subsystem states are not independent of the rest of the system. Hence weak local observability is not a special case of independent observability.

5.3.4 Example

Consider the system

$$\dot{x}(t) = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] x(t),$$

$$y(t) = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] x(t),$$

with $X_1 = \text{span}\{e_1, e_2, e_3\}$, $X_2 = \text{span}\{e_4, e_5, e_6\}$ and $X_c = \text{span}\{e_7, e_8, e_9\}$. The A-matrix here is the same as in the example for the controllability decomposition, in Section 5.2.4. From writing out the observability matrices of the different pairs of submatrices given in Definition 5.3.1, we see that

$$\mathfrak{I}_1 = \text{span}\{e_8\},\ \mathfrak{I}_2 = \text{span}\{e_7, e_8\},\ \mathfrak{I}_c = \text{span}\{e_7 - e_8\}.$$

This gives the following subspaces:

$$\mathfrak{I}_1 \cap X_1 = \begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_1 = \{0\}, \qquad \mathfrak{I}_2 \cap X_1 = \begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_2 = \{0\}$$
$$\mathfrak{I}_1 \cap X_c = \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_1 = \text{span}\{e_8\}, \qquad \mathfrak{I}_2 \cap X_c = \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_2 = \text{span}\{e_7, e_8\}.$$

This system is not subsystem observable since $\mathfrak{I}_1 \neq \{0\}$. While the subsystems are strongly locally observable because $\mathfrak{I}_i \subseteq X_c$ for $i = 1, 2$, the coordinator is not locally observable since $\mathfrak{I}_c \neq \{0\}$, and hence the overall system is not locally observable (in either the weak or the strong sense). Joint observability follows from

$$\begin{aligned} \mathfrak{I} =& (\mathfrak{I}_1 \dot{+} X_2) \cap (X_1 \dot{+} \mathfrak{I}_2) \cap (X_1 \dot{+} X_2 \dot{+} \mathfrak{I}_c) \\ &= \text{span}\{e_8\} \cap \text{span}\{e_7, e_8\} \cap \text{span}\{e_7 - e_8\} = \{0\}. \end{aligned}$$

Moreover, the system is independently observable since $\begin{bmatrix} I & 0 \end{bmatrix} \mathfrak{I}_i = \{0\}, i = 1, 2$, and

$$\begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_1 \cap \begin{bmatrix} 0 & I \end{bmatrix} \mathfrak{I}_2 \cap \mathfrak{I}_c = \text{span}\{e_8\} \cap \text{span}\{e_7, e_8\} \cap \text{span}\{e_7 - e_8\} = \{0\}.$$

5.4 Systems with inputs and outputs

Coordinated linear systems with inputs and outputs were defined in Section 2.

Since the reachability properties and indistinguishability properties of linear systems are independent of each other, a decomposition of the triple (C, A, B) according to both observability and controllability can be derived by combining the decompositions given in Tables 5.1 and 5.2. Since X_1 is decomposed into 6 subspaces according to reachability, and into 3 subspaces according to indistinguishability, combining these would lead to a decomposition of X_1 into 18 subspaces. The same holds for X_2. The coordinator state space X_c is decomposed into 3 subspaces in the controllability decomposition, and into 18 subspaces in the combined observability decomposition of X. Hence a decomposition of (C, A, B) would involve 54 subspaces of X_c, and 90 subspaces in total. In light of the size of the combined decomposition, we will only derive a decomposition of (C, A, B) for some special cases.

5.4.1 Stabilization via dynamic measurement feedback

In e.g. [63], one can find the following:

> Let a linear system
> $$\dot{x} = Ax + Bu, \; y = Cx,$$
> be given. Consider the state observer
> $$\dot{\xi} = (A - GC)\xi + Gy + Bu,$$
> with observer error $e = x - \xi$, satisfying $\dot{e} = (A - GC)e$. Couple this to the feedback $u = F\xi$. Then the closed-loop system and closed-loop error are
> $$\dot{x} = (A + BF)x + BFe, \; \dot{e} = (A - GC)e,$$
> which can be rewritten as
> $$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A - GC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}.$$

Hence the system is **stabilizable via dynamic measurement feedback** if and only if (A, B) is a stabilizable pair and (C, A) is a detectable pair.

For the class of coordinated linear systems, we need to define concepts of detectability and stabilizability in order to find equivalent conditions for the existence of a stabilizing dynamic measurement feedback.

5.4.1.1 Stabilizability

For coordinated linear systems, we define the concept of $\mathbb{R}_{\text{CLS}}$-stabilizability as follows:

5.4.1. Definition. We call a system of the form (3.1) (or, equivalently, the matrix pair (A, B)) $\mathbb{R}_{\text{CLS}}$**-stabilizable** if there exists a feedback matrix

$$F = \begin{bmatrix} F_{11} & 0 & F_{1c} \\ 0 & F_{22} & F_{2c} \\ 0 & 0 & F_{cc} \end{bmatrix} \in \mathbb{R}_{\text{CLS}}$$

such that the closed-loop system matrix $A + BF$ is stable.

The restriction of all possible stabilizing feedback matrices to the class $\mathbb{R}_{\text{CLS}}$ is necessary (and sufficient, since $\mathbb{R}_{\text{CLS}}$ is a ring) for the closed-loop system to again be a coordinated linear system: feedback matrices of any other form would destroy the underlying information structure. This restriction leads to a stronger concept of stabilizability than the one given in Section 2.3: there exist coordinated linear systems which are stabilizable via an unstructured feedback matrix F, but not stabilizable via a feedback matrix $F \in \mathbb{R}_{\text{CLS}}$.

5.4.2. Proposition. *For a system of the form (3.1), the following are equivalent:*

(1) *The system is $\mathbb{R}_{\text{CLS}}$-stabilizable,*

(2) *The matrix pairs (A_{11}, B_{11}), (A_{22}, B_{22}) and (A_{cc}, B_{cc}) are stabilizable (in the sense of Section 2.3),*

(3) *For any decomposition*

$$X_1 = \mathfrak{R}_1 \dot{+} X_1^s,\ X_2 = \mathfrak{R}_2 \dot{+} X_2^s,\ X_c = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c \dot{+} X_c^s,$$

where $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_c$ are given in Definition 5.2.1, the restriction of A to the subspace $X_1^s \dot{+} X_2^s \dot{+} X_c^s \subseteq X$ is stable.

Proof. 1 $\Leftrightarrow$ 2: By Definition 5.4.1, $\mathbb{R}_{\text{CLS}}$-stabilizability means that there exists a feedback $F \in \mathbb{R}_{\text{CLS}}$ such that the closed-loop system matrix

$$A + BF = \begin{bmatrix} A_{11} + B_{11}F_{11} & 0 & A_{1c} + B_{11}F_{1c} + B_{1c}F_{cc} \\ 0 & A_{22} + B_{22}F_{22} & A_{2c} + B_{22}F_{2c} + B_{2c}F_{cc} \\ 0 & 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix}$$

is stable. Since this matrix is upper-triangular, this is equivalent to stability of the matrices on the diagonal, which is equivalent to the existence of stabilizing feedbacks F_{11}, F_{22} and F_{cc} for the pairs (A_{11}, B_{11}), (A_{22}, B_{22}) and (A_{cc}, B_{cc}), respectively.

1 $\Leftrightarrow$ 3: As quoted in Section 2.3, for $i = 1, 2$ and with respect to any decomposition $X_i = \mathfrak{R}_i \dot{+} X_i^s$ (note that $\mathfrak{R}_i$ is fixed, and X_i^s is free to choose), the matrix pair (A_{ii}, B_{ii}) is of the form $\left(\begin{bmatrix} A_{ii}^{11} & A_{ii}^{12} \\ 0 & A_{ii}^{22} \end{bmatrix}, \begin{bmatrix} B_{ii}^{1} \\ 0 \end{bmatrix} \right)$, given in (2.4). Moreover,

$$A_{cc} \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c = \begin{bmatrix} 0 & 0 & A_{cc} \end{bmatrix} \mathfrak{R}_c = \begin{bmatrix} 0 & 0 & I \end{bmatrix} A\mathfrak{R}_c \subseteq \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$$

by the A-invariance of $\mathfrak{R}_c$, and also

$$\text{im}\, B_{cc} = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \text{im} \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix} \subseteq \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c.$$

Hence the pair (A_{cc}, B_{cc}) is also of the form (2.4). With respect to the decomposition $X = \mathfrak{R}_1 \dot{+} X_1^s \dot{+} \mathfrak{R}_2 \dot{+} X_2^s \dot{+} \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c \dot{+} X_c^s$, the pair (A, B) is now of the form

$$\left(\left[\begin{array}{cc|cc|cc} A_{11}^{11} & A_{11}^{12} & 0 & 0 & A_{1c}^{11} & A_{1c}^{12} \\ 0 & A_{11}^{22} & 0 & 0 & A_{1c}^{21} & A_{1c}^{22} \\ \hline 0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{2c}^{11} & A_{2c}^{12} \\ 0 & 0 & 0 & A_{22}^{22} & A_{2c}^{21} & A_{2c}^{22} \\ \hline 0 & 0 & 0 & 0 & A_{cc}^{11} & A_{cc}^{12} \\ 0 & 0 & 0 & 0 & 0 & A_{cc}^{22} \end{array} \right], \left[\begin{array}{c|c|c} B_{11}^{1} & 0 & B_{1c}^{1} \\ 0 & 0 & B_{1c}^{2} \\ \hline 0 & B_{22}^{1} & B_{2c}^{1} \\ 0 & 0 & B_{2c}^{2} \\ \hline 0 & 0 & B_{cc}^{1} \\ 0 & 0 & 0 \end{array} \right] \right).$$

Applying the state feedback $F = \left[\begin{array}{cc|cc|cc} F_{11}^1 & F_{11}^2 & 0 & 0 & F_{1c}^1 & F_{1c}^2 \\ \hline 0 & 0 & F_{22}^1 & F_{22}^2 & F_{2c}^1 & F_{2c}^2 \\ \hline 0 & 0 & 0 & 0 & F_{cc}^1 & F_{cc}^2 \end{array}\right] \in \mathbb{R}_{\mathrm{CLS}}$ leads to the closed-loop system matrix

$$\left[\begin{array}{cc|cc|cc} A_{11}^{11}+B_{11}^1F_{11}^1 & A_{11}^{12}+B_{11}^1F_{11}^2 & 0 & 0 & \star & \star \\ 0 & A_{11}^{22} & 0 & 0 & \star & \star \\ \hline 0 & 0 & A_{22}^{11}+B_{22}^1F_{22}^1 & A_{22}^{12}+B_{22}^1F_{22}^2 & \star & \star \\ 0 & 0 & 0 & A_{22}^{22} & \star & \star \\ \hline 0 & 0 & 0 & 0 & A_{cc}^{11}+B_{cc}^1F_{cc}^1 & A_{cc}^{12}+B_{cc}^1F_{cc}^2 \\ 0 & 0 & 0 & 0 & 0 & A_{cc}^{22} \end{array}\right],$$

where the entries denoted by $\star$ are not specified further.

Note that the restriction $\left(\begin{bmatrix} A_{11}^{11} & 0 & A_{1c}^{11} \\ 0 & A_{22}^{11} & A_{2c}^{11} \\ 0 & 0 & A_{cc}^{11} \end{bmatrix}, \begin{bmatrix} B_{11}^1 & 0 & B_{1c}^1 \\ 0 & B_{22}^1 & B_{2c}^1 \\ 0 & 0 & B_{cc}^1 \end{bmatrix}\right)$ of the pair (A, B) to $\mathfrak{R}_1 \dot{+} \mathfrak{R}_2 \dot{+} \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c$ is weakly locally controllable by Definition 5.2.11, and this is equivalent to the pairs (A_{11}^{11}, B_{11}^1), (A_{22}^{11}, B_{22}^1) and (A_{cc}^{11}, B_{cc}^1) being controllable pairs by Proposition 5.2.12. This means that there exist matrices F_{11}^1, F_{22}^1 and F_{cc}^1 such that

$$\sigma(A_{11}^{11} + B_{11}^1F_{11}^1) \cup \sigma(A_{22}^{11} + B_{22}^1F_{22}^1) \cup \sigma(A_{cc}^{11} + B_{cc}^1F_{cc}^1) \subset \mathbb{C}^-.$$

Now we have that the system is $\mathbb{R}_{\mathrm{CLS}}$-stabilizable, i.e.

$$\begin{aligned} \sigma(A + BF) =& \sigma(A_{11}^{11} + B_{11}^1F_{11}^1) \cup \sigma(A_{22}^{11} + B_{22}^1F_{22}^1) \cup \sigma(A_{cc}^{11} + B_{cc}^1F_{cc}^1) \\ & \cup \sigma(A_{11}^{22}) \cup \sigma(A_{22}^{22}) \cup \sigma(A_{cc}^{22}) \subset \mathbb{C}^-, \end{aligned}$$

if and only if $\sigma(A_{11}^{22}) \cup \sigma(A_{22}^{22}) \cup \sigma(A_{cc}^{22}) \subset \mathbb{C}^-$, and this in turn is equivalent to the restriction $\begin{bmatrix} A_{11}^{22} & 0 & A_{1c}^{22} \\ 0 & A_{22}^{22} & A_{2c}^{22} \\ 0 & 0 & A_{cc}^{22} \end{bmatrix}$ of A to $X_1^s \dot{+} X_2^s \dot{+} X_c^s$ being stable. □

5.4.1.2 Detectability

In analogy with the previous subsection, we define the concept of $\mathbb{R}_{\mathrm{CLS}}$-detectability as follows:

5.4.3. Definition. We call a system of the form (5.17) (or, equivalently, the matrix pair (C, A)) $\mathbb{R}_{\mathrm{CLS}}$**-detectable** if there exists an observer gain

$$G = \begin{bmatrix} G_{11} & 0 & G_{1c} \\ 0 & G_{22} & G_{2c} \\ 0 & 0 & G_{cc} \end{bmatrix} \in \mathbb{R}_{\mathrm{CLS}}$$

such that the observer error matrix $A - GC$ is stable.

As in the case of $\mathbb{R}_{\mathrm{CLS}}$-stabilizability, we consider only the restricted class $\mathbb{R}_{\mathrm{CLS}}$ of possible observer gains, since unstructured observer gains would lead to observer dynamics $A - GC$ which violate the information structure imposed on the system. This restriction renders the concept of $\mathbb{R}_{\mathrm{CLS}}$-detectability stronger than the usual concept of detectability, as quoted in Section 2.3.

5.4.4. Proposition. *For a system of the form (5.17), the following are equivalent:*

(1) *The system is* $\mathbb{R}_{\mathrm{CLS}}$*-detectable,*

(2) *The matrix pairs* (C_{11}, A_{11}), (C_{22}, A_{22}) *and* (C_{cc}, A_{cc}) *are detectable (in the sense of Section 2.3),*

(3) *For any decomposition*

$$X_1 = (\mathfrak{I}_1 \cap X_1) \dot{+} X_1^r,\ X_2 = (\mathfrak{I}_2 \cap X_2) \dot{+} X_2^r,\ X_c = \mathfrak{I}_c \dot{+} X_c^r,$$

where $\mathfrak{I}_1$, $\mathfrak{I}_2$ *and* $\mathfrak{I}_c$ *are given in Definition 5.3.1, the restriction of* A *to the subspace* $(\mathfrak{I} \cap X_1) \dot{+} (\mathfrak{I}_2 \cap X_2) \dot{+} \mathfrak{I}_c$ *is stable.*

Proof. $1 \Leftrightarrow 2$: Definition 5.4.3 states that $\mathbb{R}_{\mathrm{CLS}}$-detectability is equivalent to the existence of $G \in \mathbb{R}_{\mathrm{CLS}}$ such that the observer error matrix

$$A - GC = \begin{bmatrix} A_{11} - G_{11}C_{11} & 0 & A_{1c} - G_{11}C_{1c} - G_{1c}C_{cc} \\ 0 & A_{22} - G_{22}C_{22} & A_{2c} - G_{22}C_{2c} - G_{2c}C_{cc} \\ 0 & 0 & A_{cc} - G_{cc}C_{cc} \end{bmatrix}$$

is stable. This matrix is upper-triangular, and hence this reduces to the existence of stabilizing observer gains G_{11}, G_{22} and G_{cc} for the pairs (C_{11}, A_{11}), (C_{22}, A_{22}) and (C_{cc}, A_{cc}) on the diagonal.

$1 \Leftrightarrow 3$: Reversing the order of terms in the observability decomposition given in Section 2.3, we have that with respect to any decomposition of the form given in Proposition 5.4.4, the pairs (C_{jj}, A_{jj}), $j = 1, 2, c$, are of the form $\left(\begin{bmatrix} 0 & C_{jj}^2 \end{bmatrix}, \begin{bmatrix} A_{jj}^{11} & A_{jj}^{12} \\ 0 & A_{jj}^{22} \end{bmatrix}\right)$, given in (2.6). Note that the subspaces $\mathfrak{I}_j$, $j = 1, 2, c$, are fixed, while the spaces X_j^r, $j = 1, 2, c$, are free to choose. With respect to the combined decomposition $X = (\mathfrak{I}_1 \cap X_1) \dot{+} X_1^r \dot{+} (\mathfrak{I}_2 \cap X_2) \dot{+} X_2^r \dot{+} \mathfrak{I}_c \dot{+} X_c^r$, the pair (C, A) is then of the form

$$\left(\left[\begin{array}{cc|cc|cc} 0 & C_{11}^2 & 0 & 0 & C_{1c}^1 & C_{1c}^2 \\ \hline 0 & 0 & 0 & C_{22}^2 & C_{2c}^1 & C_{2c}^2 \\ \hline 0 & 0 & 0 & 0 & 0 & C_{cc}^2 \end{array} \right], \left[\begin{array}{cc|cc|cc} A_{11}^{11} & A_{11}^{12} & 0 & 0 & A_{1c}^{11} & A_{1c}^{12} \\ 0 & A_{11}^{22} & 0 & 0 & A_{1c}^{21} & A_{1c}^{22} \\ \hline 0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{2c}^{11} & A_{2c}^{12} \\ 0 & 0 & 0 & A_{22}^{22} & A_{2c}^{21} & A_{2c}^{22} \\ \hline 0 & 0 & 0 & 0 & A_{cc}^{11} & A_{cc}^{12} \\ 0 & 0 & 0 & 0 & 0 & A_{cc}^{22} \end{array} \right] \right).$$

Together with the observer gain $G = \left[\begin{array}{c|c|c} G_{11}^1 & 0 & G_{1c}^1 \\ G_{11}^2 & 0 & G_{1c}^2 \\ \hline 0 & G_{22}^1 & G_{2c}^1 \\ 0 & G_{22}^2 & G_{2c}^2 \\ \hline 0 & 0 & G_{cc}^1 \\ 0 & 0 & G_{cc}^2 \end{array} \right] \in \mathbb{R}_{\text{CLS}}$, this gives the observer error matrix

$$\left[\begin{array}{cc|cc|cc} A_{11}^{11} & A_{11}^{12} - G_{11}^1 C_{11}^2 & 0 & 0 & \star & \star \\ 0 & A_{11}^{22} - G_{11}^2 C_{11}^2 & 0 & 0 & \star & \star \\ \hline 0 & 0 & A_{22}^{11} & A_{22}^{12} - G_{22}^1 C_{22}^2 & \star & \star \\ 0 & 0 & 0 & A_{22}^{22} - G_{22}^2 C_{22}^2 & \star & \star \\ \hline 0 & 0 & 0 & 0 & A_{cc}^{11} & A_{cc}^{12} - G_{cc}^1 C_{cc}^2 \\ 0 & 0 & 0 & 0 & 0 & A_{cc}^{22} - G_{cc}^2 C_{cc}^2 \end{array} \right],$$

where the entries denoted by $\star$ are not specified further.
The restriction of the pair (C, A) to $X_1^r \dot{+} X_2^r \dot{+} X_c^r$ is

$$\left(\begin{bmatrix} C_{11}^2 & 0 & C_{1c}^2 \\ 0 & C_{22}^2 & C_{2c}^2 \\ 0 & 0 & C_{cc}^2 \end{bmatrix}, \begin{bmatrix} A_{11}^{22} & 0 & A_{1c}^{22} \\ 0 & A_{22}^{22} & A_{2c}^{22} \\ 0 & 0 & A_{cc}^{22} \end{bmatrix} \right).$$

Since $\mathfrak{I}_j \cap X_j^r = \{0\}$ by the definition of X_j^r for $j = 1, 2, c$, the restricted system is weakly locally observable in the sense of Definition 5.3.11. By Proposition 5.3.12, this is equivalent to the pairs (C_{11}^2, A_{11}^{22}), (C_{22}^2, A_{22}^{22}) and (C_{cc}^2, A_{cc}^{22}) being observable pairs, and hence there exist matrices G_{11}^2, G_{22}^2 and G_{cc}^2 such that

$$\sigma(A_{11}^{22} - G_{11}^2 C_{11}^2) \cup \sigma(A_{22}^{22} - G_{22}^2 C_{22}^2) \cup \sigma(A_{cc}^{22} - G_{cc}^2 C_{cc}^2) \subset \mathbb{C}^-.$$

Now the system is $\mathbb{R}_{\text{CLS}}$-detectable, i.e.

$$\begin{aligned}\sigma(A-GC) =&\sigma(A_{11}^{11}) \cup \sigma(A_{22}^{11}) \cup \sigma(A_{cc}^{11}) \cup \sigma(A_{11}^{22} - G_{11}^{2}C_{11}^{2}) \\ &\cup \sigma(A_{22}^{22} - G_{22}^{2}C_{22}^{2}) \cup \sigma(A_{cc}^{22} - G_{cc}^{2}C_{cc}^{2}) \subset \mathbb{C}^{-},\end{aligned}$$

if and only if $\sigma(A_{11}^{11})\cup\sigma(A_{22}^{11})\cup\sigma(A_{cc}^{11}) \subset \mathbb{C}^{-}$, and this is equivalent to the restriction $\begin{bmatrix} A_{11}^{11} & 0 & A_{1c}^{11} \\ 0 & A_{22}^{11} & A_{2c}^{11} \\ 0 & 0 & A_{cc}^{11} \end{bmatrix}$ of A to $(\mathfrak{I}_1 \cap X_1)\dot{+}(\mathfrak{I}_2 \cap X_2)\dot{+}\mathfrak{I}_c$ being stable. □

5.4.1.3 Dynamic measurement feedback

The notions of stabilizability and detectability in the setting of coordinated linear systems were described in the previous two subsections. Combining the results about stabilizability and detectability of the previous subsections gives the following result on stabilization via dynamic measurement feedback for coordinated linear systems:

5.4.5. Corollary. *For a system of the form (3.1), the following are equivalent:*

(1) *The system is $\mathbb{R}_{\mathrm{CLS}}$-stabilizable via dynamic measurement feedback with a structure-preserving observer,*

(2) *The matrix pair (C, A) is $\mathbb{R}_{\mathrm{CLS}}$-detectable and the pair (A, B) is $\mathbb{R}_{\mathrm{CLS}}$-stabilizable,*

(3) *The matrix pairs (C_{jj}, A_{jj}), $j = 1, 2, c$ are detectable and the pairs (A_{jj}, B_{jj}), $j = 1, 2, c$ are stabilizable,*

(4) *For any decomposition of the form*

$$X_1 = X_1^1\dot{+}X_1^2\dot{+}X_1^3\dot{+}X_1^4,\ X_2 = X_2^1\dot{+}X_2^2\dot{+}X_2^3\dot{+}X_2^4,\ X_c = X_c^1\dot{+}X_c^2\dot{+}X_c^3\dot{+}X_c^4,$$

where

$X_i^1 = \mathfrak{R}_i \cap (\mathfrak{I}_i \cap X_i)\,,\ i = 1, 2$
X_i^2 *is a complement of* X_i^1 *in* $\mathfrak{R}_i,\ i = 1, 2$
X_i^3 *is a complement of* X_i^1 *in* $\mathfrak{I}_i \cap X_i,\ i = 1, 2$
X_i^4 *is a complement of* $X_i^1\dot{+}X_i^2\dot{+}X_i^3$ *in* $X_i,\ i = 1, 2$
$X_c^1 = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c \cap \mathfrak{I}_c$
X_c^2 *is a complement of* X_c^1 *in* $\begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c$
X_c^3 *is a complement of* X_c^1 *in* $\mathfrak{I}_c$
X_c^4 *is a complement of* $X_c^1\dot{+}X_c^2\dot{+}X_c^3$ *in* X_c,

the restriction of A to

$$\left(X_1^1 \dot{+} X_1^3 \dot{+} X_1^4\right) \dot{+} \left(X_2^1 \dot{+} X_2^3 \dot{+} X_2^4\right) \dot{+} \left(X_c^1 \dot{+} X_c^3 \dot{+} X_c^4\right)$$

is stable.

The decomposition in item 4 of Corollary 5.4.5 is a combination of the decompositions in Propositions 5.4.2 and 5.4.4. With respect to this decomposition, the system (3.1) has the form

$$\dot{x} = \left[\begin{array}{cccc|cccc|cccc}
A_{11}^{11} & A_{11}^{12} & A_{11}^{13} & A_{11}^{14} & 0 & 0 & 0 & 0 & A_{1c}^{11} & A_{1c}^{12} & A_{1c}^{13} & A_{1c}^{14} \\
0 & A_{11}^{22} & 0 & A_{11}^{24} & 0 & 0 & 0 & 0 & A_{1c}^{21} & A_{1c}^{22} & A_{1c}^{23} & A_{1c}^{24} \\
0 & 0 & A_{11}^{33} & A_{11}^{34} & 0 & 0 & 0 & 0 & A_{1c}^{31} & A_{1c}^{32} & A_{1c}^{33} & A_{1c}^{34} \\
0 & 0 & 0 & A_{11}^{44} & 0 & 0 & 0 & 0 & A_{1c}^{41} & A_{1c}^{42} & A_{1c}^{43} & A_{1c}^{44} \\
\hline
0 & 0 & 0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{22}^{13} & A_{22}^{14} & A_{2c}^{11} & A_{2c}^{12} & A_{2c}^{13} & A_{2c}^{14} \\
0 & 0 & 0 & 0 & 0 & A_{22}^{22} & 0 & A_{22}^{24} & A_{2c}^{21} & A_{2c}^{22} & A_{2c}^{23} & A_{2c}^{24} \\
0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{33} & A_{22}^{34} & A_{2c}^{31} & A_{2c}^{32} & A_{2c}^{33} & A_{2c}^{34} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{44} & A_{2c}^{41} & A_{2c}^{42} & A_{2c}^{43} & A_{2c}^{44} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{11} & A_{cc}^{12} & A_{cc}^{13} & A_{cc}^{14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{22} & 0 & A_{cc}^{24} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{33} & A_{cc}^{34} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{44}
\end{array}\right] x$$

$$+ \left[\begin{array}{c|c|c}
B_{11}^{1} & 0 & B_{1c}^{1} \\
B_{11}^{2} & 0 & B_{1c}^{2} \\
0 & 0 & B_{1c}^{3} \\
0 & 0 & B_{1c}^{4} \\
\hline
0 & B_{22}^{1} & B_{2c}^{1} \\
0 & B_{22}^{2} & B_{2c}^{2} \\
0 & 0 & B_{2c}^{3} \\
0 & 0 & B_{2c}^{4} \\
\hline
0 & 0 & B_{cc}^{1} \\
0 & 0 & B_{cc}^{2} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] u,$$

$$y = \left[\begin{array}{cccc|cccc|cccc}
0 & C_{11}^{2} & 0 & C_{11}^{4} & 0 & 0 & 0 & 0 & C_{1c}^{1} & C_{1c}^{2} & C_{1c}^{3} & C_{1c}^{4} \\
\hline
0 & 0 & 0 & 0 & 0 & C_{22}^{2} & 0 & C_{22}^{4} & C_{2c}^{1} & C_{2c}^{2} & C_{2c}^{3} & C_{2c}^{4} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{cc}^{2} & 0 & C_{cc}^{4}
\end{array}\right] x.$$

Combining the closed-loop system matrix of Section 5.4.1.1 and the observer error dynamics of Section 5.4.1.2, we arrive at the closed-loop system and error dynamics

$$\dot{x} = \begin{bmatrix} A_{11} + B_{11}F_{11} & 0 & A_{1c} + B_{11}F_{1c} + B_{1c}F_{cc} \\ 0 & A_{22} + B_{22}F_{22} & A_{2c} + B_{22}F_{2c} + B_{2c}F_{cc} \\ 0 & 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix} x \tag{5.36}$$

$$+ \begin{bmatrix} B_{11}F_{11} & 0 & B_{11}F_{1c} + B_{1c}F_{cc} \\ 0 & B_{22}F_{22} & B_{22}F_{2c} + B_{2c}F_{cc} \\ 0 & 0 & B_{cc}F_{cc} \end{bmatrix} e, \tag{5.37}$$

$$\dot{e} = \begin{bmatrix} A_{11} - G_{11}C_{11} & 0 & A_{1c} - G_{11}C_{1c} - G_{1c}C_{cc} \\ 0 & A_{22} - G_{22}C_{22} & A_{2c} - G_{22}C_{2c} - G_{2c}C_{cc} \\ 0 & 0 & A_{cc} - G_{cc}C_{cc} \end{bmatrix} e, \tag{5.38}$$

where the diagonal entries are given by

$$A_{jj} + B_{jj}F_{jj} = \begin{bmatrix} A^{11}_{jj} + B^1_{jj}F^1_{jj} & A^{12}_{jj} + B^1_{jj}F^2_{jj} & A^{13}_{jj} + B^1_{jj}F^3_{jj} & A^{14}_{jj} + B^1_{jj}F^4_{jj} \\ B^2_{jj}F^1_{jj} & A^{22}_{jj} + B^2_{jj}F^2_{jj} & B^2_{jj}F^3_{jj} & A^{24}_{jj} + B^2_{jj}F^4_{jj} \\ 0 & 0 & A^{33}_{jj} & A^{34}_{jj} \\ 0 & 0 & 0 & A^{44}_{jj} \end{bmatrix},$$

$$B_{jj}F_{jj} = \begin{bmatrix} B^1_{jj}F^1_{jj} & B^1_{jj}F^2_{jj} & B^1_{jj}F^3_{jj} & B^1_{jj}F^4_{jj} \\ B^2_{jj}F^1_{jj} & B^2_{jj}F^2_{jj} & B^2_{jj}F^3_{jj} & B^2_{jj}F^4_{jj} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{jj} - G_{jj}C_{jj} = \begin{bmatrix} A^{11}_{jj} & A^{12}_{jj} - G^1_{jj}C^2_{jj} & A^{13}_{jj} & A^{14}_{jj} - G^1_{jj}C^4_{jj} \\ 0 & A^{22}_{jj} - G^2_{jj}C^2_{jj} & 0 & A^{24}_{jj} - G^2_{jj}C^4_{jj} \\ 0 & -G^3_{jj}C^2_{jj} & A^{33}_{jj} & A^{34}_{jj} - G^3_{jj}C^4_{jj} \\ 0 & -G^4_{jj}C^2_{jj} & 0 & A^{44}_{jj} - G^4_{jj}C^4_{jj} \end{bmatrix}.$$

Note that item 4 of Corollary 5.4.5 is equivalent to requiring that the unstable part of the system be both weakly locally observable and weakly locally controllable.

5.4.2 Example of a system with inputs and outputs

In this subsection we illustrate how to combine the decompositions according to controllability and observability found in the previous sections on an example. We choose a coordinated linear system that is subsystem observable (in the sense of Section 5.3.3.1) and independently controllable (in the sense of Section 5.2.3.5). Combining the corresponding reduced representations (5.26) and (5.15), we arrive at a representation of the form

$$C = \left[\begin{array}{cccc|cccc|cc} C_{11}^1 & C_{11}^2 & C_{11}^3 & C_{11}^4 & 0 & 0 & 0 & 0 & C_{1c}^1 & C_{1c}^2 \\ \hline 0 & 0 & 0 & 0 & C_{22}^1 & C_{22}^2 & C_{22}^3 & C_{22}^4 & C_{2c}^1 & C_{2c}^2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{cc}^1 & 0 \end{array}\right],$$

$$A = \left[\begin{array}{cccc|cccc|cc} A_{11}^{11} & A_{11}^{12} & A_{11}^{13} & A_{11}^{14} & 0 & 0 & 0 & 0 & A_{1c}^{11} & A_{1c}^{12} \\ 0 & A_{11}^{22} & A_{11}^{23} & 0 & 0 & 0 & 0 & 0 & A_{1c}^{21} & A_{1c}^{22} \\ 0 & A_{11}^{32} & A_{11}^{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{11}^{44} & 0 & 0 & 0 & 0 & A_{1c}^{41} & A_{1c}^{42} \\ \hline 0 & 0 & 0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{22}^{13} & A_{22}^{14} & A_{2c}^{11} & A_{2c}^{12} \\ 0 & 0 & 0 & 0 & 0 & A_{22}^{22} & A_{22}^{23} & 0 & A_{2c}^{21} & A_{2c}^{22} \\ 0 & 0 & 0 & 0 & 0 & A_{22}^{32} & A_{22}^{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{44} & A_{2c}^{41} & A_{2c}^{42} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{21} & A_{cc}^{22} \end{array}\right],$$

$$B = \left[\begin{array}{c|c|c} B_{11}^1 & 0 & B_{1c}^1 \\ B_{11}^2 & 0 & B_{1c}^2 \\ B_{11}^3 & 0 & 0 \\ 0 & 0 & B_{1c}^4 \\ \hline 0 & B_{22}^1 & B_{2c}^1 \\ 0 & B_{22}^2 & B_{2c}^2 \\ 0 & B_{22}^3 & 0 \\ 0 & 0 & B_{2c}^4 \\ \hline 0 & 0 & B_{cc}^1 \\ 0 & 0 & B_{cc}^2 \end{array}\right].$$

For $i = 1, 2$, the following pairs are observable pairs:

$$\left(\left[\begin{array}{cccc|cc} C_{ii}^1 & C_{ii}^2 & C_{ii}^3 & C_{ii}^4 & C_{ic}^1 & C_{ic}^2 \end{array}\right], \left[\begin{array}{cccc|cc} A_{ii}^{11} & A_{ii}^{12} & A_{ii}^{13} & A_{ii}^{14} & A_{ic}^{11} & A_{ic}^{12} \\ 0 & A_{ii}^{22} & A_{ii}^{23} & 0 & A_{ic}^{21} & A_{ic}^{22} \\ 0 & A_{ii}^{32} & A_{ii}^{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{ii}^{44} & A_{ic}^{41} & A_{ic}^{42} \\ \hline 0 & 0 & 0 & 0 & A_{cc}^{11} & 0 \\ 0 & 0 & 0 & 0 & A_{cc}^{21} & A_{cc}^{22} \end{array}\right] \right),$$

and the following pairs are controllable pairs:

$$\left(\begin{bmatrix} A_{ii}^{11} & A_{ii}^{12} & A_{ii}^{13} \\ 0 & A_{ii}^{22} & A_{ii}^{23} \\ 0 & A_{ii}^{32} & A_{ii}^{33} \end{bmatrix}, \begin{bmatrix} B_{ii}^{1} \\ B_{ii}^{2} \\ B_{ii}^{3} \end{bmatrix} \right), \left(\left[\begin{array}{cc|cc} A_{ii}^{11} & A_{ii}^{14} & A_{ic}^{11} & A_{ic}^{12} \\ 0 & A_{ii}^{44} & A_{ic}^{41} & A_{ic}^{42} \\ \hline 0 & 0 & A_{cc}^{11} & 0 \\ 0 & 0 & A_{cc}^{21} & A_{cc}^{22} \end{array} \right], \left[\begin{array}{c} B_{ic}^{1} \\ B_{ic}^{4} \\ \hline B_{cc}^{1} \\ B_{cc}^{2} \end{array} \right] \right).$$

In the case of subsystem observability, each subsystem can reconstruct the coordinator state x_c from its local output y_i via an observer. These observer estimates can then be used for feedback control: Since the local estimate of the coordinator state is available at the subsystem, it can be used for the local control input u_i.

In the case that x_c is also y_c-observable, this means that the system is *independently output controllable*: All state information that may be needed for control (i.e. x_i and x_c for subsystem i, and x_c for the coordinator) can be reconstructed locally via a stable state observer, and hence all forms of state controllability defined in Section 5.2.3 are equivalent to the corresponding forms of output controllability. Since the system was assumed to be independently (state) controllable, this means we have independent output controllability.

Note that subsystem observability, with the additional requirement that x_c be y_c-observable, is the strongest concept of observability possible for coordinated linear systems. For some concepts of state controllability to be equivalent to their corresponding forms of output controllability, weaker concepts of observability may be sufficient: For example, if a coordinated linear system is weakly locally controllable then each part of the system is controllable using local state state information, and hence strong local observability is sufficient for weak local output controllability.

5.5 Concluding remarks

In the previous sections, we refined the usual concepts of reachability and indistinguishability to better comply with the class of coordinated linear systems, a particular class of decentralized systems with several inputs and outputs. Decompositions of the state spaces of the different subsystems, with respect to the different reachable and indistinguishable subspaces corresponding to these definitions, were derived. For these decompositions according to the different state spaces, it was necessary to distinguish between independently and jointly reachable subspaces, and between completely and independently indistinguishable subspaces. These notions deviate considerably from the classical theory of linear systems.

While a generalization of our definitions of reachability and indistinguishability to other classes of decentralized systems is straightforward, the corresponding decompositions quickly become infeasible if the underlying information structure is less restrictive. However, the distinction between independent and joint reachability, and between complete and independent indistinguishability, may be relevant for other classes of decentralized linear systems as well.

When defining the concepts of controllability and observability, we again had to deviate from the classical definitions for unstructured systems: These properties of linear systems in their usual sense are of little practical use for coordinated linear systems, and cannot be verified in a decentralized manner. Instead, we introduced the slightly stronger concepts of independent controllability and observability, related to the notion of independence needed for the decompositions. In contrast to the case of unstructured linear systems, these new concepts did not suffice for pole placement and state reconstruction; for this, we needed the concepts of weak local controllability and observability, which easily carry over to other linear systems with a top-to-bottom information structure.

For stabilizability and detectability, and consequently for stabilization via dynamic measurement feedback, we again needed the concepts of weak local controllability and observability, rather than the usual concepts or their independent counterparts. This is due to the necessary restriction of admissible feedbacks and observers to the ones complying with the underlying information structure. This effect of restricting the admissible feedbacks will also play an important role in linear-quadratic optimal control for coordinated linear systems, as described in the following chapter.

6 LQ Optimal Control

In this chapter, we extend the classical LQ optimal control problem for monolithic linear systems, the formulation and solution of which was summarized in Section 2.4, to coordinated linear systems.

This chapter is submitted as [22], and the corresponding numerical algorithm and computations can be found in [70]. Based on the results derived here, the related case of LQ control for coordinated Gaussian systems was worked out in [41].

6.1 Introduction

The classical LQ (linear-quadratic) optimal control problem was first formulated and solved in [20][1], and has played a central role in system theory ever since – on the one hand because of its wide applicability, and on the other hand because of the computational and conceptual simplicity of its solution. Unfortunately, the optimal control feedback $u(\cdot) = Gx(\cdot)$, obtained by solving the Riccati equation (2.10) for a coordinated linear system, does in general not respect the underlying information structure. Hence, in order to apply the classical theory to the case of coordinated linear systems, we need to add the constraint that the state feedback should be an element of $\mathbb{R}_{\text{CLS}}$.

The main results of this chapter state that

- the LQ problem for coordinated linear systems separates into independent local problems for each subsystem and a more complex problem at the coordinator level,
- due to this separation property, the LQ problem for any hierarchical system whose information structure forms a directed tree can be separated into subproblems and approached in a bottom-to-top manner,
- and in contrast to the unstructured case, the optimal feedback matrix does depend on the initial state.

For notational reasons, and using the separability of the problem described above, we first restrict attention to the special case of leader-follower systems, i.e. hierarchical systems with two layers and only one subsystem at each layer, and then extend these results to coordinated linear systems, and more general hierarchical systems. The results are reformulated into control synthesis procedures, and the theory is illustrated in an example involving vehicle formations.

[1] According to `scholar.google.com`, this will be roughly the 1100$^{\text{th}}$ time that [20] is cited.

6.2 LQ control for leader-follower systems

In this section we consider LQ optimal control for leader-follower systems. Extensions of the results of this section to coordinated linear systems and hierarchical systems are discussed in Section 6.3.

For the purposes of this section, we denote the set of block upper-triangular matrices by

$$\mathbb{R}_{\triangle} = \left\{ \begin{bmatrix} M_{ss} & M_{sc} \\ 0 & M_{cc} \end{bmatrix},\ M_{ij} \in \mathbb{R}^{m_i \times n_j},\ i,j = s,c \right\}.$$

6.2.1 Problem formulation

The control problem considered in this section is defined for two different sets of admissible control laws:

6.2.1. Problem. We consider leader-follower systems, of the form

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_c \end{bmatrix}(t) = \begin{bmatrix} A_{ss} & A_{sc} \\ 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix}(t) + \begin{bmatrix} B_{ss} & B_{sc} \\ 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_s \\ u_c \end{bmatrix}(t),\ \begin{bmatrix} x_s \\ x_c \end{bmatrix}(0) = \begin{bmatrix} x_{0,s} \\ x_{0,c} \end{bmatrix} \quad (6.1)$$

and quadratic infinite-horizon undiscounted cost functions, of the form

$$J\left(x_0, \begin{bmatrix} u_s(\cdot) \\ u_c(\cdot) \end{bmatrix}\right) = \int_0^\infty \begin{bmatrix} x_s \\ x_c \end{bmatrix}^T \begin{bmatrix} Q_{ss} & 0 \\ 0 & Q_{cc} \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix} + \begin{bmatrix} u_s \\ u_c \end{bmatrix} \begin{bmatrix} R_{ss} & 0 \\ 0 & R_{cc} \end{bmatrix} \begin{bmatrix} u_s \\ u_c \end{bmatrix} dt, \quad (6.2)$$

where $Q_{ss} \geq 0$, $Q_{cc} \geq 0$, $R_{ss} > 0$, and $R_{cc} > 0$.

We want to minimize the cost function over the following sets of admissible control laws:[2]

$$\mathfrak{U}_{lin}^{\triangle} = \left\{ \begin{bmatrix} F_{ss} & F_{sc} \\ 0 & F_{cc} \end{bmatrix} x(t) \,\middle|\, A + BF \text{ is stable, } F \text{ may depend on } x_0 \right\},$$

$$\mathfrak{U}^{\triangle} = \left\{ \begin{bmatrix} u_s(t, x_s, x_c) \\ u_c(t, x_{s,0}, x_c) \end{bmatrix} \,\middle|\, \begin{array}{l} u_s \text{ and } u_c \text{ are stabilizing and piecewise continuous in } t, \\ u_s \text{ and } u_c \text{ depend causally on } x_s \text{ and } x_c \end{array} \right\}.$$

The control problems considered in this section are hence given by

$$\min_{u(\cdot) \in \mathfrak{U}^{\triangle}} J\left(x_0, u(\cdot)\right), \quad (6.3)$$

$$\min_{u(\cdot) = Fx(\cdot) \in \mathfrak{U}_{lin}^{\triangle}} J(x_0, Fx(\cdot)). \quad (6.4)$$

[2] In this context, we call a control law $u(\cdot)$ stabilizing if applying the control law $u(\cdot)$ leads to an exponentially decaying system state for $t \to \infty$.

The set $\mathfrak{U}^{\triangle}$ contains all (linear or nonlinear) admissible control laws which respect the underlying information structure of the system. The set $\mathfrak{U}^{\triangle}_{lin} \subset \mathfrak{U}^{\triangle}$ restricts the admissible control laws to structure-preserving linear state feedbacks.

Note that in both sets of admissible control laws, the initial state x_0 is assumed to be globally known: The coordinator system may use the subsystem initial state $x_{0,s}$, even though this does not respect the information structure. From the theory developed in this section, it will become clear that this additional knowledge has a large impact on the optimal coordinator feedback, and on the overall cost.

6.2.2 Optimal control over $\mathfrak{U}^{\triangle}$

The most straightforward option for extending the centralized solution given in Section 2.4 to the more restrictive control problem (6.3) is to include a copy of the subsystem in the coordinator. Since the state trajectory of a deterministic linear system can be reconstructed from the initial state and the closed-loop state transition matrix, the coordinator only needs to know $x_{s,0}$ and the matrix $A + BG$ in order to deduce the state $x_s(t)$ at each time t. The coordinator can then apply the optimal feedback of the centralized case, using its local copy of $x_s(t)$. With this construction, and writing $G = \begin{bmatrix} G_{ss} & G_{sc} \\ G_{cs} & G_{cc} \end{bmatrix}$, we get the coordinator control law

$$u_c(\cdot, x_{s,0}, x_c(\cdot)) = G_{cs}z(\cdot) + G_{cc}x_c(\cdot),$$

with $z(\cdot)$ given by the internal reconstruction

$$\dot{z} = (A_{ss} + B_{ss}G_{ss})z + B_{ss}G_{sc}x_c,\ z(0) = x_{s,0}$$

of $x_s(\cdot)$. This control law for the coordinator does not depend on $x_s(t)$ (but only on its local reconstruction), and hence respects the information structure imposed on the system. Together with the subsystem control law

$$u_s(\cdot, x_s(\cdot), x_c(\cdot)) = G_{ss}x_s(\cdot) + G_{sc}x_c(\cdot),$$

we have that $\begin{bmatrix} u_s \\ u_c \end{bmatrix} \in \mathfrak{U}^{\triangle}$. Since the resulting closed-loop system is equivalent to the system $\dot{x} = (A + BG)x$, this control law achieves the same optimal cost as the centralized control law $Gx(\cdot)$:

$$\min_{u(\cdot) \in \mathfrak{U}^{\triangle}} J(x_0, u(\cdot)) = J(x_0, Gx(\cdot)).$$

From this it also follows that the control law $\begin{bmatrix} u_s(\cdot) \\ u_c(\cdot) \end{bmatrix}$ constructed here is stabilizing if and only if $Gx(\cdot)$ is stabilizing, and hence the conditions for well-definedness of the control problem over $\mathfrak{U}^{\triangle}$ are the same as in the centralized case, i.e. that (A, B) be a stabilizable pair, and (Q, A) be a detectable pair.

In [56, 61] it is shown that the construction above corresponds to the control law obtained from solving the decentralized control problem in an input-output framework, and then translating the solution back to its state-space equivalent.

However the control law constructed above, based on reconstructing the subsystem state at the coordinator level, has several disadvantages:

- In a hierarchical system, the subsystem state x_s comprises all parts of the system which are influenced by the coordinator (see Section 6.3). Hence the approach used here is not scalable for large hierarchical systems: The simulation of the subsystem at the coordinator level becomes computationally very involved.
- Moreover, this approach is not extendable to systems with disturbances or parameter uncertainties: The actual subsystem state may diverge from its reconstruction by the coordinator, leading to arbitrarily large costs, and even to loss of stability. This is because after the initial time, the coordinator receives no feedback from the subsystem, and hence cannot adjust its estimate accordingly.

In Chapter 7, we present one alternative to the control law found in this section, using event-based feedback from the subsystem(s) to the coordinator. In the rest of this chapter we present another alternative, using the restriction that the coordinator control law u_c must be a linear state feedback of the form $u_c(\cdot) = F_{cc}x_c(\cdot)$.

6.2.3 Optimal control over $\mathfrak{U}^{\triangle}_{lin}$

In the following we develop the core results of the chapter – the well-definedness, separability and solution of Problem (6.4).

6.2.3.1 Conditions for well-definedness

In this subsection we introduce the concepts of $\mathbb{R}_{\triangle}$-stabilizability and $\mathbb{R}_{\triangle}$-detectability, and show that these concepts are necessary and sufficient for the well-definedness of Problem (6.4).

Stabilizability. In Section 5.4.1.1 it was shown that, for the case of coordinated linear systems, the usual concept of stabilizability needs to be restricted to the existence of a stabilizing *structure-preserving* feedback. We simplify Definition

5.4.1 of $\mathbb{R}_{\text{CLS}}$-stabilizability (for coordinated linear systems) to $\mathbb{R}_\triangle$-stabilizability (for leader-follower systems):

6.2.2. Definition. We call a system of the form (6.1) $\mathbb{R}_\triangle$**-stabilizable** if there exists a feedback matrix

$$F = \begin{bmatrix} F_{ss} & F_{sc} \\ 0 & F_{cc} \end{bmatrix} \in \mathbb{R}_\triangle$$

such that the closed-loop system matrix $A + BF$ is stable.

The following proposition is simplified from Proposition 5.4.2:

6.2.3. Proposition. *A system of the form (6.1) is $\mathbb{R}_\triangle$-stabilizable if and only if the matrix pairs (A_{ss}, B_{ss}) and (A_{cc}, B_{cc}) are stabilizable pairs.*

Detectability. Similarly, we modify Definition 5.4.3 of $\mathbb{R}_{\text{CLS}}$-detectability, which accounts for the restriction that possible observers must respect the underlying information structure. For this, we first need to define leader-follower systems with outputs:

$$\begin{aligned} \begin{bmatrix} \dot{x}_s \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A_{ss} & A_{sc} \\ 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix} + \begin{bmatrix} B_{ss} & B_{sc} \\ 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_s \\ u_c \end{bmatrix}, \\ \begin{bmatrix} y_s \\ y_c \end{bmatrix} &= \begin{bmatrix} C_{ss} & C_{sc} \\ 0 & C_{cc} \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix}. \end{aligned} \tag{6.5}$$

The role of the output matrix C will later be replaced by the cost matrix Q.

6.2.4. Definition. We call a system of the form (6.5) $\mathbb{R}_\triangle$**-detectable** if there exists an observer gain

$$K = \begin{bmatrix} K_{ss} & K_{sc} \\ 0 & K_{cc} \end{bmatrix} \in \mathbb{R}_\triangle$$

such that the observer-error matrix $A - KC$ is stable.

The following proposition is simplified from Proposition 5.4.4:

6.2.5. Proposition. *A system of the form (6.5) is $\mathbb{R}_\triangle$-detectable if and only if the matrix pairs (C_{ss}, A_{ss}) and (C_{cc}, A_{cc}) are detectable pairs.*

Conditions for well-definedness. Now we can give a sufficient condition for well-definedness of Problem (6.4), using stabilizability and detectability properties of the corresponding submatrices:

6.2.6. Proposition. *If the pairs (A_{ss}, B_{ss}) and (A_{cc}, B_{cc}) are stabilizable pairs, and if the pairs (Q_{ss}, A_{ss}) and (Q_{cc}, A_{cc}) are detectable pairs, then Problem (6.4) is well-defined, i.e. it admits a stabilizing solution, which lies inside $\mathfrak{U}^\triangle_{lin}$.*

Proof. Since (A_{ss}, B_{ss}) is a stabilizable pair and (Q_{ss}, A_{ss}) is a detectable pair, the equation

$$X_{ss}B_{ss}R_{ss}^{-1}B_{ss}X_{ss} - A_{ss}^T X_{ss} - X_{ss}A_{ss} - Q_{ss} = 0$$

has a stabilizing solution X_{ss}. Similarly, the equation

$$X_{cc}B_{cc}R_{cc}^{-1}B_{cc}X_{cc} - A_{cc}^T X_{cc} - X_{cc}A_{cc} - Q_{cc} = 0$$

has a stabilizing solution X_{cc}. Consider the feedback $F = \begin{bmatrix} F_{ss} & 0 \\ 0 & F_{cc} \end{bmatrix}$, with $F_{ss} = -R_{ss}^{-1}B_{ss}^T X_{ss}$ and $F_{cc} = -R_{cc}^{-1}B_{cc}^T X_{cc}$. To show that F is indeed a stabilizing feedback, we look at the spectrum of the closed-loop system:

$$\sigma\left(\begin{bmatrix} A_{ss} & A_{sc} \\ 0 & A_{cc} \end{bmatrix} + \begin{bmatrix} B_{ss} & B_{sc} \\ 0 & B_{cc} \end{bmatrix}\begin{bmatrix} F_{ss} & 0 \\ 0 & F_{cc} \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} A_{ss} + B_{ss}F_{ss} & A_{sc} + B_{sc}F_{cc} \\ 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix}\right)$$
$$= \sigma\left(A_{ss} + B_{ss}F_{ss}\right) \cup \sigma\left(A_{cc} + B_{cc}F_{cc}\right) \in \mathbb{C}^-.$$

We conclude that there exists a stabilizing feedback $F \in \mathfrak{U}_{lin}^{\triangle}$, leading to a finite cost (this follows directly from stability of the closed-loop system).

In order to show that the infimum over $\mathfrak{U}_{lin}^{\triangle}$ is indeed a minimum, we note that on the boundary of the set of all stabilizing feedback matrices, the closed-loop system has at least one eigenvalue on the imaginary axis, and hence one part of the closed-loop system is not exponentially stable. If the initial condition for this part is non-zero, then the state will never vanish, and by detectability of the system, the corresponding cost will be infinite. We conclude that the infimum over $\mathfrak{U}_{lin}^{\triangle}$ is attained at a point in the interior of $\mathfrak{U}_{lin}^{\triangle}$, and hence it is a minimum.
□

6.2.3.2 Conditionally-optimal solution, given F_{cc}

For the purpose of this subsection, suppose that the coordinator feedback F_{cc} is fixed. Replacing the coordinator input u_c with $F_{cc}x_c$, we then have the system

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_c \end{bmatrix}(t) = \begin{bmatrix} A_{ss} & A_{sc} + B_{sc}F_{cc} \\ 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix}\begin{bmatrix} x_s \\ x_c \end{bmatrix}(t) + \begin{bmatrix} B_{ss} \\ 0 \end{bmatrix} u_s(t), \quad \begin{bmatrix} x_s \\ x_c \end{bmatrix}(0) = \begin{bmatrix} x_{0,s} \\ x_{0,c} \end{bmatrix}. \quad (6.6)$$

The cost function becomes

$$J_{F_{cc}}(x_0, u_s(\cdot)) = \int_0^\infty \begin{bmatrix} x_s \\ x_c \end{bmatrix}^T \begin{bmatrix} Q_{ss} & 0 \\ 0 & Q_{cc} + F_{cc}^T R_{cc} F_{cc} \end{bmatrix}\begin{bmatrix} x_s \\ x_c \end{bmatrix} + u_s R_{ss} u_s \, dt.$$

For the related optimal control problem

$$\min_{u_s(\cdot) \text{ piecewise continuous}} J_{F_{cc}}(x_0, u_s(\cdot)), \tag{6.7}$$

we have the following result:

6.2.7. Theorem. *Consider the system (6.6), and assume that $A_{cc} + B_{cc}F_{cc}$ is stable, and that (A_{ss}, B_{ss}) is a stabilizable pair and (Q_{ss}, A_{ss}) is a detectable pair. Then the solution $u_s^*(x_0, F_{cc})$ of problem (6.7) is unique, and given by a linear state feedback of the form*

$$u_s^*(x_0, F_{cc}) = \begin{bmatrix} F_{ss} & F_{sc} \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix},$$

where the feedback matrices F_{ss} and F_{sc} have the following properties:

- *F_{ss} is given by $F_{ss} = -R_{ss}^{-1}B_{ss}^T X_{ss}$, where X_{ss} is the unique stabilizing solution of the Riccati equation*

  $$X_{ss}B_{ss}R_{ss}^{-1}B_{ss}^T X_{ss} - A_{ss}^T X_{ss} - X_{ss}A_{ss} - Q_{ss} = 0. \tag{6.8}$$

  In particular, F_{ss} is independent of F_{sc} and F_{cc}.

- *F_{sc} is given by $F_{sc} = -R_{ss}^{-1}B_{ss}^T X_{sc}$, where X_{sc} is the unique solution of the Lyapunov equation*

  $$(A_{ss} + B_{ss}F_{ss})^T X_{sc} + X_{sc}(A_{cc} + B_{cc}F_{cc}) + X_{ss}(A_{sc} + B_{sc}F_{cc}) = 0. \tag{6.9}$$

  X_{sc} and F_{sc} depend on X_{ss}, F_{ss} and F_{cc}.

- *Let X_{cc} be the unique solution of the Lyapunov equation*

  $$\begin{aligned} &(A_{sc} + B_{sc}F_{cc})^T X_{sc} + X_{sc}^T(A_{sc} + B_{sc}F_{cc}) - F_{sc}^T R_{ss} F_{sc} \\ &+ (A_{cc} + B_{cc}F_{cc})^T X_{cc} + X_{cc}(A_{cc} + B_{cc}F_{cc}) + Q_{cc} + F_{cc}^T R_{cc} F_{cc} = 0. \end{aligned} \tag{6.10}$$

  Then the conditionally-optimal cost, corresponding to the control law

  $$u_s^*(\cdot) = \begin{bmatrix} F_{ss} & F_{sc} \end{bmatrix} \begin{bmatrix} x_s(\cdot) \\ x_c(\cdot) \end{bmatrix}, \tag{6.11}$$

  and conditioned on F_{cc}, is given by

  $$J_{F_{cc}}(x_0, u_s(\cdot)) = \begin{bmatrix} x_{0s} \\ x_{0c} \end{bmatrix}^T \begin{bmatrix} X_{ss} & X_{sc} \\ X_{sc}^T & X_{cc} \end{bmatrix} \begin{bmatrix} x_{0s} \\ x_{0c} \end{bmatrix}. \tag{6.12}$$

Note that the linearity of u_s^* in $\begin{bmatrix} x_s \\ x_c \end{bmatrix}$ was not assumed here; it is part of the result. The linearity of $u_c = F_{cc}x_c$ however was assumed.

Proof. We assumed that $A_{cc} + B_{cc}F_{cc}$ is stable, and that (A_{ss}, B_{ss}) is a stabilizable pair and (Q_{ss}, A_{ss}) is a detectable pair. From this it follows that the pair $\left(\begin{bmatrix} A_{ss} & A_{sc} + B_{sc}F_{cc} \\ 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix}, \begin{bmatrix} B_{ss} \\ 0 \end{bmatrix} \right)$ is a stabilizable pair and that the pair $\left(\begin{bmatrix} Q_{ss} & 0 \\ 0 & Q_{cc} + F_{cc}^T R_{cc} F_{cc} \end{bmatrix}, \begin{bmatrix} A_{ss} & A_{sc} + B_{sc}F_{cc} \\ 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix} \right)$ is a detectable pair. Hence the sufficient conditions in Section 2.4 are satisfied, and we can apply the classical LQ control theory to our problem. By Section 2.4, we have existence, uniqueness and linearity of u_s^*.

Writing out the Riccati equation of Section 2.4 for our system gives

$$\begin{aligned}
&\begin{bmatrix} X_{ss} & X_{sc} \\ X_{sc}^T & X_{cc} \end{bmatrix} \begin{bmatrix} B_{ss} \\ 0 \end{bmatrix} R_{ss}^{-1} \begin{bmatrix} B_{ss}^T & 0 \end{bmatrix} \begin{bmatrix} X_{ss} & X_{sc} \\ X_{sc}^T & X_{cc} \end{bmatrix} \\
&- \begin{bmatrix} A_{ss}^T & 0 \\ (A_{sc} + B_{sc}F_{cc})^T & (A_{cc} + B_{cc}F_{cc})^T \end{bmatrix} \begin{bmatrix} X_{ss} & X_{sc} \\ X_{sc}^T & X_{cc} \end{bmatrix} \\
&- \begin{bmatrix} X_{ss} & X_{sc} \\ X_{sc}^T & X_{cc} \end{bmatrix} \begin{bmatrix} A_{ss} & A_{sc} + B_{sc}F_{cc} \\ 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix} - \begin{bmatrix} Q_{ss} & 0 \\ 0 & Q_{cc} + F_{cc}^T R_{cc} F_{cc} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

and rewriting this equation entry-wise gives the three equations

$$\begin{aligned}
&X_{ss}B_{ss}R_{ss}^{-1}B_{ss}^T X_{ss} - A_{ss}^T X_{ss} - X_{ss}A_{ss} - Q_{ss} = 0, \\
&X_{ss}B_{ss}R_{ss}^{-1}B_{ss}^T X_{sc} - A_{ss}^T X_{sc} - X_{ss}(A_{sc} + B_{sc}F_{cc}) - X_{sc}(A_{cc} + B_{cc}F_{cc}) = 0, \\
&X_{sc}^T B_{ss}R_{ss}^{-1}B_{ss}^T X_{sc} - (A_{sc} + B_{sc}F_{cc})^T X_{sc} - X_{sc}^T(A_{sc} + B_{sc}F_{cc}) \\
&- (A_{cc} + B_{cc}F_{cc})^T X_{cc} - X_{cc}(A_{cc} + B_{cc}F_{cc}) - (Q_{cc} + F_{cc}^T R_{cc}F_{cc}) = 0.
\end{aligned}$$

The first equation is the same as (6.8). Together with

$$\begin{bmatrix} F_{ss} & F_{sc} \end{bmatrix} = -R_{ss}^{-1} \begin{bmatrix} B_{ss}^T & 0 \end{bmatrix} \begin{bmatrix} X_{ss} & X_{sc} \\ X_{sc}^T & X_{cc} \end{bmatrix} = \begin{bmatrix} -R_{ss}^{-1}B_{ss}^T X_{ss} & -R_{ss}^{-1}B_{ss}^T X_{sc} \end{bmatrix},$$

the last two equations can be rewritten as

$$(A_{ss} + B_{ss}F_{ss})^T X_{sc} + X_{sc}(A_{cc} + B_{cc}F_{cc}) + X_{ss}(A_{sc} + B_{sc}F_{cc}) = 0,$$

which is equivalent to (6.9), and

$$
\begin{aligned}
&(A_{sc} + B_{sc}F_{cc})^T X_{sc} + X_{sc}^T (A_{sc} + B_{sc}F_{cc}) - F_{sc}^T R_{ss} F_{sc} \\
&+ (A_{cc} + B_{cc}F_{cc})^T X_{cc} + X_{cc}(A_{cc} + B_{cc}F_{cc}) + Q_{cc} + F_{cc}^T R_{cc} F_{cc} = 0,
\end{aligned}
$$

which is the same as (6.10). □

6.2.3.3 Control synthesis procedure

Using the result of the previous subsection, we now give a procedure for finding the optimal control law $\begin{bmatrix} u_s(\cdot) \\ u_c(\cdot) \end{bmatrix} = \begin{bmatrix} F_{ss} & F_{sc} \\ 0 & F_{cc} \end{bmatrix} \begin{bmatrix} x_s(\cdot) \\ x_c(\cdot) \end{bmatrix} \in \mathfrak{U}_{lin}^{\triangle}$:

6.2.8. Procedure.

(1) Find X_{ss} by solving the Riccati equation (6.8) numerically, and set $F_{ss} = -R_{ss}^{-1} B_{ss}^T X_{ss}$.

(2) Find F_{cc}. A numerical procedure for this is given in Section 6.2.3.5.

(3) Solve the Lyapunov equation (6.9) for X_{sc}, and set $F_{sc} = -R_{ss}^{-1} B_{ss}^T X_{sc}$.

(4) Now the Lyapunov equation (6.10) can be solved for X_{cc}, and the corresponding cost $J(x_0, u(\cdot))$ can be found by computing (6.12).

If F_{cc} is found via an iterative numerical search procedure (which is the case for numerical optimization), steps (2)-(4) will have to be iterated. Step (1) only needs to be performed once.

6.2.3.4 Uniqueness of the optimal F_{cc}

Concerning the uniqueness of the optimal coordinator feedback F_{cc}, we have the following conjecture:

6.2.9. Conjecture. *If the pairs (A_{ss}, B_{ss}) and (A_{cc}, B_{cc}) are stabilizable pairs and the pairs (Q_{ss}, A_{ss}) and (Q_{cc}, A_{cc}) are detectable pairs then there exists a unique minimizer $u^*(\cdot) \in \mathfrak{U}_{lin}^{\triangle}$ for Problem (6.4).*

Note that $u^*(\cdot) = \begin{bmatrix} F_{ss} & F_{sc} \\ 0 & F_{cc} \end{bmatrix} x(\cdot)$ by the definition of $\mathfrak{U}_{lin}^{\triangle}$, and by Theorem 6.2.7 the matrices F_{ss} and F_{sc} are unique for any given F_{cc} with $A_{cc}+B_{cc}F_{cc}$ stable. Hence, conjecturing that $u^*(\cdot) \in \mathfrak{U}_{lin}\triangle$ is unique is equivalent to conjecturing that the optimal F_{cc} is unique.

While we do not yet have a proof of this conjecture, testing randomly generated examples has not yet lead to a counterexample either.

6.2.3.5 Finding F_{cc} numerically

So far we have not found an analytical solution for the problem of finding the optimal F_{cc}.

Setting $F_{cc}^{ind} = -R_{cc}^{-1} B_{cc}^T Y_{cc}$, where Y_{cc} is the unique stabilizing solution of the local Riccati equation

$$Y_{cc} B_{cc} R_{cc}^{-1} B_{cc}^T Y_{cc} - A_{cc}^T Y_{cc} - Y_{cc} A_{cc} - Q_{cc} = 0,$$

is always an admissible option (see the proof of Proposition 6.2.6), although usually not the optimal one. It can however serve as a good initial value for numerical optimization procedures.

One approach to finding the optimal solution numerically is to use the Matlab routine `fmincon` to minimize the cost over all possible F_{cc}, with the constraint that F_{cc} must be stabilizing. This approach is implemented in [70] (for coordinated linear systems) as follows:

- The feedback F_{cc}^{ind} is used as starting value for `fmincon`,
- The local feedback F_{ss} is computed from equation (6.8),
- At each step of the optimization procedure, the cost corresponding to the current value of F_{cc} is computed using equations (6.9)-(6.12) if $A_{cc} + B_{cc} F_{cc}$ is stable, and set to ∞ otherwise.

Testing this algorithm for randomly generated examples, we found that it always converges to an admissible solution, and that the cost difference compared to the centralized feedback is in general very small, and often even negligible. The computation time scales exponentially with the problem size.

The set of all stabilizing F_{cc} is not a convex set with respect to its element-wise parametrization

$$\left\{ F_{cc} = \begin{bmatrix} f^{1,1} & \dots & f^{1,n_c} \\ \vdots & \ddots & \vdots \\ f^{m_c,1} & \dots & f^{m_c,n_c} \end{bmatrix}, \; f^{i,j} \in \mathbb{R}, \; \sigma\left(A_{cc} + B_{cc} F_{cc}\right) \subset \mathbb{C}_- \right\}.$$

Hence there is no guarantee that the algorithm described above will perform well in all situations. It is, however, a connected set, and alternative parameterizations would lead to convexity. Whether a parametrization exists which would lead to convexity of both the set of stabilizing feedbacks and the objective function restricted to this set, is an open question.

6.2.3.6 The scalar case, and properties of the solution

If the state and input spaces of the subsystem and coordinator all have dimension 1, i.e. if

$$A = \begin{bmatrix} a_{ss} & a_{sc} \\ 0 & a_{cc} \end{bmatrix},\ B = \begin{bmatrix} b_{ss} & b_{sc} \\ 0 & b_{cc} \end{bmatrix},\ Q = \begin{bmatrix} q_{ss} & 0 \\ 0 & q_{cc} \end{bmatrix},\ R = \begin{bmatrix} r_{ss} & 0 \\ 0 & r_{cc} \end{bmatrix}$$

then with $x_{ss} = \frac{r_{ss}}{b_{ss}^2}\left(a_{ss} + \sqrt{a_{ss}^2 + b_{ss}^2 \frac{q_{ss}}{r_{ss}}}\right)$ and $x_{sc} = \frac{-(a_{sc}+b_{sc}f_{cc})x_{ss}}{a_{ss}+b_{ss}f_{ss}+a_{cc}+b_{cc}f_{cc}}$, we have

$$X = \begin{bmatrix} x_{ss} & x_{sc} \\ x_{sc} & \frac{r_{ss}^{-1}b_{ss}^2 x_{sc}^2 - 2(a_{sc}+b_{sc}f_{cc})x_{sc} - q_{cc} - r_{cc}f_{cc}^2}{2(a_{cc}+b_{cc}f_{cc})} \end{bmatrix}. \tag{6.13}$$

Note that in the case where $a_{cc} + b_{cc}f_{cc}$ is a scalar, the set of stabilizing feedbacks is indeed a convex set. Numerical optimization in the scalar case is straightforward.

The cost is a rational matrix function of f_{cc}. For any stabilizing f_{ss} and f_{cc} we have $a_{ss} + b_{ss}f_{ss} + a_{cc} + b_{cc}f_{cc} < 0$ and $a_{cc} + b_{cc}f_{cc} < 0$, and hence the function $X(f_{cc})$ has no poles within the stabilizing region.

We illustrate some properties of the solution in the following example, which was worked out in [70]:

6.2.10. Example. Let the system and cost matrices be given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\quad R = \begin{bmatrix} \alpha_s & 0 \\ 0 & \alpha_c \end{bmatrix},$$

with parameters $\alpha_s, \alpha_c > 0$. The coordinator system is not controllable in this example; however, the subsystem is both locally controllable and independently controllable via the coordinator input. Hence the LQ problem for this example reduces to the problem which input (or combination of inputs) should be used to stabilize the subsystem state, taking into account that different costs (α_s and α_c) are associated to the different inputs. Using (6.13), we get

$$X = \begin{bmatrix} \alpha_s\left(1 + \sqrt{1 + \frac{1}{\alpha_s}}\right) & \alpha_s f_{cc} \\ \alpha_s f_{cc} & \frac{1}{2}\left(\alpha_s + \alpha_c\right) f_{cc}^2 \end{bmatrix}.$$

With initial state $x_0 = \begin{bmatrix} x_{0s} \\ x_{0c} \end{bmatrix}$, the cost is given by

$$J(x_0, Fx(\cdot)) = x_0^T X x_0 = \alpha_s \left(1 + \sqrt{1 + \frac{1}{\alpha_s}}\right) x_{0s}^2 + 2\alpha_s f_{cc} x_{0s} x_{0c} + \frac{1}{2}\left(\alpha_s + \alpha_c\right) f_{cc}^2 x_{0c}^2.$$

The unique minimizer of this cost is given by

$$f_{cc}^* = -\frac{2\alpha_s x_{0s}}{(\alpha_s + \alpha_c)\, x_{0c}}$$

and the corresponding minimal cost is

$$J\left(x_0, F^* x(\cdot)\right) = \alpha_s \left(1 + \sqrt{1 + \frac{1}{\alpha_s}} - \frac{2\alpha_s}{\alpha_s + \alpha_c}\right) x_{0s}^2.$$

For comparison, we also give the centralized optimum:

$$G^* = -\begin{bmatrix} \frac{\alpha_c\left(1+\sqrt{1+\frac{1}{\alpha_s}+\frac{1}{\alpha_c}}\right)}{\alpha_s+\alpha_c} & 0 \\ \frac{\alpha_s\left(1+\sqrt{1+\frac{1}{\alpha_s}+\frac{1}{\alpha_c}}\right)}{\alpha_s+\alpha_c} & 0 \end{bmatrix}, \quad J(x_0, G^* x(\cdot)) = \frac{\alpha_s \alpha_c \left(1 + \sqrt{1 + \frac{1}{\alpha_s} + \frac{1}{\alpha_c}}\right)}{\alpha_s + \alpha_c} x_{0s}^2.$$

From this, we can derive the following properties:

- The optimal coordinator feedback f_{cc}^* depends on x_0, while G^* does not.
- For $\alpha_c \to \infty$, both $J(x_0, F^* x(\cdot))$ and $J(x_0, G^* x(\cdot))$ approach $\alpha_s \left(1 + \sqrt{1 + \frac{1}{\alpha_s}}\right) x_{0s}^2$.
- For $\alpha_c \to 0$, we have that $J(x_0, F^* x(\cdot)) \to \alpha_s \left(\sqrt{1 + \frac{1}{\alpha_s}} - 1\right) x_{0s}^2$, but $J(x_0, G^* x(\cdot))$ approaches 0.

The fact that f_{cc}^* depends on the initial state is a major drawback of the LQ coordination control problem for deterministic systems considered in this chapter, and also implies that a closed-loop solution for f_{cc}^* cannot be derived directly from the matrix equations characterizing the cost, as in the centralized case.

For very large α_c, using the coordinator input for stabilizing the subsystem state is very costly compared to the local input, and hence both the centralized control law and the coordination control law converge to a local feedback law for the subsystem state. If α_c is very small then using the coordinator input is very cheap compared to the local input, and the relative cost difference $\frac{J(x_0, F^* x(\cdot)) - J(x_0, G^* x(\cdot))}{J(x_0, G^* x(\cdot))}$ of applying the coordination control law instead of the centralized control law approaches ∞.

6.3 Coordinated and hierarchical systems

In this section we discuss how to extend the control synthesis procedure introduced in the previous section to hierarchical systems with more subsystems and/or more layers.

6.3.1 LQ control of coordinated linear systems

First we extend the results of the previous section to coordinated linear systems with two subsystems and one coordinator.

6.3.1. Problem. In this subsection we consider the linear-quadratic control problem

$$\min_{u(\cdot)\in\mathfrak{U}_{lin}^{\mathrm{CLS}}} J(x_0, u(\cdot)), \tag{6.14}$$

where the system dynamics are given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_c\end{bmatrix} = \begin{bmatrix}A_{11} & 0 & A_{1c}\\ 0 & A_{22} & A_{2c}\\ 0 & 0 & A_{cc}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_c\end{bmatrix} + \begin{bmatrix}B_{11} & 0 & B_{1c}\\ 0 & B_{22} & B_{2c}\\ 0 & 0 & B_{cc}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_c\end{bmatrix}, \quad \begin{bmatrix}x_1(0)\\ x_2(0)\\ x_c(0)\end{bmatrix} = \begin{bmatrix}x_{1,0}\\ x_{2,0}\\ x_{c,0}\end{bmatrix}, \tag{6.15}$$

the cost function is given by

$$J(x_0, u(\cdot)) = \int_0^\infty \left(\begin{bmatrix}x_1\\ x_2\\ x_c\end{bmatrix}^T \begin{bmatrix}Q_{11} & 0 & 0\\ 0 & Q_{22} & 0\\ 0 & 0 & Q_{cc}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_c\end{bmatrix} + \begin{bmatrix}u_1\\ u_2\\ u_c\end{bmatrix}^T \begin{bmatrix}R_{11} & 0 & 0\\ 0 & R_{22} & 0\\ 0 & 0 & R_{cc}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_c\end{bmatrix}\right) dt \tag{6.16}$$

and the set of admissible control laws is given by

$$\mathfrak{U}_{lin}^{\mathrm{CLS}} = \left\{ \begin{bmatrix}F_{11} & 0 & F_{1c}\\ 0 & F_{22} & F_{2c}\\ 0 & 0 & F_{cc}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_c\end{bmatrix} \middle| \, A + BF \text{ is stable} \right\}. \tag{6.17}$$

The sufficient conditions for well-definedness of this problem follow from the same argumentation as in the proof of Proposition 6.2.6:

6.3.2. Corollary. *If, in the setting of Problem 6.3.1, the pairs*

$$(A_{11}, B_{11}),\ (A_{22}, B_{22}) \text{ and } (A_{cc}, B_{cc})$$

are stabilizable pairs, and the pairs

$$(Q_{11}, A_{11}),\ (Q_{22}, A_{22}) \text{ and } (Q_{cc}, A_{cc})$$

are detectable pairs, then Problem 6.3.1 has a stabilizing solution.

The following theorem is an extension of Theorem 6.2.7 from leader-follower systems to coordinated linear systems:

6.3.3. Theorem. *If we assume that F_{cc} is given, the unique conditionally-optimal solution*

$$\begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} = \begin{bmatrix} F_{11} & 0 & F_{1c} \\ 0 & F_{22} & F_{2c} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix}$$

is described by the following set of equations for X_{ii}, X_{ic}, X_{cc}, F_{ii} and F_{ic} ($i = 1, 2$):

$$\begin{aligned}
0 =& X_{11}B_{11}R_{11}^{-1}B_{11}^T X_{11} - A_{11}^T X_{11} - X_{11}A_{11} - Q_{11} && (6.18)\\
F_{11} =& -R_{11}^{-1}B_{11}^T X_{11}, && (6.19)\\
0 =& X_{22}B_{22}R_{22}^{-1}B_{22}^T X_{22} - A_{22}^T X_{22} - X_{22}A_{22} - Q_{22}, && (6.20)\\
F_{22} =& -R_{22}^{-1}B_{22}^T X_{22}, && (6.21)\\
0 =& (A_{11} + B_{11}F_{11})^T X_{1c} + X_{1c}(A_{cc} + B_{cc}F_{cc}) + X_{11}(A_{1c} + B_{1c}F_{cc}), && (6.22)\\
F_{1c} =& -R_{11}^{-1}B_{11}^T X_{1c}, && (6.23)\\
0 =& (A_{22} + B_{22}F_{22})^T X_{2c} + X_{2c}(A_{cc} + B_{cc}F_{cc}) + X_{22}(A_{2c} + B_{2c}F_{cc}), && (6.24)\\
F_{2c} =& -R_{22}^{-1}B_{22}^T X_{2c}, && (6.25)\\
0 =& (A_{1c} + B_{1c}F_{cc})^T X_{1c} + X_{1c}^T(A_{1c} + B_{1c}F_{cc}) - F_{1c}^T R_{11} F_{1c} \\
&+ (A_{2c} + B_{2c}F_{cc})^T X_{2c} + X_{2c}^T(A_{2c} + B_{2c}F_{cc}) - F_{2c}^T R_{22} F_{2c} && (6.26)\\
&+ (A_{cc} + B_{cc}F_{cc})^T X_{cc} + X_{cc}(A_{cc} + B_{cc}F_{cc}) + (Q_{cc} + F_{cc}^T R_{cc} F_{cc}).
\end{aligned}$$

Note that equations (6.18)-(6.26) can easily be solved numerically, in the order they appear here. Moreover, the subsystem feedbacks F_{11} and F_{22} can be found independently of the rest of the system: From the previous section we know that the subsystem feedbacks are independent of the coordinator, but this result tells us that they are also independent of each other.

Proof. Consider Problem 6.3.1, but with the extended set of admissible control laws

$$\mathfrak{V}_{lin}^{\text{CLS}} = \left\{ \begin{bmatrix} F_{11} & F_{12} & F_{1c} \\ F_{21} & F_{22} & F_{2c} \\ 0 & 0 & F_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} \middle| \, A + BF \text{ is stable} \right\}.$$

This extended problem is a special case of Problem 6.2.1, with

$$x_s = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; u_s = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \; A_{ss} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \; B_{ss} = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix},$$
$$Q_{ss} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}, \; R_{ss} = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix}.$$

Thus, fixing F_{cc}, we can use Theorem 6.2.7 to find the conditionally-optimal control law for the extended problem. Rewriting equation (6.8) for this case, with $X_{ss} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$, we get

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} R_{11}^{-1} & 0 \\ 0 & R_{22}^{-1} \end{bmatrix} \begin{bmatrix} B_{11}^T & 0 \\ 0 & B_{22}^T \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$$
$$- \begin{bmatrix} A_{11}^T & 0 \\ 0 & A_{22}^T \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} - \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} - \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} = 0.$$

The matrix $\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}$, with X_{11} and X_{22} the stabilizing solutions of (6.18) and (6.20), respectively, is a solution of this Riccati equation. Moreover, the corresponding feedback $F_{ss} = \begin{bmatrix} -R_{11}^{-1} B_{11}^T X_{11} & 0 \\ 0 & -R_{22}^{-1} B_{22}^T X_{22} \end{bmatrix}$ is stabilizing: The choice of X_{11} and X_{22} ensures that the closed-loop subsystem matrix $\begin{bmatrix} A_{11} - B_{11} R_{11}^{-1} B_{11}^T X_{11} & 0 \\ 0 & A_{22} - B_{22} R_{22}^{-1} B_{22}^T X_{22} \end{bmatrix}$ is stable. But the stabilizing solution of a Riccati equation is unique, and hence $\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}$ is the only stabilizing solution. From this it follows that

$$F_{ss} = -R_{ss}^{-1} B_{ss}^T X_{ss} = - \begin{bmatrix} R_{11}^{-1} & 0 \\ 0 & R_{22}^{-1} \end{bmatrix} \begin{bmatrix} B_{11}^T & 0 \\ 0 & B_{22}^T \end{bmatrix} \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}$$

is automatically block-diagonal. This means that the conditionally-optimal solution of Problem 6.3.1 over the extended set $\mathfrak{V}_{lin}^{\text{CLS}}$ is an element of $\mathfrak{U}_{lin}^{\text{CLS}}$, and since $\mathfrak{U}_{lin}^{\text{CLS}} \subseteq \mathfrak{V}_{lin}^{\text{CLS}}$, it is also the conditionally-optimal solution over $\mathfrak{U}_{lin}^{\text{CLS}}$.

Equation (6.9) splits into equations (6.22) and (6.24), and equation (6.10) reduces to (6.26). □

In light of Theorem 6.3.3, we can now extend Procedure 6.2.8 to a control synthesis procedure for coordinated linear systems:

6.3.4. Procedure.

(1) Find X_{11} and X_{22} by solving the Riccati equations (6.18) and (6.20) numerically, and set $F_{11} = -R_{11}^{-1} B_{11}^T X_{11}$ and $F_{22} = -R_{22}^{-1} B_{22}^T X_{22}$.

(2) Combine the two subsystems to one system: Set

$$A_{ss} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},\ B_{ss} = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix},\ Q_{ss} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix},\ R_{ss} = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix},$$

and fix $F_{ss} = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix}$.

(3) Use steps (2)-(4) of Procedure 6.2.8 to find the optimal coordination control feedback for the problem.

6.3.2 Extension to hierarchical systems

In this subsection we discuss how to extend the results and procedures above to more general hierarchical systems, with several layers and/or more subsystems at each layer.

6.3.5. Problem. Let a hierarchical linear system, i.e. a distributed linear system with its underlying information structure given by a directed tree, be given. We will assume that the system is both locally stabilizable and locally detectable, i.e. for each subsystem j in the hierarchy we assume that (A_{jj}, B_{jj}) is a stabilizable pair and (Q_{jj}, A_{jj}) is a detectable pair. We consider the infinite-horizon LQ control problem with Q and R block-diagonal, and restricting the set of admissible feedback matrices to those which respect the system's information structure.

For hierarchical systems, control is done in a bottom-up manner: First we find local feedbacks for the subsystems which do not act as coordinators for any other part of the system (i.e. the leaves of the corresponding directed tree), and then we find the optimal feedbacks for the coordinating systems, using the results and procedures of the previous sections.

6.3.6. Procedure.

- Pick a subsystem j, all followers[3] of which already have local feedbacks assigned to them (or no followers exist).
- For this subsystem, calculate the solution X_{jj}^{ind} of the local Riccati equation

$$X_{jj}^{ind} B_{jj} R_{jj}^{-1} B_{jj}^T X_{jj}^{ind} - A_{jj}^T X_{jj}^{ind} - X_{jj}^{ind} A_{jj} - Q_{jj} = 0.$$

- Starting with $F_{jj}^{ind} = -R_{jj}^{-1} B_{jj}^T X_{jj}^{ind}$, find the optimal F_{jj} numerically: Consider all followers of system j as one subsystem, and find F_{jj} using the numerical optimization procedure described in Section 6.2.3.5.
- Set $u_j = F_{jj} x_j$, find $\begin{bmatrix} F_{1j} \\ \vdots \\ F_{s_j j} \end{bmatrix}$ from (6.9), and apply the feedbacks $F_{kj} x_j$ to all followers k of system j.

[3]By followers, we mean all subsystems which are coordinated by system j.

Repeatedly applying the results of the previous sections, and assuming that the algorithm of Section 6.2.3.5 indeed converges to the global optimum, we find that the feedback F found by Procedure 6.3.6 is the optimal linear state feedback which respects the information structure.

6.4 Example: Vehicle formations

In order to illustrate the control synthesis procedures introduced in this paper, we apply our results to a toy example involving several autonomous vehicles. The goal of the vehicles is to maintain a fixed formation while tracking a reference signal. Even though all subsystems have the same internal dynamics, the optimal control feedback for each vehicle will be different, depending on the number and formation of its followers.

Each vehicle V_j is modeled by a very simple linear system, with its position $p_j \in \mathbb{R}^3$ and velocity $v_j \in \mathbb{R}^3$ as state variables, and its acceleration $a_j \in \mathbb{R}^3$ as control input. For vehicle V_j, this system is given by

$$\begin{bmatrix} \dot{p}_j \\ \dot{v}_j \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_j \\ v_j \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} a_j, \quad \begin{bmatrix} p_j \\ v_j \end{bmatrix} (0) = \begin{bmatrix} p_{j,0} \\ v_{j,0} \end{bmatrix}. \tag{6.27}$$

For each vehicle, we moreover have a reference position $p_j^R \in \mathbb{R}^3$ and reference velocity $v_j^R \in \mathbb{R}^3$. How these reference signals are determined depends on the formation to be kept, and will be discussed later. The optimal control problem to be solved for vehicle V_j is then the tracking problem

$$\min_{a_j(\cdot),\, p_j^R(\cdot),\, v_j^R(\cdot)} \int_0^\infty \left\| \begin{bmatrix} p_j - p_j^R \\ v_j - v_j^R \end{bmatrix} \right\|^2 + \|a_j\|^2 dt. \tag{6.28}$$

If the reference signal is given externally then p_j^R and v_j^R are fixed, and not part of the optimization problem.

Suppose the reference signal also has dynamics

$$\begin{bmatrix} \dot{p}_j^R \\ \dot{v}_j^R \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_j^R \\ v_j^R \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} a_j^R, \quad \begin{bmatrix} p_j^R \\ v_j^R \end{bmatrix} (0) = \begin{bmatrix} p_{j,0}^R \\ v_{j,0}^R \end{bmatrix},$$

for example because it is another vehicle or a moving object whose dynamics we approximate using the internal model principle, with the internal model given above. Then the difference vector occurring in the cost function has dynamics

$$\begin{bmatrix} \dot{p}_j - \dot{p}_j^R \\ \dot{v}_j - \dot{v}_j^R \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_j - p_j^R \\ v_j - v_j^R \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} (a_j - a_j^R), \ \begin{bmatrix} p_j - p_j^R \\ v_j - v_j^R \end{bmatrix} (0) = \begin{bmatrix} p_{j,0} - p_{j,0}^R \\ v_{j,0} - v_{j,0}^R \end{bmatrix}.$$

where a_j is the control input and a_j^R is a disturbance input.

We assume that the disturbance is such that the certainty equivalence property holds (see e.g. [65]).[4] Hence all reference accelerations which are not known (e.g. because they correspond to the acceleration of another vehicle in the formation) will be treated as zero. This is also necessary for the infinite-horizon undiscounted cost in (6.28) to be well-defined – taking the disturbances into account, this cost is infinite for all possible control laws. Note that in practice, our formation problem is always of finite duration (since battery power is limited), the infinite-horizon formulation is used here to avoid any emphasis on the terminal state.

In this section, we will consider the following two formations, each consisting of three vehicles:

- Formation 1 corresponds to the structure of a coordinated linear system, as defined in Section 3.1: The coordinating vehicle follows an external reference signal, and the other two vehicles follow the coordinating vehicle in a fixed formation, without interacting with each other.
- In Formation 2 the vehicles form a chain: The first vehicle follows an external reference signal, the second vehicle follows the first vehicle in a fixed formation, and the third vehicle follows the second vehicle in a fixed formation. There is no direct interaction between the first and third vehicle.

These formations are illustrated in Figure 6.1.

Formation 1 One vehicle (V_c) follows an external reference signal, and acts as a coordinator for two other vehicles (V_1 and V_2): Vehicles V_1 and V_2 regularly receive the current position and velocity of V_c, and they track the signal

$$\begin{bmatrix} p_j^R \\ v_j^R \end{bmatrix} = \begin{bmatrix} p_c \\ v_c \end{bmatrix} + \begin{bmatrix} \Delta_j \\ 0 \end{bmatrix}, \ j = 1, 2,$$

where Δ_j is a fixed, time-invariant spatial shift (i.e. V_1 is supposed to be at position $p_c + \Delta_1$, not at the same position as V_c). V_1 and V_2 do not send their state to

[4]For certainty equivalence of the classical LQ problem with disturbances, it is sufficient that the disturbances are Gaussian. Note that we have not shown that this property also holds for the coordination case.

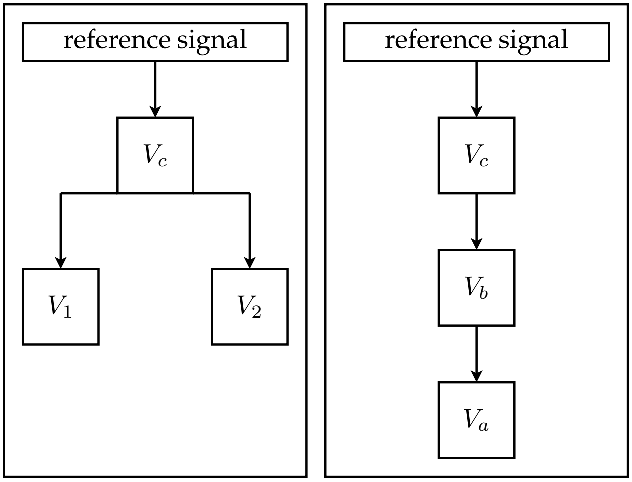

Figure 6.1: Formation 1 (left), Formation 2 (right)

the other vehicles. Moreover, we restrict attention to linear control laws for $a_1,\ a_2$ and a_c. Treating the disturbance input a_c^R as zero, the combined control problem for vehicles V_1, V_2 and V_c is now

$$\min_{\begin{bmatrix} a_1 \\ a_2 \\ a_c \end{bmatrix} \in \mathfrak{U}_{lin}^{\mathrm{CLS}}} \int_0^\infty \left\| \begin{bmatrix} p_1 - p_1^R \\ v_1 - v_1^R \\ \hline p_2 - p_2^R \\ v_2 - v_2^R \\ \hline p_c - p_c^R \\ v_c - v_c^R \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} a_1 \\ a_2 \\ a_c \end{bmatrix} \right\|^2 dt,$$

subject to

$$\begin{bmatrix} \dot{p}_1 - \dot{p}_1^R \\ \dot{v}_1 - \dot{v}_1^R \\ \hline \dot{p}_2 - \dot{p}_2^R \\ \dot{v}_2 - \dot{v}_2^R \\ \hline \dot{p}_c - \dot{p}_c^R \\ \dot{v}_c - \dot{v}_c^R \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} p_1 - p_1^R \\ v_1 - v_1^R \\ \hline p_2 - p_2^R \\ v_2 - v_2^R \\ \hline p_c - p_c^R \\ v_c - v_c^R \end{bmatrix} + \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ I & 0 & -I \\ \hline 0 & 0 & 0 \\ 0 & I & -I \\ \hline 0 & 0 & 0 \\ 0 & 0 & I \end{array}\right] \begin{bmatrix} a_1 \\ \hline a_2 \\ \hline a_c \end{bmatrix},$$

with initial condition

$$\begin{bmatrix} p_1 - p_1^R \\ v_1 - v_1^R \\ \hline p_2 - p_2^R \\ v_2 - v_2^R \\ \hline p_c - p_c^R \\ v_c - v_c^R \end{bmatrix}(0) = \begin{bmatrix} p_{1,0} - p_{c,0} - \Delta_1 \\ v_{1,0} - v_{c,0} \\ \hline p_{2,0} - p_{c,0} - \Delta_2 \\ v_{2,0} - v_{c,0} \\ \hline p_{c,0} - p_{c,0}^R \\ v_{c,0} - v_{c,0}^R \end{bmatrix}.$$

This is a control problem of the type described in Problem 6.3.1, with $Q = I_{18}$ and $R = I_9$. The local control problems (6.18) and (6.20) are solved by

$$X_{jj} = \begin{bmatrix} \sqrt{3}I & I \\ I & \sqrt{3}I \end{bmatrix},\ F_{jj} = \begin{bmatrix} -I & -\sqrt{3}I \end{bmatrix},\ i = 1, 2.$$

The coordinator feedback matrices F_{cc}, F_{1c} and F_{2c} are found numerically, using Procedure 6.3.4. Since the optimal coordinator feedback depends on the initial state of the overall system, we need to choose numerical values for the initial state. We set

$$\begin{bmatrix} p_{1,0} - p_{c,0} - \Delta_1 \\ v_{1,0} - v_{c,0} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},\ \begin{bmatrix} p_{2,0} - p_{c,0} - \Delta_2 \\ v_{2,0} - v_{c,0} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},\ \begin{bmatrix} p_{c,0} - p_{c,0}^R \\ v_{c,0} - v_{c,0}^R \end{bmatrix} = 10 * \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

i.e. the follower vehicles V_1 and V_2 are already in formation w.r.t. V_c, but the coordinating vehicle V_c is not yet tracking the reference signal.

Using Procedure 6.3.4 and the numerical algorithm described in Section 6.2.3.5, we arrive at the optimal coordinator feedback

$$\begin{bmatrix} F_{1c} \\ F_{2c} \\ F_{cc} \end{bmatrix} \approx \begin{bmatrix} 3.66 & 2.97 & -6.98 & -1.00 & -0.42 & 0.34 \\ 3.67 & 2.67 & -6.69 & 0.56 & -1.99 & 0.35 \\ 4.12 & 3.43 & -7.89 & 0.62 & -0.35 & -1.35 \\ 3.66 & 2.97 & -6.98 & -1.00 & -0.42 & 0.34 \\ 3.67 & 2.67 & -6.69 & 0.56 & -1.99 & 0.35 \\ 4.12 & 3.43 & -7.89 & 0.62 & -0.35 & -1.35 \\ -0.59 & -5.48 & 5.33 & -9.89 & -15.37 & 23.72 \\ -0.91 & -8.01 & 8.15 & 0.36 & -24.92 & 23.05 \\ 11.54 & 10.46 & -22.59 & 7.17 & -7.59 & -1.28 \end{bmatrix},$$

with corresponding cost $J(x_0, Fx(\cdot)) = 2646.92$.

The centralized cost (i.e. the minimal cost in the case that two-way communication among all vehicles is allowed) is $J(x_0, Gx(\cdot)) = 2646.89$.

Formation 2 In this formation, vehicle V_c tracks an external reference signal. Vehicle V_b regularly receives the current position and velocity of V_c and tracks the signal

$$\begin{bmatrix} p_b^R \\ v_b^R \end{bmatrix} = \begin{bmatrix} p_c \\ v_c \end{bmatrix} + \begin{bmatrix} \Delta_b \\ 0 \end{bmatrix},$$

where $\Delta_b \in \mathbb{R}^3$ is a time-invariant spatial shift parameter. Similarly, vehicle V_a has information about the current position and velocity of V_b (but not of V_c), and tracks the signal

$$\begin{bmatrix} p_a^R \\ v_a^R \end{bmatrix} = \begin{bmatrix} p_b \\ v_b \end{bmatrix} + \begin{bmatrix} \Delta_a \\ 0 \end{bmatrix},$$

with $\Delta_a \in \mathbb{R}^3$ the spatial shift parameter. The set of admissible state feedbacks respecting this information structure is given by

$$\mathfrak{U}_{lin}^{F2} = \left\{ \begin{bmatrix} F_{aa} & F_{ab} & 0 \\ 0 & F_{bb} & F_{bc} \\ 0 & 0 & F_{cc} \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix} \middle| A + BF \text{ is stable} \right\}.$$

The subscripts a, b and c for the different vehicles are chosen to avoid confusion with the roles of the vehicles in Formation 1.

The overall control problem for Formation 2 is

$$\min_{\begin{bmatrix} a_a \\ a_b \\ a_c \end{bmatrix} \in \mathfrak{U}_{lin}^{F2}} \int_0^\infty \left\| \begin{bmatrix} p_a - p_a^R \\ v_a - v_a^R \\ \hline p_b - p_b^R \\ v_b - v_b^R \\ \hline p_c - p_c^R \\ v_c - v_c^R \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} a_a \\ a_b \\ a_c \end{bmatrix} \right\|^2 dt,$$

subject to the dynamics

$$\begin{bmatrix} \dot{p}_a - \dot{p}_a^R \\ \dot{v}_a - \dot{v}_a^R \\ \hline \dot{p}_b - \dot{p}_b^R \\ \dot{v}_b - \dot{v}_b^R \\ \hline \dot{p}_c - \dot{p}_c^R \\ \dot{v}_c - \dot{v}_c^R \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} p_a - p_a^R \\ v_a - v_a^R \\ \hline p_b - p_b^R \\ v_b - v_b^R \\ \hline p_c - p_c^R \\ v_c - v_c^R \end{bmatrix} + \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ I & -I & 0 \\ \hline 0 & 0 & 0 \\ 0 & I & -I \\ \hline 0 & 0 & 0 \\ 0 & 0 & I \end{array}\right] \begin{bmatrix} a_a \\ \hline a_b \\ \hline a_c \end{bmatrix},$$

with initial condition

$$\left[\begin{array}{c} p_a - p_a^R \\ v_a - v_a^R \\ \hline p_b - p_b^R \\ v_b - v_b^R \\ \hline p_c - p_c^R \\ v_c - v_c^R \end{array}\right](0) = \left[\begin{array}{c} p_{a,0} - p_{b,0} - \Delta_a \\ v_{a,0} - v_{b,0} \\ \hline p_{b,0} - p_{c,0} - \Delta_b \\ v_{b,0} - v_{c,0} \\ \hline p_{c,0} - p_{c,0}^R \\ v_{c,0} - v_{c,0}^R \end{array}\right].$$

This problem is not dynamics-invariant, and it is not a hierarchical system in the sense of this chapter, since V_a is an indirect (but not a direct) follower of V_c. We can still apply the control procedure, by only taking into account direct followers for each subsystem.

We solve the control problem as described in Procedure 6.3.6, again choosing the initial state

$$\begin{bmatrix} p_{a,0} - p_{b,0} - \Delta_a \\ v_{a,0} - v_{b,0} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} p_{b,0} - p_{c,0} - \Delta_b \\ v_{b,0} - v_{c,0} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} p_{c,0} - p_{c,0}^R \\ v_{c,0} - v_{c,0}^R \end{bmatrix} = 10 * \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

- The solution of the local tracking problem for V_a is given by

$$X_{aa} = \begin{bmatrix} \sqrt{3}I & I \\ I & \sqrt{3}I \end{bmatrix}, \; F_{aa} = \begin{bmatrix} -I & -\sqrt{3}I \end{bmatrix}.$$

- We restrict attention to the leader-follower system involving V_a and V_b, and use Procedure 6.2.8 to find F_{bb} and F_{ab}; Since the initial state for this problem is zero, the algorithm terminates at the initial value $F_{bb} = F_{bb}^{ind}$, and hence

$$\begin{bmatrix} F_{ab} \\ F_{bb} \end{bmatrix} \approx \begin{bmatrix} -0.5 & 0 & 0 & -1.15 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & -1.15 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & -1.15 \\ -1 & 0 & 0 & -1.73 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1.73 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1.73 \end{bmatrix}.$$

- Finally, we restrict attention to the leader-follower system involving V_b and V_c, and again use Procedure 6.2.8 to solve for F_{cc} and F_{bc}, given F_{bb} from the previous step. In this case, the initial state is not zero, and numerical optimization gives

$$\begin{bmatrix} F_{bc} \\ F_{cc} \end{bmatrix} \approx \begin{bmatrix} 1.39 & -1.65 & -0.16 & -5.32 & 1.55 & 2.66 \\ 1.87 & -2.22 & -0.07 & -3.30 & -0.20 & 2.40 \\ 2.54 & -1.93 & -1.04 & -4.74 & 2.06 & 1.58 \\ -0.80 & -4.34 & 4.35 & -2.82 & -4.72 & 5.85 \\ 4.75 & -8.38 & 2.86 & 6.93 & -11.38 & 2.75 \\ -3.10 & -1.70 & 4.00 & -10.43 & 1.48 & 7.29 \end{bmatrix}.$$

The overall cost corresponding to the control law found above is

$$J(x_0, Fx(\cdot)) = 2730.82.$$

Comparison of formations From this example, we derive the following conclusions:

- Even though all vehicles have the same dynamics and the same local cost functions, the optimal feedback for each vehicle differs, depending on the number and formation of its followers.

- A disadvantage of Formation 2, compared to Formation 1, is that the nonzero initial state of V_c had no effect on the control problem for V_b. This may also be the reason for the higher costs: While the costs corresponding to Formation 1 are very close to the centralized optimum, the costs corresponding to Formation 2 are significantly higher. The total amounts of communication needed, and hence also possible communication costs, are the same for both formations.

- Formation 1 is more robust with respect to noise, communication delays, package drops, etc.: In contrast to Formation 2, possible delays and disturbances in the communication between the leading vehicle and one follower will not propagate to the other follower.

- Formation 2 is more scalable with respect to the number of vehicles in the formation: If a large number of vehicles are following the coordinating vehicle V_c in Formation 1 then the problem of finding the optimal coordinator feedback, to be solved by V_c, will get computationally more involved. A large number of vehicles arranged in a chain formation will lead to more control problems to be solved numerically (one for each vehicle), but the size of each problem remains the same.

7 LQ Control with Event-based Feedback

As an alternative to the control synthesis procedure developed in the previous chapter, we now introduce and analyze a control law which uses event-based bottom-to-top feedback.

This chapter was published as [26].

7.1 Introduction

In this chapter, we derive a control law for coordinated linear systems which combines linear state feedbacks with event-based feedbacks. While the linear feedbacks respect the system's top-to-bottom information structure, the event-based feedbacks correspond to occasional bottom-to-top communication. In Section 7.2 the control problem we consider is formulated for the special case involving only one subsystem. Our control law with event-based feedback is introduced in Section 7.3, and its extension to larger hierarchical systems is discussed in Section 7.4. Finally, the performance of the control law is illustrated in an example in Section 7.5.

7.2 Problem formulation

We will first restrict our attention to leader-follower systems, i.e. coordinated linear systems with only one subsystem, of the form

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sc} \\ 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix} + \begin{bmatrix} B_{ss} & B_{sc} \\ 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_s \\ u_c \end{bmatrix}, \quad \begin{bmatrix} x_s(t_0) \\ x_c(t_0) \end{bmatrix} = \begin{bmatrix} x_{s,0} \\ x_{c,0} \end{bmatrix}. \tag{7.1}$$

The subscript c stands for 'coordinator', and s stands for 'subsystem'. The extension of our results to larger hierarchical systems is discussed in Section 7.4.

For a system of the form (7.1), we define the following infinite-horizon quadratic cost function:

$$J\left(x_0, u(\cdot)\right) = \int_{t_0}^{\infty} \left(\begin{bmatrix} x_s \\ x_c \end{bmatrix}^T \begin{bmatrix} Q_{ss} & 0 \\ 0 & Q_{cc} \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix} + \begin{bmatrix} u_s \\ u_c \end{bmatrix}^T \begin{bmatrix} R_{ss} & 0 \\ 0 & R_{cc} \end{bmatrix} \begin{bmatrix} u_s \\ u_c \end{bmatrix} \right) dt, \tag{7.2}$$

where $Q = \begin{bmatrix} Q_{ss} & 0 \\ 0 & Q_{cc} \end{bmatrix} \geq 0$ and $R = \begin{bmatrix} R_{ss} & 0 \\ 0 & R_{cc} \end{bmatrix} > 0$.

We consider the following sets of admissible control laws:

$$\mathfrak{U}_{\uparrow\downarrow} = \left\{ \begin{bmatrix} u_s(t, x_s(t), x_c(t)) \\ u_c(t, x_s(t), x_c(t)) \end{bmatrix} \middle| \, u_s, u_c \text{ piecewise continuous in } t \right\},$$
$$\mathfrak{U}_{\downarrow} = \left\{ \begin{bmatrix} u_s(t, x_s(t), x_c(t)) \\ u_c(t, x_{s,0}, x_c(t)) \end{bmatrix} \middle| \, u_s, u_c \text{ piecewise continuous in } t \right\},$$
$$\mathfrak{U}_{r,\beta} = \left\{ \begin{bmatrix} u_s(t, x_s(t), x_c(t)) \\ u_c(t, x_{s,j}, x_c(t)) \end{bmatrix} \middle| \, \begin{matrix} u_s, u_c \text{ piecewise continuous in } t, \\ x_{s,j} = x_s(t_j),\ t_j \le t,\ \{t_0, t_1, \ldots, t_n\} \subset [t_0, \infty) \end{matrix} \right\}.$$

The subscripts r and β are used here to comply with our notation in Section 7.3, where we introduce a control law with event-based feedback $\{x_{s,j}\}_{j=0,\ldots,n}$, using a guard condition which depends on two parameters r and β. Note that $\mathfrak{U}_{\downarrow} \subseteq \mathfrak{U}_{r,\beta} \subseteq \mathfrak{U}_{\uparrow\downarrow}$.

In this chapter, we will consider the following problem:

7.2.1. Problem. Let a system of the form (7.1) and a cost function of the form (7.2) be given. Assume that

$$\left(\begin{bmatrix} A_{ss} & A_{sc} \\ 0 & A_{cc} \end{bmatrix}, \begin{bmatrix} B_{ss} & B_{sc} \\ 0 & B_{cc} \end{bmatrix} \right)$$

is a stabilizable pair and that

$$\left(\begin{bmatrix} Q_{ss} & 0 \\ 0 & Q_{cc} \end{bmatrix}, \begin{bmatrix} A_{ss} & A_{sc} \\ 0 & A_{cc} \end{bmatrix} \right)$$

is a detectable pair. Minimize the cost over $\mathfrak{U}_{r,\beta}$, i.e. find

$$\min_{u(\cdot)\in\mathfrak{U}_{r,\beta}} J\left(x_0, u(\cdot)\right). \tag{7.3}$$

In Section 7.3 we introduce a piecewise-linear control law in $\mathfrak{U}_{r,\beta}$ which leads to a finite cost for all $r \ge 0$, and approximates the centralized optimum (and hence also the solution of (7.3)) for $r \to 0$.

7.3 Control with event-based feedback

For the purpose of the following derivation, we assume that $x_c(t) \neq 0$ for all $t \ge t_0$. Set

$$T(t) = \frac{x_s(t)x_c(t)^T}{x_c(t)^T x_c(t)},$$

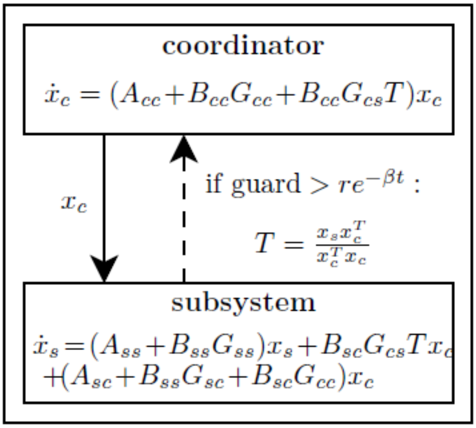

Figure 7.1: Closed-loop system using event-based feedback

then $x_s(t) = T(t)x_c(t)$, and applying the centralized feedback G found in Section 2.4 to the system in (7.1) leads to the closed-loop system

$$\begin{aligned}\begin{bmatrix}\dot{x}_s\\ \dot{x}_c\end{bmatrix} &= \left(\begin{bmatrix}A_{ss} & A_{sc}\\ 0 & A_{cc}\end{bmatrix} + \begin{bmatrix}B_{ss} & B_{sc}\\ 0 & B_{cc}\end{bmatrix}\begin{bmatrix}G_{ss} & G_{sc}\\ G_{cs} & G_{cc}\end{bmatrix}\right)\begin{bmatrix}x_s\\ x_c\end{bmatrix}\\ &= \begin{bmatrix}A_{ss}+B_{ss}G_{ss}+B_{sc}G_{cs} & A_{sc}+B_{ss}G_{sc}+B_{sc}G_{cc}\\ B_{cc}G_{cs} & A_{cc}+B_{cc}G_{cc}\end{bmatrix}\begin{bmatrix}x_s\\ x_c\end{bmatrix}\\ &= \begin{bmatrix}A_{ss}+B_{ss}G_{ss} & A_{sc}+B_{ss}G_{sc}+B_{sc}(G_{cs}T(t)+G_{cc})\\ 0 & A_{cc}+B_{cc}\left(G_{cs}T(t)+G_{cc}\right)\end{bmatrix}\begin{bmatrix}x_s\\ x_c\end{bmatrix}.\end{aligned}$$

Note that $T(t)$ still depends on $x_s(t)$. We will approximate $T(t)$ by a piecewise-constant function of t.

7.3.1 Piecewise-constant approximation of $T(t)$

Instead of sending the current subsystem state to the coordinator at all times (as in the centralized case), we define the guard condition

$$\left\|G_{cs}\left(x_s(t) - T_{j(t^-)}x_c(t)\right)\right\| \leq re^{-\beta t}, \tag{7.4}$$

with real parameters $r \geq 0$ and $\beta > 0$, and with a piecewise-constant approximation $T_{j(t)}$ of $T(t)$. We let the subsystem send its current state to the coordinator at all time points at which the guard condition is violated. At those time points, $x_s(t)$ is used to reset our approximation $T_{j(t)}$ of $T(t)$ to its current value. Note that the subsystem observes both x_s and x_c, and hence it has all the information necessary to check the guard condition at each step. Whenever the guard condition is satisfied, the coordinator uses the most recent approximation of $T(t)$, which was computed the last time the subsystem sent its state to the coordinator (or at t_0). The control law resulting from this approximation is of the form

$$g_{r,\beta}(x(t)) = \begin{bmatrix} G_{ss}x_s(t) + G_{sc}x_c(t) \\ \left(G_{cs}T_{j(t)} + G_{cc}\right) x_c(t) \end{bmatrix},$$

where T_j is given by

$$T_{j(t^+)} = \begin{cases} T_{j(t^-)} & \text{if guard} \le re^{-\beta t} \\ \frac{x_s(t)x_c^T(t)}{x_c^T(t)x_c(t)} & \text{if guard} > re^{-\beta t} \end{cases},$$

and the guard is given by

$$\text{guard} = \left\| G_{cs}\left(x_s(t) - T_{j(t^-)}x_c(t)\right)\right\|.$$

This control law is illustrated in Figure 7.1.

7.3.2 Results on the performance of $g_{r,\beta}$

Our first result on the performance of the control law $g_{r,\beta}$ states that it leads to an exponentially stable system.

7.3.1. Proposition. *For any $r \ge 0$ and $\beta > 0$, applying the control law $g_{r,\beta}(x(\cdot))$ to (7.1) leads to an exponentially stable closed-loop system.*

Proof. Applying $g_r(x(\cdot))$ leads to the closed-loop system

$$\begin{aligned} \dot{x} &= \left(A + BG + \begin{bmatrix} B_{sc} \\ B_{cc} \end{bmatrix} G_{cs}\left[-I \;\; T_j\right]\right) x \\ &= (A + BG)x + \begin{bmatrix} B_{sc} \\ B_{cc} \end{bmatrix} G_{cs}\left(T_j x_c - x_s\right) \end{aligned}$$

and hence the state trajectory is described by

$$x(t) = e^{(A+BG)(t-t_0)}x_0 + \int_{t_0}^{t} e^{(A+BG)(t-\tau)} \begin{bmatrix} B_{sc} \\ B_{cc} \end{bmatrix} G_{cs}\left(T_{j(\tau)}x_c(\tau) - x_s(\tau)\right) d\tau.$$

Since $A + BG$ is stable, there exist constants $M > 0$ and $\alpha > 0$ such that

$$\left\| e^{(A+BG)\Delta_t} \right\| \le Me^{-\alpha\Delta_t} \quad \forall \Delta_t \ge 0.$$

We pick $\alpha \neq \beta$. Now we have

$$\begin{aligned}\|x(t)\| &\le \left\|e^{(A+BG)(t-t_0)}\right\| \|x_0\| \\ &\quad + \int_{t_0}^{t} \left\|e^{(A+BG)(t-\tau)}\right\| \left\|\begin{bmatrix} B_{sc} \\ B_{cc} \end{bmatrix}\right\| \left\|G_{cs}\left(T_{j(\tau)}x_c(\tau) - x_s(\tau)\right)\right\| d\tau \\ &\le Me^{-\alpha(t-t_0)}\|x_0\| + \int_{t_0}^{t} Me^{-\alpha(t-\tau)} \left\|\begin{bmatrix} B_{sc} \\ B_{cc} \end{bmatrix}\right\| re^{-\beta\tau} d\tau \\ &= Me^{-\alpha(t-t_0)}\|x_0\| + M\frac{\left\|\begin{bmatrix} B_{sc} \\ B_{cc} \end{bmatrix}\right\| r}{\alpha - \beta}\left(e^{-\beta t} - e^{-\alpha(t-t_0)-\beta t_0}\right),\end{aligned}$$

which goes to 0 exponentially as $t \to \infty$. □

The following result states that the increase in the total cost, resulting from using a piecewise-constant approximation for $T(t)$ instead of its exact value, is bounded.

7.3.2. Proposition. *For any $r > 0$ and $\beta > 0$, the difference between the cost corresponding to $g_{r,\beta}$ and the centralized cost is bounded by*

$$J(x_0, g_{r,\beta}(x(\cdot))) - J(x_0, Gx(\cdot)) \le \left\|R^{\frac{1}{2}}\right\|^2 r^2 \frac{e^{-2\beta t_0}}{2\beta}. \tag{7.5}$$

Proof. For simplicity, we denote the difference of the control feedbacks by

$$\tilde{G}_j = \begin{bmatrix} 0 & 0 \\ -G_{cs} & G_{cs}T_j \end{bmatrix}.$$

Let $t \ge t_j$, and suppose that the most recent feedback from the subsystem was sent at time t_j. Then the system dynamics over $[t_j, t]$ are linear and time-invariant, and hence the cost corresponding to the control law $g_{r,\beta}$ over the interval $[t_j, t]$ is given by $x^T(t_j)Y_jx(t_j) - x^T(t)Y_jx(t)$, where Y_j is the solution of the Lyapunov equation

$$\left(A + B\left(G + \tilde{G}_j\right)\right)^T Y_j + Y_j\left(A + B\left(G + \tilde{G}_j\right)\right) + Q + \left(G + \tilde{G}_j\right)^T R\left(G + \tilde{G}_j\right) = 0.$$

The cost corresponding to the centralized control law $u(\cdot) = Gx(\cdot)$ was derived in Section 2.4: We have $J(x_0, Gx(\cdot)) = x_0^T X x_0$, where X is the unique solution of $XBR^{-1}B^TX - A^TX - XA - Q = 0$ such that $G = -R^{-1}B^TX$ is stabilizing (i.e.

$A - BR^{-1}B^TX$ is stable). Using $B^TX = -RG$, and in analogy with the proof of Theorem 2.4.1, we now derive a Lyapunov equation for $Y_j - X$:

$$\begin{aligned}&\left(A+B\left(G+\tilde{G}_j\right)\right)^T(Y_j-X)+(Y_j-X)\left(A+B\left(G+\tilde{G}_j\right)\right)\\&=-Q-\left(G+\tilde{G}_j\right)^TR\left(G+\tilde{G}_j\right)-(A+BG)^TX-X(A+BG)-\tilde{G}_j^TB^TX-XB\tilde{G}_j\\&=-\left(G+\tilde{G}_j\right)^TR\left(G+\tilde{G}_j\right)+G^TRG+\tilde{G}_j^TRG+G^TR\tilde{G}_j\\&=-\tilde{G}_j^TR\tilde{G}_j.\end{aligned}$$

Using this, we can now derive an expression for the difference in cost over $[t_j, t_{j+1}]$:

$$\begin{aligned}&\left(x^T(t_j)Y_jx(t_j)-x^T(t_{j+1})Y_jx(t_{j+1})\right)-\left(x^T(t_j)Xx(t_j)-x^T(t_{j+1})Xx(t_{j+1})\right)\\&=x^T(t_j)(Y_j-X)x(t_j)-x^T(t_{j+1})(Y_j-X)x(t_{j+1})\\&=\int_{t_j}^{t_{j+1}}-\frac{d}{dt}\left(x^T(t)(Y_j-X)x(t)\right)dt\\&=\int_{t_j}^{t_{j+1}}-\left(x^T(t)\left(\left(A+B\left(G+\tilde{G}_j\right)\right)(Y-X)+(Y-X)\left(A+B\left(G+\tilde{G}_j\right)\right)\right)x(t)\right)dt\\&=\int_{t_j}^{t_{j+1}}\left(x^T(t)\tilde{G}_j^TR\tilde{G}_jx(t)\right)dt\\&=\int_{t_j}^{t_{j+1}}\left\|R^{1/2}\tilde{G}_jx(t)\right\|^2dt,\end{aligned}$$

where $x(\cdot)$ is the state trajectory of the closed-loop system obtained from applying the control law $g_{r,\beta} = G + \tilde{G}_j$. Since $x(\cdot)$ is exponentially stable by Proposition 7.3.1, the last derivation also holds over $[t_{n+1}, \infty]$. Now

$$\begin{aligned}&J(x_0, g_{r,\beta}(x(\cdot)))-J(x_0, Gx(\cdot))\\&=\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left\|R^{\frac{1}{2}}G_{cs}\begin{bmatrix}-I & T_j\end{bmatrix}x(t)\right\|^2dt+\int_{t_{n+1}}^{\infty}\left\|R^{\frac{1}{2}}G_{cs}\begin{bmatrix}-I & T_{n+1}\end{bmatrix}x(t)\right\|^2dt\\&\le\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left\|R^{\frac{1}{2}}\right\|^2r^2e^{-2\beta t}dt+\int_{t_{n+1}}^{\infty}\left\|R^{\frac{1}{2}}\right\|^2r^2e^{-2\beta t}dt\\&=\left\|R^{\frac{1}{2}}\right\|^2r^2\frac{e^{-2\beta t_0}}{2\beta}.\end{aligned}$$

□

Note that Proposition 7.3.2 also implies that

$$\lim_{r \to 0} J(x_0, g_{r,\beta}(x(\cdot))) = J(x_0, Gx(\cdot)),$$

i.e. for any $\beta > 0$ the cost corresponding to $g_{r,\beta}$ approaches the optimal centralized cost for $r \to 0$.

For $r \to \infty$, we have $\mathfrak{U}_{r,\beta} \to \mathfrak{U}_{\downarrow}$, but $g_{\infty,\beta}(x(\cdot))$ is not the optimal control law over $\mathfrak{U}_{\downarrow}$ (in fact, it is not even the optimal *linear* control law over $\mathfrak{U}_{\downarrow}$, see Chapter 6). Hence, while $g_{r,\beta}$ leads to a good closed-loop performance for small r, there are better options for large r.

The following proposition verifies a property one would intuitively require of a coordination control law: If the different parts of a system are fully decoupled (i.e. the coordinator state and input do not influence the subsystem state) then the control law $g_{r,\beta}$ reduces to the optimal control law of the centralized case.

7.3.3. Proposition. *If $A_{sc} = 0$ and $B_{sc} = 0$ then $g_{r,\beta}(x(t)) = Gx(t)$ for all $t \geq t_0$, $r \geq 0$ and $\beta \geq 0$.*

Proof. For $A_{sc} = 0$ and $B_{sc} = 0$, the open-loop system is completely decoupled. Since Q and R are also chosen block-diagonal in (7.2), the solution of the Ricatti equation (2.10) is block-diagonal, and hence so is $G = -R^{-1}B^T X$. But then $G_{cs} = 0$, which gives $\left\| G_{cs}(x_s(t) - T_{j(t^-)}x_c(t)) \right\| = 0$, and hence

$$u_c(t, x_{s,j}, x_c(t)) = G_{cc}x_c(t) = u_c(t, x_{s,0}, x_c(t))$$

for all r and β. □

The following result concerning the time span between two subsequent resets of T_j has been observed for many randomly generated control problems in simulations, but has not yet been proven:

7.3.4. Conjecture. *Let a system of the form (7.1) and a cost function of the form (7.2) be given, and let*

$$\alpha = \min \{ |\text{Re}(\lambda)| | \lambda \in \sigma(A + BG) \}.$$

(a) For all $r > 0$ and $\beta \in (0, \alpha)$ there exists $\epsilon > 0$ such that $\|t_{j+1} - t_j\| > \epsilon$ for all $j \geq 0$.

(b) For all $r > 0$ there exist $\beta_{\max} \in (0, \alpha)$ and $t_N > t_0$ such that the guard condition is satisfied for all $\beta \in (0, \beta_{\max})$ and $t > t_N$.

Part (a) excludes infinite resets in finite time, and part (b) states that for β small enough, there are only finitely many resets. Assuming that Conjecture 7.3.4 holds, we have that $g_{r,\beta} \in \mathfrak{U}_{r,\beta}$. While part (a) is crucial in establishing that $g_{r,\beta}$ is useful in practice, part (b) can easily be circumvented by redefining the set $\mathfrak{U}_{r,\beta}$ to allow for countably many resets.[1]

7.4 Extension to hierarchical systems

This section explains how to extend the results of Section 7.3 to more general hierarchical systems with a top-to-bottom information structure. The cases of a system with two subsystems and a system with three layers are discussed below. Control laws for other hierarchical systems can then be derived by combining these cases.

7.4.1 Systems with several subsystems

In the following, we illustrate how to extend the control law $g_{r,\beta}$ introduced in Section 7.3 to a coordinated linear system with two subsystems and a coordinator.

We will consider the following system and cost function:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_c\end{bmatrix} = \begin{bmatrix}A_{11} & 0 & A_{1c}\\ 0 & A_{22} & A_{2c}\\ 0 & 0 & A_{cc}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_c\end{bmatrix} + \begin{bmatrix}B_{11} & 0 & B_{1c}\\ 0 & B_{22} & B_{2c}\\ 0 & 0 & B_{cc}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_c\end{bmatrix}, \quad \begin{bmatrix}x_1(t_0)\\ x_2(t_0)\\ x_c(t_0)\end{bmatrix} = \begin{bmatrix}x_{1,0}\\ x_{2,0}\\ x_{c,0}\end{bmatrix},$$

$$J(x_0, u(\cdot)) = \int_{t_0}^{\infty}\left(\begin{bmatrix}x_1\\ x_2\\ x_c\end{bmatrix}^T\begin{bmatrix}Q_{11} & 0 & 0\\ 0 & Q_{22} & 0\\ 0 & 0 & Q_{cc}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_c\end{bmatrix} + \begin{bmatrix}u_1\\ u_2\\ u_c\end{bmatrix}^T\begin{bmatrix}R_{11} & 0 & 0\\ 0 & R_{22} & 0\\ 0 & 0 & R_{cc}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_c\end{bmatrix}\right)dt.$$

The optimal centralized control law $Gx(\cdot) \in \mathfrak{U}_{\uparrow\downarrow}$ is again linear and time-invariant, with G an unstructured matrix. We again approximate $Gx(\cdot)$ by a structure-preserving piecewise-linear feedback, i.e. a feedback of the form $\begin{bmatrix}\star & 0 & \star\\ 0 & \star & \star\\ 0 & 0 & \star\end{bmatrix} x(\cdot)$, by using piecewise-constant approximations of the matrices

$$T_i(t) = \frac{x_i(t)x_c^T(t)}{x_c^T(t)x_c(t)}, \text{ for } i = 1, 2.$$

[1] In fact, if part (a) is false then applying $g_{r,\beta}$ may lead to Zeno behavior in theory – in practice the system will be unable to execute the control law, and will lose stability. Since no application is designed to run for infinite time, part (b) is irrelevant in practice; it may however be useful for bounding the long-run communication costs.

This control law is of the form

$$g_{r,\beta}(x(t)) = \begin{bmatrix} G_{11}x_1(t) + \left(G_{12}T_{2,j_2(t)} + G_{1c}\right)x_c(t) \\ G_{22}x_2(t) + \left(G_{21}T_{1,j_1(t)} + G_{2c}\right)x_c(t) \\ \left(G_{c1}T_{1,j_1(t)} + G_{c2}T_{2,j_2(t)} + G_{cc}\right)x_c(t) \end{bmatrix},$$

where T_{i,j_i} is the most recent feedback sent by subsystem i, for $i = 1, 2$:

$$T_{i,j_i(t^+)} = \begin{cases} T_{i,j_i(t^-)} & \text{if guard}_i \leq re^{-\beta t} \\ \frac{x_i(t)x_c^T(t)}{x_c^T(t)x_c(t)} & \text{if guard}_i > re^{-\beta t} \end{cases},$$

with

$$\text{guard}_1 = \left\| \begin{bmatrix} G_{21} \\ G_{c1} \end{bmatrix} \left(x_1(t) - T_{1,j_1(t^-)}x_c(t)\right) \right\|,$$

$$\text{guard}_2 = \left\| \begin{bmatrix} G_{12} \\ G_{c2} \end{bmatrix} \left(x_2(t) - T_{2,j_2(t^-)}x_c(t)\right) \right\|.$$

Note that, in addition to regularly sending its own state x_c, the coordinator also needs to send T_{1,j_1} to subsystem 2 and T_{2,j_2} to subsystem 1, whenever they are updated. The indices j_1 and j_2 are used to distinguish between the time points at which subsystem 1 and subsystem 2 send feedback to the coordinator, respectively. The closed-loop system corresponding to the control law $g_{r,\beta}$ is illustrated in Figure 7.2.

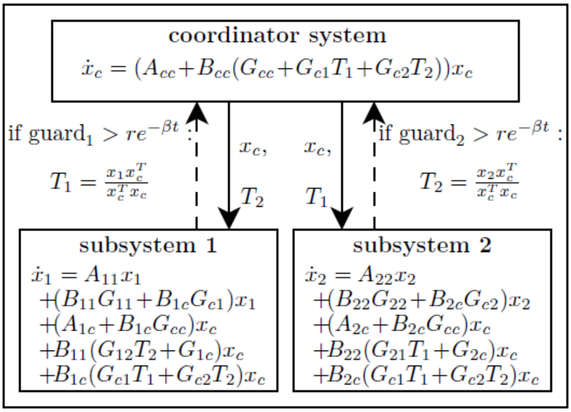

Figure 7.2: Closed-loop system with two subsystems

Proposition 7.3.1 can easily be extended to show that this closed-loop system is exponentially stable. Let $x(t)$ describe the state trajectory obtained by applying $g_{r,\beta}$, then we can bound $\|Gx(t) - g_{r,\beta}(x(t))\|^2$ in a similar manner as in Proposition 7.3.2:

$$\begin{aligned}
\|Gx(t) - g_{r,\beta}(x(t))\|^2 &= \left\| \begin{bmatrix} 0 & -G_{12} & G_{12}T_2 \\ -G_{21} & 0 & G_{21}T_1 \\ -G_{c1} & -G_{c2} & G_{c1}T_1 + G_{c2}T_2 \end{bmatrix} x(t) \right\|^2 \\
&= \left\| \begin{bmatrix} 0 \\ G_{21} \\ G_{c1} \end{bmatrix} \begin{bmatrix} -I & 0 & T_1 \end{bmatrix} x(t) + \begin{bmatrix} G_{12} \\ 0 \\ G_{c2} \end{bmatrix} \begin{bmatrix} 0 & -I & T_2 \end{bmatrix} x(t) \right\|^2 \\
&\leq 2 \left\| \begin{bmatrix} G_{21} \\ G_{c1} \end{bmatrix} (x_1(t) - T_1 x_c(t)) \right\|^2 + 2 \left\| \begin{bmatrix} G_{12} \\ G_{c2} \end{bmatrix} (x_2(t) - T_2 x_c(t)) \right\|^2 \\
&\leq 4r^2 e^{-2\beta t}.
\end{aligned}$$

Using this bound and the same argument as in the proof of Proposition 7.3.2, we find that

$$J(x_0, g_{r,\beta}(x(\cdot))) - J(x_0, Gx(\cdot)) \leq 4 \left\| R^{\frac{1}{2}} \right\|^2 r^2 \frac{e^{-2\beta t_0}}{2\beta}.$$

7.4.2 Systems with several layers

We consider the following system and cost function:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

$$\begin{aligned}
&J(x_0, u(\cdot)) \\
&= \int_{t_0}^{\infty} \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^T \begin{bmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) dt,
\end{aligned}$$

with initial state $\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{bmatrix} = \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ x_{3,0} \end{bmatrix}$. Each subsystem now corresponds to one layer in the hierarchy. We define the functions

$$T_{12} = \frac{x_1 x_2^T}{x_2^T x_2}, \quad T_{13} = \frac{x_1 x_3^T}{x_3^T x_3}, \quad T_{23} = \frac{x_2 x_3^T}{x_3^T x_3},$$

which satisfy $T_{12}x_2 = x_1$, $T_{13}x_3 = x_1$ and $T_{23}x_3 = x_2$. We again find piecewise-constant approximations to these functions by resetting $T_{ik,j_{ik}}$ whenever $\text{guard}_{ik} > re^{-\beta t}$, where

$$\begin{aligned}\text{guard}_{12} &= \left\|G_{21}\left(x_1(t) - T_{12,j_{12}(t^-)}x_2(t)\right)\right\|,\\ \text{guard}_{13} &= \left\|G_{31}\left(x_1(t) - T_{13,j_{13}(t^-)}x_3(t)\right)\right\|,\\ \text{guard}_{23} &= \left\|G_{32}\left(x_2(t) - T_{23,j_{23}(t^-)}x_3(t)\right)\right\|.\end{aligned}$$

The resulting control law is given by

$$g_{r,\beta}(x(t)) = \begin{bmatrix} G_{11}x_1(t) + G_{12}x_2(t) + G_{13}x_3(t) \\ (G_{22} + G_{21}T_{12,j_{12}})x_2(t) + G_{23}x_3(t) \\ (G_{31}T_{13,j_{13}} + G_{32}T_{23,j_{23}} + G_{33})x_3(t) \end{bmatrix},$$

and illustrated in Figure 7.3.

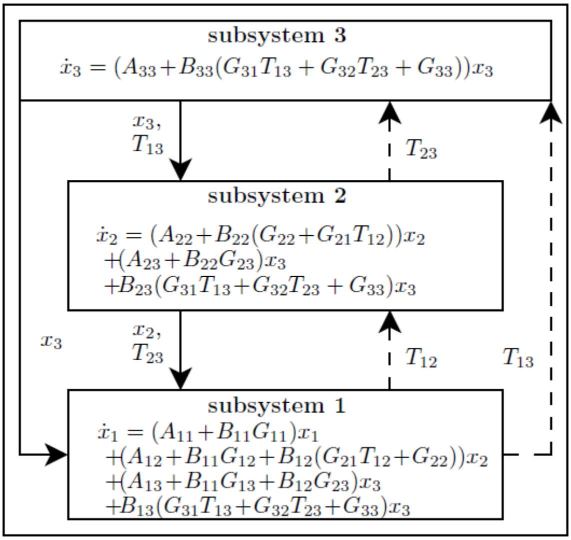

Figure 7.3: Closed-loop system with three layers

Exponential stability of the closed-loop system follows from a slight modification of Proposition 7.3.1. The difference between the corresponding cost and the centralized cost is bounded by

$$J(x_0, g_{r,\beta}(x(\cdot))) - J(x_0, Gx(\cdot)) \leq 5\left\|R^{\frac{1}{2}}\right\|^2 r^2 \frac{e^{-2\beta t_0}}{2\beta},$$

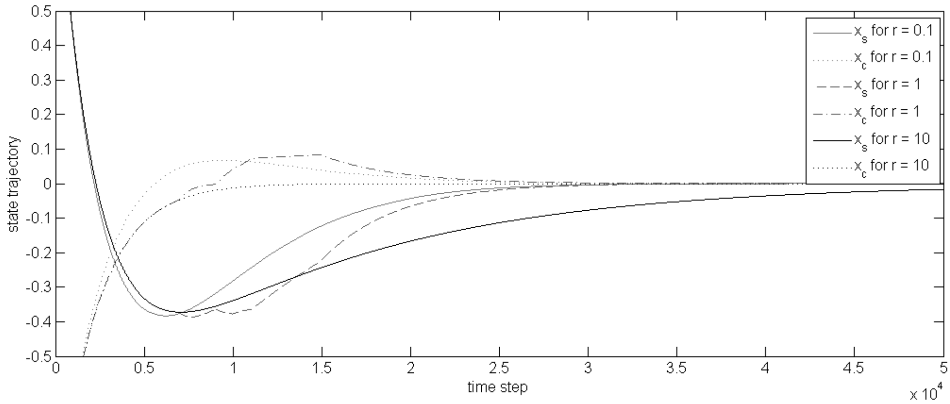

Figure 7.4: Closed-loop state trajectories for different values of r

which follows from the proof of Proposition 7.3.2, and from

$$\begin{aligned}
\|Gx(t) - g_{r,\beta}(x(t))\|^2 &= \left\| \begin{bmatrix} 0 & 0 & 0 \\ -G_{21} & G_{21}T_{12} & 0 \\ -G_{31} & -G_{32} & G_{31}T_{13} + G_{32}T_{23} \end{bmatrix} x(t) \right\|^2 \\
&= \left\| G_{21} \begin{bmatrix} -I & T_{12} & 0 \end{bmatrix} x(t) \right\|^2 + \left\| G_{31} \begin{bmatrix} -I & 0 & T_{13} \end{bmatrix} x(t) + G_{32} \begin{bmatrix} 0 & -I & -T_{23} \end{bmatrix} x(t) \right\|^2 \\
&\leq \left\| G_{21} \begin{bmatrix} -I & T_{12} & 0 \end{bmatrix} x(t) \right\|^2 + 2 \left\| G_{31} \begin{bmatrix} -I & 0 & T_{13} \end{bmatrix} x(t) \right\|^2 + 2 \left\| G_{32} \begin{bmatrix} 0 & -I & -T_{23} \end{bmatrix} x(t) \right\|^2 \\
&\leq 5r^2 e^{-2\beta t}.
\end{aligned}$$

7.5 Simulation results

In order to illustrate the performance of the control law $g_{r,\beta}$, we simulate the behavior of the corresponding closed-loop system for the following simple example:

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_s \\ u_c \end{bmatrix},$$

$$J(x_0, u) = \int_{t_0}^{\infty} \left(\begin{bmatrix} x_s \\ x_c \end{bmatrix}^T \begin{bmatrix} x_s \\ x_c \end{bmatrix} + \begin{bmatrix} u_s \\ u_c \end{bmatrix}^T \begin{bmatrix} u_s \\ u_c \end{bmatrix} \right) dt,$$

with initial state $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. For the simulation we use Matlab, with $5 * 10^4$ time steps of size 10^{-4} each (choosing smaller time steps had no influence on the cost, up to 5 significant digits).

For $\beta = 10^{-4}$, the results for different r are given in the table below. The column 'resets' gives the number of time points at which T_j was reset, upon receiving feedback from the subsystem.

r	β	$J(x_0, g_{r,\beta})$	resets
0	10^{-4}	7.3674	$5 * 10^4$
0.01	10^{-4}	7.3674	798
0.1	10^{-4}	7.3690	116
1	10^{-4}	7.5441	17
10	10^{-4}	7.6883	0

The first row with $r = 0$ corresponds to the centralized case, with feedback from the subsystem at each time step. The second and third row show that we can achieve the same cost (up to 5 significant digits) with 798 resets, and an only slightly higher cost with 116 resets. The cost increases with increasing r, with an upper bound of 7.6883, which is achieved if T_j is not reset after t_0.

The corresponding state trajectories are shown in Figure 7.4. The state trajectories for $r = 0.01$ and $u = Gx$ are very similar to the case $r = 0.1$. Changing β leads to a comparable cost/resets ratio. For smaller values of β, most resets occur earlier than for larger β. If $\beta \in (0, \alpha)$ is chosen too large then it is not apparent from our simulations whether the total number of resets will be finite.

Concluding remarks

A coordination control law was introduced, in which the bottom-to-top communication necessary for implementing the optimal centralized control law was replaced by event-based feedback, with a guard condition on the corresponding approximation error. Further research should focus on the proof of Conjecture 7.3.4, and a direct relation between the total cost and the number of resets.

8 Case Studies

In this chapter, two case studies for coordination control are described: Section 8.1 deals with a formation flying problem for autonomous underwater vehicles, and Section 8.2 discusses a coordination problem for two ramp metering devices at neighboring highway on-ramps.

8.1 Formation flying for AUVs

This section describes a coordination control approach to formation flying for AUVs, and was published as [24].

8.1.1 Introduction

This section describes a case study for coordination control, involving several autonomous underwater vehicles (AUVs): One AUV or surface vehicle should track an external reference signal, and two AUVs should follow the first vehicle in formation. This case study is strongly related to the problem statement of formation flying for AUVs formulated in [59]. The similar problem of coordinated path following control for AUVs is discussed in e.g. [11], and other approaches to formation flying using leader-follower structures are found in e.g. [7], [32].

The purpose of this case study is, on the one hand, to illustrate the theory of coordination control developed in this thesis, and on the other hand, to provide a computationally efficient control algorithm for the problem of formation flying for AUVs.

The control problem considered in this paper consists of three tracking problems, coupled by the formation to be kept, and subject to fixed bounds on the speed and acceleration of each vehicle, random waves and currents, and errors and delays in the communication among the vehicles.

Our approach adopts the linearized version of the model from [59]. In [59], a more general version of this problem is formulated, and solved using moving-horizon model predictive control on a linearized version of the model. While this approach leads to very good control laws, the on-line computations necessary for implementing these control laws exceed the on-board processing power of the AUVs considered in this setting.

The novelty of our approach lies in restricting the communication among the AUVs to a minimum by imposing a hierarchical structure on the set of vehicles, and then using LQ optimal control to solve each tracking problem separately. The navigation and communication constraints are taken into account after finding the optimal control laws. This leads to a control law which can be implemented with very little computational effort.

In a simulation study, we compare the performance of our control law to the centralized case, in which the communication among the vehicles is not restricted. Our approach leads to a slightly higher total cost for the overall tracking problem, while decreasing the total amount of information to be communicated considerably. Moreover, our approach is easily extendable to larger groups of AUVs because the total amount of information communicated among the vehicles increases linearly with the number of vehicles, while this increase is exponential in the centralized approach.

8.1.2 Description of the setting

The setting considered here concerns three vehicles, two of which are AUVs, and one may be either an AUV or a surface vehicle. The main goal is to have one vehicle track an externally given reference signal, while the other two vehicles (the AUVs) follow this vehicle in a fixed formation. This setting is illustrated in Figure 8.1.

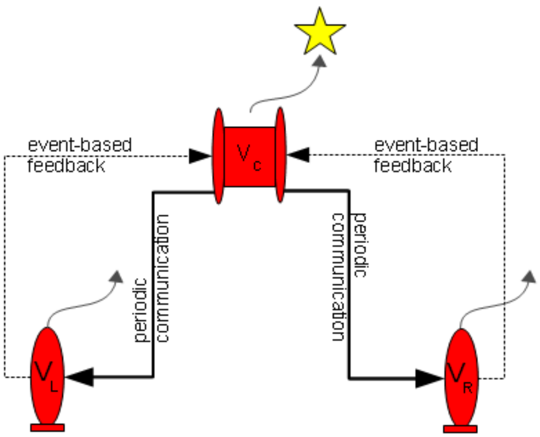

Figure 8.1: Setting

The external reference signal may belong to a fourth vehicle, or be the solution of another control problem, e.g. a search mission. In the setting considered here, the vehicle following this signal can observe the current reference position at all times.

The vehicle following the external reference signal will be called the coordinating vehicle (V_C). V_C regularly sends its position to the other two vehicles (V_1 and V_2). These vehicles use this information to follow V_C - this is modeled as a tracking problem for each follower vehicle, with as reference signal the trajectory of V_C, shifted in space by a fixed amount.

All vehicles are subject to currents and disturbances, and their velocities and accelerations are bounded in norm. Because of these restrictions, it may not always be possible for the vehicles to successfully track their reference trajectories. This leads to two possible problems: The follower vehicles V_1 and V_2 might fail to stay in formation (in the worst case, they might get lost), or the vehicles might collide. These two problems necessitate some form of communication from V_1 and V_2 to V_C in the case that the control objectives cannot be met. Since underwater communication is extremely limited, we opt for a form of event-based communication from V_1 and V_2 to V_C in exceptional circumstances: At each time step, each follower vehicle checks whether its distance to its reference position exceeds a fixed limit. This can be done internally and without additional communication since their reference position is communicated by V_C anyway. In the event that a vehicle exceeds the limit, it sends its actual position to the coordinating vehicle V_C, which then takes measures to avoid collisions, or one vehicle being left behind.

Underwater communication is modeled as being subject to random delays and packet losses. All messages sent are time-stamped, which means that at the time a message is received, the recipient knows when the message was sent. The corresponding observer can then recompute its current estimate, starting from the time given in the time stamp. The clock drift among the different vehicles is bounded for missions of limited duration, and will be ignored here.

In the setting described here, the coordinating vehicle V_C has to communicate its position regularly, while the other vehicles V_1 and V_2 do not. This means that V_C needs to use much more of its resources for communication. One possible option for ensuring that the resources of all vehicles are used in a more balanced way is to switch roles among the vehicles from time to time. In the case that V_C is a different type of vehicle than V_1 and V_2 (e.g. V_C is a ship, or an underwater vehicle with more energy available), this imbalance in the communication requirements is actually desirable.

For the purpose of comparing performances, a second setting will be considered, in which all vehicles can communicate with one another at all times. However, the communication is subject to the same delays and packet losses as described above.

8.1.3 Model with communication constraints

For ease of implementation, all dynamics involved will be approximated by discrete-time[1] linear dynamical systems, as derived in [59]. To justify this choice, we note that a linearizing feedback is commonly applied to the AUVs by a lower-level controller. The approximation errors are modeled as disturbances, together

[1]The discrete-time approximation of a coordinated linear system is obtained by replacing $\dot{x}$ with $x(t+1) - x(t)$. The sparsity structure of the system matrices, which characterizes coordinated linear systems, remains unchanged.

with possible currents and other external disturbances. All disturbances are modeled as being zero-mean disturbances in the long run.

The following notation will be used:

- V_C: coordinating vehicle
- $V_1,\ V_2$: vehicles following V_C
- R_C: external reference signal to be tracked by V_C
- $R_1,\ R_2$: reference signals to be tracked by V_1 and V_2
- $p \in \mathbb{R}^3$: position
- $s \in \mathbb{R}^3$: velocity
- $a \in \mathbb{R}^3$: acceleration
- $w \in \mathbb{R}^3$: disturbances
- $\hat{p},\ \hat{s} \in \mathbb{R}^3$: observer estimates for position and velocity
- $\Delta_1, \Delta_2 \in \mathbb{R}^3$: desired relative positions of V_1 and V_2 with respect to the position of V_C
- $\tau \in (1, \infty)$: a time constant

For each vehicle, the acceleration is the control input. The disturbances are modeled as velocities, and affect only the change in position, not the change in velocity.

These variables and their interconnections, in the case with communication constraints, are illustrated in Figure 8.2.

For the external reference system R_C, we use an internal model with the following dynamics:

$$\begin{bmatrix} p_{R_C} \\ s_{R_C} \end{bmatrix}(t+1) = \begin{bmatrix} I & I \\ 0 & \frac{\tau-1}{\tau}I \end{bmatrix} \begin{bmatrix} p_{R_C} \\ s_{R_C} \end{bmatrix}(t) + \begin{bmatrix} 0 \\ \frac{1}{\tau}I \end{bmatrix} a_{R_C}(t) + \begin{bmatrix} I \\ 0 \end{bmatrix} w_{R_C}(t).$$

The state variables of this internal model are the position p_{R_C} and velocity s_{R_C} of R_C, and the acceleration a_{R_C} is the control input. The disturbance w_{R_C} is an uncontrollable input; including w_{R_C} in the dynamics of the reference system is realistic if R_C is an actual vehicle or target to be tracked, it does not make sense if R_C is a virtual system (e.g. the solution of a control problem).

All vehicles V_1, V_2 and V_C have the following dynamics, derived in [59]:

$$\begin{bmatrix} p_{V_j} \\ s_{V_j} \end{bmatrix}(t+1) = \begin{bmatrix} I & I \\ 0 & \frac{\tau-1}{\tau}I \end{bmatrix} \begin{bmatrix} p_{V_j} \\ s_{V_j} \end{bmatrix}(t) + \begin{bmatrix} 0 \\ \frac{1}{\tau}I \end{bmatrix} a_{V_j}(t) + \begin{bmatrix} I \\ 0 \end{bmatrix} w_{V_j}(t),$$

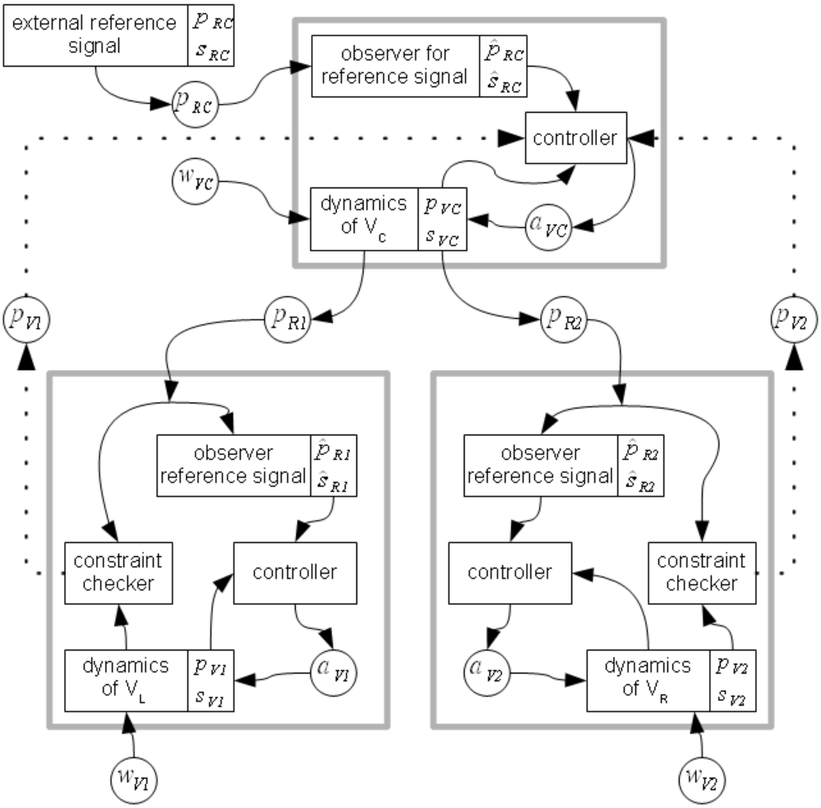

Figure 8.2: Modeling scheme

with $j = 1, 2, C$. Again, the state consists of the position and velocity of the vehicle (thus the state space is $\mathbb{R}^6$), the acceleration is the control input, and the disturbance is the uncontrollable input.

At each time step, V_C observes the current position $p_{R_C}(t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} p_{R_C} \\ s_{R_C} \end{bmatrix} (t)$ of the external reference signal. The reference trajectories R_1 and R_2 are related to the position of the coordinating vehicle V_C as follows:

$$p_{R_1}(t) = p_{V_C}(t) + \Delta_1, \quad p_{R_2}(t) = p_{V_C}(t) + \Delta_2.$$

The observer dynamics for all three observers are

$$\begin{bmatrix} \hat{p}_{R_j} \\ \hat{s}_{R_j} \end{bmatrix} (t+1) = \begin{bmatrix} I - G^p_{R_j} & I \\ -G^s_{R_j} & \frac{\tau-1}{\tau} I \end{bmatrix} \begin{bmatrix} \hat{p}_{R_j} \\ \hat{s}_{R_j} \end{bmatrix} (t) + \begin{bmatrix} G^p_{R_j} \\ G^s_{R_j} \end{bmatrix} p_{R_j}(t),$$

$j = 1, 2, C$, with p_{R_j} denoting the observations of the actual reference positions. The error dynamics are

$$\begin{bmatrix} p^{err}_{R_j} \\ s^{err}_{R_j} \end{bmatrix}(t+1) = \begin{bmatrix} I - G^p_{R_j} & I \\ -G^s_{R_j} & \frac{\tau-1}{\tau} I \end{bmatrix} \begin{bmatrix} p^{err}_{R_j} \\ s^{err}_{R_j} \end{bmatrix}(t) + \begin{bmatrix} 0 \\ \frac{1}{\tau} I \end{bmatrix} a_{R_j}(t) + \begin{bmatrix} I \\ 0 \end{bmatrix} w_{R_j}(t),$$

where a_{R_C} and w_{R_C} are the acceleration and disturbance of the external reference signal, and $a_{R_1} = a_{R_2} = a_{V_C}$ and $w_{R_1} = w_{R_2} = w_{V_C}$ because the reference systems of the follower vehicles have the same dynamics as the coordinating vehicle. a_{R_1}, a_{R_2}, w_{R_1} and w_{R_2} only play a role in the observer errors; they are not used on-board by V_1 or V_2. $G^p_{R_j}$ and $G^s_{R_j}$ are appropriate observer gains.

Combining these variables and dynamics, we arrive at the open-loop system given in Table 8.1. This is an affine system because the last term, involving Δ_1 and Δ_2, is constant.

The state variables p_{V_1}, s_{V_1}, $\hat{p}_{R_1}$, $\hat{s}_{V_1}$ belong to vehicle V_1, the variables p_{V_2}, s_{V_2}, $\hat{p}_{R_2}$, $\hat{s}_{R_2}$ are the state variables of vehicle V_2, and the state variables p_{V_C}, s_{V_C}, $\hat{p}_{V_C}$, $\hat{s}_{V_C}$ belong to the coordinating vehicle V_C. For each vehicle, the state space dimension is 12, and the state space of the overall system has dimension 36.

The internal model used for the external reference signal is not included in this open-loop system because the state variables of the external reference signal are not located in either of the vehicles. The accelerations a_{V_1}, a_{V_2}, a_{V_C} are the control inputs, the variables w_{V_1}, w_{V_2}, w_{V_C}, p_{R_C} are the external inputs, and Δ_1, Δ_2 are fixed parameters.

The open-loop system in Table 8.1 is a coordinated affine system. The coordinating vehicle corresponds to the coordinator of a coordinated system, and the follower vehicles correspond to the subsystems. Coordinated systems have the property that the coordinator influences the subsystems, while the subsystems have no influence on the coordinator, or on each other. In this case study, this corresponds to the coordinating vehicle sending its position to the other vehicles regularly. The event-based feedback from the other vehicles to the coordinating vehicle does not comply with the structure of a coordinated system, and hence the closed-loop system will only correspond to a coordinated system during the time intervals between two occurrences of this event-based feedback.

8.1.4 Control

In the formulation of the control problem, we have to consider the following control objectives:

- For each vehicle we have a tracking problem: for $j = 1, 2, C$, vehicle V_j should track its reference signal R_j.
- The vehicles should never collide.

Table 8.1: The open-loop system

$$
\left[\begin{array}{c} p_{V_1} \\ s_{V_1} \\ \hat{p}_{R_1} \\ \hat{s}_{R_1} \\ \hline p_{V_2} \\ s_{V_2} \\ \hat{p}_{R_2} \\ \hat{s}_{R_2} \\ \hline p_{V_C} \\ s_{V_C} \\ \hat{p}_{R_C} \\ \hat{s}_{R_C} \end{array}\right](t+1) =
\left[\begin{array}{cccc|cccc|cccc}
I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\tau-1}{\tau}I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I - G^p_{R_1} & I & 0 & 0 & 0 & 0 & G^p_{R_1} & 0 & 0 & 0 \\
0 & 0 & -G^s_{R_1} & \frac{\tau-1}{\tau}I & 0 & 0 & 0 & 0 & G^s_{R_1} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & I & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\tau-1}{\tau}I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I - G^p_{R_2} & I & G^p_{R_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -G^s_{R_2} & \frac{\tau-1}{\tau}I & G^s_{R_2} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau-1}{\tau}I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I - G^p_{R_C} & I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G^s_{R_C} & \frac{\tau-1}{\tau}I
\end{array}\right]
\left[\begin{array}{c} p_{V_1} \\ s_{V_1} \\ \hat{p}_{R_1} \\ \hat{s}_{R_1} \\ \hline p_{V_2} \\ s_{V_2} \\ \hat{p}_{R_2} \\ \hat{s}_{R_2} \\ \hline p_{V_C} \\ s_{V_C} \\ \hat{p}_{R_C} \\ \hat{s}_{R_C} \end{array}\right](t)
$$

$$
+ \left[\begin{array}{c|c|c}
0 & 0 & 0 \\
\frac{1}{\tau}I & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\hline
0 & 0 & 0 \\
0 & \frac{1}{\tau}I & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\hline
0 & 0 & 0 \\
0 & 0 & \frac{1}{\tau}I \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c} a_{V_1} \\ \hline a_{V_2} \\ \hline a_{V_C} \end{array}\right](t) +
\left[\begin{array}{c|c|cc}
I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & I & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & G^p_{R_C} \\
0 & 0 & 0 & G^s_{R_C}
\end{array}\right]
\left[\begin{array}{c} w_{V_1} \\ \hline w_{V_2} \\ \hline w_{V_C} \\ p_{R_C} \end{array}\right](t) +
\left[\begin{array}{c|c}
0 & 0 \\
0 & 0 \\
G^p_{R_1} & 0 \\
G^s_{R_1} & 0 \\
\hline
0 & 0 \\
0 & 0 \\
0 & G^p_{R_2} \\
0 & G^s_{R_2} \\
\hline
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{array}\right]
\left[\begin{array}{c} \Delta_1 \\ \hline \Delta_2 \end{array}\right]. \qquad (8.1)
$$

Possible solutions of the control problem are constrained by the fact that in practice, the velocities and accelerations of all vehicles are bounded in norm. The positions of the vehicles may also be constrained, e.g. by obstacles or if they should stay within a certain region. This will not be taken into account here.

The combined consideration of both control objectives and the constraint leads to a very difficult control problem. Finding an optimal control law (if one exists) would involve on-line computations of a complexity that is not feasible for the type of vehicles considered here (see [59]). Hence, our approach is to treat the objectives and constraint one-by-one; this does not lead to an optimal control law, but to an admissible control law that performs well, and that can be implemented with limited on-board computing power.

In the following, we start by solving the tracking problems for the vehicles, first for the setting with communication constraints, and then for the setting without communication constraints. We then augment the optimal control law found for the tracking problem in such a way that the bounds on the speed and acceleration are achieved. Finally we consider the problems of stability and collision: In the case with communication constraints, we have to utilize the event-based feedback from V_1 and V_2 to V_C in order to avoid collisions.

8.1.4.1 The tracking problem, with communication constraints

First we only look at the tracking problem, ignoring the collision problem and bounds. Each vehicle V_j tries to track its observed reference position, while avoiding excessive control efforts. The tracking problem for each vehicle V_j can be formulated as an LQ optimal control problem (see e.g. [63]):

$$\min_{a_{V_j}} \sum_{t=t_0}^{\infty} \left\| p_{V_j}(t) - \hat{p}_{R_j}(t) \right\|^2 + \alpha \left\| a_{V_j}(t) \right\|^2, \quad j = 1, 2, C.$$

Here, $\alpha \in \mathbb{R}$ is a parameter weighing the cost of acceleration against the cost of deviating from the reference trajectory.

The infinite-horizon formulation is chosen for simplicity, and all disturbances are ignored for now, since otherwise and without discounting, the cost would be infinite.

The difference vector $\begin{bmatrix} p_{V_j} - \hat{p}_{R_j} \\ s_{V_j} - \hat{s}_{R_j} \end{bmatrix}$ has dynamics

$$\begin{aligned} \begin{bmatrix} p_{V_j} - \hat{p}_{R_j} \\ s_{V_j} - \hat{s}_{R_j} \end{bmatrix}(t+1) &= \begin{bmatrix} I & I \\ 0 & \frac{\tau-1}{\tau} I \end{bmatrix} \begin{bmatrix} p_{V_j} - \hat{p}_{R_j} \\ s_{V_j} - \hat{s}_{R_j} \end{bmatrix}(t) \\ &+ \begin{bmatrix} 0 \\ \frac{1}{\tau} I \end{bmatrix} a_{V_j}(t) + \begin{bmatrix} I \\ 0 \end{bmatrix} w_{V_j}(t) + \begin{bmatrix} G^p_{R_j} \\ G^s_{R_j} \end{bmatrix} (\hat{p}_{R_j} - p_{R_j})(t), \end{aligned}$$

where p_{R_j} denote the observations of the actual reference position. Since $\tau > 1$, this system is controllable (see e.g. [63]).

Now the tracking problem for each vehicle can easily be solved off-line, leading to an optimal feedback

$$a_{V_j}(t) = \begin{bmatrix} F^p & F^s \end{bmatrix} \begin{bmatrix} p_{V_j} - \hat{p}_{R_j} \\ s_{V_j} - \hat{s}_{R_j} \end{bmatrix}.$$

The corresponding closed-loop system for each vehicle is then

$$\begin{bmatrix} p_{V_j} - \hat{p}_{R_j} \\ s_{V_j} - \hat{s}_{R_j} \end{bmatrix}(t+1) = \begin{bmatrix} I & I \\ \frac{1}{\tau}F^p & \frac{\tau-1}{\tau}I + \frac{1}{\tau}F^s \end{bmatrix} \begin{bmatrix} p_{V_j} - \hat{p}_{R_j} \\ s_{V_j} - \hat{s}_{R_j} \end{bmatrix}(t)$$
$$+ \begin{bmatrix} I \\ 0 \end{bmatrix} w_{V_j}(t) + \begin{bmatrix} G^p_{R_j} \\ G^s_{R_j} \end{bmatrix} (\hat{p}_{R_j} - p_{R_j})(t).$$

For the treatment of the constraints in Section 8.1.4.3, we need to rewrite the closed-loop system in terms of the original state variables:

$$\begin{bmatrix} p_{V_j} \\ s_{V_j} \\ \hat{p}_{R_j} \\ \hat{s}_{R_j} \end{bmatrix}(t+1) = \begin{bmatrix} I & I & 0 & 0 \\ \frac{1}{\tau}F^p & \frac{\tau-1}{\tau}I + \frac{1}{\tau}F^s & -\frac{1}{\tau}F^p & -\frac{1}{\tau}F^s \\ 0 & 0 & I - G^p_{R_j} & I \\ 0 & 0 & -G^s_{R_j} & \frac{\tau-1}{\tau}I \end{bmatrix} \begin{bmatrix} p_{V_j} \\ s_{V_j} \\ \hat{p}_{R_j} \\ \hat{s}_{R_j} \end{bmatrix}(t)$$
$$+ \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} w_{V_j}(t) + \begin{bmatrix} 0 \\ 0 \\ G^p_{R_j} \\ G^s_{V_j} \end{bmatrix} p_{R_j}(t).$$

The matrices characterizing the tracking problem are the same for all vehicles, and hence the feedback matrices F^p and F^s are also the same for all vehicles.

Since the reference trajectories p_{R_1} and p_{R_2} depend on the closed-loop dynamics of V_C, and observer estimates of these reference trajectories influence the control problems for V_1 and V_2, solving the tracking problem for each vehicle independently does not lead to a centralized optimum: The sum of the tracking costs for all vehicles can be decreased further by solving the combined optimization problem for all vehicles at once. However, for implementing the centralized optimum, the current states of all vehicles would need to be communicated. This alternative is used for testing the performance of our approach, and is described in the following subsection.

8.1.4.2 The tracking problem, without communication constraints

In this subsection, the same open-loop system for the motion of the vehicles is used. All communications are subject to the same uncertainties as in the setting with communication constraints. However, in this setting we do not impose any constraints on the communication of the local state observations among the vehicles. We include this setting for comparison purposes – this will allow us to quantify the costs and benefits of communication in our simulation.

In this setting, each vehicle has observers for the states of all other vehicles, so in other words each vehicle keeps a copy of the whole system in memory, with exact values for its own state, and observers for the states of the other vehicles. The control feedback for the tracking problem is the same for all vehicles: They all solve the combined tracking problem

$$\min_{a_{V_1}, a_{V_2}, a_{V_C}} \sum_{t=t_0}^{\infty} \left\| \begin{bmatrix} p_{V_1}(t) - p_{V_C}(t) - \Delta_1 \\ p_{V_2}(t) - p_{V_C}(t) - \Delta_2 \\ p_{V_C}(t) - \hat{p}_{R_C}(t) \end{bmatrix} \right\|^2 + \alpha \left\| \begin{bmatrix} a_{V_1}(t) \\ a_{V_2}(t) \\ a_{V_C}(t) \end{bmatrix} \right\|^2 .$$

The solution of this LQ-problem is

$$\begin{bmatrix} a_{V_1} \\ a_{V_2} \\ a_{V_C} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} & F_{36} \end{bmatrix} \begin{bmatrix} p_{V_1} - p_{V_C} - \Delta_1 \\ s_{V_1} - s_{V_C} \\ p_{V_2} - p_{V_C} - \Delta_2 \\ s_{V_2} - s_{V_C} \\ p_{V_C} - \hat{p}_{R_C} \\ s_{V_C} - \hat{s}_{R_C} \end{bmatrix},$$

where $F_{11}, \ldots, F_{36} \in \mathbb{R}^{3\times 3}$ can be found off-line.

Each vehicle has its own copy of the overall closed-loop system, with observer estimates for the states of the other vehicles.

8.1.4.3 The bounds on velocity and acceleration

Since the norm of the acceleration for each vehicle is penalized in the cost function of the tracking problem, the accelerations found from the state feedbacks for the two settings above will usually be small in norm. However, this does not guarantee that they stay within fixed bounds. Moreover, the velocities of the vehicles are not bounded as a result of the state feedbacks found above, and we might need a fixed bound on the speed of each vehicle for a realistic model of the settings.

With upper bounds $a_{\max} \in \mathbb{R}$ and $s_{\max} \in \mathbb{R}$ on the acceleration and speed of each vehicle, we define $\lambda_s \in \mathbb{R}$ by

$$\lambda_s = \frac{1}{\|a_{V_j}\|^2}\left((1-\tau)a_{V_j}^T s_{V_j} + \sqrt{(\tau-1)^2 a_{V_j}^T (s_{V_j} a_{V_j}^T - a_{V_j} s_{V_j}^T) s_{V_j} + \tau^2 s_{\max}^2 \|a_{V_j}\|^2}\right).$$

This variable is used for limiting the speed of the vehicle, and is derived from requiring that the velocity $\bar{s}_{V_j}(t+1)$ obtained by applying the corrected input $\lambda_s a_{V_j}(t)$ satisfies

$$\|\bar{s}_{V_j}(t+1)\|^2 = \|\frac{\tau-1}{\tau} s_{V_j}(t) + \frac{1}{\tau}\lambda_s a_{V_j}(t)\|^2 = s_{\max}^2.$$

A simple (but not necessarily optimal) way of implementing a fixed upper bound on the acceleration and speed of each vehicle is to use the following control input:

$$\bar{a}_{V_j}(t) = \min\left\{\lambda_s, \frac{a_{\max}}{\|a_{V_j}(t)\|}, 1\right\} * a_{V_j}(t),$$

where $a_{V_j}(t)$ is the optimal control feedback found in the previous two subsections, depending on the setting. This control law satisfies the bounds $\|\bar{a}_{V_j}(t)\| \leq a_{\max}$ and $\|\bar{s}_{V_j}(t+1)\| \leq s_{\max}$.

8.1.4.4 Stability and the collision constraint

In practice the speed and the acceleration of an AUV are bounded. This means that, even though both of the closed-loop systems derived in the previous subsections are output stable with respect to the output

$$y(t) = \begin{bmatrix} p_{V_1}(t) - p_{V_C}(t) - \Delta_1 \\ p_{V_2}(t) - p_{V_C}(t) - \Delta_2 \\ p_{V_C}(t) - \hat{p}_{R_C}(t) \end{bmatrix},$$

the closed-loop systems together with the constraints $\|a\| \leq a_{\max}$ and $\|s\| \leq s_{\max}$ might not be output stable.

This is interpreted as follows: If the external reference signal moves at a speed higher than $s_{\max}$ then V_C is not able to track the reference signal, and $p_{V_C} - \hat{p}_{R_C}$ increases. There is nothing that can be done about this. Another possibility is that the followers V_1 and V_2 cannot track their reference positions, because they are subjected to strong disturbances and cannot accelerate enough to compensate for that. This may lead to a follower being left behind, or a collision of two vehicles. This can be avoided if V_C is informed about the positions of V_1 and V_2, at least in the case that V_1 or V_2 are deviating too much from their reference positions. For this potential problem, we suggest three possible solutions:

- V_C receives feedback from V_1 and V_2 regularly, and includes these positions into its local tracking problem. The deviation from the formation will be small, but this involves more communication than necessary.
- V_C receives feedback from V_1 or V_2 only in the case that a follower vehicle is too far from its reference position, i.e. if $\|p_{V_j} - (\hat{p}_{V_C} + \Delta_j)\| \geq r$ for some fixed $r > 0$. In this way, the communication from V_1 and V_2 to V_C is kept minimal. If the safety regions of radius r around V_1 and V_2 are chosen far enough from each other then this approach avoids collision.
- We set the maximum speed of V_C well below the actual maximum speed of V_1 and V_2. The follower vehicles have a better chance at tracking their reference signal. No additional communication is necessary, however V_C cannot fly at its maximum speed, and hence might have more difficulties tracking the external reference signal.

In this case study we choose the second option: At each time instant, the follower vehicles check whether their position deviates from their observed reference position by more than r. If that is the case, they send their position p_{V_j} to V_C.

There are several possibilities for V_C to use this information in order to help the follower vehicle get back into formation. One option, which turned out to be successful in simulations, is to have the V_C track the signal

$$\hat{p}_{R_C} + \frac{(\hat{p}_{V_1} - \Delta_1 - p_{V_C})\mathbb{I}_1 + (\hat{p}_{V_2} - \Delta_2 - p_{V_C})\mathbb{I}_2}{W}$$

instead of the signal $\hat{p}_{R_C}$, where $\mathbb{I}_j = 1$ if V_C received p_{V_j} from V_j during this time step, and $\mathbb{I}_j = 0$ otherwise. The second term is a weighted average of the deviations of the vehicle positions from their reference positions, with weight parameter $W > 0$. This average deviation has to be computed by V_C. At most times, V_C does not know the positions of the follower vehicles because the follower vehicles are within a radius r of their reference positions. In this case, the tracking signal is $\hat{p}_{R_C}$.

The collision problem is automatically solved by our approach if the distance between the uncertainty regions $\mathbb{D}_r(p_{V_j})$ for the two follower vehicles is large enough - this can be made more precise by taking into account the maximum speed and acceleration.

8.1.4.5 The control algorithm

We summarize the control algorithm described in the previous subsections:
For the case with communication constraints, we have

$$a_{V_j} = \begin{bmatrix} F^p & F^s \end{bmatrix} \begin{bmatrix} p_{V_j} - \hat{p}_{R_j} \\ s_{V_j} - \hat{s}_{R_j} \end{bmatrix}, \; j = 1, 2,$$

$$a_{V_C} = \begin{bmatrix} F^p & F^s \end{bmatrix} \begin{bmatrix} p_{V_j} - \hat{p}_{R_j} - \frac{(\hat{p}_{V_1} - \Delta_1 - p_{V_C})\mathbb{I}_1 + (\hat{p}_{V_2} - \Delta_2 - p_{V_C})\mathbb{I}_2}{W} \\ s_{V_j} - \hat{s}_{R_j} \end{bmatrix}.$$

For the case without communication constraints, we found

$$\begin{bmatrix} a_{V_1} \\ a_{V_2} \\ a_{V_C} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} & F_{36} \end{bmatrix} \begin{bmatrix} p_{V_1} - p_{V_C} - \Delta_1 \\ s_{V_1} - s_{V_C} \\ p_{V_2} - p_{V_C} - \Delta_2 \\ s_{V_2} - s_{V_C} \\ p_{V_C} - \hat{p}_{R_C} \\ s_{V_C} - \hat{s}_{R_C} \end{bmatrix},$$

with observer values where actual values are not available.

For both cases, the control feedback to be implemented is then given by

$$\bar{a}_{V_j}(t) = \min\left\{\lambda_s, \frac{a_{\max}}{\|a_{V_j}(t)\|}, 1\right\} * a_{V_j}(t), \quad j = 1, 2, C.$$

As discussed in the previous subsections, this control law meets the control objectives and satisfies the constraint. Since the feedback and observer gains can be computed offline, the computational burden on the AUVs is very low.

8.1.5 Simulation results

We test the performance of the control law and communication scheme described above using MATLAB simulations. Simulation 1 implements the control law with communication constraints, on the linearized version of the model and with noise. Simulation 2 implements the system without communication constraints.

8.1.5.1 Settings and parameters

Our simulations ran over 1000 time steps, each of length $1s$. For the external reference trajectory we chose a circular path, starting at a distance of $40m$ from the vehicles. For the vehicles, we used $s_{\max} = 3m/s$, $a_{\max} = 0.3m/s^2$ and $\tau = 5$. The disturbances were chosen to be Gaussian with mean 0 and $\sigma = 0.3$, and we used uncertainty radius $r = 7m$ around the follower vehicles. Messages were modeled to arrive with a probability of 0.9, and with an average delay of $2.4s$. The weights for the tracking problems were chosen to be $\alpha = 10$ and $W = 7$.

8.1.5.2 Performance and comparison

Both simulations show that the control objectives and constraints are met. Figure 8.3 illustrates this for Simulation 1: While the distances of the vehicles to their observed reference positions quickly drop below $10m$, the distances between the three vehicles stay between $20m$ and $40m$ at all times. Feedback from V_1 and V_2 occurred at 110 time steps.

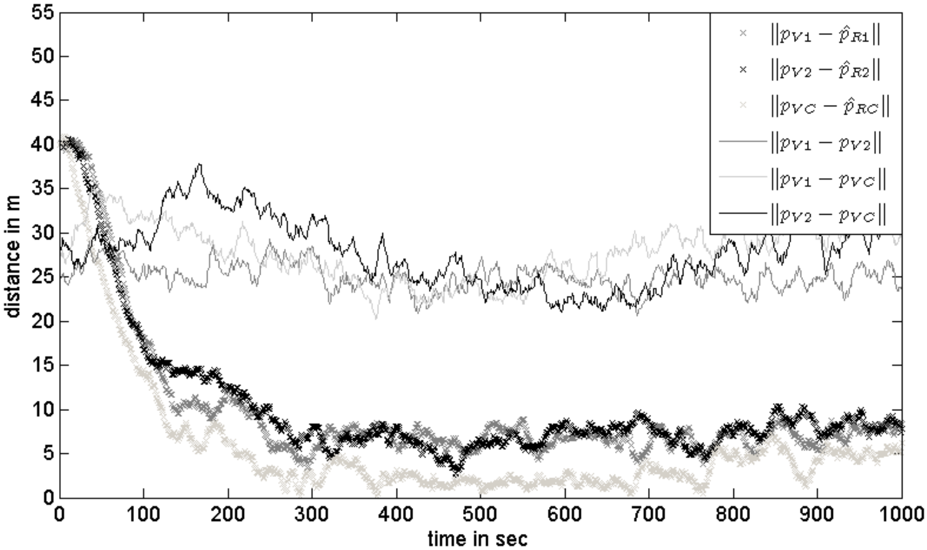

Figure 8.3: Simulation results

For comparing performances, we evaluate the cost function from the tracking problems (note that the cost function is the same in both cases). Based on one representative run, we found that the total costs are:

- Simulation 1: $1.37 * 10^5$,
- Simulation 2: $1.23 * 10^5$.

This means that our control law with communication constraints leads to an increase by around 11%, compared to the control law with unconstrained communication.

We can take into account communication costs by specifying a fixed cost C_C per message broadcast by the coordinating vehicle V_C (a message is an element of $\mathbb{R}^3$ or $\mathbb{R}^2$), and a fixed cost C_F per message broadcast by one of the follower vehicles V_1 and V_2. Now the total communication costs are:

- Simulation 1: $1000C_C + 110C_F$,

- Simulation 2: $2000C_C + 2000C_F$.

The communication costs for Simulation 1 depend strongly on the disturbances, and on the radius r of the uncertainty regions.

8.1.6 Concluding remarks

In this section, we described a control algorithm and a communication scheme for the problem of formation flying for AUVs. This approach is implementable with low on-board computing power, and it requires very little communication among the vehicles. In a simulation, we compared the performances of this approach and a similar approach with unlimited communication. While the total cost increased slightly with our communication scheme, the total amount of communication decreased considerably.

Another case study was worked out for the problem of collision avoidance for several unmanned aerial vehicles, coordinated by a control tower. Since the approaches and results are very similar to the case study discussed in this section, the case study for unmanned aerial vehicles is not included here.

8.2 Coordinated ramp metering

This section describes an application of coordination control, concerning the coordinated control of several ramp metering devices at highway on-ramps.

8.2.1 Introduction

Ramp metering devices are traffic lights at the on-ramps of a highway, which allow one vehicle to get onto the highway every s seconds, where $s \in [0, \infty)$ can be chosen by the controller (i.e. the road authorities). Ramp metering is employed in order to achieve two control objectives: flow control and temporization. Flow control concerns the regulation of the on-flow onto the highway in the case that the overall demand exceeds the highway capacity: it is used to avoid traffic jams on the highway, at the expense of creating queues at the on-ramp. The rationale behind this choice is based on a hierarchical ordering of – possibly conflicting – control objectives: If the overall traffic in a network exceeds the overall capacity, queues cannot be avoided. Our first priority in that case is to keep the largest roads, the highways, queue-free, at the expense of the smaller roads. Temporization is a process which 'evens out' the traffic flow: In the city network, vehicles typically move in batches (or platoons) from intersection to intersection. By letting vehicles onto the highway at equally-spaced time points, the ramp metering device removes this effect, thus reducing the variance of the traffic flow on the highway. In the following, we will only consider the objective of flow control.

The need for coordination of different ramp metering devices arises when several neighboring on-ramps, leading vehicles onto the same direction of the highway, have a combined demand exceeding the downstream capacity of the highway – in that case, local control would lead to long queues at the downstream on-ramps since the upstream on-ramps ignore downstream demand when determining their on-flow[2].

Coordination schemes for ramp metering were first implemented in Los Angeles and Delft. Later, research on coordinated ramp metering has focused on the highway rings of Amsterdam and Paris, for which historical traffic data is available. The highway traffic on the Amsterdam ring is currently monitored and controlled from the traffic control center of North-Holland. In [30, 42], a centralized approach (i.e. all on-ramp flows are determined by the traffic control center) to coordinated ramp metering was tested on several non-linear traffic models, and applied in simulations to the highway rings of Amsterdam and Paris. The control objectives were to minimize the total time spent by all drivers in the network, and to have equal queue lengths at the on-ramps. The centralized approach used was computationally feasible for the simulation studies[3], but did not necessarily lead to a global optimum.

A decentralized approach to coordinated ramp metering was suggested in [73–75], and applied in simulations to the Amsterdam ring: Each on-ramp used a local controller, and only when its local maximum queue length was exceeded, it would tell the next upstream on-ramp to reduce its on-flow accordingly. During peak hours, this coordination process would then naturally propagate upstream. This approach has the advantage of high scalability, but does not perform well in all situations (e.g. if there is no demand at the next upstream on-ramp).

In the following, we will apply the concepts of coordination control derived in earlier chapters to this problem, using a simple linear traffic model. The purpose of this section is to illustrate the theory of coordinated linear systems, but also to give some insight into the separation of this problem into global and local subproblems.

8.2.2 Modeling

For the purpose of this case study, we use a strongly simplified linear version of the traffic model found in [66]. In particular, we make the following modeling assumptions:

- the speed is constant for all vehicles and at all times,
- we restrict attention to two neighboring on-ramps,

[2]In urban areas, this can lead to vehicles driving back to the next upstream on-ramp via the city network – a detour on their part, and an unnecessary load on city traffic.

[3]Extendability is of course limited in the centralized approach – a national highway network may be too large.

- the system dynamics are modeled by a discrete-time linear system, using the different flows and queue lengths as state variables,
- all in-flows into the system are estimated from historical data.

The discrete-time formulation of the problem is chosen because the system contains considerable delays (i.e. the travel times from one on-ramp to another etc.): These delays are intrinsic to the problem, and with the assumption of constant speeds they can easily be handled by a finite-dimensional discrete-time system (but not by its continuous-time counterpart). Moreover, in practice the measurements from the detection loops are sent in aggregated form once every minute, rather than being continuously available.

For each on-ramp $i = 1, 2$, our model involves the following variables:

$q_i^{\text{in}} \in [0, \infty)$ is the in-flow into the ramp area from upstream, in $\frac{\text{vehicles}}{\text{time unit}}$

$q_i^{\text{out}} \in [0, \infty)$ is the out-flow from the ramp area to downstream, in $\frac{\text{vehicles}}{\text{time unit}}$

$q_i^{\text{on}} \in [0, \infty)$ is the on-flow onto the highway at the on-ramp, in $\frac{\text{vehicles}}{\text{time unit}}$

$q_i^{\text{off}} \in [0, \infty)$ is the off-flow off the highway via the off-ramp, in $\frac{\text{vehicles}}{\text{time unit}}$

$q_i^{\text{arr}} \in [0, \infty)$ is the arrival flow to the queue at the on-ramp, in $\frac{\text{vehicles}}{\text{time unit}}$

$Q_i \in [0, \infty)$ is the length of the queue formed at the on-ramp, measured in number of vehicles

These variables are illustrated in Figure 8.4.

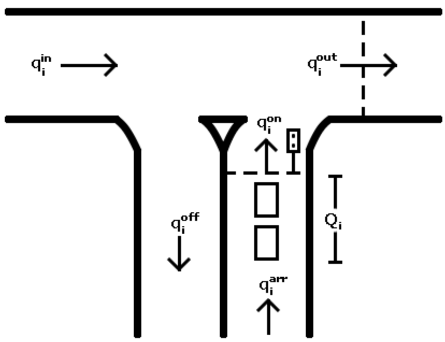

Figure 8.4: Setting and state variables for one on-ramp

Considering each on-ramp separately for now, we have that q_i^{in} and q_i^{arr} are external inputs to our system: They correspond to the in-flows into the ramp area from other parts of the highway and the city network, and while we cannot measure them directly, we assume that historical estimates of these quantities are available at the traffic control center (i.e. the centralized controller).

The state variables of the system will be q_i^{out}, q_i^{off} and Q_i: For the out-flow q_i^{out}, regular up-to-date measurements from the detection loops downstream of the on-ramp are available locally, at the ramp metering device. The off-flow q_i^{off} cannot be measured directly, and we assume that it is a certain fraction of the in-flow, and that historical estimates of this fraction exist. The queue length is assumed to be measurable at the traffic control center (via cameras); this assumption is necessary since we would otherwise need to rely entirely on the historical predictions for the arrival flow, and any form of closed-loop control for the queue length would be impossible. If the queue lengths were observable locally at the on-ramps, the local problems could be to regulate the on-flow such that a target queue length, prescribed by the traffic control center, is not exceeded.

The on-flow q_i^{on} onto the highway is the control input: provided that the demand at the on-ramp is high enough, we can determine the on-flow by setting the time period s between two successive cars being let onto the highway accordingly. If the demand (i.e. the arrival flow q_i^{arr}) is lower than our target on-flow then we can still bound the on-flow from above by setting s accordingly, but we need to use the downstream detection loop measurements to estimate the actual on-flow.[4]

Since part of our control objective is to avoid exceeding the highway capacity, we will need an estimate of the downstream capacity. This estimate will be denoted by $q^{\text{cap}}(t) \in [0, \infty)$, and can be obtained from historical data at the traffic control center. It is modeled as time-varying because the highway capacity strongly depends on the weather conditions.

For a highway stretch with several on-ramps, the in-flows and out-flows of the different on-ramps are related by the fact that all vehicles which are on the highway after passing an upstream on-ramp will enter the section containing the next downstream on-ramp a fixed number of time steps later.[5] Thus only the in-flow into the first upstream on-ramp is an external input to the system, which we will denote by q^{U}.

For simplicity of notation, and in line with the other parts of this thesis, we limit attention to highway stretches with two on-ramps, and point out that an extension to more on-ramps is conceptually straightforward. Since each on-ramp is only related to the next upstream and downstream on-ramps via its in-flow and out-flow, a model for a highway stretch with n on-ramps would have the form of a nearest-neighbor system with n subsystems, as described in Section 3.3.

[4] If there is less demand than the highway capacity would allow for, then there is no risk of congestion in that region, and traffic control measures such as ramp metering are unnecessary.

[5] This is due to the principle of conservation of vehicles – no vehicles can enter or leave the highway between successive on-ramps.

Moreover, we always restrict attention to one direction of the highway: The two different directions of a highway are modeled as independent.

The different state and input variables are illustrated in Figure 8.5.

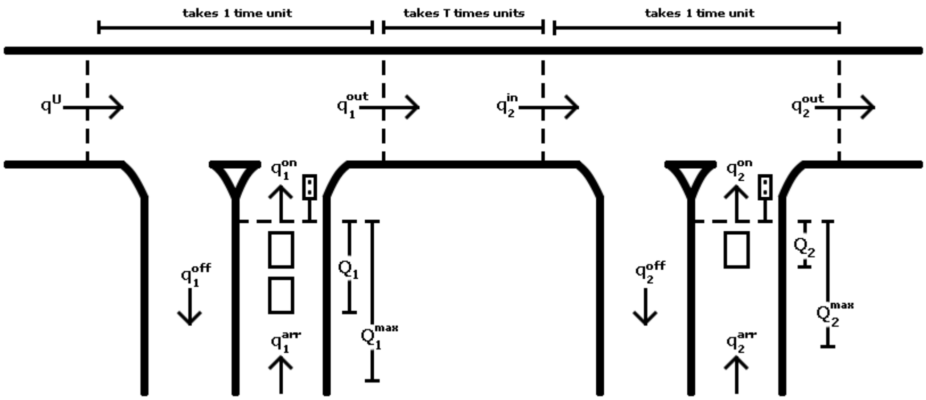

Figure 8.5: Setting and state variables for the coordinated ramp metering problem

For the evolution of the state variables of the system with two on-ramps shown in Figure 8.5, and for a fixed delay $T \in \mathbb{N}$, we derive the following set of equations:

$$q_1^{\text{out}}(t+1) = q^{\text{U}}(t) - q_1^{\text{off}}(t) + q_1^{\text{on}}(t) \tag{8.2}$$

$$q_2^{\text{out}}(t+1) = q_2^{in}(t) - q_2^{\text{off}}(t) + q_2^{\text{on}}(t) \tag{8.3}$$

$$q_1^{\text{off}}(t+1) = \beta_1 q^{\text{U}}(t) \tag{8.4}$$

$$q_2^{\text{off}}(t+1) = \beta_2 q_2^{\text{in}}(t) \tag{8.5}$$

$$Q_1(t+1) = Q_1(t) + \Delta_t(q_1^{\text{arr}}(t) - q_1^{\text{on}}(t)) \tag{8.6}$$

$$Q_2(t+1) = Q_2(t) + \Delta_t(q_2^{\text{arr}}(t) - q_2^{\text{on}}(t)) \tag{8.7}$$

$$q_2^{\text{in}}(t+T) = q_1^{\text{out}}(t) \tag{8.8}$$

We choose our time steps Δ_t in such a way that it takes exactly one time step to pass an on-ramp area, and assume for simplicity that all on-ramp areas have the same length. On the Amsterdam ring, the detection loops in the road are roughly equally-spaced at 500m, so a natural choice would be to define an on-ramp area to consist of the highway section between the nearest detection loops upstream and downstream of the on-ramp.

Equations (8.2) and (8.3) describe the out-flows out of the two ramp areas: These are given by adding up the upstream inflow one time step earlier (q^{U} for ramp 1 and q_2^{in} for ramp 2) and the on-flow via the respective on-ramp (q_i^{on}), and deducting the off-flow at the respective off-ramp (q_i^{off}).

The off-flows q_1^{off} and q_2^{off} are modeled as fixed fractions of the in-flows into the respective ramp areas. The parameters $\beta_1, \beta_2 \in [0, \infty)$ appearing in equations (8.4) and (8.5) were assumed to be estimated from historical data; they are treated as time-invariant in our setting, but may in principle be slowly time-varying (e.g. they may differ for the morning and evening rush hours).

In equations (8.6) and (8.7), the queue lengths at the two ramp metering devices are described to grow by the number of new vehicles arriving at the on-ramp over one time step, and to shrink by the number of vehicles which were let onto the highway by the ramp metering device over the same time period. Since Δ_t is the length of one time step, these two quantities can easily be estimated by $\Delta_t q_1^{\text{arr}}(t)$ and $\Delta_t q_2^{\text{on}}(t)$. Note that these equations do not account for the fact that queue lengths can never be negative in practice – again we justify this choice by arguing that if the demand is far below the highway capacity then there is no risk of congestion, and the ramp metering device does not need to restrict the on-flow.

Equation (8.8) models the interconnection between the two on-ramps, as described above: Under the assumption of constant speed, the out-flow out of ramp area 1 will reach ramp area 2 T time units later. Incorporating equation (8.8) into our system requires us to keep the last $T - 1$ values for q_2^{in} in memory: Suppose $T = 3$, then the equation can be rewritten in the form

$$\begin{bmatrix} q_2^{\text{in},+2} \\ q_2^{\text{in},+1} \\ q_2^{\text{in}} \end{bmatrix} (t+1) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_2^{\text{in},+2} \\ q_2^{\text{in},+1} \\ q_2^{\text{in}} \end{bmatrix} (t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} q_1^{\text{out}}(t), \tag{8.9}$$

with auxiliary variables $q_2^{\text{in},+2}$ and $q_2^{\text{in},+1}$ satisfying

$$q_2^{\text{in},+2}(t) = q_2^{\text{in}}(t+2) \quad \text{and} \quad q_2^{\text{in},+1}(t) = q_2^{\text{in}}(t+1).$$

Combining equations (8.2)-(8.8) and (8.9), and re-ordering the state and input variables according to their locations (i.e. ramp 1 or ramp 2), we arrive at the open-loop system given in Table 8.2.2. This system is a nearest-neighbor system with a directed information structure, or equivalently, a hierarchical system with a chain structure. In order to incorporate in the model which state variable is observable from which location, we define an output vector $\begin{bmatrix} y_1 \\ y_2 \\ y_c \end{bmatrix}$, where y_i consists of all variables which are observable at ramp i ($i = 1, 2$), and y_c contains all observations available at the traffic control center.

$$
\begin{aligned}
\left[\begin{array}{c} q_1^{\mathrm{out}} \\ q_1^{\mathrm{off}} \\ Q_1 \\ \hline q_2^{\mathrm{in},+2} \\ q_2^{\mathrm{in},+1} \\ q_2^{\mathrm{in}} \\ \hline q_2^{\mathrm{out}} \\ q_2^{\mathrm{off}} \\ Q_2 \end{array}\right](t+1) = {} & \left[\begin{array}{ccc|ccc|ccc}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c} q_1^{\mathrm{out}} \\ q_1^{\mathrm{off}} \\ Q_1 \\ \hline q_2^{\mathrm{in},+2} \\ q_2^{\mathrm{in},+1} \\ q_2^{\mathrm{in}} \\ \hline q_2^{\mathrm{out}} \\ q_2^{\mathrm{off}} \\ Q_2 \end{array}\right](t) \\
& + \left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\beta_1 & 0 & 0 & 0 \\
0 & \Delta_t & 0 & 0 \\ \hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \Delta_t & 0
\end{array}\right]
\left[\begin{array}{c} q^{\mathrm{U}} \\ q_1^{\mathrm{arr}} \\ q_2^{\mathrm{arr}} \\ q^{\mathrm{cap}} \end{array}\right](t) +
\left[\begin{array}{c|c}
1 & 0 \\
0 & 0 \\
-\Delta_t & 0 \\ \hline
0 & 0 \\
0 & 0 \\
0 & 0 \\ \hline
0 & 1 \\
0 & 0 \\
0 & -\Delta_t
\end{array}\right]
\left[\begin{array}{c} q_1^{\mathrm{on}} \\ \hline q_2^{\mathrm{on}} \end{array}\right](t)
\end{aligned}
\tag{8.10}
$$

Table 8.2: The open-loop system, for two on-ramps

We arrive at the output equation

$$
\left[\begin{array}{c} y_1 \\ \hline y_2 \\ \hline y_{c,1} \\ y_{c,2} \\ y_{c,3} \\ y_{c,4} \\ y_{c,5} \\ y_{c,6} \end{array}\right]
=
\left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c} q_1^{\mathrm{out}} \\ q_1^{\mathrm{off}} \\ Q_1 \\ \hline q_2^{\mathrm{in},+2} \\ q_2^{\mathrm{in},+1} \\ q_2^{\mathrm{in}} \\ \hline q_2^{\mathrm{out}} \\ q_2^{\mathrm{off}} \\ Q_2 \end{array}\right]
+
\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c} q^{\mathrm{U}} \\ q_1^{\mathrm{arr}} \\ q_2^{\mathrm{arr}} \\ q^{\mathrm{cap}} \end{array}\right].
$$

8.2.3 Control problem

As already mentioned above, our control objective will be to have equal queue lengths while ensuring that the downstream flow does not exceed the highway capacity. In the following we justify this choice, discuss possible ways of formalizing this objective, and split the overall objective into local control problems for the ramp metering devices, and a coordination control problem for the traffic control center.

8.2.3.1 Discussion of possible control objectives

A common control objective in traffic control is to minimize the *total time spent* by all drivers in the network. In our setting, this would not lead to equal queue lengths: The queue at ramp 1 would be kept shorter than the queue at ramp 2 since –according to the model– a fraction of β_2 of the vehicles entering the highway at ramp 1 will leave the highway at ramp 2. Hence the total time spent can be decreased by having a higher on-flow at ramp 1, without exceeding $q^{\text{cap}}(t)$ downstream of ramp 2. In practice, however, vehicles are unlikely to leave the highway right after entering it.

Another alternative to the objective of equal queue lengths is to have *equal waiting times* at the on-ramps: The waiting time $\tau_i(t)$ for each vehicle can be estimated from the queue length at the time of arrival in the queue, and the on-flow, i.e. $\tau_i(t) = \frac{Q_i(t)}{q_i^{\text{on}}(t)}$ at ramp i. This control objective would lead to a non-linear control law, and is hence not considered here.

For the objective of equal queue lengths, we need to choose *at which times* we compare the queue lengths:

- On the one hand, a good option would be to compare $Q_1(t)$ with $Q_2(t+T)$, since the vehicles entering the highway via ramp 1 at time t will pass ramp area 2 at time $t+T$, and will thus pass the detection loops downstream of ramp 2 at the same time as the vehicles entering the highway via ramp 2 at time $t+T$. This objective would, however, require the controller to have predictions of the state variables up to T time units ahead – since many variables are estimated from historical data this may not be a problem for the setting with two on-ramps; for larger networks with several on-ramps and larger delays this may be impractical, and for networks with loops (such as highway rings) this would be impossible.

- A comparison of $Q_1(t)$ with $Q_2(t)$, on the other hand, is feasible at time t since all required data is available at that time, and hence we will choose this option for this case study. However, in some situations this objective may perform worse than the first option: If many vehicles want to reach the same downstream destination at the same time, then the queue at ramp 1 will have a peak T time units before the queue at ramp 2, and hence a comparison of the peak values at t and $t+T$ would be preferable.

To summarize, our control objective at time t is to choose $q_1^{\text{on}}(t)$ and $q_2^{\text{on}}(t)$ in such a way that

$$Q_1(t+1) = Q_2(t+1) \text{ and } q_2^{\text{out}}(t+1) = q^{\text{cap}}(t), \tag{8.11}$$

if the overall demand at time t exceeds the highway capacity.

In practice, it may also be of interest to keep the queue lengths below a fixed threshold value (denoted by $Q_1^{\max}$ and $Q_2^{\max}$ in Figure 8.5): Incorporating this bound, we could avoid that the queues become so long that upstream intersections in the city network may get blocked. In particular, if long queues would have a strong effect on city traffic at one on-ramp but not on the other on-ramp, it may be preferable to allow for longer queues at one ramp, rather than keeping the queue lengths equal. Since this additional constraint may conflict with both objectives given in (8.11) and depends strongly on the topology of the city network, we will not take this additional consideration into account in our control problem.

8.2.3.2 Separation into two layers

We suggest the following distribution of the overall control task over the different locations:

- The traffic control center assigns target out-flows $\bar{q}_1^{\text{out}}$ and $\bar{q}_2^{\text{out}}$, such that the control objective is achieved, based on its historical estimates for the in- and out-flows into and out of the network, and its estimate for the highway capacity.
- Each local controller then tracks the target out-flow assigned by the traffic control center, based on its local measurements at the downstream detection loops.

Since the queue lengths are only measured at the traffic control center, and since coordination is required for the objective of keeping the queues equally long, determining the target flows to be tracked by the local systems is the task of the traffic control center. Note that the traffic control center could in principle also assign target on-flows directly, instead of assigning target out-flows and leaving the problem of choosing appropriate on-flows to the local systems; however, the estimates from historical data used at the traffic control center will be less accurate than the direct measurements at the on-ramps. For this reason we opt for a feedback loop at each on-ramp, using the local measurements.

8.2.4 Control synthesis and the closed-loop system

Having settled on the control objective of keeping the queue lengths at the two on-ramps equally long at each given time, we can now derive the corresponding control laws and the closed-loop system.

8.2.4.1 Control law for the coordinator

The target out-flows $\bar{q}_1^{\text{out}}$ and $\bar{q}_2^{\text{out}}$ to be determined by the traffic control center can be derived directly from the objectives given in (8.11). The variables which are observable at the traffic control center are illustrated in Figure 8.6. We get

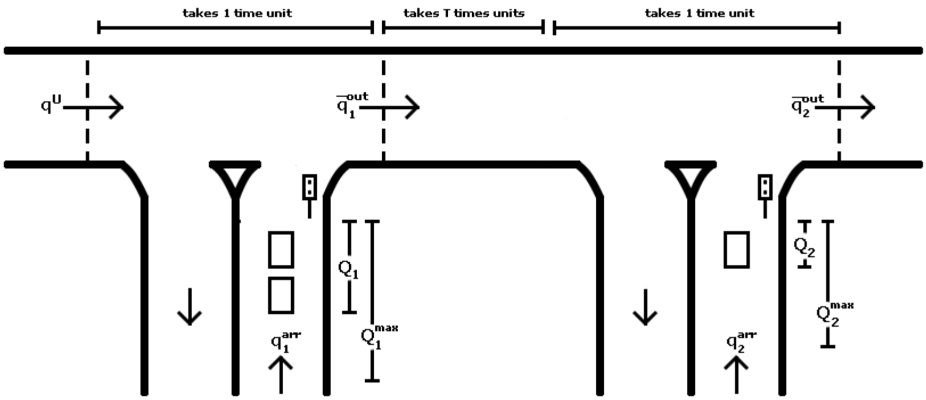

Figure 8.6: Variables observable at the traffic control center

$\bar{q}_2^{\text{out}}(t+1) = q^{\text{cap}}(t)$, and $\bar{q}_1^{\text{out}}(t+1)$ can be derived from the objective $Q_1(t+1) = Q_2(t+1)$, using the system dynamics: Noting that

$$q_i^{\text{on}}(t) = \bar{q}_i^{\text{out}}(t+1) - q_i^{\text{in}}(t) + q_i^{\text{off}}(t) = \bar{q}_i^{\text{out}}(t+1) - (1-\beta_i)q_i^{\text{in}}(t)$$

and $q_2^{\text{in}}(t) = \bar{q}_1^{\text{out}}(t-T)$, we get

$$\begin{aligned}
& Q_1(t+1) = Q_2(t+1) \\
\Leftrightarrow\ & Q_1(t) + \Delta_t\left(q_1^{\text{arr}}(t) - q_1^{\text{on}}(t)\right) = Q_2(t) + \Delta_t\left(q_2^{\text{arr}}(t) - q_2^{\text{on}}(t)\right) \\
\Leftrightarrow\ & Q_1(t) + \Delta_t\left(q_1^{\text{arr}}(t) - \bar{q}_1^{\text{out}}(t+1) + (1-\beta_1)q_1^{\text{in}}(t)\right) \\
& \quad = Q_2(t) + \Delta_t\left(q_2^{\text{arr}}(t) - \bar{q}_2^{\text{out}}(t+1) + (1-\beta_2)q_2^{\text{in}}(t)\right) \\
\Leftrightarrow\ & \bar{q}_1^{\text{out}}(t+1) = \frac{1}{\Delta_t}\left(Q_1(t) - Q_2(t)\right) + \left(q_1^{\text{arr}}(t) - q_2^{\text{arr}}(t)\right) + (1-\beta_1)q^U(t) \\
& \quad - (1-\beta_2)\bar{q}_1^{\text{out}}(t-T) + q^{\text{cap}}(t).
\end{aligned}$$

Implementing this control law will require us to keep the last T target out-flows $\bar{q}_1^{\text{out}}$ in memory. The control law for the traffic control center can be rewritten as

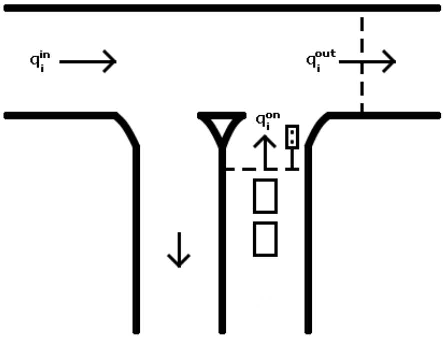

Figure 8.7: Variables observable at on-ramp i

$$\begin{bmatrix}\bar{q}_1^{\mathrm{out}}\\ \bar{q}_2^{\mathrm{out}}\end{bmatrix}(t+1) = \begin{bmatrix}\frac{1}{\Delta_t} & -\frac{1}{\Delta_t}\\ 0 & 0\end{bmatrix}\begin{bmatrix}Q_1\\ Q_2\end{bmatrix}(t) + \begin{bmatrix}1-\beta_1 & 1 & -1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}q^U\\ q_1^{\mathrm{arr}}\\ q_2^{\mathrm{arr}}\\ q^{\mathrm{cap}}\end{bmatrix}(t) + \begin{bmatrix}\beta_2-1\\ 0\end{bmatrix}\bar{q}_1^{\mathrm{out}}(t-T).$$

8.2.4.2 Control law for the local systems

The task of the local controllers is to choose $q_i^{\mathrm{on}}(t)$ in such a way that $q_i^{\mathrm{out}}(t+1) = \bar{q}_i^{\mathrm{out}}(t+1)$. The variables which are observable at the on-ramp are illustrated in Figure 8.7. For q_i^{in} we do not have local measurements; instead, we will estimate $q_i^{\mathrm{in}}(t)$ by $q_i^{\mathrm{in}}(t-1)$, which in turn can be derived from the measured quantity $q_i^{\mathrm{out}}(t)$ and the last control input $q_i^{\mathrm{on}}(t-1)$, using the system dynamics. Our estimate $\hat{q}_i^{\mathrm{in}}$ of q_i^{in} is then given by

$$\hat{q}_i^{\mathrm{in}}(t) = q_i^{\mathrm{in}}(t-1) = \frac{1}{1-\beta_i}\left(q_i^{\mathrm{out}}(t) - q_i^{\mathrm{on}}(t-1)\right).$$

Plugging the requirement $q_i^{\mathrm{out}}(t+1) = \bar{q}_i^{\mathrm{out}}(t+1)$ into the system dynamics, and using the estimate $\hat{q}_i^{\mathrm{in}}$, we arrive at

$$\begin{aligned}
&\bar{q}_i^{\mathrm{out}}(t+1) = \hat{q}_i^{\mathrm{in}}(t) - q_i^{\mathrm{off}}(t) + q_i^{\mathrm{on}}(t)\\
\Leftrightarrow\quad &\bar{q}_i^{\mathrm{out}}(t+1) = (1-\beta_i)\hat{q}_i^{\mathrm{in}}(t) + q_i^{\mathrm{on}}(t)\\
\Leftrightarrow\quad &\bar{q}_i^{\mathrm{out}}(t+1) = q_i^{\mathrm{out}}(t) - q_i^{\mathrm{on}}(t-1) + q_i^{\mathrm{on}}(t)\\
\Leftrightarrow\quad &q_i^{\mathrm{on}}(t) = \bar{q}_i^{\mathrm{out}}(t+1) - q_i^{\mathrm{out}}(t) + q_i^{\mathrm{on}}(t-1).
\end{aligned}$$

Thus the on-flow to be allowed onto the highway by the local ramp metering device is given by the difference between the target out-flow and the last measured out-flow, and by the on-flow that was allowed onto the highway during the last time step. For implementing this, the most recent value of the on-flow, $q_i^{\text{on}}(t-1)$, needs to be kept in memory at the local on-ramp controller.

8.2.4.3 Closed-loop system

In the previous subsection, the variables $\bar{q}_1^{\text{out}}(t+1)$ and $\bar{q}_2^{\text{out}}(t+1)$ were introduced as auxiliary control inputs for the subsystems, and the variables $q_1^{\text{on}}(t-1)$, $q_2^{\text{on}}(t-1)$ and $\bar{q}^{\text{out}}(t-T)$ denote the past values of the corresponding state variables to be kept in memory at ramp 1, ramp 2 and the coordinator, respectively. The variables q_i^{in} and their past values can now be replaced by $\bar{q}_1^{\text{out}}(t-T)$. Using these additional variables, we can rewrite the open-loop system (8.10) as

$$
\begin{aligned}
\left[\begin{array}{c} q_1^{\text{out}} \\ q_1^{\text{off}} \\ Q_1 \\ \hline q_2^{\text{out}} \\ q_2^{\text{off}} \\ Q_2 \end{array}\right](t+1) &=
\left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]
\left[\begin{array}{c} q_1^{\text{out}} \\ q_1^{\text{off}} \\ Q_1 \\ \hline q_2^{\text{out}} \\ q_2^{\text{off}} \\ Q_2 \end{array}\right](t) +
\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \\ 0 & \Delta_t & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_t & 0 \end{array}\right]
\begin{bmatrix} q^{\text{U}} \\ q_1^{\text{arr}} \\ q_2^{\text{arr}} \\ q^{\text{cap}} \end{bmatrix}(t) \\
&+ \left[\begin{array}{c|c|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\Delta_t & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\Delta_t & 0 & 0 \end{array}\right]
\left[\begin{array}{c} q_1^{\text{on}}(t) \\ \hline q_2^{\text{on}}(t) \\ \hline \bar{q}_1^{\text{out}}(t+1) \\ \bar{q}_2^{\text{out}}(t+1) \end{array}\right] +
\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & \beta_2 \\ 0 & 0 & 0 \end{array}\right]
\left[\begin{array}{c} q_1^{\text{on}}(t-1) \\ \hline q_2^{\text{on}}(t-1) \\ \hline \bar{q}_1^{\text{out}}(t-T) \end{array}\right].
\end{aligned}
\tag{8.12}
$$

We first close the subsystem loops, using the local control laws

$$q_i^{\text{on}}(t) = \bar{q}_i^{\text{out}}(t+1) - q_i^{\text{out}}(t) + q_i^{\text{on}}(t-1)$$

for $i = 1, 2$. The system is then given by

$$
\begin{aligned}
\left[\begin{array}{c} q_1^{\text{out}} \\ q_1^{\text{off}} \\ Q_1 \\ \hline q_2^{\text{out}} \\ q_2^{\text{off}} \\ Q_2 \end{array}\right](t+1) &=
\left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_t & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_t & 0 & 1 \end{array}\right]
\left[\begin{array}{c} q_1^{\text{out}} \\ q_1^{\text{off}} \\ Q_1 \\ \hline q_2^{\text{out}} \\ q_2^{\text{off}} \\ Q_2 \end{array}\right](t) +
\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \\ 0 & \Delta_t & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_t & 0 \end{array}\right]
\begin{bmatrix} q^{\text{U}} \\ q_1^{\text{arr}} \\ q_2^{\text{arr}} \\ q^{\text{cap}} \end{bmatrix}(t) \\
&+ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ -\Delta_t & 0 \\ \hline 0 & 1 \\ 0 & 0 \\ 0 & -\Delta_t \end{array}\right]
\begin{bmatrix} \bar{q}_1^{\text{out}}(t+1) \\ \bar{q}_2^{\text{out}}(t+1) \end{bmatrix} +
\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\Delta_t & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & \beta_2 \\ 0 & -\Delta_t & 0 \end{array}\right]
\left[\begin{array}{c} q_1^{\text{on}}(t-1) \\ \hline q_2^{\text{on}}(t-1) \\ \hline \bar{q}_1^{\text{out}}(t-T) \end{array}\right].
\end{aligned}
$$

Plugging in the coordinator control law, we arrive at the closed-loop system

$$
\left[\begin{array}{c} q_1^{\mathrm{out}} \\ q_1^{\mathrm{off}} \\ Q_1 \\ \hline q_2^{\mathrm{out}} \\ q_2^{\mathrm{off}} \\ Q_2 \end{array}\right](t+1) = \left[\begin{array}{ccc|ccc} -1 & -1 & \frac{1}{\Delta_t} & 0 & 0 & -\frac{1}{\Delta_t} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_t & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_t & 0 & 1 \end{array}\right] \left[\begin{array}{c} q_1^{\mathrm{out}} \\ q_1^{\mathrm{off}} \\ Q_1 \\ \hline q_2^{\mathrm{out}} \\ q_2^{\mathrm{off}} \\ Q_2 \end{array}\right](t)
$$
$$
+ \left[\begin{array}{cccc} 2-\beta_1 & 1 & -1 & 1 \\ \beta_1 & 0 & 0 & 0 \\ \Delta_t(\beta_1-1) & 0 & \Delta_t & -\Delta_t \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_t & -\Delta_t \end{array}\right] \left[\begin{array}{c} q^{\mathrm{U}} \\ q_1^{\mathrm{arr}} \\ q_2^{\mathrm{arr}} \\ q^{\mathrm{cap}} \end{array}\right](t) + \left[\begin{array}{c|c|c} 1 & 0 & \beta_2-1 \\ 0 & 0 & 0 \\ -\Delta_t & 0 & \Delta_t(1-\beta_2) \\ \hline 0 & 1 & 1 \\ 0 & 0 & \beta_2 \\ 0 & -\Delta_t & 0 \end{array}\right] \left[\begin{array}{c} q_1^{\mathrm{on}}(t-1) \\ \hline q_2^{\mathrm{on}}(t-1) \\ \hline \bar{q}_1^{\mathrm{out}}(t-T) \end{array}\right].
$$

Note that the local dynamics at the two on-ramps are no longer independent: The dynamics at ramp 1 now depends on $Q_2(t)$. Since $Q_2(t)$ is observable at the traffic control center, we can include the measured value $\hat{Q}_2(t)$ of $Q_2(t)$ in the vector of estimates available at the traffic control center. This leads to the coordinated linear system

$$
\left[\begin{array}{c} q_1^{\mathrm{out}} \\ q_1^{\mathrm{off}} \\ Q_1 \\ \hline q_2^{\mathrm{out}} \\ q_2^{\mathrm{off}} \\ Q_2 \end{array}\right](t+1) = \left[\begin{array}{ccc|ccc} -1 & -1 & \frac{1}{\Delta_t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_t & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_t & 0 & 1 \end{array}\right] \left[\begin{array}{c} q_1^{\mathrm{out}} \\ q_1^{\mathrm{off}} \\ Q_1 \\ \hline q_2^{\mathrm{out}} \\ q_2^{\mathrm{off}} \\ Q_2 \end{array}\right](t)
$$
$$
+ \left[\begin{array}{ccccc} 2-\beta_1 & 1 & -1 & 1 & -\frac{1}{\Delta_t} \\ \beta_1 & 0 & 0 & 0 & 0 \\ \Delta_t(\beta_1-1) & 0 & \Delta_t & -\Delta_t & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_t & -\Delta_t & 0 \end{array}\right] \left[\begin{array}{c} q^{\mathrm{U}} \\ q_1^{\mathrm{arr}} \\ q_2^{\mathrm{arr}} \\ q^{\mathrm{cap}} \\ \hat{Q}_2 \end{array}\right](t) \tag{8.13}
$$
$$
+ \left[\begin{array}{c|c|c} 1 & 0 & \beta_2-1 \\ 0 & 0 & 0 \\ -\Delta_t & 0 & \Delta_t(1-\beta_2) \\ \hline 0 & 1 & 1 \\ 0 & 0 & \beta_2 \\ 0 & -\Delta_t & 0 \end{array}\right] \left[\begin{array}{c} q_1^{\mathrm{on}}(t-1) \\ \hline q_2^{\mathrm{on}}(t-1) \\ \hline \bar{q}_1^{\mathrm{out}}(t-T) \end{array}\right].
$$

8.2.5 Concluding remarks

In this section, we motivated and discussed a coordinated ramp metering problem. The traffic flows and queue lengths in the on-ramp areas were modeled as a linear system, and the control objective was discussed and formalized.

Whether the control objective can be achieved in a decentralized manner depends on which partial observations are available at which locations. For one possible setting, we derived a suitable coordination control law and the corresponding closed-loop system.

9 Concluding Remarks

In this concluding chapter, we briefly summarize the main contributions of this thesis, and discuss several possible extensions.

9.1 Summary of the main results

One of the main contributions of this thesis is the derivation of explicit procedures for the transformation of interconnected or monolithic linear systems into coordinated linear systems in Chapter 4: Finding appropriate system decompositions if no a priori decomposition is given (as in the monolithic case) or the given decomposition is unsuitable for control purposes (as in the interconnected case) is a major problem in decentralized control. Three concepts of minimality of decompositions were introduced in order to identify decompositions which are 'as decentralized as possible', and many of our decomposition procedures were shown to produce such minimal decompositions.

Another relevant contribution to decentralized control theory is the refinement of the standard concepts of reachability and indistinguishability to reflect which input/output is used, and which part of the system is affected, and the subsequent distinction between jointly and independently reachable subspaces, and between completely and independently indistinguishable subspaces in Chapter 5. These concepts helped us identify and characterize concepts of controllability and observability which are more meaningful for coordination control than the corresponding standard concepts for monolithic systems.

The main result of this thesis is Theorem 6.2.7, together with its immediate consequences for coordinated and hierarchical linear systems: When restricting the set of admissible control laws for an LQ control problem to structure-preserving static state feedback, the optimal feedback for each subsystem only depends on the subsystem itself, and on its followers, but not on the rest of the hierarchy. This result allows us to approach the problem in a bottom-to-top manner, finding the optimal feedback for each subsystem numerically, at each step using an algorithm derived from Theorem 6.2.7 and the optimal feedbacks found for its follower systems in previous steps.

Finally, this control synthesis procedure for LQ control problems is adjusted to allow for event-based bottom-to-top feedback and to significantly reduce its computational complexity, in Chapter 7. The cost corresponding to this control law approximates the centralized optimum arbitrarily well, at the expense of an increased need for bottom-to-top feedback.

9.2 Extensions

In the following, we discuss some possible extensions of the concepts and results derived in this thesis.

Coordinated non-linear systems

A straightforward generalization of the definition of coordinated linear systems to the non-linear setting is given by the following:

9.2.1. Definition. Given a system of the form

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x,u), \end{aligned}$$

and given decompositions $X = X_1 \dot{+} X_2 \dot{+} X_c$, $U = U_1 \dot{+} U_2 \dot{+} U_c$ and $Y = Y_1 \dot{+} Y_2 \dot{+} Y_c$ of the state, input and output spaces, we call this system a **coordinated system** if there exists a representation of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_c, u_1, u_c) \\ f_2(x_2, x_c, u_2, u_c) \\ f_c(x_c, u_c) \end{bmatrix},$$
$$\begin{bmatrix} y_1 \\ y_2 \\ y_c \end{bmatrix} = \begin{bmatrix} h_1(x_1, x_c, u_1, u_c) \\ h_2(x_2, x_c, u_2, u_c) \\ h_c(x_c, u_c) \end{bmatrix}.$$

The information available to each subsystem is restricted in the same way as for coordinated linear systems: The coordinator receives no information from the subsystems, and each subsystem only has its own local information and information about the coordinator state and input, but no information about the other subsystem.

If the coordinated non-linear system of Definition 9.2.1 is linearizable around a point $(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_c \end{bmatrix}, \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_c \end{bmatrix} \right)$ then its local linearization around this point is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & 0 & \frac{\partial f_c}{\partial x_1} \\ 0 & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_c}{\partial x_2} \\ 0 & 0 & \frac{\partial f_c}{\partial x_c} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & 0 & \frac{\partial f_c}{\partial u_1} \\ 0 & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_c}{\partial u_2} \\ 0 & 0 & \frac{\partial f_c}{\partial u_c} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_c \end{bmatrix}$$

where $\frac{\partial f_j}{\partial x_k}$, $j, k = 1, 2, c$ stands for the total derivative of f_j with respect to x_k at the point $(\bar{x}, \bar{u})$. Thus the coordinated non-linear system is approximated by a coordinated linear system in a neighborhood of the point $(\bar{x}, \bar{u})$, and our results concerning minimality, controllability, observability and optimal control apply locally around this point.

Extendability to hierarchical linear systems

The extendability of various results to hierarchical systems with more subsystems and/or several layers was already discussed in the previous chapters; in particular, Sections 6.3 and 7.4 are devoted to the extension of the LQ control synthesis algorithms developed in this thesis to more general hierarchies.

Further work is needed on the generalization of the construction procedures in Sections 4.1 and 4.2 to interconnections of several subsystems or hierarchical systems with several layers – the most straightforward extension would be to apply the existing procedures repeatedly, for the different interconnections or the different parts of the system. The concept of minimal communication introduced in Section 4.3 can be straightforwardly extended to hierarchical systems, but the concept of a minimal coordinator would have to be changed, taking into account the coordinating systems on all levels of the hierarchy.

An extension of the concepts of local observability and local controllability amounts to a mere reformulation of the existing concepts, and the related concepts of local stabilizability and local detectability are equivalent to the existence of structured stabilizing state feedbacks and converging state observers for any hierarchical linear system with a top-to-bottom information structure.

While our refinement of the concept of reachability is easily extendable to hierarchical systems, with each local input affecting the corresponding local system and all its followers, a controllability decomposition of the system according to independent and jointly reachable subspaces as in Section 5.2 will in general be infeasible. In how far the different concepts of controllability can be extended to hierarchical systems, is an open question. The same considerations apply to the concepts of indistinguishability and observability in Section 5.3.

Almost coordinated linear systems

In practice, and in particular if the model parameters are estimated from numerical data, it may be impossible to find a non-trivial system decomposition with conditionally independent subsystems, corresponding to zero blocks in the system matrices. Instead, one may want to extend the concept of a coordinated linear system to structured systems in which the corresponding blocks are not required to be exactly zero, but very small in norm. The concept of an almost invariant subspace provides such a generalization from exact invariance to approximate invariance.

Almost invariant subspaces were first introduced in [69], and can be defined as follows:

Given a linear system of the form $\dot{x} = Ax,\ x(t_0) = x_0$, we call a subspace V_a of the state space X an **almost invariant subspace** if for all $x_0 \in V_a$ and $\epsilon > 0$ we have $d(x(t), V_a) < \epsilon$ for all $t \geq t_0$.

Replacing the invariance and independence conditions imposed on the state, input and output spaces of a coordinated linear system in Definition 3.1.1 by almost-invariance and almost-independence conditions may help in finding a meaningful definition of an almost coordinated linear system.

Coordinated differential-algebraic equations

Differential-algebraic equations provide a framework for the combined consideration of dynamical systems and algebraic constraints (see [31]). Non-linear differential-algebraic equations are of the form

$$F(t, x(t), \dot{x}(t)) = 0, \quad x(t_0) = x_0,$$

with state trajectory $x(\cdot) : [t_0, \infty) \to X$ and initial state $x_0 \in X$, and with $t \in [t_0, \infty)$ and $F : (X \times X \times \mathbb{R}) \to \mathbb{R}^m$.

Given such a system, and given a decomposition $X = X_1 \dot{+} X_2 \dot{+} X_c$ with dimensions $\dim X_1 = n_1$, $\dim X_2 = n_2$ and $\dim X_c = n_c$, we call it a **coordinated differential-algebraic equation** if there exists a representation of the form

$$\begin{bmatrix} F_1(t, x_1, x_c, \dot{x}_1, \dot{x}_c) \\ F_2(t, x_2, x_c, \dot{x}_c, \dot{x}_c) \\ F_c(t, x_c, \dot{x}_c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} (t_0) = \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,c} \end{bmatrix},$$

with $F_1 : ([t_0, \infty) \times X_1 \times X_c \times X_1 \times X_c) \to \mathbb{R}^{n_1}$, $F_2 : ([t_0, \infty) \times X_2 \times X_c \times X_2 \times X_c) \to \mathbb{R}^{n_2}$ and $F_c : ([t_0, \infty) \times X_c \times X_c) \to \mathbb{R}^{n_c}$.

If we have a coordinated differential-algebraic equation, and F is differentiable with respect to $(t, x, \dot{x})$ in a point $(\bar{t}, \bar{x}, \bar{y})$ then the equation can be locally approximated by its linearization around $(\bar{t}, \bar{x}, \bar{y})$:

$$\left[\begin{array}{c|ccc|ccc} \frac{\partial F_1}{\partial t} & \frac{\partial F_1}{\partial x_1} & 0 & \frac{\partial F_1}{\partial x_c} & \frac{\partial F_1}{\partial \dot{x}_1} & 0 & \frac{\partial F_1}{\partial \dot{x}_c} \\ \frac{\partial F_2}{\partial t} & 0 & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_c} & 0 & \frac{\partial F_2}{\partial \dot{x}_2} & \frac{\partial F_2}{\partial \dot{x}_c} \\ \frac{\partial F_c}{\partial t} & 0 & 0 & \frac{\partial F_c}{\partial x_c} & 0 & 0 & \frac{\partial F_c}{\partial \dot{x}_c} \end{array}\right] \left[\begin{array}{c} t \\ \hline x_1 \\ x_2 \\ x_c \\ \hline \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{array}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} -\frac{\partial F_1}{\partial \dot{x}_1} & 0 & -\frac{\partial F_1}{\partial \dot{x}_c} \\ 0 & -\frac{\partial F_2}{\partial \dot{x}_2} & -\frac{\partial F_2}{\partial \dot{x}_c} \\ 0 & 0 & -\frac{\partial F_c}{\partial \dot{x}_c} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & 0 & \frac{\partial F_1}{\partial x_c} \\ 0 & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_c} \\ 0 & 0 & \frac{\partial F_c}{\partial x_c} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} \frac{\partial F_1}{\partial t} \\ \frac{\partial F_2}{\partial t} \\ \frac{\partial F_c}{\partial t} \end{bmatrix} t,$$

with $\frac{\partial F_j}{\partial x_j}, \frac{\partial F_j}{\partial \dot{x}_j} \in \mathbb{R}^{n_j \times n_j}$ for $j = 1, 2, c$. If the matrix on the left is invertible then our system is locally approximated by a coordinated affine system (i.e. a coordinated linear system with an additional affine term which only depends on t). This special case motivates the consideration that this class of systems may show a similar behavior as coordinated linear systems, in the sense that some of the results of this thesis may be extendable to the differential-algebraic case.

Other possible extensions

Another direction in which the theory of coordinated linear systems may be extended is the introduction of disturbances into the system dynamics: The case of coordinated linear systems with Gaussian disturbances was considered in [41], where some of the results of this thesis were already shown to carry over to the stochastic case. Other types of stochastic disturbances, especially related to communication errors and communication delays among the different parts of the system, may also be interesting to consider.

Moreover, in this thesis we restricted attention to LQ optimal control problems. Other types of control problems, such as robust control or an optimal control formulation which includes communication costs in the objective function, could also be useful for practical purposes. In particular, an interesting question is whether the separation of the overall problem into conditionally-independent subproblems shown in Theorem 6.2.7 carries over to these classes of control problems.

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Samenvatting (Dutch Summary)

Gecoördineerde regeling van lineaire systemen

Gecoördineerde lineaire systemen zijn decentrale lineaire systemen met een specifieke structuur: Zij bestaan uit drie of meer deelsystemen, waarvan er één de rol van een coördinator speelt voor de andere deelsystemen. De communicatiestructuur weerspiegelt deze rolverdeling: De coördinator kan de andere deelsystemen beïnvloeden, maar de andere deelsystemen hebben geen invloed op de coördinator of op elkaar. Een gecoördineerd lineair systeem representeert dus één aftakking in een hierarchisch lineair systeem, met een gerichtte boom als communicatiestructuur.

Mogelijke toepassingen van gecoördineerde lineaire systemen omvatten decentrale systemen met een inherente hierarchische structuur, zoals verkeersnetwerken of elektriciteitsnetwerken, maar ook andere decentrale systemen die een hierarchische aanpak toelaten, zoals groepen of formaties van gedeeltelijk-autonome voertuigen. Bovendien kunnen ongestructureerde lineaire systemen vaak in meerdere deelsystemen met een hierarchische structuur worden ontbonden, om de complexiteit van bijbehorende regelproblemen to reduceren.

Het in dit proefschrift beschreven onderzoek focust op de volgende vragen:

(1) Hoe kunnen ongestructureerde lineaire systemen, of decentrale systemen met een niet-hierarchische communicatiestructuur, in gecoördineerde lineaire systemen worden onbonden of getransformeerd? Is een gegeven gecoördineerd lineair systeem 'zo decentraal mogelijk', of kan het –binnen de hierarchische structuur– nog verder worden ontbonden?

(2) Welk deelsysteem is regelbaar door welke ingangsvariabele – is een lokale regelaar voldoende, of is er coördinatie nodig om het gewenste regelgedrag te bereiken? Zijn alle voor de implementatie van de gewenste regelaar nodige meetwaardes lokaal waarneembaar, of is communicatie van deze meetwaardes vereist?

(3) Kunnen wij voor een gegeven gecoördineerd lineair systeem een regelaar vinden die tot het gewenste regelgedrag leidt maar ook de hierarchische communicatiestructuur respecteert? Hoe presteert deze regelaar, vergeleken met ongestructureerde regelaars? Kan de prestatie verbeterd worden door gebeurtenis-gebaseerde terugkoppeling van de gecoördineerde deelsystemen naar de coördinator toe te laten?

In de hoofdstukken 1 en 2 wordt het onderwerp van gecöordineerde lineaire systemen gemotiveerd en geïntroduceerd, en relevante concepten en resultaten uit de klassieke systeem- en regeltheorie worden samengevat. De definitie en sommige basiseigenschappen van gecoördineerde lineaire systemen worden in hoofdstuk 3 beschreven en met gerelateerde decentrale systemen vergeleken.

Vraag (1) is het onderwerp van hoofdstuk 4: Hier worden expliciete procedures voor de transformatie van ongestructureerde of niet-hierarchische lineaire systemen naar gecoördineerde lineaire systemen gegeven. Mogelijke definities van minimaliteit van een gecoördineerd lineair systeem worden voorgesteld, om het concept van een 'zo decentraal mogelijk' systeem te formaliseren.

De in vraag (2) bedoelde verdere onderverdeling van de klassieke concepten van regelbaarheid en waarneembaarheid ten opzichte van de verschillende deelsystemen, ingangsvariabelen en uitgangsvariabelen wordt in hoofdstuk 5 beschreven. Hierbij onderscheiden wij tussen onafhankelijk en gezamenlijk regelbare en waarneembare deelruimtes, om ook het niet-bedoelde effekt van de coördinator op de andere deelsystemen te identificeren. Decomposities van de toestandsruimtes van de deelsystemen volgens deze verfijnde concepten geven inzicht in de vraag welke delen lokaal regelbaar of waarneembaar zijn, en waar coördinatie of communicatie vereist is.

Vanwege de algemeenheid van de formulering, is vraag (3) makkelijker gesteld dan beantwoord: een wiskundige analyse van het regelgedrag is alleen voor bepaalde regelproblemen mogelijk. Hoofdstukken 6 en 7 behandelen de beperking van vraag (3) tot LQ (lineair-kwadratische) regelproblemen: In hoofdstuk 6 construëren wij een lineaire terugkoppeling voor een gecoördineerd lineair systeem die een kwadratisch kostenkriterium minimaliseert maar ook de communicatiestructuur respecteert. De uitbreiding van de toelaatbare regelaars naar lineaire terugkoppelingen die, naast de binnen het raam van de communicatiestructuur beschikbare toestandswaardes, ook over stuksgewijs-konstante benaderingen van de andere toestandswaardes beschikken, is het onderwerp van hoofdstuk 7. De hierarchische structuur van het systeem heeft ten gevolg dat de lokale regelproblemen voor de deelsystemen onafhankelijk van elkaar op te lossen zijn; alleen voor het regelprobleem voor de coördinator is de rest van het systeem van belang.

Ter illustratie van de in dit proefschrift ontwikkelde theorie zijn in hoofdstuk 8 twee toepassingen uitgewerkt: Eén betreft de formatievlucht van drie onbemande onderwatervoertuigen – vanwege de hoge kosten en beperking van onderwatercommunicatie, is het voordelig één voertuig de rol van de coördinator toe te wijzen, die zijn positie regelmatig aan de andere twee voetuigen stuurt. Het formatieprobleem is een voorbeeld van het in hoofdstuk 6 besproken LQ probleem. Om de robuustheid tegen storingen en communicatieproblemen te verbeteren, is het voor de andere twee deelsystemen mogelijk feedback aan de coördinator te sturen als zij het referentiesignaal niet kunnen volgen.

De andere toepassing betreft de coördinatie van instroombeperkingen aan naburige opritten van een snelweg: Als de gezamenlijke toestroom de capaciteit overschrijdt, leidt de instroombeperking tot wachtrijen bij de oprit. Bij een uitsluitend lokale regeling van de instroom kunnen de wachttijden bij naburige opritten uiteenlopen – een effect dat door coördinatie te voorkomen is.