SUPEREXTENSIONS

By

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In [1] J. de Groot introduced the notion of a superextension of a space with respect to a closed subbase for the space. Our purpose here is to list some of the basic properties of superextensions and to indicate the invariance of various topological properties under suitable restrictions on the space and subbase.

Let X be a topological space and $\mathcal{S}$ a closed subbase for X; we only consider closed subbases for the topology. We say that

(i) $\mathcal{S}$ is a T_1- subbase iff for each $x \in X, \{x\} = \bigcap\{S \in \mathcal{S} \mid x \in S\}$ and for each $x \in X$ and $S \in \mathcal{S}$ with $x \notin S$, there exists $T \in \mathcal{S}$ such that $x \in T$ and $S \cap T = \emptyset$.

(ii) $\mathcal{S}$ is normal in case for each $S, T \in \mathcal{S}$ with $S \cap T = \emptyset$, there exists $S_1, T_1 \in \mathcal{S}$ with $S_1 \cup T_1 = X$, $S_1 \cap T = \emptyset$ and $S \cap T_1 = \emptyset$ (i. e. in the terminology of [2], S and T are screened by S_1 and T_1).

We observe that:

(1) X is T_1 iff X has a T_1-subbase.

(2) X is T_1 and completely regular iff it has a normal T_1-subbase (c. f. [2] for a proof.)

To construct the superextension of X with respect to a T_1- subbase $\mathcal{S}$ of X, we need the notion of a linked system of $\mathcal{S}$; a linked system of $\mathcal{S}$ is a subcollection of $\mathcal{S}$ with the property that every pair of elements of the subcollection have nonempty intersection.

If $\mathcal{S}$ is a T_1- subbase for X, then we let $\lambda_{\mathcal{S}} X$ be the set of all maximal linked systems (m.l.s.) of $\mathcal{S}$; $\mathcal{M}, \mathcal{N}$ will denote elements of $\lambda_{\mathcal{S}} X$. If X is any T_1- space, then we let λX be the set of all maximal linked systems of the base of all closed sets of X.

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* Presented by this author.

For $A \subset X$, define

$$A^+ = \{\mathcal{M} \in \lambda_{\mathcal{S}}X \mid \exists\ S \in \mathcal{M} \text{ with } S \subset A\}.$$

$\{S^+ \mid S \in \mathcal{S}\}$ is a subbase fora topology on $\lambda_{\mathcal{S}}X$ and $\lambda_{\mathcal{S}}X$ with this topology is called the superextension of X with respect to $\mathcal{S}$.

The following proposition contains some easily proved consequences of the definitions.

PROPOSITION 1. Let $\mathcal{S}$ be a T_1- subbase for X.

(i) If A, $B \subset X$, then $A \cap B = \emptyset$ iff $A^+ \cap B^+ = \emptyset$.

(ii) If $A \subset B \subset X$, then $A^+ \subset B^+$.

(iii) If $S \in \mathcal{S}$, then $S^+ \cup (X \setminus S)^+ = \lambda_{\mathcal{S}}X$.

(iv) If $\mathcal{M} \in \lambda_{\mathcal{S}}X$ with $\cap \mathcal{M} \neq \emptyset$, then there exists $x \in X$ such that $\cap \mathcal{M} = \{x\}$.

(v) If $x \in X$, then $\{S \in \mathcal{S} \mid x \in S\} \in \lambda_{\mathcal{S}}X$.

COROLLARY. The mapping $x \longrightarrow \{S \in \mathcal{S} \mid x \in S\}$ is an embedding of X in $\lambda_{\mathcal{S}}X$.

THEOREM 1. If $\mathcal{S}$ is a T_1- subbase for X, then $\lambda_{\mathcal{S}}X$ is a compact T_1- space; indeed, $\lambda_{\mathcal{S}}X$ is supercompact [1].

In general, it is not the case that $\lambda_{\mathcal{S}}X$ is Hausdorff even when X is a compact Hausdorff space.

THEOREM 2. Let X be a space with a normal T_1- subbase $\mathcal{S}$. Then the following hold:

(1) $\lambda_{\mathcal{S}}X$ is compact Hausdorff.

(2) If Y is a T_1- space and f is a continuous mapping of Y onto X, then f has a continuous extension from λY onto $\lambda_{\mathcal{S}}X$.

(3) If X is compact and if the weight of X is infinite, then the weight of $\lambda_{\mathcal{S}}X$ is equal to the weight of X.

(4) If X is compact and zero-dimensional and $\mathcal{S}$ contains all of the clopen sets of X, then $\lambda_{\mathcal{S}}X$ is zero-dimensional.

COROLLARY. If X is a Cantor space, then λX is a Cantor space of the same weight. In particular, if X is the Cantor discontinuum, then λX is the Cantor discontinuum.

Suppose now that $\mathcal{S}$ is a T_1- subbase for X containing all of the finite subsets of X. An m.l.s. $\mathcal{M}$ of $\mathcal{S}$ is called finite if there exists a finite set $F \subset X$ and an m.l.s. $\mathcal{N}$ of 2^F such that $\mathcal{N} \subset \mathcal{M}$.

PROPOSITION 2. If $\mathcal{S}$ is a T_1- subbase for X containing all of the finite subsets of X, then the collection of all finite maximal linked systems is dense in $\lambda_{\mathcal{S}} X$.

With the aid of the finite maximal linked systems, we can prove the following:

THEOREM 3 [A. Verbeek]. *If* X *is connected and* $\mathcal{S}$ *is a* T_1- *subbase containing all of the finite subsets of* X *or if* $\mathcal{S}$ *is a normal* T_1- *subbase, then* $\lambda_{\mathcal{S}} X$ *is connected and locally connected*.

PROBLEM. If I is the unit interval, is λ I the Hilbert cube? We know already that λ I is a Peano continuum (i. e. is compact metrizable, connected, and locally connected.)

REMARK. Superextensions for completely regular spaces X with respect to the base of all zerosets contain the Čech-Stone compactification βX of X (i. e. $X \subset \mathrm{cl}\ X = \beta X \subset \lambda_{\mathcal{Z}} X$), and mappings can be extended over the superextension in a natural way. Furthermore, $\lambda_{\mathcal{S}} X$ is always supercompact.

Proofs of the results here will appear later.

REFERENCES.

1. J. de Groot, *Superextensions and supercompactness*, Berlin Topology Symposium, 1967.
2. J. de Groot and J. Aarts, *Complete regularity as a separation axiom*, to appear in Can. J. Math..

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