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MATHEMATICS

ON ELFVING'S PROBLEM OF IMBEDDING A TIME-DISCRETE MARKOV-CHAIN IN A TIME-CONTINUOUS ONE FOR FINITELY MANY STATES. I

BY

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(Communicated by Prof. J. POPKEN at the meeting of May 26, 1962)

Call a stochastic matrix (i.e. a square matrix of order n with elements $p_{jk} \geqslant 0$ for all $j, k \in \{1, 2, \ldots, n\}$ and $\sum_{k=1}^{n} p_{jk} = 1$ for all $j \in \{1, 2, \ldots, n\}$) a P-matrix. Then the following problem (further called Elfving's problem) is a specialization of a problem posed in CHUNG [1958] and CHUNG [1960] and due to ELFVING (cf. KINGMAN [1962], where an alternative approach to this problem is given).

ELFVING'*s problem*: Find conditions in order that to a given P-matrix P_1 there exists a matrix function $P(t)$ such that $P(t)$ is a P-matrix for each $t \geqslant 0$, for which [2])

$$(1) \qquad \begin{cases} \lim\limits_{t \downarrow 0} P(t) = I, \\ P(s+t) = P(s)P(t) \text{ for all real } s, t > 0, \\ P(1) = P_1. \end{cases}$$

For $n=2$ Elfving's problem has been solved by FRÉCHET [1952], page 255, and BELLMAN [1960]. In my thesis (RUNNENBURG [1960]) the case $n=3$ was considered.

It is well-known (cf. DOOB [1953], BIRKHOFF and VARGA [1958]), that if $P(t)$ exists, then

$$(2) \qquad P(t) = e^{Q_1 t} \quad \text{for all real } t \geqslant 0,$$

where Q_1 is a Q-matrix (i.e. a square matrix of order n with elements q_{jk} satisfying $q_{jk} \geqslant 0$ if $j \neq k$ for all $j, k \in \{1, 2, \ldots, n\}$ and $\sum_{k=1}^{n} q_{jk} = 0$ for all $j \in \{1, 2, \ldots, n\}$).

Furthermore, if $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of Q_1, then by the Lévy–Hadamard theorem (cf. BODEWIG [1959], page 67) these eigenvalues lie inside the closed domain G consisting of all circles K_j with

1) Report S 299 of the Statistical Department, Mathematical Centre, Amsterdam.

2) I is the $n \times n$ identity matrix.

centres q_{jj} and radii $-q_{jj}$. Hence in particular $\operatorname{Re} \mu_j \leqslant 0$ for all j. The eigenvalues $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$ of $P(t)$ are given by

(3) $$\lambda_j(t) = e^{\mu_j t} \text{ for all real } t \geqslant 0 \text{ and all } j.$$

For any Q-matrix Q_1 and any real $t \geqslant 0$ we always have that $P(t) = e^{Q_1 t}$ is a P-matrix.

According to Gantmacher [1959], in 1938 Kolmogorov posed the problem: Characterize the complex numbers z with $|z| \leqslant 1$ which can occur as eigenvalues of an nth order stochastic matrix. This problem was partly solved in Dmitriev and Dynkin [1946] and definitely in Karpelewitsch [1951]. Making use of their results, the next theorem can be proved.

Theorem: The eigenvalues $\mu, \mu_2, \ldots, \mu_n$ of a Q-matrix (of order $n \geqslant 3$) satisfy

(4) $$\left(\frac{1}{2} + \frac{1}{n}\right)\pi \leqslant \arg \mu_j \leqslant \left(\frac{3}{2} - \frac{1}{n}\right)\pi \quad \text{for } 1 \leqslant j \leqslant n;$$

the only Q-matrices $Q_1{}^*$ (with elements $q_{jk}{}^*$) with at least one $\mu_j \neq 0$ on the boundary of this region are given (after a suitable renumbering of states, i.e. rows and columns at the same time) by

(5) $$q_{jk}{}^* = \begin{cases} -\alpha \text{ for } j = k, \\ \alpha \text{ for } j \equiv k-1 \pmod{n}, \\ 0 \quad \text{otherwise,} \end{cases}$$

where α is an arbitrary positive number.

Remark 1: From this theorem we conclude, that for $n \geqslant 3$ all eigenvalues $\lambda_j(t)$ of $P(t) = e^{Q_1 t}$ satisfy

(6) $$\lambda_j(t) \in H_n,$$

where H_n is a heart-shaped region in the complex plane, contained in the unit circle and symmetric with respect to the real axis, with the curve

(7) $$\exp\left(-1 + \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}\right)t \qquad \left(\text{where } 0 \leqslant t \leqslant \frac{\pi}{\sin 2\pi/n}\right)$$

as boundary in the region $\operatorname{Im} z \geqslant 0$. Hence Elfving's problem can only be solved for those P_1 for which all eigenvalues belong to the region H_n. The only matrices e^{Q_1} with a Q-matrix Q_1 for which at least one eigenvalue $\lambda \neq 1$ lies on the boundary curve of the region H_n are given by

(8) $$P_1{}^* = e^{Q_1{}^*}.$$

Proof of the theorem: If P_1 is a P-matrix, then $Q_1 = P_1 - I$ is a Q-matrix; if Q_1 is a Q-matrix, then $P_{1,\beta} = I + \beta Q_1$ (where $\beta > 0$ is chosen in such a way that $0 \leqslant 1+\beta q_{jj} \leqslant 1$ for $1 \leqslant j \leqslant n$, e.g. $\beta = -(\min_{1\leqslant j\leqslant n} q_{jj})^{-1}$ if some $q_{jj} < 0$) is a P-matrix. Hence if for an arbitrary Q-matrix Q_1 we have

$$\det(Q_1 - \mu I) = 0, \tag{9}$$

then

$$\det(P_{1,\beta} - (\beta\mu + 1)\, I) = 0. \tag{10}$$

Now Karpelewitsch has shown that the roots of the characteristic equation

$$\det(P_1 - \lambda I) = 0 \tag{11}$$

for an arbitrary P-matrix P_1 satisfy

$$\left(\frac{1}{2} + \frac{1}{n}\right)\pi \leqslant \arg(\lambda - 1) \leqslant \left(\frac{3}{2} - \frac{1}{n}\right)\pi \tag{12}$$

for $n \geqslant 3$, and so with $\lambda = \beta\mu + 1$, because of $\beta > 0$ we have

$$\left(\frac{1}{2} + \frac{1}{n}\right)\pi \leqslant \arg\mu \leqslant \left(\frac{3}{2} - \frac{1}{n}\right)\pi. \tag{13}$$

If $Q_1{}^*$ has an eigenvalue μ_j for which $\arg\mu_j = (1/2 + 1/n)\,\pi$ with $\mu_j \neq 0$, i.e. $\mu_j = \gamma(e^{2\pi i/n} - 1)$ with a $\gamma > 0$, then $P_{1,\beta}{}^*$ has an eigenvalue λ_j with $\arg(\lambda_j - 1) = (1/2 + 1/n)\,\pi$ and $\lambda_j - 1 \neq 0$. As Dmitriev and Dynkin have shown, in that case the elements $p_{jk}{}^*$ of $P_{1,\beta}{}^*$ satisfy (after a suitable renumbering of the rows and columns at the same time)

$$p_{jk}{}^* = \begin{cases} 1 - \alpha_j{}' & \text{for } j = k, \\ \alpha_j{}' & \text{for } j \equiv k - 1 \pmod{n}, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where $0 \leqslant \alpha_j{}' \leqslant 1$ for each $j \in \{1, 2, \ldots, n\}$. Therefore the elements $q_{jk}{}^*$ of $Q_1{}^*$ are given by

$$q_{jk}{}^* = \begin{cases} -\alpha_j & \text{for } j = k, \\ \alpha_j & \text{for } j \equiv k - 1 \pmod{n}, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where we have written α_j instead of $\alpha_j{}'/\beta$, and so $\alpha_j \geqslant 0$ for each $j \in \{1, 2, \ldots, n\}$.

For the characteristic equation of $Q_1{}^*$ we obtain

$$(\alpha_1 + \mu)(\alpha_2 + \mu) \ldots (\alpha_n + \mu) = \alpha_1 \alpha_2 \ldots \alpha_n. \tag{16}$$

In order that $\mu = \gamma(e^{2\pi i/n} - 1)$, with a constant $\gamma > 0$, is a root of this equation, all α_j must be positive. We shall prove that

$$\alpha_j = \gamma \text{ for all } j \in \{1, 2, \ldots, n\}, \tag{17}$$

for which it is sufficient to consider $\gamma = 1$.

If we introduce the finite positive numbers $\theta_j = 1/\alpha_j$ in (16) and substitute $e^{2\pi i/n} - 1$ for μ, we obtain

$$\{\theta_1 e^{2\pi i/n} + (1-\theta_1)\}\{\theta_2 e^{2\pi i/n} + (1-\theta_2)\} \ldots \{\theta_n e^{2\pi i/n} + (1-\theta_n)\} = 1. \tag{18}$$

Consider a line in the complex plane which does not pass through the origin. Let $r(\varphi)$ be the absolute value of the complex number on this line with argument φ. Then $\log r(\varphi)$ is a strictly convex analytic function of φ for finite $r(\varphi)$. Hence if $\varphi_1 \neq \varphi_2$, we have for any $p_1, p_2 > 0$ with $p_1 + p_2 = 1$

$$p_1 \log r(\varphi_1) + p_2 \log r(\varphi_2) > \log r(p_1\varphi_1 + p_2\varphi_2). \tag{19}$$

Therefore

$$r(\varphi_1)\, r(\varphi_2) \ldots r(\varphi_n) > r\left(\frac{\varphi_1 + \varphi_2 + \ldots + \varphi_n}{n}\right)^n, \tag{20}$$

unless $\varphi_1 = \varphi_2 = \ldots = \varphi_n$.

Now consider for $n \geqslant 3$ the points $\theta_j \cdot e^{2\pi i/n} + (1-\theta_j) \cdot 1$ for finite $\theta_j > 0$, which lie on the line passing through 1 and $e^{2\pi i/n}$. Then $0 < \varphi_j < (1/2 + 1/n)\pi$. Under the restriction $\varphi_1 + \varphi_2 + \ldots + \varphi_n \equiv 0 \pmod{2\pi}$, the product $r(\varphi_1) r(\varphi_2) \ldots r(\varphi_n)$ assumes its smallest value for $\varphi_1 = \varphi_2 = \ldots = \varphi_n = 2\pi/n$. Any other choice of values for the φ_j leads to a larger value for the product. Hence if we introduce polar coordinates in (18) by writing

$$\theta_j e^{2\pi i/n} + 1 - \theta_j = r(\varphi_j)\, e^{i\varphi_j}, \tag{21}$$

we find that under the condition "all θ_j are finite positive numbers" the left-hand side of (18) has smallest positive value 1, where the value 1 is only obtained if $\varphi_j = 2\pi/n$ for all j. Therefore, if equation (18) holds, we must have $\theta_j = 1$ for all j or $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1$.

Remark 2: For $n = 3$ one may easily verify that

$$P_1 = \begin{pmatrix} \alpha & 1-\alpha & 0 \\ 0 & \beta & 1-\beta \\ 1-\gamma & 0 & \gamma \end{pmatrix} \tag{22}$$

has characteristic equation

$$(1-\lambda)\{\lambda^2 + \lambda(1-\alpha-\beta-\gamma) + 1 - \alpha - \beta - \gamma + \alpha\beta + \beta\gamma + \gamma\alpha\} = 0. \tag{23}$$

If we take $\alpha + \beta + \gamma = 1$, then $\lambda_1 = 1$ and $\lambda_{2,3} = \pm i\sqrt{\beta\gamma + (\beta+\gamma)(1-\beta-\gamma)}$. Hence λ_1, λ_2 and λ_3 lie inside H_3 if β and γ are sufficiently small positive numbers. From DOOB [1953], page 239, we know that if Elfving's problem can be solved for a P_1, then $p_{ij}(t)$ for $i \neq j$ either vanishes identically or never, except when $t=0$. Now here $p_{21}(1)=0$ and $p_{21}(2)=(1-\beta)(1-\gamma)>0$ for small positive β and γ. Thus there exists a P_1 of order 3 for which Elfving's problem can not be solved even though all eigenvalues lie inside H_3.

Remark 3: Any P-matrix of order 3 which is also a circulant can be written

$$C_1 = \begin{pmatrix} \frac{1}{3}-2a-2b & \frac{1}{3}+2a & \frac{1}{3}+2b \\ \frac{1}{3}+2b & \frac{1}{3}-2a-2b & \frac{1}{3}+2a \\ \frac{1}{3}+2a & \frac{1}{3}+2b & \frac{1}{3}-2a-2b \end{pmatrix}, \tag{24}$$

where $a \geqslant -\frac{1}{6}$, $b \geqslant -\frac{1}{6}$ and $a+b \leqslant \frac{1}{6}$. This matrix has eigenvalues

$$\begin{cases} \lambda_1 = 1, \\ \lambda_{2,3} = -3(a+b) \pm i(a-b)\sqrt{3}. \end{cases} \tag{25}$$

Hence C_1 always has one eigenvalue 1 and two conjugate eigenvalues. If we prescribe these eigenvalues and choose them inside H_3, then there are exactly two matrices C_1 with these eigenvalues, if $a \neq b$. Otherwise there is exactly one. The second one can be obtained from the first by exchanging the values of a and b.

If we compute e^{Q_1}, where

$$Q_1 = \begin{pmatrix} -2\alpha-2\beta & 2\alpha & 2\beta \\ 2\beta & -2\alpha-2\beta & 2\alpha \\ 2\alpha & 2\beta & -2\alpha-2\beta \end{pmatrix} \tag{26}$$

with $\alpha \geqslant 0$ and $\beta \geqslant 0$, then we obtain a P-matrix of type C_1 with

$$\begin{cases} a = \frac{1}{3} e^{-3(\alpha+\beta)} \cos\left(-2\pi/3 + (\alpha-\beta)\sqrt{3}\right), \\ b = \frac{1}{3} e^{-3(\alpha+\beta)} \cos\left(2\pi/3 + (\alpha-\beta)\sqrt{3}\right). \end{cases} \tag{27}$$

We must have $\det C_1 > 0$ (FRÉCHET [1952], footnote 2 on page 210) for this C_1 and hence $9(a+b)^2+3(a-b)^2>0$. Therefore $a=b=0$ can not occur.

Now Q_1 has eigenvalues $\mu_1=0$, $\mu_{2,3}=-3(\alpha+\beta) \pm i(\alpha-\beta)\sqrt{3}$. Independently we may verify that μ_1, μ_2 and μ_3 are always inside H_3. It is easy to see that a suitable choice of α and β leads to $\lambda_i = e^{\mu_i}$ for $i=1, 2, 3$, where the λ_i are prescribed eigenvalues inside H_3 (with $\lambda_1=1$, λ_2 and λ_3 conjugate, $\lambda_2 \neq 0$). Hence for that choice of α and β we obtain one of the matrices C_1 upon computing e^{Q_1}. If there is a second one, i.e. if $a \neq b$, then that matrix is obtained by exchanging α and β in Q_1. Clearly $a=b$ if $\alpha=\beta$, but $a=b$ may occur if $\alpha \neq \beta$.

We conclude that to every P-matrix P_1 of order 3 which is a circulant and has its eigenvalues inside H_3, Elfving's problem has at least one solution $P(t)=e^{Q_1 t}$ where Q_1 is a circulant (!), except for the one P-matrix having all its elements equal to $\frac{1}{3}$. The number of solutions of this kind can also be found.

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MATHEMATICS

ON ELFVING'S PROBLEM OF IMBEDDING A TIME-DISCRETE MARKOV CHAIN IN A TIME-CONTINUOUS ONE FOR FINITELY MANY STATES. II

BY

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1. The problem, considered in the preceding note (RUNNENBURG [1]), of finding a domain H_n to which the eigenvalues of a matrix of transition probabilities of a continuous parameter Markov chain must necessarily belong, is solved there by using a result in the corresponding problem for discrete parameter Markov chains, obtained by DMITRIEV and DYNKIN [2] and KARPELEWITSCH [3]. For a detailed statement of the problem we refer to [1].

n this note we want to show that the domain H_n can be found by using only the characterization of a Markov matrix given by (1) below.

We use the fact that a Markov-chain on a finite set consisting of n states $A_1, A_2, \ldots, A_n$, induces an abelian semi-group G of real linear transformations on an R_n: with the Markov matrix (p_{jk}) (where $p_{jk} = P\{\underline{x}_{n+1} = A_k | \underline{x}_n = A_j\}$) corresponds the linear transformation π defined by

$$\pi e_j = (p_{j1}, p_{j2}, \ldots, p_{tn}),$$

where e_j is the j'th unit vector.

A necessary and sufficient condition that the matrix of a given linear transformation φ is a Markov matrix is

$$(1) \qquad \varphi W \subset W,$$

where W is the set of all probability distributions on $\{A_1, A_2, \ldots, A_n\}$

$$(2) \qquad W = \{x | x = (\xi_1, \xi_2, \ldots, \xi_n);\ \forall_j\, \xi_j \geqq 0;\ \sum_{j=1}^{n} \xi_j = 1\}.$$

2. Now let G be a continuous one-parameter abelian semi-group of linear transformations satisfying (1) (ι is the identical transformation on R_n)

$$G = \{\varphi_t | t \in (0, \infty);\ \forall_t^{(0,\infty)} \varphi_t W \subset W;\ \forall_t^{(0,\infty)} \forall_s^{(0,\infty)} \varphi_t \varphi_s = \varphi_{t+s};\ \lim_{t \downarrow 0} \varphi_t = \iota\}.$$

It is well-known (cf. e.g. FRÉCHET [4]) that under these conditions the eigenvalues of φ_t are $1, \lambda_1^t, \lambda_2^t, \ldots, \lambda_m^t$ $(m \leqq n-1)$, where $1, \lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalues of φ_1, for some choice of the arguments of $\lambda_1, \ldots, \lambda_m$. Moreover there is at least one vector $p \in W$, which is invariant under G.

Let $\lambda = \varrho e^{i\psi}$ $(0 \leqq \varrho \leqq 1)$ be an eigenvalue of φ_1. We suppose that ψ is chosen in such a way that $\varrho^t e^{it\psi}$ is, for all $t > 0$, an eigenvalue of φ_t. We shall show below that

$$(A) \qquad \varrho \leqq e^{-\psi_0 \operatorname{tg} \pi/n}$$

holds, where

$$\psi_0 = \min\{\psi_0', 2\pi - \psi_0'\} \text{ and } \psi_0' = \psi - [\psi/2\pi]\, 2\pi,$$

i.e. ψ_0 is the minimum of all non-negative arguments of λ and $\bar{\lambda}$. It is easy to verify that the set of all complex numbers satisfying (A) is equal to the set H_n defined by (7) in [1]. Therefore (A) is equivalent to (6) in [1]. Furthermore (A) is trivially satisfied for $\psi = 0$ (for then $\psi_0 = 0$ also) and for $\varrho = 0$. Therefore we suppose henceforward $\psi \neq 0$ and $\varrho \neq 0$. As changing the unit of time amounts to multiplying ψ by a positive number we can always choose this unit in such a way that $\psi \neq 0$ entails $\lambda \notin R$ (R is the set of real numbers). Finally we assume $n \geqq 3$.

Let

$$z_0 = x_0 + iy_0,$$

where x_0 and y_0 are real vectors, be an eigenvector of φ_1 belonging to the eigenvalue λ. As $\lambda \notin R$, it follows that x_0 and y_0 are independent. Then the two-dimensional subspace V spanned by x_0 and y_0

$$V = \{u | u = \alpha x_0 + \beta y_0;\ \alpha, \beta \text{ real}\}$$

is invariant under G, and the movement induced by G in V is easily seen to be described by the equations

$$(3) \qquad \begin{cases} \varphi_t x_0 = \varrho^t (x_0 \cos t\psi - y_0 \sin t\psi), \\ \varphi_t y_0 = \varrho^t (x_0 \sin t\psi + y_0 \cos t\psi). \end{cases}$$

Introducing

$$\begin{aligned} W_0 &= \{x | x \in W;\ \mathit{\Pi}_j \xi_j = 0\}, \\ U &= W \cap (p + V) \qquad , \\ U_0 &= W_0 \cap (p + V) \qquad , \end{aligned}$$

the condition (1) together with the invariance op p and V yields

$$(4) \qquad V_{t^{(0,\infty)}}\, \varphi_t\, U \subset U.$$

As U is a closed convex polygon [1]) in the plane $p + V$ and U_0 its boundary

[1]) It might seem that this statement is false for $n = 3$. However in that case our hypothesis $\lambda \notin R$ entails that 1 is the only real eigenvalue of φ_t (and hence of the adjoint transformation). Therefore $\{x | \xi_1 + \xi_2 + \xi_3 = 0\}$ is the only possible real invariant two-dimensional subspace and thus $V = \{x | \xi_1 + \xi_2 + \xi_3 = 0\}$, implying $W \subset p + V$, whence $U = W$ and $U_0 = W_0$.

the condition (4) is satisfied if and only if

$$F_t^{(0,\infty)}\ \varphi_t\, U_0 \subset U \tag{5}$$

is satisfied.

To derive a geometric condition equivalent with (5), consider a vector $w \in U_0$. The curve $\{\varphi_t w | t > 0\}$ must, on account of (5), be in U. Hence a vector v_1 tangent to [1] $\{\varphi_t w | t \geqq 0\}$ in w must point from w to the interior of U. To make this condition more precise consider a vector v_2 pointing from w along U_0 in such a way that v_1 lies inside the angle [2] formed by $p-w$ and v_2.

If now we define

$$\theta_w = \sphericalangle\,(p-w, v_1);\ \theta_{0,w} = \sphericalangle\,(p-w, v_2), \tag{6}$$

then it is easily seen that (5) is satisfied if and only if

$$F_{w \in U_0}\ \theta_w \leqq \theta_{0,w} \tag{7}$$

is true.

3. It does not seem possible to deduce useful results directly from (7), as θ_w is in general a complicated expression depending on w. The only case in which θ_w is independent of w obtains when $|x_0| = |y_0|$ and the inner product $(x_0, y_0) = 0$, which will be shown below. The general case can be reduced to this case by means of the following considerations:

Let σ be a real non-singular linear transformation on R_n. Then the semi-group

$$G^* = \sigma G \sigma^{-1}$$

consists of those linear transformations φ_t^* for which

$$\varphi_t^* = \sigma\varphi_t\sigma^{-1}.$$

These φ_t^* have the same eigenvalues as φ_t and eigenvectors

$$z^* = \sigma z, \text{ with } z^* = x^* + iy^*;\ x^* = \sigma x;\ y^* = \sigma y,$$

where $z = x + iy$ is an eigenvector of φ_t.

Application of the transformation σ to $W, p, x_0, y_0, V, U\ U_0, v_1, v_2$ yields $W^*, p^*, x_0^*, y_0^*, V^*, U^*, U_0^*, v_1^*, v_2^*$, i.e. $W^* = \sigma W$, etc

We can now repeat the argument of 2 and arrive at equivalent conclusions. In particular, if we define

$$\theta_w^* = \sphericalangle\,(p^*-w, v_1^*);\ \theta_{0,w}^* = \sphericalangle\,(p^*-w, v_2^*), \tag{6*}$$

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[1] Where $\varphi_0 \overset{\text{df}}{=} \iota$.

[2] There is a slight difficulty here on account of the fact that p may be in U_0. This case does not occur if $\psi \neq 0, \varrho \neq 0$: On account of (3) any vector $w-p$ performs a complete rotation around p in the time interval $(0, 2\pi/\psi]$, and does not vanish in this time interval. Hence (5) and $p \in U_0$ are inconsistent.

hen

(7*) $$V_{w \in U_0^*} \theta_w^* \leqq \theta^*_{0,w}$$

is equivalent with (7).

In view of the remark at the beginning of this section, we choose σ in such a way that

(8) $$|x_0^*| = |y_0^*| \text{ and } (x_0^*, y_0^*) = 0.$$

Using (8) it is easy to calculate θ_w^*. For an arbitrary $w \in p^* + V^*$ we have

(9) $$w = p^* + \alpha(x_0^* \sin\gamma + y_0^* \cos\gamma) \qquad (\alpha, \gamma \text{ real}).$$

Consider $\{\varphi_t w | t \geqq 0\}$. A vector tangent to this curve in w is given by

(10) $$v_1^* = \lim_{h \downarrow 0} \frac{\varphi_h^* w - w}{h}.$$

Substitution of (9) and

(3*) $$\begin{cases} \varphi_t^* x_0^* = \varrho^t(x_0^* \cos t\psi - y_0^* \sin t\psi) \\ \varphi_t^* y_0^* = \varrho^t(x_0^* \sin t\psi + y_0^* \cos t\psi) \end{cases}$$

in (10) gives after some calculation

(11) $$v_1^* = -\alpha A[x_0^* \sin(\gamma-\chi) + y_0^* \cos(\gamma-\chi)],$$

where

(12) $$A^2 = \psi^2 + \log^2 \varrho;\ 0 < \chi < \pi \text{ and } ctn\,\chi = -\psi^{-1} \log \varrho.$$

Then, using (8), (11) and (9), it follows that

(13) $$\cos\theta_w^* = \frac{(p^* - w, v_1^*)}{|p^* - w| \cdot |v_1^*|} = \cos\chi,$$

and hence (as θ_w^* and χ are both in $[0, \pi]$)

(14) $$\theta_w^* = \chi.$$

We thus see that θ_w^* is indeed independent of w. Our condition (7*) yields

(15) $$\chi \leqq \inf_{w \in U_0^*} \theta^*_{0,w}.$$

4. As σ is non-singular, W^* is like W an $(n-1)$-simplex. Hence U_0^*, being the boundary of the intersection of W^* with a plane that has at least one point (namely p^*) in common with W^*, is a convex polygon with at most n vertices, say $a_1, a_2, \ldots, a_n$. It is then easy to see, that the infinum in the right-hand side of (15) is reached if w is in one of the vertices of U_0^*. Therefore, if we introduce [1])

$$\left.\begin{aligned} \alpha_j &= \sphericalangle\, p^* a_j a_{j+1} \\ \alpha_j' &= \sphericalangle\, p^* a_j a_{j-1} \end{aligned}\right\} \ (j = 1, 2, \ldots, n;\ \text{indices taken mod } n)$$

[1]) For any three vectors a, b and c we use the following notation:

$$\sphericalangle\, abc \overset{\text{df}}{=} \sphericalangle\,(a-b,\ c-b).$$

then

(16) $$\inf_{w \in U_0^*} \theta^*_{0,w} = \min \alpha_j \text{ or } \min \alpha_j',$$

depending on whether the sense of rotation in $p^* + V^*$ given by p^*, w, v_1^* in that order is the same as the rotational sense given by $a_1, a_2, \ldots, a_n$ or that given by $a_n, a_{n-1}, \ldots, a_1$ respectively.

In **5** we shall give a proof of the fact that

(17) $$\min_j \alpha_j \leqq \frac{n-2}{2n}\pi; \ \min_j \alpha_j' \leqq \frac{n-2}{2n}\pi,$$

with equality in at least one of these two relations if and only if the polygon under consideration is regular and p^* its centre.

Combination of (15), (16) and (17) yields

$$\chi \leqq \frac{n-2}{2n}\pi,$$

from which it follows that

(18) $$\varrho \leqq e^{-\psi \operatorname{tg} \pi/n}.$$

Now ψ is of the form $\psi_0' + 2k\pi$ with $0 \leqq \psi_0' < 2\pi$ and k an integer. As $\bar{\lambda}$ is also an eigenvalue of φ_1, we may conclude

(19) $$\exists_k \ \varrho \leqq e^{-(\psi_0' + 2k\pi) \operatorname{tg} \pi/n} \text{ and } \varrho \leqq e^{-(-\psi_0' - 2k\pi) \operatorname{tg} \pi/n}.$$

If we put

$$\psi_0 = \min \{\psi_0', 2\pi - \psi_0'\},$$

than either $\psi_0' + 2k\pi \geqq \psi_0$ or $-\psi_0' - 2k\pi \geqq \psi_0$, hence (A) holds for all eigenvalues λ of φ_1 for which $\psi \neq 0$, $\varrho \neq 0$, and therefore for all eigenvalues of φ_1 (cf. the remarks at the beginning of section 2).

5. It remains to prove (17). To this purpose suppose, with the notation of section 4

(21) $$\forall_j \ \alpha_j \geqq \frac{n-2}{2n}\pi.$$

Then it is possible to find vectors [1] $p_1 \in \langle p^*, a_1 \rangle$, $p_2 \in \langle p^*, a_2 \rangle, \ldots,$ $p_n \in \langle p^*, a_n \rangle$ in such a way that

$$\sphericalangle\, p^* a_n p_1 = \sphericalangle\, p^* p_1 p_2 = \ldots = \sphericalangle\, p^* p_{n-1} p_n = \frac{n-2}{2n}\pi.$$

Then, as $p_n \in \langle p^* a_n \rangle$

$$\frac{|p_n - p^*|}{|a_n - p^*|} \leqq 1,$$

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[1]) We denote by $\langle b_1, b_2, \ldots, b_k \rangle$ the closed convex hull of $\{b_1, \ldots, b_k\}$:

$$\langle b_1, b_2, \ldots, b_k \rangle = \{x | x = \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_k b_k;\ \forall_j \alpha_j \geqq 0;\ \sum_{j=1}^{k} \alpha_j = 1\}.$$

with equality if and only if $\forall_j p_j = a_j$. Introducing $\alpha_j'' = \sphericalangle p^* p_j p_{j-1}$ ($j=2, \ldots, n$) and $\alpha_1'' = \sphericalangle p^* p_1 a_n$, we find

$$1 \geqq \frac{|p_n - p^*|}{|a_n - p^*|} = \frac{|p_n - p^*|}{|p_{n-1} - p^*|} \cdot \frac{|p_{n-1} - p^*|}{|p_{n-2} - p^*|} \cdots \frac{|p_1 - p^*|}{|a_n - p^*|} = \prod_{=1}^{n} \frac{\sin \frac{n-2}{2n}\pi}{\sin \alpha_j''}.$$

Hence

$$\prod_{j=1}^{n} \sin \alpha_j'' \geqq \sin^n \frac{n-2}{2n}\pi, \tag{22}$$

with equality if and only if $\forall_j \alpha_j'' = \alpha_j'$.

On the other hand, as the sum of the angles ($n\pi$) of the n triangles $\langle a_n, p_1, p^* \rangle, \langle p_1, p_2, p^* \rangle, \ldots, \langle p_{n-1}, p_n, p^* \rangle$ is equal to

$$\sum_{=1}^{n} \alpha_j'' + 2\pi + n \cdot \frac{n-2}{2n}\pi,$$

we have

$$\sum_{j=1}^{n} \alpha_j'' = n\pi - 2\pi - n \cdot \frac{n-2}{2n}\pi = \frac{n-2}{2}\pi.$$

As $-\log \sin x$ is a convex function for $0 < x < \pi$, we have

$$\prod_{j=1}^{n} \sin \alpha_j'' \leqq \sin^n \frac{1}{n} \sum_{j=1}^{n} \alpha_j'' = \sin^n \frac{n-2}{2n}\pi \tag{23}$$

with equality if and only if

$$\forall_j \alpha_j'' = \frac{n-2}{2n}\pi.$$

Combination of (22) and (23) yields:

$$\forall_j \alpha_j' = \alpha_j'' = \frac{n-2}{2n}\pi. \tag{24}$$

We can now repeat this argument, starting from

$$\forall_j \alpha_j' \geqq \frac{n-2}{2n}\pi, \tag{25}$$

which follows from (24). We then arrive at the conclusion

$$\forall_j \alpha_j = \frac{n-2}{2}\pi. \tag{26}$$

Hence, either of (21) or (25) entails (24) and (26). In other words Either

$$\forall_j \alpha_j = \alpha_j' = \frac{n-2}{2n}\pi,$$

in which case $\langle a_1, a_2, \ldots, a_n \rangle$ is regular with centre p^*, or

$$\min_j \alpha_j < \frac{n-2}{2n}\pi \text{ and } \min \alpha_j' < \frac{n-2}{2n}\pi,$$

which proves (17).

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