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A QUALITY CONTROL PROBLEM,
AN APPLICATION OF MARKOV-PROGRAMMING

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3.1. Problem formulation *)

Consider a machine that produces each day a product whose quality will be identified with one of the integers $1,\dots,M$. At the beginning of each day the machine may be inspected. The costs of an inspection are equal to J. An inspection is done when the decision maker wants to find out the quality of the product that will be produced that day. **) After an inspection he knows this quality. When he thinks that this quality (say i) is not acceptable, he decides to a revision of the machine. The revision costs are $R(i)$. Both the time needed to inspect the machine and the time needed to revise the machine may be neglected. After a revision the machine produces that day a product of quality M. The production costs incurred at a day are given by $p(i)$ when the machine produces that day a product of quality i, $(1 \le i \le M)$.

If the machine produces at day t a product of quality i, then, with probability p_{ij}, at the beginning of day t+1 the machine will be in a condition to produce a product of quality j $(1 \le j \le M)$, and, with probability p_{i0}, the machine will be defect at the beginning of day t+1. We assume $p_{i0} < 1$ and $p_{i0} + p_{i1} + \dots + p_{iM} = 1$ for all $1 \le i \le M$. When the machine is defect at the beginning of a day, the machine is repaired. The repair costs are $R(0)$ and the repair time may be neglected. After a repair the machine produces that day a product of quality M.

It is assumed that the machine becomes eventually defect when never a revision occurs.

The decision maker wants to determine a strategy for inspecting and revising the machine such that the long-run average costs per day are minimal.

*) This report presents the elaboration of chapter 3 in [1].

**) It is assumed that the quality of the finished product cannot be checked.

3.2. The state space, the natural process and the functions k(x;d) and t(x;d).

We first introduce some notation. Let $p_{00} = 1$, and let $p_{0j} = 0$ for $1 \le j \le M$. Define the Markov matrix P by $P = (p_{ij})$, $i,j = 1,\ldots,M$. State 0 is an absorbing state of P. Further the states $1,\ldots,M$ are transient states of P, since the machine becomes eventually defect when it is never revised. Define for $i,j = 1,\ldots,M$,

$$p_{ij}^{(1)} = p_{ij} \quad \text{and} \quad p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}\, p_{kj}^{(n-1)} \qquad \text{for } n \ge 2.$$

Observe $p_{i0}^{(n)} < 1$ for all $i \ge 1$. For $i,j = 1,\ldots,M$ and $n \ge 1$, let

$$\tilde{p}_{ij}^{(n)} = p_{ij}^{(n)}/\{1 - p_{i0}^{(n)}\} \quad \text{and} \quad q(i,n) = \sum_{j=1}^{M} \tilde{p}_{ij}^{(n)}(1-p_{j0}).$$

We may interprete these probabilities as follows. For that purpose we consider the situation in which neither the machine is inspected nor is revised nor is repaired. Then, $p_{i0}^{(t)}$ (respectively $p_{ij}^{(t)}$, $j \ge 1$) is the probability that at day t+1 the machine is defect (respectively will produce a product of quality j), given that at day 1 the machine produces a product of quality i. Further, $\tilde{p}_{ij}^{(t)}$ is the probability that at day t+1 the machine produces a product of quality j, given that the machine is not defect at the beginning of day t+1 and that at day 1 the machine produced a product of quality i. Finally, q(i,t) is the probability that the machine is not defect at the beginning of day t+2, given that the machine is not defect at the beginning of day t+1 and that at day 1 the machine produced a product of quality i.

For this problem it suffices to specify only the state of the system at the beginning of each day. We take as state space

$$X = \{0,1,\ldots,M\} \cup \{(i,n) \mid i = 1,\ldots,M;\ n = 1,2,\ldots\}.$$

State 0 corresponds to the situation where the machine is defect, while state $j \ge 1$ corresponds to the situation where the machine will produce a product of quality j provided that the machine will not be revised. Finally, state (i,n) corresponds to the situation in which the machine is not defect and n days ago the quality of production was known for the last time where this quality was i.

Next we define a natural process. We assume that in the natural process neither the machine is inspected nor is revised nor is repaired. Hence the natural process remains forever in state 0 as soon as this state is taken on. When the natural process starts in state (i,n), the next state is (i,n+1) if the machine does not get defect (the probability of this event is q(i,n)), otherwise 0 is the next state. When the natural process starts in state $i \geq 1$, the next state is 0 if the machine gets defect (the probability of this event is p_{i0}), otherwise (i,1) is the next state.

Finally, we define the feasible decisions. In state 0 the machine has to be repaired. By this intervention the system is transferred instantaneously into state M. In the states i = 1,...,M-1 both the null-decision and the intervention "revise the machine" are feasible. By the latter intervention the system is transferred instantaneously into state M. In state M the null-decision is the only feasible decision. Let us assume that for each i = 1,...,M we may choose a positive integer T_i such that in state (i,n) with $n \geq T_i$ the only feasible decision is to inspect the machine. In the other states (i,n) both the null-decision and the intervention "inspect the machine" are feasible. By the latter intervention in state (i,n) the system is transferred instantaneously into one of the states 1,...,M where $\tilde{p}_{ij}^{(n)}$ is the probability that the system takes on state j.

In each state there is at most one feasible intervention. Let us denote any intervention by d = 1.

Clearly, point (D) on p.2 in [1] is satisfied for

$$A_0 = \bigcup_{i=1}^{M} \{(i,n) \mid n \geq T_i\} \cup \{0\}.$$

To determine the functions k(x;d) and t(x;d), we choose (see p.5 in [1]),

$$A_{01} = A_{02} = \{0\}.$$

Since $A_{01} = A_{02}$, we write $\underline{w}_0 = \underline{w}_{0i}$ and $\underline{w}_d = \underline{w}_{di}$, $i = 1,2$. Consider first the walk $\underline{w}_0$ with initial state $i \geq 0$. It is easy to see

$$(3.1) \qquad k_0(0) = t_0(0) = 0$$

$$(3.2) \qquad k_0(i) = p(i) + \sum_{j=1}^{M} p_{ij}\, k_0(j) \qquad \text{for } i = 1,\ldots,M$$

$$(3.3) \qquad t_0(i) = 1 + \sum_{j=1}^{M} p_{ij}\, t_0(j) \qquad \text{for } i = 1,\ldots,M.$$

The systems of linear equations (3.2) and (3.3) have a unique solution, since $1,\ldots,M$ are transient states of P. Consider next the walk $\underline{w}_1$ with initial state $i < M$. By the intervention $d = 1$ the system is transferred instantaneously into state M. It now follows that $k_1(i;1) = R(i) + k_0(M)$ and $t_1(i;1) = t_0(M)$. Hence for $i = 0,\ldots,M-1$,

$$(3.4) \qquad k(i;1) = R(i) + k_0(M) - k_0(i), \quad t(i;1) = t_0(M) - t_0(i).$$

Consider now the walk $\underline{w}_0$ with initial state (i,n). The probability that at the first day of this walk the machine produces a product of quality j is $\tilde{p}_{ij}^{(n)}$. Hence,

$$(3.5) \qquad k_0((i,n)) = \sum_{j=1}^{M} \tilde{p}_{ij}^{(n)}\, k_0(j), \quad t_0((i,n)) = \sum_{j=1}^{M} \tilde{p}_{ij}^{(n)}\, t_0(j).$$

Finally, consider the walk $\underline{w}_1$ with initial state (i,n). By the intervention "inspect the machine" the system is transferred instantaneously into state j with probability $\tilde{p}_{ij}^{(n)}$. Hence, for $1 \leq i \leq M$ and $n \geq 1$,

$$(3.6) \quad k_1((i,n);1) = J + \sum_{j=1}^{M} \tilde{p}_{ij}^{(n)}\, k_0(j), \quad t_1((i,n);1) = \sum_{j=1}^{M} \tilde{p}_{ij}^{(n)}\, t_0(j).$$

From (3.5) and (3.6)

$$(3.7) \qquad k((i,n);1) = J \text{ and } t((i,n);1) = 0 \quad \text{for } 1 \leq i \leq M;\ n \geq 1.$$

So the functions $k(x;d)$ and $t(x;d)$ can be calculated from (3.1) - (3.4) and (3.7).

3.3. The functional equations.

Any strategy z can be characterized by M positive integers $t_1(z),\dots,t_M(z)$ and by a set $R_z \subseteq \{1,\dots,M-1\}$, where strategy z dictates the null-decision in the states (i,n) with $n < t_i(z)$, strategy z dictates the intervention "inspect the machine" in the state $(i,t_i(z))$ for $i = 1,\dots,M$, and where R_z consists of the states in which strategy z dictates the intervention "revise the machine". Clearly, under such a strategy z the states (i,n) with $n > t_i(z)$ are transient states. Hence it is no restriction to assume that strategy z dictates an intervention in the states (i,n) with $n > t_i(z)$. Observe that for any strategy the associated Markov chain $\{\underline{I}_n\}$ has no two disjoint ergodic sets, since state M can be reached from each state. Also, observe that in this problem an intervention in a state (i,n) in A_z may transfer the system instantaneously into a state j in A_z (cf. p.4 in [1]). Nevertheless the theory of chapter 1 applies since the number of interventions in a finite time is finite and any intervention leads to a state outside A_0.

Fix a strategy z which is characterized as above. To determine a solution of the functional equation (9) on p.8 in [1] *), we put

(3.8) $$c(z;0) = 0.$$

The intervention d = 1 in state i transfers the system into state M. Hence, by (3.9), $c(z;0) = k(0;1) - r(z)\ t(0;1) + c(z;M)$, so

(3.10) $$c(z;M) = -k(0;1) + r(z)\ t(0;1).$$

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*) We will use repeatedly the following useful relation:

(3.9) $$c(z;x) = k(x;z(x)) - r(z)\ t(x;z(x)) + Ec(z;\underline{v}) \qquad \text{for } x \in A_z,$$

where $\underline{v}$ is the state in which the system is transferred by the intervention z(x) in state x. This relation can be easily deduced from (9) and (10) in [1].

Further,

$$c(z;i) = k(i;1) - r(z)\ t(i;1) + c(z;M) \qquad \text{for } i \in R_z. \tag{3.11}$$

For initial state $i \notin R_z$ the next intervention state is one of the states $(i,t_i(z))$ and 0, where the probability that 0 is the next intervention state equals $p_{i0}^{(t_i(z))}$. Hence, by (10) in [1] and (3.8),

$$c(z;i) = \{1 - p_{i0}^{(t_i(z))}\}\ c(z;(i,t_i(z))) \qquad \text{for } i \notin R_z. \tag{3.12}$$

Using (3.7) and (3.9), we have for $1 \le i \le M$,

$$c(z;(i,n)) = J + \sum_{j=1}^{M} \tilde{p}_{ij}^{(n)}\ c(z;j) \qquad \text{for } n \ge t_i(z). \tag{3.13}$$

Finally, for initial state (i,n) with $n < t_i(z)$ the next state is one of the states $(i,n+1)$ and 0, where $q(i,n)$ is the probability that $(i,n+1)$ is the next state. Hence *), by (3.8), for $1 \le i \le M$,

$$c(z;(i,n)) = q(i,n)\ c(z;(i,n+1)) \qquad \text{for } n < t_i(z). \tag{3.14}$$

We shall now demonstrate that in fact we need only to solve a system of $|\bar{R}_z| + 1$ linear equations, where

$$\bar{R}_z = \{1,\dots,M\} \setminus R_z. \tag{3.15}$$

From (3.13) with $n = t_i(z)$, (3.12), (3.11) and the definition of $\tilde{p}_{ij}^{(n)}$ it follows after some straightforward calculations that

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*) Use the following relation. If $A \supseteq A_z$, then $c(z;x) = Ec(z;\underline{a})$ for $x \notin A_z$, where $\underline{a}$ is the first state in A taken on by the system when the system is subjected to the natural process and x is the initial state. This relation follows directly from (10) in [1].

$$(3.16)\qquad c(z;i) = \{1 - p_{i0}^{(t_i(z))}\}\, J + \sum_{j\in R_z} p_{ij}^{(t_i(z))} \{k(j;1) - k(0;1)\} +$$

$$- r(z) \sum_{j\in R_z} p_{ij}^{(t_i(z))} \{t(j;1) - t(0;1)\} +$$

$$+ \sum_{j\in \bar{R}_z} p_{ij}^{(t_i(z))} c(z;j) \qquad \text{for } i \in \bar{R}_z.$$

By (3.10) and (3.16) the quantities $r(z)$ and $c(z;i)$ for $i \in \bar{R}_z$ are determined uniquely (observe that state M belongs always to $\bar{R}_z$). When these quantities have been calculated, $c(z;x)$ for $x \notin \bar{R}_z$ can be calculated successively from (3.11), (3.13) and (3.14).

3.4. The policy improvement operation and the cutting mechanism.

For strategy z, let $(r(z),c(z;x))$ be determined as described in section 3.3. First we shall specify the policy improvement operation for the determination of strategy z'. Using (3.7), we have by (15), (12) and (11) in [1] that state (i,n) with $n < t_i(z)$ belongs to $A_{z'}$ if and only if

$$(3.17)\qquad c(1.z;(i,n)) = J + \sum_{j=1}^{M} \tilde{p}_{ij}^{(n)} c(z;j) < c(z;(i,n)).$$

Further, state i with $1 \le i < M$ and $i \notin A_z$ belongs to $A_{z'}$ if and only if

$$(3.18)\qquad c(1.z;i) = k(i;1) - r(z)\, t(i;1) + c(z;M) < c(z;i).$$

For any state $x \in A_z$ we have by (12) and the agreement below 15 in [1] that $z'(x) = z(x)$. Hence

$$(3.19)\qquad c^*(z;x) = c(z;x) \qquad \text{for } x \in A_z,$$

and, by (3.17) and (3.18), for $x \in A_{z'} \setminus A_z$,

$$(3.20) \qquad c^*(z;x) = \begin{cases} J + \sum_{j=1}^{M} \tilde{p}_{ij}^{(n)} \, c(z;j) & \text{if } x = (i,n), \\ k(i;1) - r(z)\, t(i;1) + c(z;M) & \text{if } x = i. \end{cases}$$

Before we specify the cutting mechanism for the determination of the set $A'_{z'}$, we consider the following optimal stopping problem. We have a Markov chain with a finite state space S and with one-step transition probabilities q_{st}. Let $S = S_0 \cup S_1 \cup S_2$, where S_i and S_j are disjoint for $i \neq j$. The Markov chain has to be stopped on S_0, the Markov chain may be stopped on S_1 and the Markov chain cannot be stopped on S_2. It is assumed that the set S_0 will be reached with probability 1 from each initial state. When the Markov chain is stopped in state s, a cost c(s) is incurred. For this stopping problem, let M(s) be the minimal expected cost when the initial state is s. Then *),

$$(3.21) \qquad M(s) = c(s) \qquad \text{for } s \in S_0,$$

$$(3.22) \qquad M(s) = \min[c(s), \sum_{t \in S} q_{st} \, M(t)] \qquad \text{for } s \in S_1,$$

$$(3.23) \qquad M(s) = \sum_{t \in S} q_{st} \, M(t) \qquad \text{for } s \in S_2.$$

Also, the smallest optimal stopping set is given by

$$(3.24) \qquad S_0 \cup \{s \in S_1 \mid c(s) < \sum_{t \in S} q_{st} \, M(t)\}.$$

Let us now return to the determination of the set $A'_{z'}$ (see pp. 11-12 in [1]). This set is determined by the natural process and the cost function $c^*(z;x)$, $x \in A_{z'}$. Let us recall that when the natural process starts in state (i,n), the next state of the natural process is one of the states (i,n+1) and 0, where q(i,n) is the probability that the next state is

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*) C.f. chapter 8 in C. Derman, Finite State Markovian Decision Processes, Academic Press, New York, 1970.

(i,n+1). When the natural process starts in state $i > 0$ the next state of the natural process is one of the states (i,1) and 0, where p_{i0} is the probability that the next state is 0. Now, we have that the set $A'_{z'}$ is the smallest optimal stopping set for the stopping problem in which (cf. p.12 in [1] and cf. [2])

$$S_0 = A_0, \quad S_1 = A_{z'} \setminus A_0, \quad S_2 = X \setminus A_{z'},\ ^{*)}$$

$$q_{(i,n),(i,n+1)} = q(i,n), \quad q_{(i,n),0} = 1 - q(i,n),$$

$$q_{i,(i,1)} = 1 - p_{i0}, \; q_{i0} = p_{i0}, \quad \text{and} \quad c(x) = c^*(z;x).$$

Due to the special form of the transition probabilities q_{ij} we can deduce from (3.21) - (3.24) a simple algorithm to construct $A'_{z'}$, since these states belong to A_0. By (3.21), (3.20) and (3.8),

$$(3.25) \qquad M((i,T_i)) = c(z;(i,T_i)) \qquad \text{for } 1 \le i \le M \text{ and } M(0) = 0.$$

Now fix i with $1 \le i \le M$. To determine which states (i,n) with $n < T_i$ belong to $A'_{z'}$, we proceed as follows. Consider successively the states $(i,T_i-1),\dots,(i,1)$. By (3.24) and $M(0) = 0$, state (i,T_i-k) belongs to $A'_{z'}$ if and only if this state belongs to $A'_{z'}$ and

$$(3.26) \qquad c^*(z;(i,T_i-k)) < q(i,T_i-k)\, M((i,T_i-k+1)).$$

By (3.22) and (3.23),

$$(3.27) \qquad M((i,T_i-k)) = \begin{cases} c^*(z;(i,T_i-k)), & \text{if } (i,T_i-k) \in A'_{z'}, \\ q(i,T_i-k)\, M((i,T_i-k+1)), & \text{otherwise.} \end{cases}$$

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*) The states (i,n) with $n > T_i$ need not be considered in the stopping problem.

When we have classified state (i,T_i-k), then we can next classify state (i,T_i-k-1).

Finally, by (3.24) and $M(0) = 0$, state $i \neq 0$ belong to A'_z, if and only if $i \in A_z$, and

$$(3.28) \qquad c^*(z;i) < (1-p_{i0})\, M((i,1)).$$

An examination of the policy improvement operation and the cutting mechanism shows that for this problem these procedures can be combined. This will be done in the next section where an algorithm for the determination of an optimal strategy will be given.

3.5. The algorithm

The n-th step of the algorithm runs as follows.

n-th step

(a) Let $z = z^{(n-1)}$ be the strategy obtained at the end of the (n-1)-th step (start step 1 with an arbitrary strategy). The strategy z is characterized by the set R_z and the integers $t_i(z)$, $1 \leq i \leq M$ (cf. section 3.3). Determine the unique solution $(r(z);c(z;i),\ i \in \bar{R}_z)$ of (3.10) and (3.16). Next compute $c(z;i)$ for $i \in R_z$ from (3.11).

(b) To determine the integers $t_i(z^{(n)})$, $1 \leq i \leq M$, corresponding to stategy $z^{(n)}$, we perform for each fixed i, $1 \leq i \leq M$, the following procedure. Let

$$\alpha_i(T_i) = J + \sum_{j=1}^{M} \tilde{p}_{ij}^{(T_i)}\, c(z;j) \quad \text{and} \quad u_i(T_i) = T_i.$$

Successively for $k = 1,\dots,T_i-t_i(z)$, we compute

$$a_{k1} = J + \sum_{j=1}^{M} \tilde{p}_{ij}^{(T_i-k)}\, c(z;j), \quad a_{k2} = q(i,T_i-k)\, \alpha_i(T_i-k+1),$$

and, if $a_{k2} > a_{k1}$, we put

$$\alpha_i(T_i-k) = a_{k1} \quad \text{and} \quad u_i(T_i-k) = T_i - k,$$

otherwise, we put

$$\alpha_i(T_i-k) = a_{k2} \quad \text{and} \quad u_i(T_i-k) = u_i(T_i-k+1).$$

When this step has been done for $k = 1,\ldots,T_i-t_i(z)$, we proceed as follows. Let

$$\beta_i(t_i(z)) = J + \sum_{j=1}^{M} \tilde{p}_{ij}^{(t_i(z))} c(z;j).$$

Successively for $k = T_i-t_i(z)+1,\ldots,T_i-1$, we compute

$$\beta_i(T_i-k) = q(i,T_i-k)\ \beta_i(T_i-k+1); \quad b_{k1} = J + \sum_{j=1}^{M} \tilde{p}_{ij}^{(T_i-k)} c(z;j);$$

and $$b_{k2} = q(i,T_i-k)\ \alpha_i(T_i-k+1),$$

and, if $b_{k1} < \beta_i(T_i-k)$ and if $b_{k1} < b_{k2}$, we put

$$\alpha_i(T_i-k) = b_{k1} \quad \text{and} \quad u_i(T_i-k) = T_i - k,$$

otherwise, we put

$$\alpha_i(T_i-k) = b_{k2} \quad \text{and} \quad u_i(T_i-k) = u_i(T_i-k+1).$$

The final number $u_i(1)$ equals the integer $t_i(z^{(n)})$ corresponding to strategy $z^{(n)}$. Next we determine the set $R_{z^{(n)}}$. To do this, we subject each of the states $i = 1,\ldots,M-1$ to the following test where we distinguish between $i \in R_z$ and $i \notin R_z$.

State $i \in R_z$ belongs to $R_{z^{(n)}}$ if and only if

$$c(z;i) < (1-p_{i0})\ \alpha_i(1),$$

while state $i \notin R_z$ belongs to $R_{z^{(n)}}$ if and only if both

$$c(z;i) > k(i;1) - r(z)\ t(i;1) + c(z;M)$$

and

$$(1-p_{i0})\ \alpha_i(1) > k(i;1) - r(z)\ t(i;1) + c(z;M).$$

<u>End of the n-th step</u>

3.6. <u>Numerical example</u>

The following numerical data are given

$$M = 10,\ T_i = 25 \quad \text{for } i = 1,\dots,10,\ J = 30$$

i	p(i)	R(i)	p_{ij}, j = 0,...,10										
0		130	1	0	0	0	0	0	0	0	0	0	0
1	10	40	.5	.5	0	0	0	0	0	0	0	0	0
2	9	40	.2	.2	.6	0	0	0	0	0	0	0	0
3	8	40	0	.2	.2	.6	0	0	0	0	0	0	0
4	7	40	0	0	.2	.2	.6	0	0	0	0	0	0
5	6	40	0	0	0	.2	.2	.6	0	0	0	0	0
6	5	35	0	0	0	0	.1	.2	.7	0	0	0	0
7	4	35	0	0	0	0	0	.1	.2	.7	0	0	0
8	3	35	0	0	0	0	0	0	.1	.2	.7	0	0
9	3	35	0	0	0	0	0	0	0	.1	.2	.7	0
10	3		0	0	0	0	0	0	0	0	0	.2	.8

<u>1st step</u>

We begin with strategy $z^{(1)}$ which is characterized by

$$R_{z^{(1)}} = \{1,\dots,9\} \quad \text{and} \quad t_i(z^{(1)}) = 25 \quad \text{for } i = 1,\dots,10.$$

We find $r(z^{(1)}) = 9{,}76$. The function $c(z^{(1)};i)$ and the strategy $z^{(2)}$ are given in table 1, where the states of $R_{z^{(2)}}$ are marked with (*).

i	$c(z^{(1)};i)$	$R_{z^{(2)}}$	$t_i(z^{(2)})$
1	-90.48	(*)	1
2	-88.34	(*)	1
3	-85.01	(*)	1
4	-79.78	(*)	1
5	-73.00	(*)	3
6	-64.39	(*)	5
7	-49.73	(*)	7
8	-32.08		9
9	-15.43		11
10	-16.64		16

Table 1

End of the 1st step

2nd step

We find $r(z^{(2)}) = 8{,}96$. The function $c(z^{(2)};i)$ and strategy $z^{(3)}$ are given in table 2.

i	$c(z^{(2)};i)$	$R_{z^{(3)}}$	$t_i(z^{(3)})$
1	-92.08	(*)	1
2	-91.14	(*)	1
3	-89.21	(*)	1
4	-85.28	(*)	1
5	-79.85	(*)	2
6	-73.46	(*)	4
7	-60.73	(*)	6
8	-44.03	(*)	8
9	-41.33		10
10	-35.65		15

Table 2

End of the 2nd step

3^{rd} step

We find $r(z^{(3)}) = 8{,}93$. The function $c(z^{(3)};i)$ and strategy $z^{(4)}$ are given in table 3.

i	$c(z^{(3)},i)$	$R_{z^{(4)}}$	$t_i(z^{(4)})$
1	-92.14	(*)	1
2	-91.25	(*)	1
3	-89.38	(*)	1
4	-85.50	(*)	1
5	-80.12	(*)	2
6	-73.82	(*)	4
7	-61.16	(*)	6
8	-45.62	(*)	8
9	-41.71		10
10	-36.40		15

Table 3

End of the 3^{rd} step

Since $z^{(3)} = z^{(4)}$, we have that strategy $z^{(4)}$ is optimal.

References

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