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On the Stability of Multistep Formulas for Volterra Integral Equations of the Second Kind

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Abstract — Zusammenfassung

On the Stability of Multistep Formulas for Volterra Integral Equations of the Second Kind. The purpose of this paper is to analyse the stability properties of a class of multistep methods for second kind Volterra integral equations. Our approach follows the usual analysis in which the kernel function is a priori restricted to a special class of test functions. We consider the class of finitely decomposable kernels. Stability conditions will be derived and compared with those obtained with the simple test equation. It turns out that the new criteria are more severe than the conventional conditions. The practical value is tested by numerical experiments with the trapezoidal rule.

Key words and phrases: Numerical analysis, Volterra integral equations of the second kind, stability.

Über die Stabilität des Mehrschrittverfahrens für Volterrasche Integralgleichungen zweiter Art. Ziel dieser Arbeit ist es, die Stabilitätseigenschaften einer Klasse Volterrascher Integralgleichungen zweiter Art zu untersuchen. Unsere Behandlung ist der üblichen Stabilitätsanalyse ähnlich, in der die Kernfunktionen zu einer im voraus beschränkten Klasse von Testfunktionen gehören. Wir haben die Klasse der „endlich zerlegbaren" Kerne betrachtet. Stabilitätsbedingungen werden abgeleitet und verglichen mit den Bedingungen für die einfache Testgleichung. Es zeigt sich, daß die neuen Kriteria einschränkender sind als die konventionellen Bedingungen. Der praktische Wert wird getestet durch numerische Experimente mit der Trapezregel.

1. Introduction

Suppose we are given the system of non-linear Volterra integral equations

$$f(x)=g(x)+\int_{x_0}^{x} K(x,y,f(y))\,dy, \qquad x_0 \le x \le X, \tag{1.1}$$

where g and K are given vector functions and f is the unknown vector function.

Several numerical methods have been proposed to solve this equation, the most familiar ones of which are based on a direct quadrature rule. These methods have the form

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$$f_{n+1}=g(x_{n+1})+\sum_{j=0}^{n+1} w_{n+1,j} K(x_{n+1}, x_j, f_j), \qquad n\geq k-1, \tag{1.2}$$

where $f_0, f_1, \ldots$ are approximations to $f(x_0), f(x_1), \ldots$ and $w_{n,j}$ are given weight parameters. It will be assumed that $f_0, f_1, \ldots, f_{k-1}$ are prescribed.

In the literature, the stability analysis of this and other methods is carried out either for $h_n \to 0$ ($h_n = x_{n+1} - x_n$), where general kernel functions K are admitted (cf. [8] and [12]), or for fixed $h \neq 0$, where the kernel is of the form $K=af$, yielding the test equation (cf. [11], [2])

$$f(x)=g(x)+a\int_0^x f(y)\,dy, \qquad a\in\mathbb{C}. \tag{1.3}$$

The main idea behind this last approach is that the kernel function is chosen in such a way that the numerical scheme can be rewritten as a recurrence relation with a *fixed* number of terms, which is due to the possibility of reducing (1.3) to the differential equation $f'=af$. In this connection, a remark of Kershaw in [4, p. 159] about the use of this kernel function may be quoted: "... it is obviously convenient, however its true relevance to the integral equation situation does not appear to have been thoroughly examined." It is possible, however, to extend the analysis of (1.3) to more general kernel functions to obtain a firmer foundation for the stability conditions derived for (1.3). In particular, we want to consider kernel functions which depend upon the independent variable x in order to treat integral equations which are more general than the integrated form of a first-order ordinary differential equation.

The present paper is based on two earlier institute reports [6] and [7] and reproduced here in a condensed and slightly modified version. In [6] the kernel functions $K(x,y,f)$ were allowed to be of the form

$$K(x,y,f)=(a+b\,x)\,f, \tag{1.4}$$

where a and b are constants, and in [7] we considered the class of *finitely decomposable kernels* (cf. [3]), i.e., kernels of the form

$$K(x,y,f)=\sum_{i=1}^{r} Q_i(x)\,B_i(y,f), \tag{1.5}$$

where the Q_i are matrices only depending on x and where the B_i are vectors which only depend on (y,f) and which are differentiable with respect to f. In this paper K will be assumed of the form (1.5).

Although our stability analysis applies to rather general integral equations with kernels of the form (1.5), it turns out that the resulting stability conditions resemble those obtained for integral equations with the simple kernel $a(y)f$. Thus, the stability behaviour found for ordinary differential equations with the test equation $f'=a(x)f$ is indicative of the local stability behaviour for Volterra integral equations of the type (1.5).

2. Derivation of Recurrence Relations

The first-order variational equation of (1.2) is of the form

$$\Delta f_{n+1} = \sum_{j=0}^{n+1} w_{n+1,j} \frac{\partial K}{\partial f}(x_{n+1}, x_j, f_j)\, \Delta f_j, \qquad n \geq k-1. \tag{2.1}$$

In order to obtain a fixed-term recurrence relation for the perturbations Δf_j, we use two properties of the kernel function K and the weights w_{nj}, respectively. Firstly, by virtue of (1.5), the arguments x, y and f in the Jacobian matrix can be assumed to be separable according to the formula

$$\frac{\partial K}{\partial f}(x, y, f) = \sum_{i=1}^{r} Q_i(x)\, R_i(y, f), \tag{2.2}$$

where the R_i are arbitrary matrices only depending on y and f, and the Q_i are arbitrary matrices only depending on x.

Secondly, we use a property of w_{nj} that holds for all quadrature rules which are reducible to a linear multistep method with coefficients $\{a_l, b_l\}_{l=0}^{k}$ (cf. [10], [13]). This property reads

$$\begin{aligned} &\sum_{l=0}^{k} a_l\, w_{n+1-l,j} = \begin{cases} 0 & \text{for } j=0,1,\ldots,n-k, \quad n \geq 2k-1 \\ -h_n b_{n+1-j} & \text{for } j=n-k+1,\ldots,n+1 \end{cases}, \\ &\sum_{l=0}^{k} a_l = 0, \end{aligned} \tag{2.3}$$

where the parameters a_l and b_l are independent of j, and k is a positive integer.

Apart from the properties (2.2) and (2.3) we need the quantities

$$\Delta G_n^{(i)} = \sum_{j=0}^{n} w_{n,j}\, R_i(x_j, f_j)\, \Delta f_j, \qquad n=0,1,\ldots;\ i=1,2,\ldots,r. \tag{2.4}$$

It is possible to derive a fixed-term recurrence relation for the perturbations Δf_j and $\Delta G_n^{(i)}$. Before doing this, however, we shortly discuss the meaning of the perturbations $\Delta G_n^{(i)}$ by relating them to quantities introduced by Bownds [3]. Following Bownds (see also [1]) we substitute (1.5) into the integral equation to obtain (in our notation)

$$f(x) = g(x) + \sum_{i=1}^{r} Q_i(x)\, G^{(i)}(x), \qquad G^{(i)}(x) = \int_{x_0}^{x} B_i(y, f(y))\, dy. \tag{2.5}$$

The functions $G^{(i)}(x)$ satisfy the system of differential equations

$$\frac{d}{dx} G^{(i)}(x) = B_i\Big(x, g(x) + \sum_{j=1}^{r} Q_j(x)\, G^{(j)}(x)\Big), \qquad i=1,2,\ldots,r. \tag{2.6}$$

Hence, by solving the $G^{(i)}(x)$ from these equations the function $f(x)$ can be found. Returning to our perturbations $\Delta G_n^{(i)}$, it is easily seen that they just are the discrete analogues of the perturbations $\Delta G^{(i)}(x)$ of the functions $G^{(i)}(x)$, defined by (2.5).

Substitution of (2.2) into (2.1) leads to the relation

$$\Delta f_{n+1}=\sum_{i=1}^{r} Q_i(x_{n+1})\,\Delta G_{n+1}^{(i)}. \qquad (2.7$$

In addition, we have from (2.3) for the perturbations $\Delta G_n^{(i)}$ the recurrence relations

$$\sum_{l=0}^{k} a_l\,\Delta G_{n+1-l}^{(i)}+h_n\sum_{l=0}^{k} b_l\,R_i(x_{n+1-l},f_{n+1-l})\,\Delta f_{n+1-l}=0, \qquad i=1,2,\ldots,r. \qquad (2.8)$$

Writing $\Delta f_n=\Delta G_n^{(0)}$ the recurrence relations (2.7) and (2.8) for the $\Delta G_n^{(i)}$ can be compactly written as a two-term recursion of the form

$$\Delta V_{n+1}=A_n\,\Delta V_n, \qquad (2.9)$$

where A_n is a matrix operator and

$$\Delta V_n=(\Delta G_n,\ldots,\Delta G_{n+1-k})^T$$

with

$$\Delta G_n=(\Delta G_n^{(0)},\ldots,\Delta G_n^{(r)})^T.$$

The magnitude of the eigenvalues of A_n give an indication of the propagation of the perturbations $\Delta G_n^{(i)}$ and therefore also of the errors Δf_n. Hence, the characteristic equation of A_n is of importance. Following the analysis given in [7] we arrive at the result:

Theorem 2.1: *The non-trivial eigenvalues of the matrix A_n satisfy the characteristic equation*

$$\det\left[\sum_{l=0}^{k}\left(a_l I+b_l h_n\frac{\partial K}{\partial f}(x_{n+1},x_{n+1-l},f_{n+1-l})\right)\zeta^{k-l}\right]=0. \qquad \Box\,(2.10)$$

First of all we emphasize that the characteristic equation (2.10) is completely expressed in terms of $\partial K/\partial f$ at a number of points (x, y, f) in the neighbourhood of $(x_{n+1}, y_{n+1}, f_{n+1})$. This indicates that only the *possibility to separate* $\partial K/\partial f$ according to (2.2) is used in the analysis; the specific form of this separation, however, is not reflected in the characteristic equation. Further we note that, due to the *local* character of our analysis, the jacobian $\partial K/\partial f$ is evaluated only at $x=x_{n+1}$. Therefore, if $\partial K/\partial f$ is independent of x, then (2.10) is equivalent with the characteristic equation of the linear multistep method for solving ordinary differential equations applied to the test equation $y'(t)=\lambda(t)\,y(t)$.

By comparing (2.10) with the characteristic equation obtained if the test equation (1.3) is used in the stability analysis, i.e., the "linear multistep" equation (cf. [2])

$$\sum_{l=0}^{k}(a_l+b_l z)\,\zeta^{k-l}=0, \qquad z=a\,h, \qquad (2.11)$$

one can quantify the extent of the simplifications which are introduced if stability considerations are based on (2.11) instead of (2.10). The use of (2.11) seems justified for slowly varying Jacobian matrices, but may give wrong stability conditions if $\partial K/\partial f$ is rapidly varying with x, y or f.

3. Stability Regions

In this section we assume that the Jacobian matrices occurring in (2.10) can be diagonalized by the same transformation. In that case (2.10) can be factorized and leads to the equation

$$\sum_{l=0}^{k} (a_l + b_l z_{n+1,n+1-l}) \zeta^{k-l} = 0, \tag{3.1}$$

where $z_{n+1,n+1-l}$ is an eigenvalue of $h_n \frac{\partial K}{\partial f}(x_{n+1}, x_{n+1-l}, f_{n+1-l})$. From (3.1) stability conditions can be derived. It is more convenient, however, to determine the *stability region*, defined as the set of points in the space $\{z_{n+1,n+1-l}\}_{l=0}^{k}$, where the solutions ζ of (3.1) (the *amplification factors*) are inside the unit circle. We illustrate this by deriving the stability region of the trapezoidal rule. This rule is defined by

$$2\, w_{n0} = w_{n1} = \ldots = w_{n,n-1} = 2\, w_{nn} = h. \tag{3.2}$$

This quadrature rule satisfies (2.3) with $k=1$, $a_0 = -a_1 = -1$ and $b_0 = b_1 = \frac{1}{2}$, so that equation (3.1) assumes the form

$$(1 - \tfrac{1}{2} z_{n+1,n+1}) \zeta - (1 + \tfrac{1}{2} z_{n+1,n}) = 0. \tag{3.3}$$

Let the eigenvalues z be real; then by the Hurwitz criterion we arrive at the stability region

$$(z_{n+1,n} + z_{n+1,n+1})(4 + z_{n+1,n} - z_{n+1,n+1}) < 0. \tag{3.4}$$

Comparing (3.4) with the condition $z<0$ resulting from (2.11) when the test equation (1.3) is used, we observe that (3.4) is more restrictive in the sense that it takes into account the variation $z_{n+1,n} - z_{n+1,n+1}$.

However, methods exist for which there is no distinction between stability regions based on (2.11) and (3.1). As an example of such methods we mention the backward differentiation formulas [9] defined by

$$a_0 = -1;\ b_l = 0,\ l \neq 0;\ \sum_{l=1}^{k} (1-l)^j a_l + j\, b_0 = 1, \qquad j = 0, 1, \ldots, k.$$

The stability regions of these formulas contain the whole left half plane for $k \leq 2$ and almost the whole left half plane (except for a small region near the imaginary axis) for $k = 3, 4, 5$ and 6. In order to make use of these excellent stability properties one should find the corresponding weights $w_{n,j}$ by solving the relations (2.3). In [13] solutions are given and the resulting quadrature formulas are investigated.

In case of the backward differentiation formulas, (3.1) takes the form

$$(a_0 + b_0 z_{n+1,n+1}) \zeta^k + \sum_{l=1}^{k} a_l \zeta^{k-l} = 0$$

and it is readily seen that the region of stability is exactly the same as the one obtained with (2.11).

4. Numerical Illustration

Finally, we investigate the practical value of the theoretical stability conditions derived from Theorem 2.1. We will illustrate this by performing a few experiments with the trapezoidal rule. From the characteristic equation (3.3) it follows that its amplification factor is given by

$$\zeta_{n+1}=(1+\tfrac{1}{2}z_{n+1,n})/(1-\tfrac{1}{2}z_{n+1,n+1}). \tag{4.1}$$

This factor describes the propagation of the perturbations

$$\Delta V_n=(\Delta f_n, \Delta G_n^{(1)}, \ldots, \Delta G_n^{(r)})^T,$$

if the starting value f_0 of the trapezoidal rule is perturbed. In practice, however, we are interested in the behaviour of Δf_n instead of ΔV_n. In the tables of results below we list the values of Δf_n due to a perturbation $\Delta f_0={}_{10}-3$ and in addition the values of the theoretical amplification factor ζ_n which were computed during the integration process. Furthermore, we also performed an experiment where we introduced instead of the *isolated* perturbation Δf_0 (cf. [2]) the perturbations $\Delta f_0, \Delta f_1, \ldots, \Delta f_9$ obtained by rounding $f_0, \ldots, f_9$ from the unperturbed experiment to 4 significant digits.

We have chosen the frequently quoted equation of de Hoog and Weiss [5]

$$\begin{aligned} f(x)=&[1+(1+x)e^{-10x}]^{\frac{1}{2}}+(1+x)[10\ln(1+x)+1-e^{-10x}]+\\ &-10\int_0^x \frac{1+x}{1+y} f^2(y)\,dy. \end{aligned} \tag{4.2}$$

In order to avoid the initial phase of this stiff problem we started the integration at $x=1$, taking the exact solution $f(x)=[1+(1+x)\exp(-10x)]^{\frac{1}{2}}$ in the interval [0, 1]. From the Tables 4.1 and 4.2 we may conclude that the perturbation $|\Delta f_n|$ decreases if the amplification factor $|\zeta_n|$ is continuously less than 1 and that

Table 4.1. *Δf_n due to an isolated perturbation Δf_0 in* (4.4)

x	$h=1$		$h=3$	
	Δf_n	ζ_n	Δf_n	ζ_n
1	$-1.0_{10}-3$		$-1.0_{10}-3$	
2	$+1.4_{10}-3$	-1.31		
3	$-1.5_{10}-3$	-1.07		
4	$+1.6_{10}-3$	-1.09	$+3.2_{10}-3$	-3.13
5	$-1.5_{10}-3$	-0.95		
6	$+1.5_{10}-3$	-1.02		
7	$-1.3_{10}-3$	-0.91	$-2.9_{10}-3$	-0.92
8	$+1.2_{10}-3$	-0.97		
9	$-1.1_{10}-3$	-0.89		
10	$+1.0_{10}-3$	-0.94	$+8.9_{10}-3$	-3.11
11	$-8.8_{10}-4$	-0.87		
12	$+8.0_{10}-4$	-0.92		
13	$-6.9_{10}-4$	-0.87	$-3.8_{10}-3$	-0.44

Table 4.2. *Δf_n due to perturbing $\Delta f_0, \ldots, \Delta f_9$ in* (4.4)

x	$h=1$		$h=2$	
	Δf_n	ζ_n	Δf_n	ζ_n
21	$+4.1_{10}-4$	-0.86	$+1.3_{10}-3$	-0.65
22	$-3.5_{10}-4$	-0.87		
23	$+3.0_{10}-4$	-0.85	$-1.9_{10}-3$	-1.50
24	$-2.5_{10}-4$	-0.86		
25	$+2.1_{10}-4$	-0.85	$+1.2_{10}-3$	-0.65
26	$-1.8_{10}-4$	-0.86		
27	$+1.5_{10}-4$	-0.85	$-1.8_{10}-3$	-1.45
28	$-1.3_{10}-4$	-0.85		
29	$+1.1_{10}-4$	-0.85	$+1.2_{10}-3$	-0.65
30	$-9.4_{10}-5$	-0.85		
31	$+7.9_{10}-5$	-0.85	$-1.6_{10}-3$	-1.39
32	$-6.7_{10}-5$	-0.85		
33	$+5.6_{10}-5$	-0.85	$+1.1_{10}-3$	-0.67

$|\Delta f_n|$ is oscillating if $|\zeta_n|$ is alternately less than and greater than 1. (In the experiment with 10 independently perturbed f_i-values we got divergence for $h=3$ and $x>19$.)

Since in this problem $\partial K/\partial f$ is always negative, application of the analysis based on (1.3) would predict a decrease in Δf_n for all stepsizes h. This is contradicted by our experiments.

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