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ORTHOGONAL POLYNOMIALS IN TWO VARIABLES. A FURTHER ANALYSIS OF THE POLYNOMIALS ORTHOGONAL OVER A REGION BOUNDED BY TWO LINES AND A PARABOLA*

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Abstract. Some new results are obtained for the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$, introduced by Koornwinder [4] which are orthogonal over a region bounded by two straight lines and a parabola. The most important results are a Rodrigues-type formula and the recurrence relations for $up_{n,k}^{\alpha,\beta,\gamma}(u, v)$ and $vp_{n,k}^{\alpha,\beta,\gamma}(u, v)$. These recurrence relations contain 5 and 9 terms, respectively. Furthermore, the quadratic norm of $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ and the value of $p_{n,k}^{\alpha,\beta,\gamma}(2, 1)$ are explicitly given.

1. Introduction. In this paper, the analysis of the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ introduced by T. H. Koornwinder [4] will be continued.

In many respects, this class of orthogonal polynomials in two variables can be compared with the important class of Jacobi polynomials. In this analysis, some properties of the Jacobi polynomials are generalized to the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$.

The polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ form an orthogonal set over a region bounded by two perpendicular straight lines, $1-u+v=0$, $1+u+v=0$ and by the parabola $u^2-4v=0$ touching these lines, with respect to the weight function $(1-u+v)^{\alpha}(1+u+v)^{\beta}(u^2-4v)^{\gamma}$ which is singular at the boundary of the orthogonality region. For reasons of convergence, it is required that $\alpha,\beta,\gamma>-1$ and $\alpha+\gamma+3/2$, $\beta+\gamma+3/2>0$. The main results of Koornwinder's paper are summarized in § 2.

In the subsequent sections, a further analysis is given, using as the main tools a number of partial differential operators. In [4] it is proved that the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ are eigenfunctions of a second order operator $D_1^{\alpha,\beta,\gamma}$ and a fourth order operator $D_2^{\alpha,\beta,\gamma}$, which are algebraically independent. Furthermore, two second order operators D_-^{γ} and $D_+^{\alpha,\beta,\gamma}$ are derived with the property that $D_-^{\gamma}p_{n,k}^{\alpha,\beta,\gamma}(u, v) = \text{const.}\ p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(u, v)$ and $D_+^{\alpha,\beta,\gamma}p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(u, v) = \text{const.}\ p_{n,k}^{\alpha,\beta,\gamma}(u, v)$. Then $D_2^{\alpha,\beta,\gamma}$ is given by $D_2^{\alpha,\beta,\gamma}=D_+^{\alpha,\beta,\gamma}\circ D_-^{\gamma}$.

In § 4 of this paper, another pair of differential operators is derived: these operators $E_-^{\alpha,\beta}$ and $E_+^{\alpha,\beta,\gamma}$ have the property that $E_-^{\alpha,\beta}p_{n,k}^{\alpha,\beta,\gamma}(u, v) = \text{const.}\ p_{n-1,k}^{\alpha,\beta,\gamma}(u, v)$ and $E_+^{\alpha,\beta,\gamma}p_{n-1,k}^{\alpha,\beta,\gamma+1}(u, v) = \text{const.}\ p_{n,k}^{\alpha,\beta,\gamma}(u, v)$. Then another fourth order operator, which has the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ as eigenfunctions, can be defined by $D_3^{\alpha,\beta,\gamma}=E_+^{\alpha,\beta,\gamma}\circ E_-^{\alpha,\beta}$. This operator is explicitly expressed as a polynomial in $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$. The operators D_-^{γ} and $E_-^{\alpha,\beta}$ together play a similar role for the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ to that played by the operator d/dx for the Jacobi polynomials.

One of the first problems which arise is to find an explicit expression for $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$. We have succeeded in finding an explicit expression of the Rodrigues-type by using the second order operators D_+ and E_+. Expressing D_+ and E_+ in $(D_-)^*$ and $(E_-)^*$ respectively, we obtain a formula for $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ which is similar to the Rodrigues formula for the Jacobi polynomials, but with the two second

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order operators $(D_-)^*$ and $(E_-)^*$ instead of d/dx (§ 5). However, the expression derived by us is rather complicated, and so we have tried to find other expressions for $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. If $\gamma=-\frac{1}{2}$ and $\gamma=+\frac{1}{2}$, the polynomials can be expressed as symmetric ([4], § 2) or antisymmetric (§ 3) products of Jacobi polynomials. In § 10, the polynomials with $\alpha,\beta=\pm\frac{1}{2}$ are expressed in terms of Jacobi polynomials. The case $\gamma=+\frac{1}{2}$ is comparable with the determinants of orthogonal polynomials treated by Karlin and McGregor [3]. The orthogonal set of 2×2 determinants of Jacobi polynomials gives $p_{n,k}^{\alpha,\beta,+1/2}(x+y,xy)$ after dividing by $(x-y)$.

In § 6, the explicit value of the quadratic norm for the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ is given. The quadratic norm is important for finding coefficients in Fourier expansions with respect to the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ and will be used for the computation of some of the coefficients in the recurrence relations (§ 9).

Without knowing an explicit expression for $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$, it is possible to find the value of $p_{n,k}^{\alpha,\beta,\gamma}(2,1)$ by using the operators D_+ and E_+ (§ 7). The point $(u,v)=(2,1)$ is a vertex of the orthogonality region, which probably plays a similar role for the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ to that played by the point $x=1$ for the Jacobi plynomials $P_n^{(\alpha,\beta)}(x)$. The (unproved) hypothesis is that $p_{n,k}^{\alpha,\beta,\gamma}(2,1)$ is the absolute maximum of $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ if $\alpha\geqq\beta\geqq-\frac{1}{2}$ and $\gamma\geqq-\frac{1}{2}$. For $\gamma=-\frac{1}{2}$, this maximum property follows directly from the explicit expression of $p_{n,k}^{\alpha,\beta,-(1/2)}(u,v)$ and the maximum property of the Jacobi polynomials.

The analysis of these polynomials suggests that not all powers $\leqq(n,k)$ of u and v appear in $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. This is proved in § 8 and it has a number of consequences. An immediate consequence is that some theorems which give alternative definitions for $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ can be derived. Another is that the number of terms in the recurrence relations is uniformly bounded, while for general polynomials in more than one variable this number depends on the degree of the polynomial. In § 9, the recurrence relations are explicitly given. For $up_{n,k}^{\alpha,\beta,\gamma}(u,v)$ and $vp_{n,k}^{\alpha,\beta,\gamma}(u,v)$, we obtain a five-term and a nine-term recurrence relation, respectively. To build up $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ using the recurrence relations we need the formula for $vp_{n,k}^{\alpha,\beta,\gamma}(u,v)$ only if $n=k$, and then six terms remain.

Finally, in § 10, two quadratic transformation formulas are given for the case $\alpha=\beta$. These formulas, together with the explicit expressions for $\gamma=+\frac{1}{2}$, $\gamma=-\frac{1}{2}$ yield explicit expressions for the cases that α and β are $+\frac{1}{2}$ or $-\frac{1}{2}$.

2. Preliminaries. In this section, the main results obtained by Koornwinder [4] are summarized.

Let $\mathcal{N}$ be the set of pairs of integers (n,k), $n\geqq k\geqq 0$, with a lexicographic ordering defined by

$$(m,l)\leqq(n,k)\Leftrightarrow\{m<n\vee(m=n\wedge l\leqq k)\}. \tag{2.1}$$

A polynomial $q(u,v)$ is said to have *degree* $(n,k)\in\mathcal{N}$ if

$$q(u,v)=\sum_{(m,l)\leqq(n,k)} c_{m,l}u^{m-l}v^l,\quad \text{with } c_{n,k}\neq 0.$$

The region with the properties $1-u+v>0$, $1+u+v>0$ and $u^2-4v>0$, is denoted by R (cf. Fig. 1). In the region R the weight function $\mu^{\alpha,\beta,\gamma}(u, v)$ is defined by

$$\mu^{\alpha,\beta,\gamma}(u, v) = (1-u+v)^{\alpha}(1+u+v)^{\beta}(u^2-4v)^{\gamma}. \tag{2.2}$$

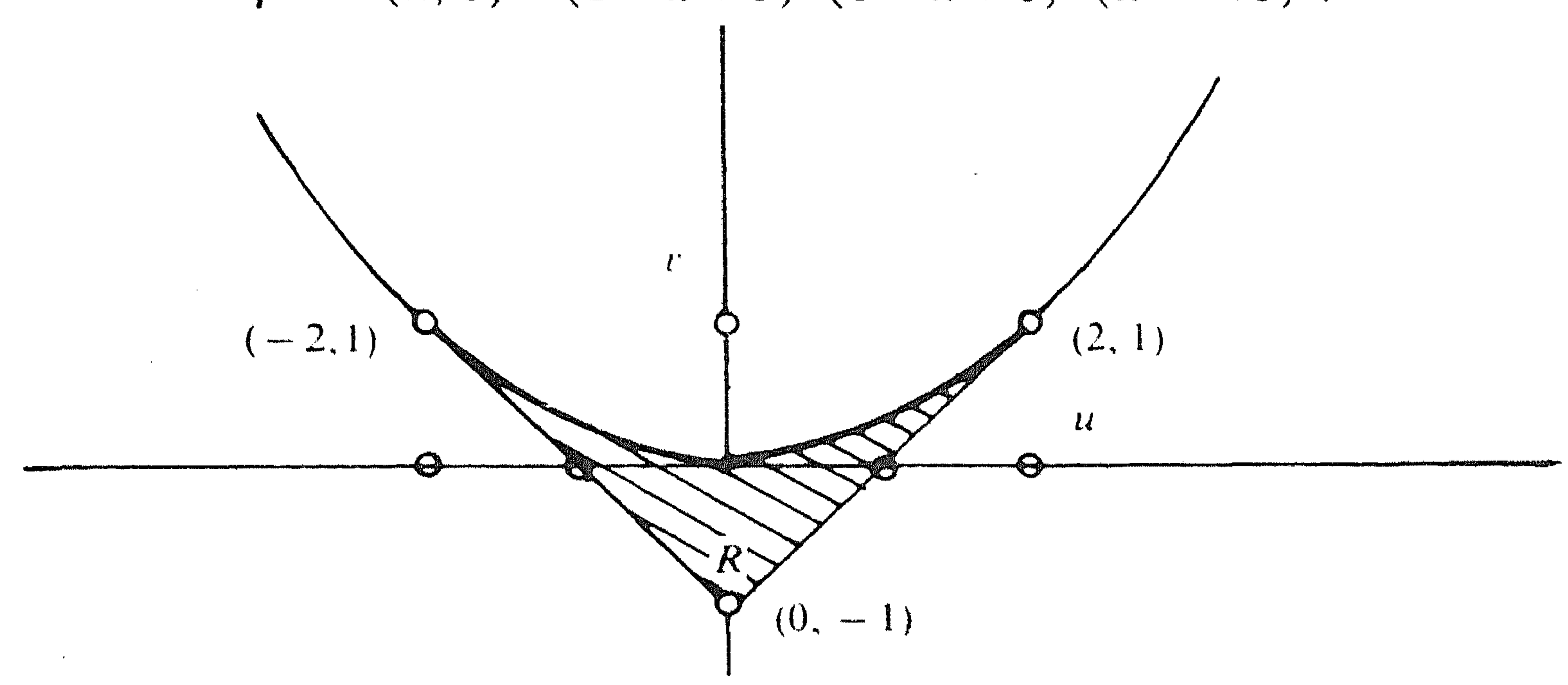

FIG. 1

DEFINITION 2.1. For $(n, k) \in \mathcal{N}$ and $\alpha,\beta,\gamma > -1$, $\alpha+\gamma+\frac{3}{2}$, $\beta+\gamma+\frac{3}{2} > 0$ the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ are given by

(i) $p_{n,k}^{\alpha,\beta,\gamma}(u, v) = u^{n-k}v^{k} + \text{a polynomial of degree lower than } (n, k)$.

(ii) $\iint_R p_{n,k}^{\alpha,\beta,\gamma}(u, v)\, u^{m-l}v^{l}\mu^{\alpha,\beta,\gamma}(u, v)\, du\, dv = 0$ if $(m, l) < (n, k)$.

Then $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ satisfies

$$D_1^{\alpha,\beta,\gamma} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = -[n(n+\alpha+\beta+2\gamma+2)+k(k+\alpha+\beta+1)]p_{n,k}^{\alpha,\beta,\gamma}(u, v), \tag{2.3}$$

$$D_2^{\alpha,\beta,\gamma} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = k(k+\alpha+\beta+1)(n+\gamma+\tfrac{1}{2})(n+\alpha+\beta+\gamma+\tfrac{3}{2})p_{n,k}^{\alpha,\beta,\gamma}(u, v), \tag{2.4}$$

$$D_-^{\gamma} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = \begin{cases} k(n+\gamma+\frac{1}{2})p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(u, v) & \text{if } k > 0, \\ 0 & \text{if } k = 0, \end{cases} \tag{2.5}$$

$$D_+^{\alpha,\beta,\gamma} p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(u, v) = (k+\alpha+\beta+1)(n+\alpha+\beta+\gamma+\tfrac{3}{2})p_{n,k}^{\alpha,\beta,\gamma}(u, v) \quad \text{if } k > 0. \tag{2.6}$$

The operators are defined by

$$\begin{aligned} D_1^{\alpha,\beta,\gamma} = {} & (-u^2+2v+2)\frac{\partial^2}{\partial u^2} - 2u(v-1)\frac{\partial^2}{\partial u\,\partial v} + (u^2-2v^2-2v)\frac{\partial^2}{\partial v^2} \\ & + [-(\alpha+\beta+2\gamma+3)u + (2\beta-2\alpha)]\frac{\partial}{\partial u} \\ & + [(\beta-\alpha)u - (2\alpha+2\beta+2\gamma+5)v - (2\gamma+1)]\frac{\partial}{\partial v}, \end{aligned} \tag{2.7}$$

$$D_-^{\gamma} = \frac{\partial^2}{\partial u^2} + u\frac{\partial^2}{\partial u\,\partial v} + v\frac{\partial^2}{\partial v^2} + (\gamma+\tfrac{3}{2})\frac{\partial}{\partial v}, \tag{2.8}$$

$$\begin{aligned} D_+^{\alpha,\beta,\gamma} = {} & (1-u+v)^{-\alpha}(1+u+v)^{-\beta} D_-^{\gamma} \circ (1-u+v)^{\alpha+1}(1+u+v)^{\beta+1} \\ = {} & (1-u+v)(1+u+v)\left(\frac{\partial^2}{\partial u^2} + u\frac{\partial^2}{\partial u\,\partial v} + v\frac{\partial^2}{\partial v^2}\right) \\ & + [(\alpha-\beta)(u^2-2v-2) + (\alpha+\beta+2)u(v-1)]\frac{\partial}{\partial u} \end{aligned}$$

(2.9)
$$+[(\alpha+\beta+\gamma+\tfrac{7}{2})(-u^2+2v)+(\alpha-\beta)u(v-1)+(2\alpha+2\beta+\gamma+\tfrac{11}{2})v^2$$
$$+(\gamma+\tfrac{3}{2})]\frac{\partial}{\partial v}+(\alpha-\beta)(\alpha+\beta+\gamma+\tfrac{5}{2})u+(\alpha+\beta+2)(\alpha+\beta+\gamma+\tfrac{5}{2})v$$
$$+(\alpha-\beta)^2+(\gamma+\tfrac{1}{2})(\alpha+\beta+2),$$

(2.10)
$$D_2^{\alpha,\beta,\gamma}=D_+^{\alpha,\beta,\gamma}\circ D_-^{\gamma}.$$

Consideration of $(D_-^\gamma)^*$, the adjoint operator to D_-^γ, yields

(2.11)
$$(D_-^\gamma)^*=D_-^{-\gamma}=(u^2-4v)^\gamma D_-^\gamma\circ(u^2-4v)^{-\gamma}.$$

Hence

(2.12)
$$D_+^{\alpha,\beta,\gamma}=\{\mu^{\alpha,\beta,\gamma}(u,v)\}^{-1}(D_-^\gamma)^*\circ\mu^{\alpha+1,\beta+1,\gamma}(u,v).$$

The operators $D_+^{\alpha,\beta,\gamma}$ and D_-^γ are related by

(2.13)
$$\iint_R (D_+^{\alpha,\beta,\gamma}p(u,v))q(u,v)\mu^{\alpha,\beta,\gamma}(u,v)\,du\,dv$$
$$=\iint_R p(u,v)(D_-^\gamma q(u,v))\mu^{\alpha+1,\beta+1,\gamma}(u,v)\,du\,dv,$$

for any two polynomials $p(u,v)$ and $q(u,v)$.

Let

(2.14)
$$p_n^{\alpha,\beta}(x)=\frac{2^n n!}{(n+\alpha+\beta+1)_n}P_n^{(\alpha,\beta)}(x),$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of order (α,β) (for Jacobi polynomials see Erdélyi [2] or Szegö [6]). Then

(2.15)
$$p_{n,k}^{\alpha,\beta,-1/2}(x+y,xy)=\begin{cases}p_n^{\alpha,\beta}(x)p_k^{\alpha,\beta}(y)+p_k^{\alpha,\beta}(x)p_n^{\alpha,\beta}(y) & \text{if } n>k,\\ p_n^{\alpha,\beta}(x)p_k^{\alpha,\beta}(y) & \text{if } n=k.\end{cases}$$

3. The polynomials $p_{n,k}^{\alpha,\beta,+1/2}(u,v)$ as an antisymmetric product of Jacobi polynomials. Consider the antisymmetric product of Jacobi polynomials:

(3.1)
$$f_{n,k}^{\alpha,\beta}(x,y)=p_{n+1}^{\alpha,\beta}(x)p_k^{\alpha,\beta}(y)-p_k^{\alpha,\beta}(x)p_{n+1}^{\alpha,\beta}(y),$$

where $p_n^{\alpha,\beta}(x)$ is defined by (2.14).

The polynomials $f_{n,k}^{\alpha,\beta}(x,y)$ form an orthogonal set of antisymmetric polynomials over the simplex $-1\leqq y\leqq x\leqq 1$ with respect to the weight function $((1-x)(1-y))^\alpha((1+x)(1+y))^\beta$ (cf. [3]). Then $(x-y)^{-1}f_{n,k}^{\alpha,\beta}(x,y)$ is a symmetric polynomial in x and y which can be uniquely expressed as a polynomial in $x+y=u$ and $xy=v$ (see van der Waerden [7, § 33]).

LEMMA 3.1. $(x-y)^{-1}f_{n,k}^{\alpha,\beta}(x,y)=p_{n,k}^{\alpha,\beta,+1/2}(x+y,xy)$.

Proof. Application of the definition of $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ (Definition 2.1) yields

(i) $$(x-y)^{-1}f_{n,k}^{\alpha,\beta}(x, y) = \sum_{(m,l)\leqq(n,k)} c_{m,l}\frac{x^{m+1}y^l - x^l y^{m+1}}{x-y} \quad \text{with } c_{n,k}=1$$

$$= (x+y)^{n-k}(xy)^k + \text{a polynomial in } (x+y) \text{ and } xy \text{ of degree lower than } (n, k).$$

(ii) $\{(x-y)^{-1}f_{n,k}^{\alpha,\beta}(x, y)\}$ is an orthogonal set with respect to the measure

$$(x-y)^2((1-x)(1-y))^\alpha((1+x)(1+y))^\beta\, dx\, dy =$$

$$\text{const.}\ (1-u+v)^\alpha(1+u+v)^\beta(u^2-4v)^{+1/2}\, du\, dv.$$

Hence

(3.2) $$p_{n,k}^{\alpha,\beta,+1/2}(x+y, xy) = (x-y)^{-1}\{p_{n+1}^{\alpha,\beta}(x)p_k^{\alpha,\beta}(y) - p_k^{\alpha,\beta}(x)p_{n+1}^{\alpha,\beta}(y)\}.$$

4. A pair of differential operators which change n and γ. A pair of differential operators which change n and γ can be found by using (2.15) and (3.2) and the differential operators for the Jacobi polynomials. Let us define

(4.1) $$D_{(x)}^{\alpha,\beta} = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{\partial}{\partial x}(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{\partial}{\partial x};$$

then

$$D_{(x)}^{\alpha,\beta}p_n^{\alpha,\beta}(x) = c_n p_n^{\alpha,\beta}(x), \quad \text{where } c_n = -n(n+\alpha+\beta+1).$$

Hence

(4.2)
$$(x-y)^{-1}\{D_{(x)}^{\alpha,\beta} - D_{(y)}^{\alpha,\beta}\}p_{n,k}^{\alpha,\beta,-1/2}(x+y, xy) = \begin{cases}(c_n - c_k)p_{n-1,k}^{\alpha,\beta,+1/2}(x+y, xy) & \text{if } n > k,\\ 0 & \text{if } n = k,\end{cases}$$

and

(4.3)
$$\{D_{(x)}^{\alpha,\beta} - D_{(y)}^{\alpha,\beta}\}\circ(x-y)p_{n-1,k}^{\alpha,\beta,+1/2}(x+y, xy) = (c_n - c_k)p_{n,k}^{\alpha,\beta,-1/2}(x+y, xy) \quad \text{if } n > k.$$

Formulas (4.2) and (4.3) now suggest the following definition:

(4.4)
$$E_-^{\alpha,\beta} = (x-y)^{-1}\{D_{(x)}^{\alpha,\beta} - D_{(y)}^{\alpha,\beta}\}$$
$$= u\frac{\partial^2}{\partial u^2} + 2(v+1)\frac{\partial^2}{\partial u\,\partial v} + u\frac{\partial^2}{\partial v^2} + (\beta-\alpha)\frac{\partial}{\partial v} + (\alpha+\beta+2)\frac{\partial}{\partial u}.$$

If $(E_-^{\alpha,\beta})^*$ is the adjoint operator to $E_-^{\alpha,\beta}$, then

(4.5) $$(E_-^{\alpha,\beta})^* = u\frac{\partial^2}{\partial u^2} + 2(v+1)\frac{\partial^2}{\partial u\,\partial v} + u\frac{\partial^2}{\partial v^2} - (\beta-\alpha)\frac{\partial}{\partial v} - (\alpha+\beta-2)\frac{\partial}{\partial u}.$$

Note that

(4.6) $$(E_-^{\alpha,\beta})^* = E_-^{-\alpha,-\beta} = (1-u+v)^\alpha(1+u+v)^\beta E_-^{\alpha,\beta}\circ(1-u+v)^{-\alpha}(1+u+v)^{-\beta}.$$

Now we can define the operator $E_+^{\alpha,\beta,\gamma}$ as

$$
\begin{aligned}
E_+^{\alpha,\beta,\gamma} &= \{\mu^{\alpha,\beta,\gamma}(u,v)\}^{-1}(E_-^{\alpha,\beta})^* \circ \mu^{\alpha,\beta,\gamma+1}(u,v) \\
&= (u^2-4v)\left(u\frac{\partial^2}{\partial u^2} + 2(v+1)\frac{\partial^2}{\partial u\,\partial v} + u\frac{\partial^2}{\partial v^2}\right) \\
&\quad + [(\alpha+\beta+4\gamma+6)(u^2-4v)+8(\gamma+1)(v-1)]\frac{\partial}{\partial u} \\
&\quad + [(\beta-\alpha)(u^2-4v)+4(\gamma+1)u(v-1)]\frac{\partial}{\partial v} \\
&\quad + 2(\gamma+1)(\alpha+\beta+2\gamma+3)u - 4(\gamma+1)(\beta-\alpha).
\end{aligned} \tag{4.7}
$$

LEMMA 4.1. *If*

$$q_{n,k}(u,v) = u^{n-k}v^k + \text{a polynomial of degree lower than } (n,k)$$

and

$$q_{n-1,k}(u,v) = u^{n-k-1}v^k + \text{a polynomial of degree lower than } (n-1,k),$$

then

$$
E_-^{\alpha,\beta} q_{n,k}(u,v) = \begin{cases} (n-k)(n+k+\alpha+\beta+1)u^{n-k-1}v^k + \text{a polynomial of} \\ \text{degree lower than } (n-1,k) \quad \text{if } n>k, \\ \text{a polynomial of degree equal or lower than} \\ (n-1,n-1) \quad \text{if } n=k, \end{cases}
$$

and

$$
\begin{aligned}
E_+^{\alpha,\beta,\gamma} q_{n-1,k}(u,v) = (n-k+2\gamma+1)(n+k+\alpha+\beta+2\gamma+2)u^{n-k}v^k + \text{a polynomial} \\
\text{of degree lower than } (n,k) \quad \text{if } n>k.
\end{aligned}
$$

Proof. Lemma 4.1 follows immediately from (4.4) and (4.7). □

Koornwinder proved the following.

LEMMA 4.2. *Let R be a bounded region in $\mathbb{R}^2$ such that certain polynomials $w_1(x,y), w_2(x,y), \cdots, w_k(x,y)$ are positive over R and the product $w_1 \cdot w_2, \cdots, w_k$ is zero at the boundary ∂R.*

Let $X^{\alpha_1,\cdots,\alpha_k}$ be a partial differential operator in x, and y, its coefficients being polynomials in $x, y, \alpha_1, \cdots, \alpha_k$.

Let the operator $Y^{\alpha_1,\cdots,\alpha_k}$ be defined by

$$Y^{\alpha_1,\cdots,\alpha_k} = w_1^{-\alpha_1}\cdots w_k^{-\alpha_k}(X^{\alpha_1,\cdots,\alpha_k})^* \circ w_1^{\alpha_1+i_1}\cdots w_k^{\alpha_k+i_k},$$

for certain nonnegative integers $i_1, \cdots, i_k$.

If this operator also has coefficients that are polynomials in $x, y, \alpha_1, \cdots, \alpha_k$, then

$$
\begin{aligned}
&\iint_R p(Y^{\alpha_1,\cdots,\alpha_k}q)w_1^{\alpha_1}\cdots w_k^{\alpha_k}\,dx\,dy \\
&\qquad = \iint_R (X^{\alpha_1,\cdots,\alpha_k}p)q w_1^{\alpha_1+i_1}\cdots w_k^{\alpha_k+i_k}\,dx\,dy,
\end{aligned}
$$

for any two polynomials p and q, and for all real $\alpha_1, \cdots, \alpha_k$ such that

$$\iint_R w_1^{\alpha_1} \cdots w_k^{\alpha_k}\, dx\, dy < \infty.$$

Proof. For sufficiently large $\alpha_1, \alpha_2, \cdots, \alpha_k$ the equality follows from partial integration because the function $w_1^{\alpha_1} \cdots w_k^{\alpha_k}$ and its partial derivatives up to a certain order are zero at the boundary ∂R. By analytic continuation the equality follows for all $\alpha_1, \cdots, \alpha_k$ such that

$$\iint_R w_1^{\alpha_1} \cdots w_k^{\alpha_k}\, dx\, dy < \infty. \qquad \square$$

Rewriting this lemma for $E_-^{\alpha,\beta}$ and $E_+^{\alpha,\beta,\gamma}$ we obtain

$$(4.8) \qquad \iint_R p(E_+^{\alpha,\beta,\gamma} q)\mu^{\alpha,\beta,\gamma}(u, v)\, du\, dv = \iint_R (E_-^{\alpha,\beta} p) q \mu^{\alpha,\beta,\gamma+1}(u, v)\, du\, dv,$$

for any two polynomials p and q.

From Lemma 4.1, formula (4.8) and Definition 2.1 the following can be proved.

COROLLARY.

$$(4.9) \quad E_-^{\alpha,\beta} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = \begin{cases} (n-k)(n+k+\alpha+\beta+1) p_{n-1,k}^{\alpha,\beta,\gamma+1}(u, v) & \text{if } n > k, \\ 0 & \text{if } n = k, \end{cases}$$

and

$$(4.10) \quad E_+^{\alpha,\beta,\gamma} p_{n-1,k}^{\alpha,\beta,\gamma+1}(u, v) = (n-k+2\gamma+1) \cdot (n+k+\alpha+\beta+2\gamma+2) p_{n,k}^{\alpha,\beta,\gamma}(u, v) \quad \text{if } n > k$$

(cf. the proof of Theorem 5.4 in [4]).

Let us define the fourth order operator

$$(4.11) \qquad D_3^{\alpha,\beta,\gamma} = E_+^{\alpha,\beta,\gamma} \circ E_-^{\alpha,\beta}.$$

The polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ are eigenfunctions of $D_3^{\alpha,\beta,\gamma}$:

$$(4.12) \quad D_3^{\alpha,\beta,\gamma} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = (n-k)(n-k+2\gamma+1)(n+k+\alpha+\beta+1) \cdot (n+k+\alpha+\beta+2\gamma+2) p_{n,k}^{\alpha,\beta,\gamma}(u, v).$$

Hence $D_3^{\alpha,\beta,\gamma}$ can be uniquely expressed as a polynomial in $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$ (cf. [4, Thm. 6.5]). By considering the eigenvalues it is clear that

$$(4.13) \qquad D_3^{\alpha,\beta,\gamma} = (D_1^{\alpha,\beta,\gamma})^2 - 4D_2^{\alpha,\beta,\gamma} - (2\gamma+1)(\alpha+\beta+1) D_1^{\alpha,\beta,\gamma}.$$

5. A Rodrigues-type formula. Using (2.6) and (4.10), $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ can be expressed in terms of polynomials of lower degree.

In (2.6) and (4.10) we write $D_+^{\alpha,\beta,\gamma}$ and $E_+^{\alpha,\beta,\gamma}$ respectively as

$$(2.11) \qquad D_+^{\alpha,\beta,\gamma} = \{\mu^{\alpha,\beta,\gamma}(u, v)\}^{-1} (D_-^{\gamma})^* \circ \mu^{\alpha+1,\beta+1,\gamma}(u, v)$$

and

$$E_{+}^{\alpha,\beta,\gamma}=\{\mu^{\alpha,\beta,\gamma}(u,v)\}^{-1}(E_{-}^{\alpha,\beta})^{*}\circ\mu^{\alpha,\beta,\gamma+1}(u,v). \tag{4.7}$$

An $(n-k)$-fold application of (4.10) and a k-fold application of (2.6) to $p_{0,0}^{\alpha+k,\beta+k,\gamma+n-k}(u,v)\equiv 1$ yields

$$\begin{aligned}(k+\alpha+\beta+1)_k(n+\alpha+\beta+\gamma+\tfrac{3}{2})_k(n-k+2\gamma+1)_{n-k}(n+k+\alpha+\beta\\ +2\gamma+2)_{n-k}p_{n,k}^{\alpha,\beta,\gamma}(u,v)\\ =(1-u+v)^{-\alpha}(1+u+v)^{-\beta}(u^2-4v)^{-\gamma}\\ \cdot\left\{\frac{\partial^2}{\partial u^2}+u\frac{\partial^2}{\partial u\,\partial v}+v\frac{\partial^2}{\partial v^2}-\left(\gamma-\frac{3}{2}\right)\frac{\partial}{\partial v}\right\}^k\\ \circ\left\{u\frac{\partial^2}{\partial u^2}+2(v+1)\frac{\partial^2}{\partial u\,\partial v}+u\frac{\partial^2}{\partial v^2}-(\beta-\alpha)\frac{\partial}{\partial v}-(\alpha+\beta+2k-2)\frac{\partial}{\partial u}\right\}^{n-k}\\ \circ(1-u+v)^{\alpha+k}(1+u+v)^{\beta+k}(u^2-4v)^{\gamma+n-k}.\end{aligned} \tag{5.1}$$

This is a Rodrigues-type formula for $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. So far it is the only "explicit" expression for $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ in the case of general α, β, γ.

6. The quadratic norm of $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. The quadratic norm $h_{n,k}^{\alpha,\beta,\gamma}$ of the polynomial $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ is defined by

$$h_{n,k}^{\alpha,\beta,\gamma}=\iint_R\{p_{n,k}^{\alpha,\beta,\gamma}(u,v)\}^2\mu^{\alpha,\beta,\gamma}(u,v)\,du\,dv. \tag{6.1}$$

The explicit value of $h_{n,k}^{\alpha,\beta,\gamma}$ is important for calculating the coefficients in Fourier expansions with respect to the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ (cf. § 9).

From (2.13) and (4.8) we obtain the following recurrence relations for $h_{n,k}^{\alpha,\beta,\gamma}$:

$$h_{n,k}^{\alpha,\beta,\gamma}=\frac{k(n+\gamma+\frac{1}{2})}{(k+\alpha+\beta+1)(n+\alpha+\beta+\gamma+\frac{3}{2})}h_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}, \tag{6.2}$$

and

$$h_{n,k}^{\alpha,\beta,\gamma}=\frac{(n-k)(n+k+\alpha+\beta+1)}{(n-k+2\gamma+1)(n+k+\alpha+\beta+2\gamma+2)}h_{n-1,k}^{\alpha,\beta,\gamma+1}. \tag{6.3}$$

By repeated application of (6.2) and (6.3) we find

$$\begin{aligned}h_{n,k}^{\alpha,\beta,\gamma}=\\ \frac{k!(n-k)!(n-k+\gamma+\frac{3}{2})_k(2k+\alpha+\beta+2)_{n-k}}{(k+\alpha+\beta+1)_k(n+\alpha+\beta+\gamma+\frac{3}{2})_k(n-k+2\gamma+1)_{n-k}(n+k+\alpha+\beta+2\gamma+2)_{n-k}}\\ \cdot h_{0,0}^{\alpha+k,\beta+k,\gamma+n-k}.\end{aligned} \tag{6.4}$$

LEMMA 6.1.

(6.5)
$$h_{0,0}^{\alpha,\beta,\gamma}=\frac{2^{2\alpha+2\beta+4\gamma+3}}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+\frac{5}{2})}\,\frac{\Gamma(\alpha+\gamma+\frac{3}{2})\Gamma(\beta+\gamma+\frac{3}{2})}{\Gamma(\alpha+\beta+2\gamma+3)}.$$

Proof. $p_{0,0}^{\alpha,\beta,\gamma}(u,v)\equiv 1$, thus
$$h_{0,0}^{\alpha,\beta,\gamma}=\iint_R (1-u+v)^\alpha(1+u+v)^\beta(u^2-4v)^\gamma\,du\,dv.$$
This transforms under the substitution
$$u=x+y,\quad v=xy$$
into
$$h_{0,0}^{\alpha,\beta,\gamma}=\int_{x=-1}^{1}\left\{\int_{y=-1}^{x}(1-y)^\alpha(1+y)^\beta(x-y)^{2\gamma+1}\,dy\right\}(1-x)^\alpha(1+x)^\beta\,dx.$$
By making the substitution $t=(1+x)^{-1}(1+y)$ and using
$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z),$$
it follows that
$$h_{0,0}^{\alpha,\beta,\gamma}=2^\alpha\frac{\Gamma(\beta+1)\Gamma(2\gamma+2)}{\Gamma(\beta+2\gamma+3)}\int_{-1}^{+1}(1-x)^\alpha(1+x)^{2\beta+2\gamma+2}$$
$$\cdot\,{}_2F_1\left(-\alpha,\beta+1;\beta+2\gamma+3;\frac{1+x}{2}\right)dx$$
$$=2^{2\alpha+2\beta+2\gamma+3}\frac{\Gamma(\beta+1)\Gamma(2\gamma+2)}{\Gamma(\beta+2\gamma+3)}\int_0^1(1-s)^\alpha s^{2\beta+2\gamma+2}$$
$$\cdot\,{}_2F_1(-\alpha,\beta+1;\beta+2\gamma+3;s)\,ds$$
$$=2^{2\alpha+2\beta+2\gamma+3}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\gamma+2)\Gamma(2\beta+2\gamma+3)}{\Gamma(\beta+2\gamma+3)\Gamma(\alpha+2\beta+2\gamma+4)}$$
$$\cdot\,{}_3F_2(-\alpha,\beta+1,2\beta+2\gamma+3;\ \beta+2\gamma+3,\alpha+2\beta+2\gamma+4;1).$$

This ${}_3F_2$ function is of type ${}_3F_2(a,b,c;1+a-b,1+a-c;1)$ with $a=2\beta+2\gamma+3$, $b=\beta+1$ and $c=-\alpha$ and so the theorem of Dixon can be applied (see Bailey [1, Chap. 3.1] or Slater [5, (2.3.3)]). This proves the lemma. □

COROLLARY. *The quadratic norm $h_{n,k}^{\alpha,\beta,\gamma}$ is equal to*

(6.6)
$$h_{n,k}^{\alpha,\beta,\gamma}=\frac{2^{4n+2\alpha+2\beta+4\gamma+3}k!(n-k)!(n-k+\gamma+\frac{3}{2})_k(2k+\alpha+\beta+2)_{n-k}}{\sqrt{\pi}(k+\alpha+\beta+1)_k(n+\alpha+\beta+\gamma+\frac{3}{2})_k(n-k+2\gamma+1)_{n-k}(n+k+\alpha+\beta+2\gamma+2)_{n-k}}$$
$$\cdot\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)\Gamma(n-k+\gamma+1)\Gamma(n+\alpha+\gamma+\frac{3}{2})\Gamma(n+\beta+\gamma+\frac{3}{2})}{\Gamma(n+k+\alpha+\beta+\gamma+\frac{5}{2})\Gamma(2n+\alpha+\beta+2\gamma+3)}.$$

7. The value of $p_{n,k}^{\alpha,\beta,\gamma}(2,1)$. It is possible to find the value of $p_{n,k}^{\alpha,\beta,\gamma}(2,1)$ by using the operators $D_+^{\alpha,\beta,\gamma}$ and $E_+^{\alpha,\beta,\gamma}$. It is of interest to know this value because of the hypothesis that for $\alpha \geqq \beta \geqq -\frac{1}{2}$ and $\gamma \geqq -\frac{1}{2}$ the inequality

$$|p_{n,k}^{\alpha,\beta,\gamma}(u,v)| \leqq p_{n,k}^{\alpha,\beta,\gamma}(2,1)$$

is valid. This hypothesis was proved for $\gamma = -\frac{1}{2}$. If $\gamma \geqq -\frac{1}{2}$, then it is true if $\alpha = \beta = -\frac{1}{2}$. Further it holds for the polynomials $p_{n,n}^{\alpha,\beta,+1/2}(u,v)$, $p_{n,n-1}^{\alpha,\alpha,+1/2}(u,v)$ and $p_{n,0}^{+1/2,-1/2,\gamma}(u,v)$.

Considering (2.9) and (4.7) we obtain the following equalities:

$$(D_+^{\alpha,\beta,\gamma}p)(2,1) = 4(\alpha+1)(\alpha+\gamma+\tfrac{3}{2})p(2,1)$$

and

$$(E_+^{\alpha,\beta,\gamma}p)(2,1) = 8(\gamma+1)(\alpha+\gamma+\tfrac{3}{2})p(2,1),$$

for any polynomial $p(u,v)$.

Hence

$$p_{n,k}^{\alpha,\beta,\gamma}(2,1) = \frac{4(\alpha+1)(\alpha+\gamma+\frac{3}{2})}{(k+\alpha+\beta+1)(n+\alpha+\beta+\gamma+\frac{3}{2})} p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(2,1) \tag{7.1}$$

and

$$p_{n,k}^{\alpha,\beta,\gamma}(2,1) = \frac{8(\gamma+1)(\alpha+\gamma+\frac{3}{2})}{(n-k+2\gamma+1)(n+k+\alpha+\beta+2\gamma+2)} p_{n-1,k}^{\alpha,\beta,\gamma+1}(2,1). \tag{7.2}$$

From (7.1), (7.2) and $p_{0,0}^{\alpha,\beta,\gamma}(u,v) \equiv 1$ it follows that

$$p_{n,k}^{\alpha,\beta,\gamma}(2,1) = \frac{2^{3n-k}(\alpha+1)_k(\gamma+1)_{n-k}(\alpha+\gamma+\frac{3}{2})_n}{(k+\alpha+\beta+1)_k(n+\alpha+\beta+\gamma+\frac{3}{2})_k(n-k+2\gamma+1)_{n-k}(n+k+\alpha+\beta+2\gamma+2)_{n-k}}. \tag{7.3}$$

Remark. The relation

$$p_{n,k}^{\alpha,\beta,\gamma}(-2,1) = (-1)^{n-k} p_{n,k}^{\beta,\alpha,\gamma}(2,1) \quad \text{(equation (10.1))}$$

immediately gives the value of $p_{n,k}^{\alpha,\beta,\gamma}(-2,1)$.

8. The coefficients in the power series of $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. For the coefficients $a_{ij}(n,k,\alpha,\beta,\gamma)$ in the expansion

$$p_{n,k}^{\alpha,\beta,\gamma}(u,v) = \sum_{(i,j)\leqq(n,k)} a_{i,j}(n,k,\alpha,\beta,\gamma)u^{i-j}v^j \tag{8.1}$$

the following theorem holds.

THEOREM 8.1. *$a_{i,j}(n,k,\alpha,\beta,\gamma) = 0$ if $i+j > n+k$ or $i > n$.*

At this point it is useful to define the following partial ordering for $\mathcal{N} = \{(n,k) | n \geqq k \geqq 0,\ n, k \in \mathbb{N}\}$:

$$(i,j) < (n,k) \quad \text{iff } i \leqq n \quad \text{and} \quad i+j \leqq n+k. \tag{8.2}$$

Thus

$$(i, j) \prec (n, k) \Leftrightarrow ((i, j) \leqq (n, k) \wedge i + j \leqq n + k).$$

Theorem 8.1 is equivalent to

$$p_{n,k}^{\alpha,\beta,\gamma}(u, v) = \sum_{(i,j) \prec (n,k)} a_{i,j}(n, k, \alpha, \beta, \gamma) u^{i-j} v^j. \tag{8.3}$$

Proof of Theorem 8.1. The second statement is a consequence of the definition of $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$, because if $i > n$, then $(i, j) > (n, k)$. The first statement is trivially true for the polynomials $p_{n,n}^{\alpha,\beta,\gamma}(u, v)$, because in that case, $i + j > n + n$ implies $i > n$. It is clear from (4.7) that

$$E_+^{\alpha,\beta,\gamma} u^{m-l} v^l = \sum_{(i,j) \prec (m+1,l)} c_{i,j} u^{i-j} v^j,$$

for certain constants $c_{i,j}$. By repeated application of the operators E_+ to $p_{n,n}^{\alpha,\beta,\gamma}(u, v)$ and by using (4.10) the theorem follows. □

Corollaries of Theorem 8.1 are the next two theorems.

THEOREM 8.2. *Let*

(i) $$p(u, v) = \sum_{(m,l) \prec (n,k)} c_{m,l} u^{m-l} v^l,$$

for certain constants $c_{m,l}$, *with* $c_{n,k} = 1$, *and*

(ii) $$\iint_R p(u, v) u^{m-l} v^l \mu^{\alpha,\beta,\gamma}(u, v)\, du\, dv = 0 \quad \textit{if } (m, l) \precneqq (n, k)$$

Then

$$p(u, v) = p_{n,k}^{\alpha,\beta,\gamma}(u, v).$$

THEOREM 8.3. *Let*

(i) $$p(u, v) = \sum_{(m,l) \prec (n,k)} c_{m,l} u^{m-l} v^l,$$

for certain constants $c_{m,l}$, *with* $c_{n,k} = 1$, *and*

(ii) $$D_1^{\alpha,\beta,\gamma} p(u, v) = \lambda p(u, v) \quad \textit{for some } \lambda \in \mathbb{R}.$$

Then

$$p(u, v) = p_{n,k}^{\alpha,\beta,\gamma}(u, v)$$

and

$$\lambda = -[n(n + \alpha + \beta + 2\gamma + 2) + k(k + \alpha + \beta + 1)].$$

Proof of Theorem 8.2. From (i) it follows that $p(u, v)$ can be uniquely expressed as

$$p(u, v) = \sum_{(m,l) \prec (n,k)} c'_{m,l} p_{m,l}^{\alpha,\beta,\gamma}(u, v) \quad \text{with } c'_{n,k} = 1.$$

Then (ii) yields

$$c'_{m,l} = (h_{m,l}^{\alpha,\beta,\gamma})^{-1} \iint_R p(u,v) p_{m,l}^{\alpha,\beta,\gamma}(u,v) \mu^{\alpha,\beta,\gamma}(u,v)\, du\, dv = 0 \quad \text{if } (m,l) \underset{\neq}{<} (n,k).$$

This proves the theorem. □

For the proof of Theorem 8.3 we need the following lemma.

LEMMA 8.1. *If* $(m, l) \underset{\neq}{<} (n, k)$, *then* $\lambda_{m,l} \neq \lambda_{n,k}$, *with*

$$\lambda_{m,l} = -[m(m+\alpha+\beta+2\gamma+2) + l(l+\alpha+\beta+1)].$$

Proof. The parameters α, β, γ satisfy $\alpha, \beta, \gamma > -1$, $\alpha+\gamma+\frac{3}{2}$, $\beta+\gamma+\frac{3}{2} > 0$. Suppose that $(m, l) \underset{\neq}{<} (n, k)$ and $\lambda_{m,l} = \lambda_{n,k}$. Then

$$(n-m)(n+m+\alpha+\beta+2\gamma+2) = (l-k)(l+k+\alpha+\beta+1)$$

and the factors $n+m+\alpha+\beta+2\gamma+2$ and $l+k+\alpha+\beta+1$ are positive. Hence $n-m>0$ and $l-k>0$. Observe that $n+m+\alpha+\beta+2\gamma+2 \geqq 2l+1+\alpha+\beta+2\gamma+2 > 2l+\alpha+\beta+1 \geqq l+k+\alpha+\beta+1$. Thus $n-m<l-k$, contradicting the hypothesis $(m, l) < (n, k)$. □

Proof of Theorem 8.3. From (i) it follows that $p(u, v)$ can be uniquely expressed as

$$p(u,v) = \sum_{(m,l)<(n,k)} c'_{m,l} p_{m,l}^{\alpha,\beta,\gamma}(u,v) \quad \text{with } c'_{n,k} = 1.$$

Then (ii) yields

$$\sum_{(m,l)<(n,k)} \lambda c'_{m,l} p_{m,l}^{\alpha,\beta,\gamma}(u,v) = \sum_{(m,l)<(n,k)} \lambda_{m,l} c'_{m,l} p_{m,l}^{\alpha,\beta,\gamma}(u,v).$$

From $c'_{n,k} = 1$ it follows that

$$\lambda = \lambda_{n,k} = -[n(n+\alpha+\beta+2\gamma+2) + (k+\alpha+\beta+1)],$$

and from $\lambda_{n,k} c'_{m,l} = \lambda_{m,l} c'_{m,l}$ and Lemma 8.1 it follows that $c'_{m,l} = 0$ for $(m, l) \underset{\neq}{<} (n, k)$. □

Application of $D_1^{\alpha,\beta,\gamma}$ to (8.3) and comparison of the coefficients of equal powers of u and v give the following explicit values for some of the coefficients $a_{i,j}(n, k, \alpha, \beta, \gamma)$ in (8.3), which will be used in § 9 for the computation of the coefficients in the recurrence relations:

(8.4a) $\quad a_{n,k}(n, k, \alpha, \beta, \gamma) = 1,$

(8.4b) $\quad a_{n,k-1}(n, k, \alpha, \beta, \gamma) = -(\beta-\alpha)k/(2k+\alpha+\beta),$

(8.4c) $\quad a_{n,k-2}(n, k, \alpha, \beta, \gamma) = -\frac{1}{2}k(k-1) \cdot \{1-(\beta-\alpha)^2/(2k+\alpha+\beta)\}/(2k+\alpha+\beta-1),$

(8.4d) $\quad a_{n-1,k+1}(n, k, \alpha, \beta, \gamma) = -(n-k)(n-k-1)/(n-k+\gamma-\frac{1}{2}),$

$$a_{n-1,k}(n,k,\alpha,\beta,\gamma)=\frac{(\beta-\alpha)(n-k)}{(2n+\alpha+\beta+2\gamma+1)}\left\{\frac{2k(n-k+1)}{2k+\alpha+\beta}+\frac{(n-k-1)(k+1)}{n-k+\gamma-\frac{1}{2}}-2\right\}, \tag{8.4e}$$

$$\begin{aligned}&-2(n+k+\alpha+\beta+\gamma+\tfrac{1}{2})a_{n-1,k-1}(n,k,\alpha,\beta,\gamma)\\ &=(\beta-\alpha)ka_{n-1,k}+k(k+1)a_{n-1,k+1}+2(n-k+2)\\ &\quad\cdot(n-k+1)a_{n,k-2}+2(\beta-\alpha)(n-k+1)a_{n,k-1}\\ &\quad+2(n-2k-\gamma+\tfrac{1}{2})k,\end{aligned} \tag{8.4f}$$

$a_{n,k-3}(n,k,\alpha,\beta,\gamma)$ and $a_{n,k-4}(n,k,\alpha,\beta,\gamma)$ do not depend on n and γ.

9. The recurrence relations. For a further analysis of the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$, it is useful to have formulas for the series expansions of $up_{n,k}^{\alpha,\beta,\gamma}(u,v)$ and $vp_{n,k}^{\alpha,\beta,\gamma}(u,v)$ in terms of $p_{i,j}^{\alpha,\beta,\gamma}(u,v)$. These formulas give $p_{n+1,k}^{\alpha,\beta,\gamma}(u,v)$ and $p_{n+1,k+1}^{\alpha,\beta,\gamma}(u,v)$ as linear combinations of lower degree polynomials.

Case I. Expansion of $up_{n,k}^{\alpha,\beta,\gamma}(u,v)$. Consider the following equality:

$$\begin{aligned}up_{n,k}^{\alpha,\beta,\gamma}(u,v)&=\sum_{(m,l)<(n,k)}a_{m,l}u^{m-l+1}v^{l}\\ &=\sum_{(m,l)<(n+1,k)}a'_{m,l}u^{m-l}v^{l}\\ &=\sum_{(m,l)<(n+1,k)}b_{m,l}(n,k,\alpha,\beta,\gamma)p_{m,l}^{\alpha,\beta,\gamma}(u,v),\end{aligned} \tag{9.1}$$

with

$$b_{m,l}(n,k,\alpha,\beta,\gamma)=\{h_{m,l}^{\alpha,\beta,\gamma}\}^{-1}\iint_R up_{n,k}^{\alpha,\beta,\gamma}(u,v)p_{m,l}^{\alpha,\beta,\gamma}(u,v)\mu^{\alpha,\beta,\gamma}(u,v)\,du\,dv. \tag{9.2}$$

From symmetry it follows that

$$b_{m,l}(n,k,\alpha,\beta,\gamma)=h_{n,k}^{\alpha,\beta,\gamma}\{h_{m,l}^{\alpha,\beta,\gamma}\}^{-1}b_{n,k}(m,l,\alpha,\beta,\gamma). \tag{9.3}$$

Hence $b_{m,l}(n,k,\alpha,\beta,\gamma)\neq 0$ only if $(m,l)\succ(n-1,k)$. And so the summation in (9.1) at most runs through $(m,l)\in\{(n+1,k),\ (n+1,k-1),\ (n+1,k-2),\ (n,k+1),(n,k),(n,k-1),(n-1,k+2),(n-1,k+1),(n-1,k)\}$. The coefficients can be computed by means of (8.4), (9.1) and (9.3). The coefficients $b_{n+1,k-1}$, $b_{n-1,k+1}$, $b_{n+1,k-2}$ and $b_{n-1,k+2}$ turn out to be zero.

For the five remaining coefficients in (9.1) we obtain

$$b_{n+1,k}(n,k,\alpha,\beta,\gamma)=1, \tag{9.4a}$$

$$\begin{aligned}&b_{n-1,k}(n,k,\alpha,\beta,\gamma)\\ &=\frac{4(n+\gamma+\frac{1}{2})(n+\alpha+\gamma+\frac{1}{2})(n+\beta+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+\frac{1}{2})}{(2n+\alpha+\beta+2\gamma)_3(2n+\alpha+\beta+2\gamma+1)}\\ &\quad\cdot\frac{(n-k)(n-k+2\gamma)(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+1)}{(n-k+\gamma-\frac{1}{2})(n-k+\gamma+\frac{1}{2})(n+k+\alpha+\beta+\gamma+\frac{1}{2})(n+k+\alpha+\beta+\gamma+\frac{3}{2})},\end{aligned} \tag{9.4b}$$

$$b_{n,k+1}(n, k, \alpha, \beta, \gamma) = \frac{(n-k)(n-k+2\gamma)}{(n-k+\gamma-\frac{1}{2})(n-k+\gamma+\frac{1}{2})}, \tag{9.4c}$$

$$\begin{aligned} &b_{n,k-1}(n, k, \alpha, \beta, \gamma) \\ &= \frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+1)}{(2k+\alpha+\beta-1)_3(2k+\alpha+\beta)(n+k+\alpha+\beta+\gamma+\frac{1}{2})(n+k+\alpha+\beta+\gamma+\frac{3}{2})}, \end{aligned} \tag{9.4d}$$

$$\begin{aligned} b_{n,k}(n, k, \alpha, \beta, \gamma) &= (\beta-\alpha)(\alpha+\beta) \\ &\cdot \left\{ \frac{1}{(2n+\alpha+\beta+2\gamma+1)(2n+\alpha+\beta+2\gamma+3)} \right. \\ &+ \left. \frac{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4\gamma+2)}{(2n+\alpha+\beta+2\gamma+1)(2n+\alpha+\beta+2\gamma+3)(2k+\alpha+\beta)(2k+\alpha+\beta+2)} \right\}. \end{aligned} \tag{9.4e}$$

If we define

$$p_{n,k}^{\alpha,\beta,\gamma}(u, v) \equiv 0 \quad \text{if } n < k \quad \text{or if} \quad k < 0, \tag{9.5}$$

then the following five-term formula holds for $up_{n,k}^{\alpha,\beta,\gamma}(u, v)$ for all $n \geqq k \geqq 0$:

$$\begin{aligned} up_{n,k}^{\alpha,\beta,\gamma}(u, v) = p_{n+1,k}^{\alpha,\beta,\gamma}(u, v) &+ b_{n,k+1}(n, k, \alpha, \beta, \gamma) p_{n,k+1}^{\alpha,\beta,\gamma}(u, v) \\ &+ b_{n,k}(n, k, \alpha, \beta, \gamma) p_{n,k}^{\alpha,\beta,\gamma}(u, v) \\ &+ b_{n,k-1}(n, k, \alpha, \beta, \gamma) p_{n,k-1}^{\alpha,\beta,\gamma}(u, v) \\ &+ b_{n-1,k}(n, k, \alpha, \beta, \gamma) p_{n-1,k}^{\alpha,\beta,\gamma}(u, v), \end{aligned} \tag{9.6}$$

with $b_{m,l}(n, k, \alpha, \beta, \gamma)$ given by (9.4).

It follows that

$$\begin{aligned} p_{n+1,k}^{\alpha,\beta,\gamma}(u, v) = &-b_{n,k+1}(n, k, \alpha, \beta, \gamma) p_{n,k+1}^{\alpha,\beta,\gamma}(u, v) \\ &+ (u - b_{n,k}(n, k, \alpha, \beta, \gamma)) p_{n,k}^{\alpha,\beta,\gamma}(u, v) \\ &- b_{n,k-1}(n, k, \alpha, \beta, \gamma) p_{n,k-1}^{\alpha,\beta,\gamma}(u, v) \\ &- b_{n-1,k}(n, k, \alpha, \beta, \gamma) p_{n-1,k}^{\alpha,\beta,\gamma}(u, v) \end{aligned} \tag{9.7}$$

if $n \geqq k \geqq 0$.

Remark. By application of the quadratic transformation formulas (10.5) and (10.6) to (9.6), repeated application of D_-^γ and analytic continuation, it can be proved that

(9.8)

$$p_{n,k}^{\alpha,\beta,\gamma}(u, v) = p_{n,k}^{\alpha,\beta+1,\gamma}(u, v) + Ap_{n,k-1}^{\alpha,\beta+1,\gamma}(u, v) + Bp_{n-1,k}^{\alpha,\beta+1,\gamma}(u, v) + Cp_{n-1,k-1}^{\alpha,\beta+1,\gamma}(u, v) \quad \text{if } n \geqq k \geqq 0,$$

with A, B and C being functions of n, k, α, β and γ to be determined from the coefficients of $u^{m-l}v^l$.

Case II. Expansion of $vp_{n,k}^{\alpha,\beta,\gamma}(u, v)$. Consider the equality

$$vp_{n,k}^{\alpha,\beta,\gamma}(u, v) = \sum_{(m,l)<(n+1,k+1)} c_{m,l}(n, k, \alpha, \beta, \gamma) p_{m,l}^{\alpha,\beta,\gamma}(u, v), \tag{9.9}$$

with

$$(9.10)\quad c_{m,l}(n,k,\alpha,\beta,\gamma)=\{h_{m,l}^{\alpha,\beta,\gamma}\}^{-1}\iint_R v p_{n,k}^{\alpha,\beta,\gamma}(u,v)p_{m,l}^{\alpha,\beta,\gamma}(u,v)\mu^{\alpha,\beta,\gamma}(u,v)\,du\,dv.$$

From symmetry it follows that

$$(9.11)\quad c_{m,l}(n,k,\alpha,\beta,\gamma)=h_{n,k}^{\alpha,\beta,\gamma}\{h_{m,l}^{\alpha,\beta,\gamma}\}^{-1}c_{n,k}(m,l,\alpha,\beta,\gamma).$$

Hence $c_{m,l}(n,k,\alpha,\beta,\gamma)\neq 0$ only if $(m,l)>(n-1,k-1)$. And so the summation in (9.9) at most runs through $(m,l)\in\{(n+1,k+1),(n+1,k),(n+1,k-1),(n+1,k-2),(n+1,k-3),(n,k+2),(n,k+1),(n,k),(n,k-1),(n,k-2),(n-1,k+3),(n-1,k+2),(n-1,k+1),(n-1,k),(n-1,k-1)\}$. The coefficients can be computed by means of (8.4), (9.9) and (9.10), and by comparison with the case $\gamma=-\frac{1}{2}$. The coefficients $c_{n+1,k-2}$, $c_{n-1,k+2}$, $c_{n+1,k-3}$, $c_{n-1,k+3}$, $c_{n,k+2}$ and $c_{n,k-2}$ turn out to be zero.

For the nine remaining coefficients in (9.9) we obtain

$$(9.12a)\quad c_{n+1,k+1}(n,k,\alpha,\beta,\gamma)=1,$$

$$(9.12b)\quad \begin{aligned} &c_{n-1,k-1}(n,k,\alpha,\beta,\gamma)\\ &=\frac{2^4(n+\gamma+\frac{1}{2})(n+\alpha+\gamma+\frac{1}{2})(n+\beta+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+\frac{1}{2})}{(2n+\alpha+\beta+2\gamma)_3(2n+\alpha+\beta+2\gamma+1)}\\ &\cdot\frac{k(k+\alpha)(k+\beta)(k+\alpha+\beta)(n+k+\alpha+\beta)_2(n+k+\alpha+\beta+2\gamma)_2}{(2k+\alpha+\beta-1)_3(2k+\alpha+\beta)(n+k+\alpha+\beta+\gamma-\frac{1}{2})_3(n+k+\alpha+\beta+\gamma+\frac{1}{2})},\end{aligned}$$

$$(9.12c)\quad c_{n+1,k}(n,k,\alpha,\beta,\gamma)=\frac{(\beta-\alpha)(\alpha+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)},$$

$$(9.12d)\quad \begin{aligned} &c_{n-1,k}(n,k,\alpha,\beta,\gamma)\\ &=\frac{4(\beta-\alpha)(\alpha+\beta)(n+\gamma+\frac{1}{2})(n+\alpha+\gamma+\frac{1}{2})(n+\beta+\gamma+\frac{1}{2})}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)(2n+\alpha+\beta+2\gamma)_3}\\ &\cdot\frac{(n+\alpha+\beta+\gamma+\frac{1}{2})(n-k)(n-k+2\gamma)(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+1)}{(2n+\alpha+\beta+2\gamma+1)(n-k+\gamma-\frac{1}{2})(n-k+\gamma+\frac{1}{2})(n+k+\alpha+\beta+\gamma+\frac{1}{2})_2},\end{aligned}$$

$$(9.12e)\quad c_{n+1,k-1}(n,k,\alpha,\beta,\gamma)=\frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta-1)_3(2k+\alpha+\beta)},$$

$$(9.12f)\quad \begin{aligned} &c_{n-1,k+1}(n,k,\alpha,\beta,\gamma)\\ &=\frac{4(n+\gamma+\frac{1}{2})(n+\alpha+\gamma+\frac{1}{2})(n+\beta+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+\frac{1}{2})}{(2n+\alpha+\beta+2\gamma)_3(2n+\alpha+\beta+2\gamma+1)}\\ &\cdot\frac{(n-k-1)_2(n-k+2\gamma-1)_2}{(n-k+\gamma-\frac{3}{2})_3(n-k+\gamma-\frac{1}{2})},\end{aligned}$$

$$c_{n,k+1}(n,k,\alpha,\beta,\gamma) = \frac{(\beta-\alpha)(\alpha+\beta)(n-k)(n-k+2\gamma)}{(2n+\alpha+\beta+2\gamma+1)(2n+\alpha+\beta+2\gamma+3)(n-k+\gamma-\frac{1}{2})(n-k+\gamma+\frac{1}{2})}, \tag{9.12g}$$

$$c_{n,k-1}(n,k,\alpha,\beta,\gamma) = \frac{4(\beta-\alpha)(\alpha+\beta)k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2n+\alpha+\beta+2\gamma+1)(2n+\alpha+\beta+2\gamma+3)(2k+\alpha+\beta-1)_3(2k+\alpha+\beta)} \cdot \frac{(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+1)}{(n+k+\alpha+\beta+\gamma+\frac{1}{2})_2}, \tag{9.12h}$$

$$\begin{aligned} c_{n,k}(n,k,\alpha,\beta,\gamma) &= a_{n-1,k-1}(n,k,\alpha,\beta,\gamma) - a_{n,k}(n+1,k+1,\alpha,\beta,\gamma) \\ &\quad - c_{n+1,k}(n,k,\alpha,\beta,\gamma)a_{n,k}(n+1,k,\alpha,\beta,\gamma) - c_{n+1,k-1}(n,k,\alpha,\beta,\gamma) \\ &\quad \cdot a_{n,k}(n+1,k-1,\alpha,\beta,\gamma) - c_{n,k+1}(n,k,\alpha,\beta,\gamma)a_{n,k}(n,k+1,\alpha,\beta,\gamma). \end{aligned} \tag{9.12i}$$

If $\gamma = -\frac{1}{2}$, then $c_{n,k}(n,k,\alpha,\beta,-\frac{1}{2})$ is given by

$$c_{n,k}(n,k,\alpha,\beta,-\tfrac{1}{2}) = \frac{(\alpha+\beta)^2(\beta-\alpha)^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)(2k+\alpha+\beta)(2k+\alpha+\beta+2)}. \tag{9.12i$'$}$$

Formula (9.9) holds, with the coefficients given by (9.12), for all $n \geqq k \geqq 0$, where the convention (9.5) is used again.

Formulas (9.6) and (9.9) together give an algorithm for calculating $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$. If $n \neq k$, then $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ can be expressed in terms of lower degree polynomials by the five-term relation (9.6). If $n = k$, then (9.9) provides a six-term relation which expresses $p_{n,n}^{\alpha,\beta,\gamma}(u, v)$ in terms of lower degree polynomials.

10. A quadratic transformation. The reflection $u \to -u$ maps the region R onto itself and transforms the weight function $\mu^{\alpha,\beta,\gamma}(u, v)$ into $\mu^{\beta,\alpha,\gamma}(u, v)$. Hence, in view of Definition 2.1, the following equality holds:

$$p_{n,k}^{\alpha,\beta,\gamma}(-u, v) = (-1)^{n-k} p_{n,k}^{\beta,\alpha,\gamma}(u, v). \tag{10.1}$$

If $\alpha = \beta$, then (10.1) becomes

$$p_{n,k}^{\alpha,\alpha,\gamma}(-u, v) = (-1)^{n-k} p_{n,k}^{\alpha,\alpha,\gamma}(u, v). \tag{10.2}$$

Formula (10.2) means that if $(n-k)$ is even, then $p_{n,k}^{\alpha,\alpha,\gamma}(u, v)$ is a polynomial in u^2 and v, and if $(n-k)$ is odd, then $u^{-1}p_{n,k}^{\alpha,\alpha,\gamma}(u, v)$ is a polynomial in u^2 and v.

Consider now the new variables

$$u' = 2v, \quad v' = u^2 - 2v - 1. \tag{10.3}$$

These variables satisfy the following properties:

(i) Each polynomial in u^2 and v is a polynomial in u' and v'.

(ii) The half region R given by $R \cap \{(u, v) | u > 0\}$ is mapped onto

$$\tilde{R} = \{(u', v') | (1+u'+v') > 0 \wedge (1-u'+v') > 0 \wedge ((u')^2 - 4v') > 0\}.$$

(iii) If $(u, v) \doteq (2, 1)$, then $(u', v') = (2, 1)$.

(The transformation of variables $u' = -2v$ and $v' = u^2 - 2v - 1$ also satisfies (i) and (ii)).

From (10.3) we obtain

$$\begin{aligned} &u = \sqrt{1+u'+v'}, \quad v = \tfrac{1}{2}u', \\ (10.4) \qquad &(1+u+v)(1-u+v) = \tfrac{1}{4}((u')^2 - 4v'), \qquad u^2 - 4v = 1 - u' + v', \\ &du\, dv = \tfrac{1}{4}(1+u'+v')^{-1/2}\, du'\, dv'. \end{aligned}$$

If $\alpha = \beta$, the following quadratic transformation formulas hold.

THEOREM 10.1.

$$(10.5) \qquad p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v) = 2^{-n+k} p^{\gamma,-1/2,\alpha}_{n,k}(u', v'),$$

and

$$(10.6) \qquad u^{-1} p^{\alpha,\alpha,\gamma}_{n+k+1,n-k}(u, v) = 2^{-n+k} p^{\gamma,+1/2,\alpha}_{n,k}(u', v'),$$

with u' and v' given by (10.3).

Proof.

$$p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v) = \sum_{(i,j) \prec (n+k,n-k)} a_{ij} u^{i-j} v^j.$$

If $(i-j)$ is odd, then $a_{i,j} = 0$, so we can substitute $i - j = 2l$ and $i + j = 2m$. By (8.2), $(i, j) \prec (n+k, n-k)$ iff $(m, l) \prec (n, k)$. Hence

$$\begin{aligned} p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v) &= \sum_{(m,l) \prec (n,k)} a'_{m,l} (u^2)^l v^{m-l} \\ &= \sum_{(m,l) \prec (n,k)} a''_{m,l} (u^2 - 2v - 1)^l (2v)^{m-l} \\ &= \sum_{(m,l) \prec (n,k)} a''_{m,l} (u')^{m-l} (v')^l, \quad \text{with } a''_{n,k} = 2^{-n+k}. \end{aligned}$$

With respect to the orthogonality the following holds:

$$\begin{aligned} 0 &= \iint_R p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v)(2v)^{m-l}(u^2 - 2v - 1)^l \mu^{\alpha,\alpha,\gamma}(u, v)\, du\, dv \\ &= \text{const.} \iint_R p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v)(u')^{m-1}(v')^l \mu^{\gamma,-1/2,\alpha}(u', v')\, du'\, dv', \end{aligned}$$

$$\text{if } (m, l) \underset{\neq}{\prec} (n, k).$$

Application of Theorem 8.2 proves (10.5). A similar proof can be given for (10.6). □

If $(u, v)=(2, 1)$, then $(u', v')=(2, 1)$; hence (10.5) and (10.6) can also be written as

$$\frac{p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v)}{p^{\alpha,\alpha,\gamma}_{n+k,n-k}(2, 1)}=\frac{p^{\gamma,-1/2,\alpha}_{n,k}(2v, u^2-2v-1)}{p^{\gamma,-1/2,\alpha}_{n,k}(2, 1)}, \tag{10.5$'$}$$

and

$$\frac{2u^{-1}p^{\alpha,\alpha,\gamma}_{n+k+1,n-k}(u, v)}{p^{\alpha,\alpha,\gamma}_{n+k+1,n-k}(2, 1)}=\frac{p^{\gamma,+1/2,\alpha}_{n,k}(2v, u^2-2v-1)}{p^{\gamma,+1/2,\alpha}_{n,k}(2, 1)}. \tag{10.6$'$}$$

Formulas (10.5) and (10.6) in combination with (2.15) and (3.2) give an explicit expression for the polynomials $p^{\alpha,\beta,\gamma}_{n,k}(u, v)$ if α and β are $+\frac{1}{2}$ or $-\frac{1}{2}$:

$$\begin{aligned} &p^{-1/2,-1/2,\gamma}_{n,k}(2xy, x^2+y^2-1) \\ &\quad=\begin{cases} 2^{n-k}\{p^{\gamma,\gamma}_{n+k}(x)p^{\gamma,\gamma}_{n-k}(y)+p^{\gamma,\gamma}_{n-k}(x)p^{\gamma,\gamma}_{n+k}(y)\} & \text{if } k>0, \\ 2^n p^{\gamma,\gamma}_n(x)p^{\gamma,\gamma}_n(y) & \text{if } k=0, \end{cases} \end{aligned} \tag{10.7}$$

$$\begin{aligned} &p^{+1/2,-1/2,\gamma}_{n,k}(2xy, x^2+y^2-1) \\ &\quad=2^{n-k}(x-y)^{-1}\{p^{\gamma,\gamma}_{n+k+1}(x)p^{\gamma,\gamma}_{n-k}(y)-p^{\gamma,\gamma}_{n-k}(x)p^{\gamma,\gamma}_{n+k+1}(y)\}, \end{aligned} \tag{10.8}$$

$$\begin{aligned} &p^{-1/2,+1/2,\gamma}_{n,k}(2xy, x^2+y^2-1) \\ &\quad=2^{n-k}(x+y)^{-1}\{p^{\gamma,\gamma}_{n+k+1}(x)p^{\gamma,\gamma}_{n-k}(y)+p^{\gamma,\gamma}_{n-k}(x)p^{\gamma,\gamma}_{n+k+1}(y)\}, \end{aligned} \tag{10.9}$$

$$\begin{aligned} &p^{+1/2,+1/2,\gamma}_{n,k}(2xy, x^2+y^2-1) \\ &\quad=2^{n-k}(x^2-y^2)^{-1}\{p^{\gamma,\gamma}_{n+k+2}(x)p^{\gamma,\gamma}_{n-k}(y)-p^{\gamma,\gamma}_{n-k}(x)p^{\gamma,\gamma}_{n+k+2}(y)\}. \end{aligned} \tag{10.10}$$

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