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EXTENSION OF COLOURINGS OF THE EDGES OF A COMPLETE (UNIFORM HYPER)GRAPH

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Extension of colourings of the edges of a complete (uniform hyper)graph

by

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ABSTRACT

Let $1 \leq m < n$ and consider the complete graph on 2m points K_{2m} as a subgraph of K_{2n}. We prove that if an edge-colouring of K_{2m} (with 2m-1 colours) is given, this colouring can be extended to a colouring of K_{2n} (with 2n-1 colours) iff $2m \leq n$. The corresponding problem for complete h-uniform hypergraphs is discussed, the case h = 3 is solved completely and asymptotic results are given for arbitrary h.

KEYWORDS & PHRASES: *parallelism*

0. INTRODUCTION

Let X be a finite set and let $P_h(X)$ be the collection of all h-element subsets of X. A *parallelism* on $P_h(X)$ is an equivalence relation of $P_h(X)$ such that the members of each equivalence class form a partition of X. Obviously for the existence of a parallelism $h \mid \#X$ is necessary, and in Baranyai [1] it is shown that this condition suffices. A subset Y of X (provided with a given parallelism) is called a *subspace* when the restriction of the equivalence relation on $P_h(X)$ to $P_h(Y)$ yields a parallelism on Y [-in other words, when it never happens that $H_1 \,//\, H_2$ and $H_1 \subset Y$ but H_2 intersects both Y and X\Y]. Cameron [5] remarked that if Y is a proper subspace of X then $2\,\#Y \le \#X$, and in Brouwer [3] it is shown that if $2h \mid \#X$ then there exists a //ism on X with a subspace Y such that $\#Y = \frac{1}{2}\,\#X$. More generally it can be shown in the same way that if $th \mid \#X$ then there exists a //ism on X with a subspace Y such that $\#Y = \frac{1}{t} \cdot \#X$ (see [2], [4]). We conjecture that the requirements $2\,\#Y \le \#X$ and $\#Y \equiv \#X \equiv 0 \pmod h$ suffice in all cases for the existence of a //ism on X with subspace Y. In this note we prove this conjecture for h = 2 or 3 and for h arbitrary, n sufficiently large.

0A. Graph theoretic terminology and upper bound.

These results can be phrased in the language of (hyper)graphs as follows:

A parallelism on $P_h(X)$, where $\#X = n$, is a colouring of the complete h-uniform hypergraph on n vertices with $\frac{h}{n}\binom{n}{h} = \binom{n-1}{h-1}$ colours, where edges with the same colour are disjoint. If Y is a subspace of X, where $\#Y = m$, then any such colouring of Y (with $\binom{m-1}{h-1}$ colours) can be extended to a colouring of X with $\binom{n-1}{h-1}$ colours. A necessary condition for this to be possible is that $m \le \frac{1}{2}n$ [for: the $\binom{m-1}{h-1}$ colours used to colour the h-subsets of Y colour $\frac{n-m}{h}\binom{m-1}{h-1}$ h-subsets of X\Y, so that $\frac{n-m}{h}\binom{m-1}{h-1} \le \binom{n-m}{h}$ hence $\binom{m-1}{h-1} \le \binom{n-m-1}{h-1}$, and consequently $m \le n-m$].

OB. A general existence theorem.

Define for fixed X and Y (where $Y \subset X$, $\#X = n$, $\#Y = m$) the *weight* of an h-subset H of X as $\#(H \cap Y)$. In order to prove the existence of a parallelism on X with subspace Y it suffices to indicate a suitable weight distribution of the parallel classes (by the theorem quoted below). If the parallel classes are $F_z (z=1,\ldots,\binom{n-1}{h-1})$ and F_z contains X_{gz} elements of weight g ($0 \le g \le h$) then obviously the X_{gz} satisfy

(1) $$\sum_g X_{gz} = \frac{n}{h}$$

(2) $$\sum_g gX_{gz} = m$$

(3) $$\sum_z X_{gz} = \binom{m}{g}\binom{n-m}{h-g}.$$

Conversely, given a matrix (X_{gz}) satisfying these equations (where the X_{gz} are nonnegative integers), there exists a parallelism on X with this weight distribution. In particular if for $\binom{m-1}{h-1}$ values of z we have $X_{0z} = \frac{n-m}{h}$ and $X_{hz} = \frac{m}{h}$ and $X_{gz} = 0$ ($1 \le g \le h-1$), then Y will be a subspace of this parallelism.

That the above is true can be proved in the same way as it was proved in the case $n = 2m$ in [3]; on the other hand, it is a special case of a very general theorem in [2].

1. THE CASE h = 2

By what was stated in section OB we have to find nonnegative integers X_{gz} such that for $z = 1,\ldots,n-1$ we have

$$\sum_{g=0}^{2} X_{gz} = \tfrac{1}{2}n$$

$$\sum_{g=0}^{2} gX_{gz} = m$$

$$\sum_z X_{gz} = \binom{m}{g}\binom{n-m}{2-g} \qquad (g=0,1,2).$$

and for m-1 values of z we have

$$X_{0z} = \tfrac{1}{2}(n-m), \quad X_{2z} = \tfrac{1}{2}m \text{ and } X_{1z} = 0.$$

The unique solution is

$$X_{0z} = \frac{n}{2} - m,\ X_{1z} = m,\ X_{2z} = 0 \text{ for } n-m \text{ values of } z$$

and

$$X_{0z} = \tfrac{1}{2}(n-m),\ X_{1z} = 0,\ X_{2z} = \tfrac{1}{2}m \text{ for } m-1 \text{ values of } z.$$

In particular there is a solution.

2. THE CASE $m \mid n$

Suppose $n = mt$. Then (as already remarked in [2] and [4]) a solution exists. For any ordered t-tuple $(h_1, \ldots, h_t)$ with $\sum h_j = h$ take $\frac{h}{n} \prod_j \binom{m}{h_j}$ columns z with ($X_{gz} = 0$ if g does not occur among the h_j and)

$$X_{gz} = \sum_{g=h_j} \frac{m}{h}.$$

Obviously

$$\sum_g X_{gz} = t \cdot \frac{m}{h} = \frac{n}{h},$$

$$\sum_g gX_{gz} = \sum_j h_j \frac{m}{h} = m; \text{ also}$$

$$\sum_z X_{gz} = \sum_{\substack{(h_1 \ldots h_t) \\ \Sigma h_j = h}} \left(\sum_{g=h_j} \frac{m}{h}\right) \cdot \frac{h}{n} \prod_j \binom{m}{h_j} =$$

$$= \frac{1}{t} \sum_{\substack{(h_1, \ldots, h_t) \\ \Sigma h_j = h}} \left(\sum_{g=h_j} 1\right) \cdot \prod_j \binom{m}{h_j} =$$

$$= \sum_{\substack{(h_1, \ldots, h_t) \\ \Sigma h_j = h,\, g = h_1}} \prod_j \binom{m}{h_j} = \binom{m}{g}\binom{n-m}{h-g},$$

Hence this yields a solution of (1) - (3).

Perhaps you remark that

$$\frac{h}{n} \Pi_j \binom{m}{h_j}$$

need not be an integer; but, since the t-tuples $(h_1,\ldots,h_t)$, $(h_t,h_1,\ldots,h_{t-1})$, $,\ldots,(h_2,\ldots,h_t,h_1)$ yield the same columns all we need is that

$$\frac{h o(\sigma)}{n} \Pi_j \binom{m}{h_j}$$

is an integer, where $o(\sigma)$ is the order of the cyclic permutation $(h_1,\ldots,h_t) \to (h_t,h_1,\ldots,h_{t-1})$. Now

$$\frac{h o(\sigma)}{n} \Pi_j \binom{m}{h_j} = \sum_{i=1}^{t} \frac{o(\sigma)}{t} (\Pi_{j\neq i} \binom{m}{h_j}) \cdot \binom{m-1}{h_i-1} =$$

$$\sum_{i=j}^{o(\sigma)} (\Pi_{j\neq i} \binom{m}{h_j}) \cdot \binom{m-1}{h_i-1},$$ which is an integer.

It remains to prove that for $\binom{m-1}{h-1}$ values of z we have $X_{hz} = \frac{m}{h}$ and $X_{gz} = 0$ $(1 \leq g \leq h-1)$. We obtain such solutions from the t-tuples $(0\ldots h\ldots 0)$. The number of solutions of this type is $\frac{h o(\sigma)}{n} \Pi_j \binom{m}{h_j} = \binom{m-1}{h-1}$ as required.

3. THE CASE h = 3

For arbitrary h we can somewhat simplify our equations: If we let $X_{0z} = \frac{n-m}{h}$, $X_{hz} = \frac{m}{h}$ and $X_{gz} = 0$ $(1 \leq g \leq h-1)$ for $z = \binom{n-1}{h-1} - \binom{m-1}{h-1} + 1$, $,\ldots,$ $\binom{n-1}{h-1}$ then we have to solve

(1') $$\sum_{g=1}^{h-1} X_{gz} \leq \frac{n}{h}$$

(2') $$\sum_{g=1}^{h-1} gX_{gz} = m$$

(3') $$\sum_z X_{gz} = \binom{m}{g}\binom{n-m}{h-g} \qquad (1 \leq g \leq h-1)$$

where now z runs from 1 up to $\binom{n-1}{h-1} - \binom{m-1}{h-1}$ [since for these z we can define X_{0z} and X_{hz} by $X_{0z} = \frac{n}{h} - \sum_{g=1}^{h-1} X_{gz}$ and $X_{hz} = 0$ and from the other equations it follows that (3) holds, that is, $\sum_{\text{all } z} X_{0z} = \binom{m}{0}\binom{n-m}{h-0}$].

Note that

(4') $$\sum_z 1 = \binom{n-1}{h-1} - \binom{m-1}{h-1}$$

follows from (1') - (3').
Our solutions (X_{gz}) will contain many identical columns say columns (Y_{gi}) with multiplicity N_i. Rewriting (1') - (3') we get:

(1") $$\sum_{g=1}^{h-1} Y_{gi} \le \frac{n}{h}$$

(2") $$\sum_{g=1}^{h-1} gY_{gi} = m$$

(3") $$\sum_i N_i Y_{gi} = \binom{m}{g}\binom{n-m}{h-g}$$

(4") $$\sum_i N_i = \binom{n-1}{h-1} - \binom{m-1}{h-1}.$$

In the special case h = 3 we need two different columns; in the table below we give N_1 and Y_{2i} (i=1,2) - then $Y_{1i} = m - 2Y_{2i}$, and $N_2 = \binom{n-1}{2} - \binom{m-1}{2} - N_1$.

Case	N_1	Y_{21}	Y_{22}
$m \le \frac{n}{3}$, m even	$\frac{1}{2}(n-m)(n-m-1)$	0	$\frac{m}{2}$
$m \le \frac{n}{3}$, m odd	$\frac{1}{2}(n-m)(n-2m)$	0	$\frac{m}{3}$
$m \ge \frac{n}{3}$:			
$n \equiv m \equiv 0 \pmod 2$	$\frac{3}{2}\, m(n-m-1)$	$\frac{1}{6}(4m-n)$	$\frac{m}{2}$
$n \equiv 1,\ m \equiv 0 \pmod 2$	$\frac{3}{2}\, m(n-m)$	$\frac{1}{6}(4m-n+1)$	$\frac{m}{2}$
$n \equiv m \equiv 1 \pmod 2$	$\frac{3}{2}\,(m-1)(n-m-3)$	$\frac{1}{6}(4m-n-3)$	$\frac{m-1}{2}$
$n \equiv 0,\ m \equiv 1 \pmod 2$	$\frac{3}{2}\,(m-1)(n-m)$	$\frac{1}{6}(4m-n)$	$\frac{m-1}{2}$

That this is indeed a solution can be readily verified.

3. THE CASE h = 4

In this case we have an easy solution for $n \geq 4m$; we did not bother to look for solutions if $2m < n < 4m$.
Here the matrices $(Y_{gi})_{1 \leq g \leq 3,\ 1 \leq i \leq 3}$ can be taken as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & \frac{1}{2}m & 0 \\ 0 & 0 & \frac{1}{3}m \end{bmatrix} \quad \text{if } 3|m \quad \text{and}$$

$$\begin{bmatrix} m & 0 & \frac{1}{4}m \\ 0 & \frac{1}{2}m & 0 \\ 0 & 0 & \frac{1}{4}m \end{bmatrix} \quad \text{otherwise.}$$

The multiplicities N_i are uniquely determined from the (Y_{gi}) and (3"), (4").

5. ASYMPTOTIC RESULTS

For n large (for instance $n \geq mh^{3/2}$) we can give an explicit solution as follows:
We define the matrix (Y_{gi}) and multiplicities N_i $1 \leq g \leq h-1$, $1 \leq i \leq h-1$ with help of the numbers Y_g $2 \leq g \leq h-1$ which are to be chosen later.
The matrix (Y_{gi}) will contain 0'-s except in the first row and the main diagonal - this explains why the indices g and i will be a little bit mixed.
Let $Y_{gi} = \delta_{gi} Y_g$ for $2 \leq g \leq h-1$ and $1 \leq i \leq h-1$

$$Y_{1i} = m - \sum_{g=2}^{h-1} gY_{gi} = m - iY_i \quad (\text{supposing } Y_1 = 0)$$

$$N_g = \frac{1}{Yg} \binom{m}{g}\binom{n-m}{h-g}$$

$$N_1 = \binom{n-1}{h-1} - \binom{m-1}{h-1} - \sum_{i=2}^{h-1} N_i$$

For this to be a solution first of all the Y_{gi} and the N_i must be nonnegative integers, that is,

(5) $$0 \le Y_i \le \frac{m}{i},$$

(6) $$Y_g \mid \binom{m}{g}\binom{n-m}{h-g},$$

(7) $$\binom{n-1}{h-1} - \binom{m-1}{h-1} - \sum_{g=2}^{h-1} \frac{1}{Y_g} \binom{m}{g}\binom{n-m}{h-g} \ge 0$$

and in order to satisfy (1") we need $n \ge mh$, while (2") - (4") are satisfied automatically.

One possible choice would be to take $Y_g = 1$ for all g. This satisfies (5) and (6), and since (7) is a polynomial in n of degree $h-1$ with leading coefficient $\frac{1}{(h-1)!} > 0$ this surely yields a solution when n is large enough. To get a bound that is linear in m we have to do some work:
Choose $Y_2 = [\frac{m}{2}]$; note that this satisfies (5) and (6) (since $[\frac{m}{2}] \mid \binom{m}{2}$).
If $g \mid m$ then choose $Y_g = \frac{m}{g}$; again this is OK.
In the general case choose

$$Y_g = \frac{m(h,g)}{h(m,g)} .$$

This choice satisfies (5) since $h \mid m$ so that

$$(h,g) \le (m,g) \text{ and } Y_g \le \frac{m}{h} < \frac{m}{g}.$$

also (6) is satisfied, for if $(m,g) = am + bg$ then

$$\frac{h}{m}\frac{(m,g)}{(h,g)} \binom{m}{g} = a\frac{h}{(h,g)} \binom{m}{g} + b\frac{h}{(h,g)} \binom{m-1}{g-1}$$

is integral.
Note that

$$Y_g \ge \frac{m}{h} \cdot \frac{1}{\frac{1}{2}g} = \frac{2m}{gh}$$

in this case (since if $g \nmid m$ then $(g,m) \le \frac{1}{2}g$), while also if $g \mid m$ then $Y_g = \frac{m}{g} \ge \frac{2m}{gh}$.

Now concerning (7) we find

$$\binom{n-1}{h-1} - \binom{m-1}{h-1} - \sum_{g=2}^{h-1} \frac{1}{Y_g} \binom{m}{g}\binom{n-m}{h-g} \geq$$

$$\binom{n-1}{h-1} - \binom{m-1}{h-1} - \frac{1}{Y_2}\binom{m}{2}\binom{n-m}{h-2} - \frac{h}{2}\sum_{g=3}^{h-1} \binom{m-1}{g-1}\binom{n-m}{h-g} \geq$$

$$\binom{n-1}{h-1} - \binom{m-1}{h-1} - m\binom{n-m}{h-2} - \frac{h}{2}\binom{m-1}{2}\binom{n-m}{h-3} - \frac{h}{2}\binom{m-1}{3}\binom{n-m}{h-4}$$

$$- \frac{h}{2}(h-5) \binom{m-1}{4}\binom{n-m}{h-5} \geq$$

$$\binom{n-1}{h-1}\left\{1 - \frac{1}{2^{h-1}} - \frac{m(h-1)}{(n-1)} - \frac{h(h-1)(h-2)}{2}\binom{m-1}{2}\frac{1}{(n-1)(n-2)} - \right.$$

$$\frac{h(h-1)(h-2)(h-3)}{2}\binom{m-1}{3}\frac{1}{(n-1)(n-2)(n-3)} -$$

$$\left. - \frac{h(h-1)(h-2)(h-3)(h-4)(h-5)}{2(n-1)(n-2)(n-3)(n-4)}\binom{m-1}{4}\right\}$$

$$\geq \binom{n-1}{h-1}\left\{1 - \frac{1}{2^{h-1}} - \frac{mh}{n} - \frac{m^2h^3}{4n^2} - \frac{m^3h^4}{12n^3} - \frac{m^4h^6}{48n^4}\right\}$$

$$\geq \binom{n-1}{h-1}\left\{1 - \frac{1}{8} - \frac{1}{2} - \frac{1}{4} - \frac{1}{24} - \frac{1}{48}\right\} = \binom{n-1}{h-1} \cdot \frac{3}{48} > 0$$

where we used $n \geq mh^{3/2}$ and $h \geq 4$ (and the facts that $\binom{m-1}{4}\binom{n-m}{h-5}$ is larger than $\binom{m-1}{g-1}\binom{n-m}{h-g}$ for $g \geq 6$, and that $\frac{a-1}{b-1} < \frac{a}{b}$ if $a < b$).
This proves that a solution exists when $n > mh^{3/2}$.

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